

# Credit Risk, III.

**Summer School in Financial Mathematics**

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### **III. Hazard process Approach**

1. General case
2.  $(\mathcal{H})$ -Hypothesis
3. Representation theorem
4. Partial information
5. Intensity approach

## General case

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We denote by  $\mathcal{G}_t \stackrel{def}{=} \mathcal{F}_t \vee \mathcal{H}_t$ .

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Let  $X$  be an  $\mathcal{F}_T$ -measurable integrable r.v. Then,

$$\mathbb{E}(X \mathbb{1}_{\tau < \tau} | \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}(X \mathbb{1}_{\{\tau > T\}} | \mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t)} = \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E}(X e^{-\Gamma_\tau} | \mathcal{F}_t).$$

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Let  $h$  be an  $\mathbb{F}$ -predictable process. Then,

$$\mathbb{E}(h_\tau \mathbb{1}_{\tau < T} | \mathcal{G}_t) = h_\tau \mathbb{1}_{\{\tau < t\}} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left( \int_t^T h_u dF_u | \mathcal{F}_t \right).$$

## Martingales

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$$M_t \stackrel{\text{def}}{=} H_t - \int_0^{t \wedge \tau} \frac{dA_u}{1 - F_u}$$

is a  $\mathbb{G}$ -martingale.

Proofs: *The process  $L_t = (1 - H_t)e^{\Gamma_t}$  is a  $\mathbb{G}$ -martingale.*

From the key lemma, for  $t > s$

$$\mathbb{E}(L_t | \mathcal{G}_s) = \mathbb{E}(\mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} | \mathcal{G}_s) = \mathbb{1}_{\{\tau > s\}} e^{\Gamma_s} \mathbb{E}(\mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} | \mathcal{F}_s)$$

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 &= \mathbb{1}_{\{\tau > s\}} e^{\Gamma_s} = L_s
 \end{aligned}$$

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$$M_t \stackrel{def}{=} \int_{]0,t[} dH_u - \int_{]0,t[} (1 - H_u)d\Gamma_u = - \int_{]0,t[} e^{-\Gamma_u}dL_u$$

and is a  $\mathbb{G}$ -martingale since  $L$  is  $\mathbb{G}$ -martingale.

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Let  $s < t$ . We give the proof in two steps, using the Doob-Meyer decomposition of  $F$  as  $F_t = Z_t + A_t$ .

First step: we prove

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In a second step, we prove that, setting  $\Lambda_u = \int_0^u \frac{dA_s}{G_s}$ ,

$$\mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) = \Lambda_{s \wedge \tau} + \mathbb{1}_{s < \tau} \frac{1}{G_s} \mathbb{E}(A_t - A_s | \mathcal{F}_s)$$

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We now use IP formula, using that  $\Lambda$  is bounded variation and continuous

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hence

$$\begin{aligned} \int_s^t \Lambda_u dF_u + \Lambda_t G_t &= -\Lambda_t G_t + \Lambda_s G_s + A_t - A_s + \Lambda_t G_t \\ &= \Lambda_s G_s + A_t - A_s \end{aligned}$$

From

$$\mathbb{E}(\Lambda_{t \wedge \tau} | \mathcal{G}_s) = \Lambda_{s \wedge \tau} \mathbb{1}_{\tau \leq s} + \mathbb{1}_{s < \tau} \frac{1}{G_s} \mathbb{E} \left( \int_s^t \Lambda_u dF_u + \Lambda_t G_t | \mathcal{F}_s \right)$$

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From

$$\mathbb{E}(H_t | \mathcal{G}_s) = H_s + \mathbb{1}_{s < \tau} \frac{1}{G_s} \mathbb{E}(A_t - A_s | \mathcal{F}_s)$$

and

$$\mathbb{E}(A_{t \wedge \tau} | \mathcal{G}_s) = A_{s \wedge \tau} + \mathbb{1}_{s < \tau} \frac{1}{G_s} \mathbb{E}(A_t - A_s | \mathcal{F}_s)$$

we deduce

$$\mathbb{E}(H_t - A_{t \wedge \tau} | \mathcal{G}_s) = H_s - A_{s \wedge \tau}$$

If  $A$  is absolutely continuous w.r.t. the Lebesgue measure, there exists an  $\mathbb{F}$ -adapted process  $\gamma$ , called the intensity such that the process

$$H_t - \int_0^{t \wedge \tau} \gamma_u du = H_t - \int_0^t (1 - H_u) \gamma_u du$$

is a  $\mathbb{G}$ -martingale. The process  $\gamma$  satisfies

$$\gamma_t = \lim_{h \rightarrow 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t + h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)}$$

## Computation in a restricted filtration

Let  $\tilde{\mathbb{F}} \subset \mathbb{F}$  and  $\tilde{\mathcal{G}}_t = \tilde{\mathcal{F}}_t \vee \mathcal{H}_t$ .

From

$$F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$$

we deduce

$$\tilde{F}_t = \mathbb{P}(\tau \leq t | \tilde{\mathcal{F}}_t) = \mathbb{E}(F_t | \tilde{\mathcal{F}}_t)$$

The computation of the intensity is more difficult, the  $\tilde{\mathbb{F}}$ -intensity in the restricted filtration is not the conditional expectation of the  $\mathbb{F}$ -intensity

## Conditional independence

We now introduce the notion of conditional independence. Let  $\mathcal{F}$ ,  $\mathcal{G}$  and  $\mathcal{H}$  be three  $\sigma$ -algebra. The  $\sigma$ -algebra  $\mathcal{G}$  and  $\mathcal{H}$  are said to be conditionally independent with respect to  $\mathcal{F}$  if

$$\mathbb{E}(\xi \eta \mid \mathcal{F}) = \mathbb{E}(\xi \mid \mathcal{F}) \mathbb{E}(\eta \mid \mathcal{F})$$

for any bounded,  $\mathcal{G}$ -measurable random variable  $\xi$  and bounded,  $\mathcal{H}$ -measurable random variable  $\eta$ .

Let  $\mathbb{F}$  and  $\mathbb{G}$  be two filtrations with  $\mathbb{F} \subset \mathbb{G}$ . The  $\sigma$ -fields  $\mathcal{F}_\infty$  and  $\mathcal{G}_t$  are conditionally independent given  $\mathcal{F}_t$  if and only if one of the following conditions holds

(i) For any  $t \in \mathbb{R}_+$  and any bounded,  $\mathcal{F}_\infty$ -measurable random variable  $\xi$ :  
$$\mathbb{E}(\xi | \mathcal{G}_t) = \mathbb{E}(\xi | \mathcal{F}_t).$$

(ii) For any  $t \in \mathbb{R}_+$ , and any bounded,  $\mathcal{G}_t$ -measurable random variable  $\eta$ :  
$$\mathbb{E}(\eta | \mathcal{F}_t) = \mathbb{E}(\eta | \mathcal{F}_\infty).$$

PROOF: (a) Let us assume that  $\mathcal{F}_\infty$  and  $\mathcal{G}_t$  are conditionally independent given  $\mathcal{F}_t$ . Note that  $\mathbb{E}(\xi|\mathcal{F}_t)$  is  $\mathcal{F}_t$  hence  $\mathcal{G}_t$ -measurable. To establish (i), we shall prove that

$$E(\eta_t \mathbb{E}(\xi|\mathcal{G}_t)) = \mathbb{E}(\eta_t \mathbb{E}(\xi|\mathcal{F}_t)) \quad \forall \xi \in \mathcal{F}_\infty \quad \forall \eta_t \in \mathcal{G}_t$$

or equivalently

$$E(\xi \eta_t) = \mathbb{E}(\eta_t \mathbb{E}(\xi|\mathcal{F}_t))$$

The rules of conditional expectation yield to the equalities

$$\begin{aligned} E(\xi \eta_t) &= \mathbb{E}\{\mathbb{E}(\xi \eta_t | \mathcal{F}_t)\} = \mathbb{E}\{\mathbb{E}(\xi | \mathcal{F}_t) \mathbb{E}(\eta_t | \mathcal{F}_t)\} \\ &= \mathbb{E}\{\mathbb{E}[\mathbb{E}(\eta_t \mathbb{E}(\xi | \mathcal{F}_t)) | \mathcal{F}_t]\} = \mathbb{E}\{\mathbb{E}[\eta_t \mathbb{E}(\xi | \mathcal{F}_t)]\} = \mathbb{E}(\eta_t \mathbb{E}(\xi | \mathcal{F}_t)) \end{aligned}$$

(b) Let us prove that (i) implies (ii). Note that  $\mathbb{E}(\eta_t | \mathcal{F}_t)$  is  $\mathcal{F}_t$  hence  $\mathcal{F}_\infty$ -measurable. From the definition of conditional expectation (ii) is equivalent to: for any bounded  $\mathcal{F}_\infty$ -measurable r.v.  $\xi$

$$\mathbb{E}(\xi \mathbb{E}(\eta_t | \mathcal{F}_t)) = \mathbb{E}(\xi \eta_t)$$

From (i)

$$\mathbb{E}(\xi \eta_t) = \mathbb{E}(\eta_t \mathbb{E}(\xi | \mathcal{G}_t)) = \mathbb{E}(\eta_t \mathbb{E}(\xi | \mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(\eta_t | \mathcal{F}_t) \mathbb{E}(\xi | \mathcal{F}_t)) = \mathbb{E}(\xi \mathbb{E}(\eta_t | \mathcal{F}_t))$$

It remains to prove that (ii) implies the conditional independence. Let  $\xi$  be any bounded  $\mathcal{F}_\infty$ -measurable random variable and  $\eta$  any bounded  $G_t$ -measurable r.v. Then

$$\mathbb{E}(\xi\eta|\mathcal{F}_t) = \mathbb{E}(\xi\eta|G_t|\mathcal{F}_t) = \mathbb{E}(\xi\mathbb{E}(\eta|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(\xi|\mathcal{F}_t)\mathbb{E}(\eta|\mathcal{F}_t)$$

Note that (i) is equivalent to : any bounded  $\mathbb{F}$ -martingale is a bounded  $G$ -martingale.

## $(\mathcal{H})$ Hypothesis

## Complete model case

Let  $S$  be a semi-martingale on  $(\Omega, \mathcal{G}, \mathbb{P})$  such that there exists a **unique** probability  $\mathbb{Q}$ , equivalent to  $\mathbb{P}$  on  $\mathcal{F}_T$ , where  $\mathcal{F}_t = \mathcal{F}_t^S = \sigma(S_s, s \leq t)$  such that  $(\tilde{S}_t = S_t R_t, 0 \leq t \leq T)$  is an  $\mathbb{F}^S$ -martingale under the probability  $\mathbb{Q}$ .

We assume that there exists a probability  $\tilde{\mathbb{Q}}$ , equivalent to  $\mathbb{P}$  on  $\mathcal{G}_T$  such that  $(\tilde{S}_t, 0 \leq t \leq T)$  is a  $\mathbb{G}$ -martingale under the probability  $\tilde{\mathbb{Q}}$ .

Then, **any  $(\mathbb{F}, \mathbb{Q})$ -martingale is a  $(\mathbb{G}, \tilde{\mathbb{Q}})$ -martingale** and the restriction of  $\tilde{\mathbb{Q}}$  to  $\mathcal{F}_T$  is equal to  $\mathbb{Q}$ .

## Definition and Properties of immersion

We shall now examine the immersion property (or  $(\mathcal{H})$ -hypothesis) which reads:

**$(\mathcal{H})$  Every  $\mathbb{F}$  square-integrable martingale is a  $\mathbb{G}$  square integrable martingale.**

This hypothesis implies that the  $\mathbb{F}$ -Brownian motion remains a Brownian motion in the enlarged filtration and that every  $\mathbb{F}$ -local martingale is a  $\mathbb{G}$ -local martingale .

Assume that  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ , where  $\mathbb{F}$  is an arbitrary filtration and  $\mathbb{H}$  is generated by the process  $H_t = \mathbb{1}_{\{\tau \leq t\}}$ . Then the following conditions are equivalent to the hypothesis  $(\mathcal{H})$ .

(i) For any  $t \in \mathbb{R}_+$ , we have

$$\mathbb{P}(\tau \leq t \mid \mathcal{F}_t) = \mathbb{P}(\tau \leq t \mid \mathcal{F}_\infty).$$

(ii) For any  $t \in \mathbb{R}_+$ , the  $\sigma$ -fields  $\mathcal{F}_\infty$  and  $\mathcal{G}_t$  are conditionally independent given  $\mathcal{F}_t$  under  $\mathbb{P}$ , that is,

$$\mathbb{E}_{\mathbb{P}}(\xi \eta \mid \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{F}_t) \mathbb{E}_{\mathbb{P}}(\eta \mid \mathcal{F}_t)$$

for any bounded,  $\mathcal{F}_\infty$ -measurable random variable  $\xi$  and bounded,  $\mathcal{G}_t$ -measurable random variable  $\eta$ .

(iii) For any  $t \in \mathbb{R}_+$  and any bounded,  $\mathcal{F}_\infty$ -measurable random variable  $\xi$ :  $\mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \mid \mathcal{F}_t)$ .

## **Change of a probability measure**

Kusuoka shows, by means of a counter-example, that the hypothesis ( $\mathcal{H}$ ) is not invariant with respect to an equivalent change of the underlying probability measure, in general.

Let  $\mathbb{Q}$  be a probability measure equivalent to  $\mathbb{P}$  on  $(\Omega, \mathcal{G}_t)$  for every  $t \in \mathbb{R}_+$ , with the associated Radon-Nikodým density process  $\eta$ . If the **density process  $\eta$  is  $\mathbb{F}$ -adapted** then we have

$$\mathbb{Q}(\tau \leq t \mid \mathcal{F}_t) = \mathbb{P}(\tau \leq t \mid \mathcal{F}_t)$$

for every  $t \in \mathbb{R}_+$ . Hence, the hypothesis  $(\mathcal{H})$  is also valid under  $\mathbb{Q}$  and the  $\mathbb{F}$ -intensities of  $\tau$  under  $\mathbb{Q}$  and under  $\mathbb{P}$  coincide.

PROOF:

$$\begin{aligned}\mathbb{Q}(\tau \leq t \mid \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \mid \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \mid \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \leq t\}} \mid \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_t \mid \mathcal{F}_{\infty})} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(\eta_{\infty} \mathbb{1}_{\{\tau \leq t\}} \mid \mathcal{F}_{\infty})}{\mathbb{E}_{\mathbb{P}}(\eta_{\infty} \mid \mathcal{F}_{\infty})} = \mathbb{P}(\tau \leq t \mid \mathcal{F}_{\infty}).\end{aligned}$$

If  $(\mathcal{H})$  holds,  $F$  is increasing.

The converse is not true. In fact  $F$  is increasing if and only if, for any bounded  $\mathbb{F}$ -martingale  $m$ , one has  $\mathbb{E}(m_\tau) = m_0$  or if and only if for any  $\mathbb{F}$ -martingale  $m$ , the stopped process  $m_{t \wedge \tau}$  is a  $\mathbb{G}$ -martingale.

## Stochastic Barrier

Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where  $\Gamma$  is an arbitrary continuous strictly increasing  $\mathbb{F}$ -adapted process.

There exists a random variable  $\Theta$ , independent of  $\mathcal{F}_\infty$ , with exponential law of parameter 1, such that  $\tau \stackrel{law}{=} \inf\{t \geq 0 : \Gamma_t > \Theta\}$ . In fact  $\Theta \stackrel{def}{=} \Gamma_\tau$ .

PROOF: : Suppose that

$$P(\tau \leq t | \mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

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Let us set  $\Theta \stackrel{def}{=} \Gamma_\tau$ . Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where  $C$  is the right inverse of  $\Gamma$ , so that  $\Gamma_{C_t} = t$ .

PROOF: : Suppose that

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$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where  $C$  is the right inverse of  $\Gamma$ , so that  $\Gamma_{C_t} = t$ . Therefore

$$P(\Theta > u | \mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of  $\Theta$  and its independence of the  $\sigma$ -field  $\mathcal{F}_\infty$ . Furthermore,  $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$ .

## **Representation theorem**

Kusuoka establishes the following representation theorem. Under  $(\mathcal{H})$ , any  $G$ -square integrable martingale admits a representation as the sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale  $M$ .

Suppose that hypothesis  $(\mathcal{H})$  holds under  $\mathbb{P}$  and that any  $\mathbb{F}$ -martingale is continuous. Then, the martingale  $M_t^h = \mathbb{E}_{\mathbb{P}}(h_\tau | \mathcal{G}_t)$ , where  $h$  is an  $\mathbb{F}$ -predictable process such that  $\mathbb{E}(h_\tau) < \infty$ , admits the following decomposition

$$M_t^h = m_0^h + \int_0^{t \wedge \tau} e^{\Gamma_u} dm_{u^h} + \int_{]0, t \wedge \tau]} (h_u - J_u) dM_u,$$

where  $m^h$  is the continuous  $\mathbb{F}$ -martingale

$$m_t^h = \mathbb{E}_{\mathbb{P}} \left( \int_0^\infty h_u dF_u \mid \mathcal{F}_t \right),$$

$J_t = e^{\Gamma_t} (m_t^h - \int_0^t h_u dF_u)$  and  $M$  is the discontinuous  $\mathbb{G}$ -martingale

$$M_t = H_t - \Gamma_{t \wedge \tau}.$$

PROOF: : We know that

$$\begin{aligned} M_t^h &= \mathbb{E}(h_{\tau} | \mathcal{G}_t) \\ &= \mathbb{1}_{\{\tau \leq t\}} h_{\tau} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left( \int_t^{\infty} h_u dF_u \mid \mathcal{F}_t \right) \end{aligned}$$

PROOF: : We know that

$$\begin{aligned}
M_t^h &= \mathbb{E}(h_{\tau} | \mathcal{G}_t) \\
&= \mathbb{1}_{\{\tau \leq t\}} h_{\tau} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \mathbb{E} \left( \int_t^{\infty} h_u dF_u \mid \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{\tau \leq t\}} h_{\tau} + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \left( m_t^h - \int_0^t h_u dF_u \right) \\
&= \int_0^t h_u dH_u + \mathbb{1}_{\{\tau > t\}} J_t.
\end{aligned}$$

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&= \mathbb{1}_{\{\tau \leq t\}} h_\tau + \mathbb{1}_{\{\tau > t\}} e^{\Gamma_t} \left( m_t^h - \int_0^t h_u dF_u \right) \\
&= \int_0^t h_u dH_u + \mathbb{1}_{\{\tau > t\}} J_t.
\end{aligned}$$

From the facts that  $\Gamma$  is an increasing process

$m^h$  a continuous martingale

and using the integration by parts formula, we deduce that

$$dJ_t = e^{\Gamma_t} dm_t^h + (J_t - h_t) \frac{dF_t}{G_t}$$

## Partial information

As pointed out by Jamschidian, “one may wish to apply the general theory perhaps as an intermediate step, to a subfiltration that is not equal to the default-free filtration. In that case,  $\mathbb{F}$  rarely satisfies hypothesis  $(\mathcal{H})$ ”.

## Information at discrete times

Assume that

$$dW_t = V_t(\mu dt + \sigma dW_t), \quad V_0 = v$$

i.e.,  $V_t = ve^{\sigma(W_t + \nu t)}$ . The default time is assumed to be the first hitting time of  $\alpha$  with  $\alpha < v$ , i.e.,

$$\tau = \inf\{t : V_t \leq \alpha\} = \inf\{t : X_t \leq a\}$$

where  $a = \sigma^{-1} \ln(\alpha/v)$ .

Here,  $\mathbb{F}$  is the filtration of the observations of  $V$  at discrete times  $t_1, \dots, t_n$  where  $t_n \leq t < t_{n+1}$ , i.e.,

$$\mathcal{F}_t = \sigma(V_{t_1}, \dots, V_{t_n}, t_i \leq t)$$

The process  $F_t = P(\tau \leq t | \mathcal{F}_t)$   $F$  is **continuous and increasing** in  $[t_i, t_{i+1}[$  but is **not increasing**.

**Lemma 0.1** *The process  $\zeta$  defined by*

$$\zeta_t = \sum_{i, t_i \leq t} \Delta F_{t_i}.$$

*is an  $\mathbb{F}$ -martingale.*

The Doob-Meyer decomposition of  $F$  is

$$F_t = \zeta_t + (F_t - \zeta_t),$$

where  $\zeta$  is an  $\mathbb{F}$ -martingale and  $F_t - \zeta_t$  is a predictable increasing process.

From

$$P(\inf_{s \leq t} X_s > z) = \Phi(\nu, t, z),$$

where

$$\begin{aligned} \Phi(\nu, t, z) &= \mathcal{N}\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} \mathcal{N}\left(\frac{z + \nu t}{\sqrt{t}}\right), & \text{for } z < 0, t > 0, \\ &= 0, & \text{for } z \geq 0, t \geq 0, \\ \Phi(\nu, 0, z) &= 1, & \text{for } z < 0 \end{aligned}$$

we obtain (we skip the parameter  $\nu$  in the definition of  $\Phi$ ) for  $t_1 < t < t_2$  and  $X_{t_1} > a$

$$F_t = 1 - \Phi(t - t_1, a - X_{t_1}) \left[ 1 - \exp\left(-\frac{2a}{t_1} (a - X_{t_1})\right) \right].$$

The case  $X_{t_1} \leq a$  corresponds to default: for  $X_{t_1} \leq a$ ,  $F_t = 1$ .

Another example, related with Parisian stopping times is presented in Çetin et al.

## Delayed information

Guo et al. suggested to start from a structural model with delayed information, i.e. the reference filtration is  $\mathcal{F}_t = \sigma(S_u, u \leq t - \delta)$ . In that case, (H) hypothesis is not satisfied.

## Intensity approach

In the so-called intensity approach, the default time  $\tau$  is a  $\mathbb{G}$ -stopping time. The intensity is defined as any non-negative process  $\lambda$ , such that

$$M_t \stackrel{\text{def}}{=} H_t - \int_0^{t \wedge \tau} \lambda_s ds$$

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$$A_t \mathbb{1}_{t \geq \tau} = A_\tau \mathbb{1}_{t \geq \tau}.$$

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The intensity exists only if  $\tau$  is a totally inaccessible stopping time.

We emphasize that, in that setting the intensity is not well defined after time  $\tau$ , i.e., if  $\lambda$  is an intensity, for any non-negative predictable process  $g$  the process  $\tilde{\lambda}_t = \lambda_t \mathbb{1}_{t \leq \tau} + g_t \mathbb{1}_{\{t > \tau\}}$  is also an intensity.

If the process  $Y_t = \mathbb{E} \left( X \exp \left( - \int_0^T \lambda_u du \right) \middle| \mathcal{G}_t \right)$  is continuous at time  $\tau$ , then, setting  $L_t = \mathbb{1}_{\{t < \tau\}} e^{\Gamma_t}$

$$\mathbb{E}(X \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t) = \mathbb{1}_{\{t < \tau\}} \mathbb{E} \left( X \exp \left( - \int_t^T \lambda_u du \right) \middle| \mathcal{G}_t \right) = L_t Y_t$$

If  $Y$  is not continuous

$$\mathbb{E}(X \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t) = L_t Y_t - \mathbb{E}(\Delta Y_\tau \mathbb{1}_{\tau < T} | \mathcal{G}_t).$$

It can be mentioned that the continuity of the process depends on the choice of  $\lambda$  after time  $\tau$ .

If the process  $Y$  is not continuous, then setting

$$U_t = L_t Y_t = \mathbb{1}_{t < \tau} \exp \left( \int_0^t \lambda_s ds \right) \mathbb{E} \left( X \exp \left( - \int_0^T \lambda_u du \right) \middle| \mathcal{G}_t \right)$$

we have  $U_T = X \mathbb{1}_{\{T < \tau\}}$  and

$$dU_t = L_{t-} dY_t + Y_{t-} dL_t + d[L, Y]_t = L_{t-} dY_t + Y_{t-} dL_t + \Delta L_t \Delta Y_t$$

and

$$\mathbb{E}(U_T | \mathcal{G}_t) = \mathbb{E}(X \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t) = U_t - \mathbb{E}(e^{\Delta \tau} \Delta Y_\tau \mathbb{1}_{\tau < T} | \mathcal{G}_t).$$

Then, for any  $X \in \mathcal{G}_T$  :

$$\mathbb{E}(X \mathbb{1}_{T < \tau} | \mathcal{G}_t) = \mathbb{1}_{\tau > t} \left( e^{\Lambda_t} \mathbb{E}(e^{-\Lambda_T} X | \mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau} \Delta Y_\tau \mathbb{1}_{\tau < T} | \mathcal{G}_t) \right)$$

where  $Y_t = \mathbb{E}(X \exp(-\Lambda_T) | \mathcal{G}_t)$  and  $\Lambda_t = \int_0^t \lambda_u du$

## CDS Price, General case

The ex-dividend price of a credit default swap, with a rate process  $\kappa$  and a protection payment  $\delta_\tau$  at default, equals, for every  $t \in [0, T]$

$$S_t(\kappa) = \mathbb{1}_{\{t < \tau\}} \frac{B_t}{G_t} \mathbb{E} \left( \int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa) du \mid \mathcal{F}_t \right)$$

and thus the cumulative price of a CDS equals, for any  $t \in [0, T]$ ,

$$S_t^{\text{cum}}(\kappa) = \mathbb{1}_{\{t < \tau\}} \frac{B_t}{G_t} \mathbb{E} \left( \int_t^T B_u^{-1} G_u (\delta_u \lambda_u - \kappa) du \mid \mathcal{F}_t \right) + B_t \int_{[0, t]} B_u^{-1} dD_u.$$

The dividend process  $D(\kappa, \delta, T, \tau)$  of a CDS equals

$$D_t = \int_{]0, t \wedge T]} \delta_u dH_u - \kappa \int_{]0, t \wedge T]} (1 - H_u) du = \delta_\tau \mathbb{1}_{\{\tau \leq t\}} - \kappa(t \wedge T \wedge \tau).$$

We now assume that **(H) hypothesis holds** between  $\mathbb{F}$  and  $\mathbb{G}$ , that is  $\mathbb{F}$ -martingales are  $\mathbb{G}$ -martingales. Then,  $F$  is increasing and the process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u du,$$

with  $\gamma_t dt = \frac{dF_t}{G_t}$  is a  $\mathbb{G}$ -martingale.

The dynamics of the ex-dividend price  $S_t(\kappa)$  are

$$d_t S_t(\kappa) = -S_{t-}(\kappa) dM_t + (1 - H_t) B_t G_t^{-1} dm_t + (1 - H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t) dt,$$

where  $m$  is the  $(Q, \mathbb{F})$ -martingale given by

$$m_t = \mathbb{E}_Q \left( \int_0^T B_u^{-1} \delta_u G_u \gamma_u du - \kappa \int_0^T B_u^{-1} G_u du \mid \mathcal{F}_t \right).$$

## Hedging defaultable claims

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

using two CDS with maturities  $T_i$ , rates  $\kappa_i$  and protection payment  $\delta^i$ . We assume  $r = 0$ . Let  $\zeta_t^i$  defined as

$$m_t^i = \mathbb{E}_Q \left( \int_0^T \delta_u^i G_u \gamma_u du - \kappa_i \int_0^T G_u du \mid \mathcal{F}_t \right), \quad dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q \left( - \int_0^\infty Z_u dG_u + G_T X \mid \mathcal{F}_t \right), \quad dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist  $\mathbb{F}$ -predictable processes  $\phi^1, \phi^2$  such that

$$\sum_{i=1}^2 \phi_t^i (\delta_t^i - \tilde{S}_t^i(\kappa_i)) = Z_t - \tilde{y}_t, \quad \sum_{i=1}^2 \phi_t^i \zeta_t^i = \zeta_t,$$

where  $\tilde{y}$  is given by

$$\tilde{y}_t = \frac{1}{G_t} \mathbb{E}_{\mathcal{Q}} \left( - \int_t^T Z_u dG_u + G_T X \mid \mathcal{F}_t \right).$$

Let  $\phi_t^0 = V_t(\phi) - \sum_{i=1}^2 \phi_t^i S_t^i(\kappa_i)$ , where the process  $V(\phi)$  is given by

$$dV_t(\phi) = \sum_{i=1}^2 \phi_t^i (dS_t^i(\kappa_i) + dD_t^i)$$

with the initial condition  $V_0(\phi) = \mathbb{E}_{\mathcal{Q}}(Y)$ . Then the self-financing trading strategy  $\phi = (\phi^0, \phi^1, \phi^2)$  is admissible and it is a replicating strategy for a defaultable claim  $(X, 0, Z, \tau)$ .