

# UP AND DOWN CREDIT RISK

Tomasz R. Bielecki \*  
Department of Applied Mathematics  
Illinois Institute of Technology  
Chicago, IL 60616, USA

Stéphane Crépey\*  
Département de Mathématiques  
Université d'Évry Val d'Essonne  
91025 Évry Cedex, France

Monique Jeanblanc†  
Département de Mathématiques  
Université d'Évry Val d'Essonne  
91025 Évry Cedex, France  
and  
Europlace Institute of Finance

November 7, 2008

## Abstract

This paper discusses the main modeling approaches that have been developed so far for handling portfolio credit derivatives. In particular the so called top, top down and bottom up approaches are considered. We first provide an overview of these approaches. We give some mathematical insights to the fact that information, namely, the choice of a relevant model filtration, is the major modeling issue. In this regard, we examine the notion of thinning that was recently advocated for the purpose of hedging a multi-name derivative by single-name derivatives. We then give a further analysis of the various approaches using simple models, discussing in each case the issue of hedging. Finally we explain by means of numerical simulations (semi-static hedging experiments) why and when the portfolio loss process may not be a sufficient statistics for the purpose of valuation and hedging of portfolio credit risk.

---

\*The research of T.R. Bielecki was supported by NSF Grant 0604789.

†The research of S. Crépey and M. Jeanblanc benefited from the support of the 'Chaire Risque de crédit', Fédération Bancaire Française, and of the Europlace Institute of Finance.

# Contents

<b>1</b>	<b>Introduction</b>	<b>3</b>
1.1	Outline of the paper . . . . .	3
<b>2</b>	<b>Set-up</b>	<b>3</b>
2.1	Ordered versus Unordered Defaults . . . . .	4
2.2	Information is it! . . . . .	5
<b>3</b>	<b>Top, Top-Down and Bottom-Up Approaches: an Overview</b>	<b>5</b>
3.1	Top and Top-Down Approaches . . . . .	5
3.2	Bottom-Up Approaches . . . . .	6
3.2.1	Interacting Particles Approaches . . . . .	6
3.3	Discussion . . . . .	6
<b>4</b>	<b>Thinning</b>	<b>7</b>
4.1	Motivation . . . . .	7
4.1.1	Pricing . . . . .	7
4.1.2	Hedging . . . . .	8
4.1.3	Multi-Name versus Single-Name Credit . . . . .	8
4.2	Ordered Thinning of $\Lambda$ . . . . .	9
4.3	$\mathbb{F}$ -Thinning of $\Lambda$ . . . . .	9
4.4	$\widehat{\mathbb{F}}^i$ -Thinning of $\Lambda$ . . . . .	10
4.4.1	Calibration Issues . . . . .	10
4.5	The case when $\tau_i$ 's are not $\mathbb{F}$ -stopping times . . . . .	11
<b>5</b>	<b>Toy Models</b>	<b>11</b>
5.1	Bottom-Up Approaches . . . . .	12
5.1.1	Pure Bottom-Up Approaches . . . . .	12
5.1.2	Adding a Reference Filtration . . . . .	16
5.2	Top Approaches . . . . .	18
5.2.1	Pure Top Approaches . . . . .	19
5.2.2	Adding a Reference Filtration . . . . .	20
<b>6</b>	<b>Sufficient Statistics</b>	<b>21</b>
6.1	Homogeneous Groups Model . . . . .	22
6.1.1	Pricing in the Homogeneous Groups Model . . . . .	23
6.1.2	Hedging in the Homogeneous Groups Model . . . . .	23
6.2	Numerical Results . . . . .	24
6.2.1	Model Simulation . . . . .	24
6.2.2	Pricing Results . . . . .	24

6.2.3	Hedging Results . . . . .	25
6.2.4	Fully Homogeneous Case . . . . .	29
<b>7</b>	<b>Conclusions</b>	<b>32</b>
<b>A</b>	<b>A glimpse of General Theory</b>	<b>32</b>
A.1	Optional Projections . . . . .	32
A.2	Dual Predictable Projections and Compensators . . . . .	32
A.3	Proof of Proposition 2.1 . . . . .	33

## 1 Introduction

Presently, most if not all credit portfolio derivatives have cash flows that are determined solely by the evolution of the cumulative loss process generated by the underlying portfolio. Thus, as of today, credit portfolio derivatives can be considered as derivatives of the cumulative loss process  $L$ . The consequence of this is that, as of today, most of the models of portfolio credit risk, and related derivatives, focus on eventual modeling of the dynamics of the process  $L$ , or, directly on modeling of the dynamics of the related conditional probabilities, such as

$$\text{Prob}(L \text{ takes some values at future time(s) given present information}).$$

In this paper we shall study various methodologies that have been developed for this purpose, particularly the so called *top*, *tow down* and *bottom up* approaches. In addition, we shall discuss the issue of *hedging* of loss process derivatives, and we shall argue that loss process may not provide a sufficient basis for this, in the sense described later in the paper. In fact, we engage in some in depth study of the *role of information* with regard to valuation and hedging of derivatives written on the loss process.

### 1.1 Outline of the paper

The paper is organized as follows. Section 2 introduces the set-up and emphasizes the role of information. In Section 3 we provide an overview of the main modeling approaches that have been developed so far for handling portfolio credit derivatives. In Section 4 we revisit the notion of *thinning* that was recently advocated for the purpose of hedging a multi-name derivative by single-name derivatives. In Section 5 we give a further analysis of the various approaches using simple models, discussing in each case the issue of *hedging*. In Section 6 we explain by means of numerical simulations (semi-static hedging experiments) why and when the portfolio loss process may not be a *sufficient statistics* for the purpose of valuation and hedging of portfolio credit risk. Conclusions are drawn in Section 7. Finally we gathered in an Appendix definitions and results from the theory of processes that we use repeatedly in this paper, such as, for instance, the definition of the *compensator* (see section A.2) of a non-decreasing adapted process.

## 2 Set-up

Let us first introduce some standing notation:

- If  $X$  is a given process, we denote by  $\mathbb{F}^X$  its natural filtration satisfying usual conditions (perhaps after completion and augmentation);
- By the  $(\mathbb{F})$ -compensator of an  $\mathbb{F}$ -stopping time  $\tau$ , we mean the  $(\mathbb{F})$ -compensator of the (non-decreasing) one point process  $\mathbb{1}_{\tau \leq t}$ ;
- For every  $d, k \in \mathbb{N}$ , we denote  $\mathbb{N}_k = \{0, \dots, k\}$ ,  $\mathbb{N}_k^* = \{1, \dots, k\}$  and  $\mathbb{N}_k^d = \{0, \dots, k\}^d$ .

From now on,  $t$  will denote the present time, and  $T > t$  will denote some future time. Suppose that  $\xi$  represents a future payment at time  $T$ , which will be derived from the evolution of the loss process  $L$  on a credit portfolio, and representing a specific (stylized<sup>1</sup>) credit portfolio derivative claim. We may have at least two tasks at hand:

- to compute the time- $t$  *price* of the claim, given the information that we may have available and we are willing to use at time  $t$ ;
- to *hedge* the claim at time  $t$ . By this, we mean computing hedging sensitivities of the claim with respect to hedging instruments that are available and that we may want to use.

For simplicity we shall assume that we use spot martingale measure, say  $\mathbb{P}$ , for pricing, and that the interest rate is zero. Thus, denoting by  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  a filtration that represents flow of information we use for pricing, and by  $\mathbb{E}$  the expectation relative to  $\mathbb{P}$ , the pricing task amounts to computation of the conditional expectation  $\mathbb{E}(\xi | \mathcal{F}_t)$  ( $\xi$  being assumed  $\mathcal{F}_T$ -measurable and  $\mathbb{P}$ -integrable).

More specifically, on a standard stochastic basis  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ , we consider a (strictly) increasing sequence of stopping times (representing the *ordered default times* of the names of the credit pool)  $t_i$ 's, for  $i \in \mathbb{N}_n^*$ , and we define the ( $\mathbb{F}$ -adapted) portfolio loss process  $L$  by, for  $t \geq 0$ :

$$L_t = \sum_{i=1}^n \mathbb{1}_{t_i \leq t} \quad (1)$$

(assuming for simplicity zero recoveries). So  $L$  is a non-decreasing càdlàg process stopped at time  $t_n$ , taking its values in  $\mathbb{N}_n$ , with jumps of size one ( $L$  is in particular a *point process*, see, e.g., Brémaud [9], Last and Brandt [35]).

We shall then consider (stylized) portfolio loss derivatives with payoff  $\xi = \pi(L_T)$ .

In all the paper we work under the **standing assumption** that the  $t_i$ 's are *totally inaccessible*  $\mathbb{F}$ -stopping times, which is tantamount to assuming that their compensators  $\Lambda^i$ 's are continuous processes (and are therefore stopped at the  $t_i$ 's, cf. Appendix A.2). The compensator  $\Lambda = \sum_{i=1}^n \Lambda_i$  of  $L$  is therefore in turn continuous and stopped at  $t_n$ .

## 2.1 Ordered versus Unordered Defaults

Let  $\tau_i$ ,  $i \in \mathbb{N}_n^*$ , denote an arbitrary collection of (mutually avoiding) random (*not necessarily stopping*) times on  $(\Omega, \mathcal{F}, \mathbb{P})$ , and let  $\tau_{(i)}$ ,  $i \in \mathbb{N}_n^*$ , denote the corresponding ordered sequence, that is  $\tau_{(1)} < \tau_{(2)} < \dots < \tau_{(n)}$ . We denote  $H_t^i = \mathbb{1}_{\tau_i \leq t}$ . Accordingly, we set  $H_t^{(i)} = \mathbb{1}_{\tau_{(i)} \leq t}$ . So, obviously,  $\sum_{i=1}^n H^i = \sum_{i=1}^n H^{(i)}$ .

The following Remark is, of course, elementary,

**Remark 2.1** One has

$$L = \sum_{i=1}^n H^i \quad (2)$$

if and only if

$$t_i = \tau_{(i)}, \quad i \in \mathbb{N}_n^* \quad (3)$$

(in which case the  $\tau_{(i)}$ 's are of course  $\mathbb{F}$ -stopping times).

It is clear that for a given process  $L$  there may be multiple families of random times  $(\tau_i)_{1 \leq i \leq n}$  for which equation (2) is satisfied. For example, in the case where  $n = 2$  and  $t_1$  and  $t_2$  are *constants* with  $t_1 < t_2$ , the particular choice

$$\tau_1 = t_1 \mathbb{1}_{\omega \in \Omega_1} + t_2 \mathbb{1}_{\omega \in \Omega_2}, \quad \tau_2 = t_2 \mathbb{1}_{\omega \in \Omega_1} + t_1 \mathbb{1}_{\omega \in \Omega_2},$$

<sup>1</sup>Of course most credit products are swapped and involve therefore coupon streams, so in general we need to consider a *cumulative ex-dividend cash flow*  $\xi^t$  on the time interval  $(t, T]$ .

where  $\{\Omega_1, \Omega_2\}$  is any measurable partition of  $\Omega$ , gives a family of times  $\tau_i, i = 1, 2$ , such that

$$\tau_{(1)} = \tau_1 \wedge \tau_2 = t_1, \quad \tau_{(2)} = \tau_1 \vee \tau_2 = t_2.$$

From now on we assume (2). The random times  $\tau_i$  can thus be interpreted as the default times of the pool names, and  $H_t^i = \mathbb{1}_{\tau_i \leq t}$  and  $J_t^i = 1 - H_t^i = \mathbb{1}_{t < \tau_i}$  as the *default* and *non-default indicator processes* of name  $i$  ( $\mathbb{F}$ -raw processes, not necessarily  $\mathbb{F}$ -adapted; we stress that any of the random times  $\tau_i$  may or may not be an  $\mathbb{F}$ -stopping time, even assuming (2), though all the  $\tau_{(i)}$ 's are  $\mathbb{F}$ -stopping times in this case).

## 2.2 Information is it!

Various approaches to valuation of derivatives written on credit portfolios differ between themselves depending on what is the content of filtration  $\mathbb{F}$ . Thus, loosely speaking, these approaches differ between themselves depending on what they take (presume) to be sufficient information so to price, and consequently to hedge, credit portfolio derivatives.

The choice of a filtration is of course a crucial modeling issue. In particular the *compensator*  $\Lambda$  of an (adapted) non-decreasing (and bounded, say) process  $K$ , defined as the *predictable non-decreasing Doob-Meyer component* of  $K$  (see section A.2), is an information- (filtration-) dependent quantity. So is therefore the *intensity* process (time-derivative of  $\Lambda$ , assumed to exist) of  $K$  (see, for instance, Proposition 5.7 and Figure 1 in section 5.1 of the paper).

Let thus  $K$  denote an  $\widehat{\mathbb{F}}$ -adapted non-decreasing process where  $\widehat{\mathbb{F}} \subseteq \mathbb{F}$ . Let  $\Lambda$  and  $\widehat{\Lambda}$  denote the  $\mathbb{F}$ -compensator and the  $\widehat{\mathbb{F}}$ -compensator of  $K$ , respectively. The following general result, which is proved in section A.3, establishes the relation between  $\Lambda$  and  $\widehat{\Lambda}$  (and the related  $\mathbb{F}$ - and  $\widehat{\mathbb{F}}$ -intensity processes  $\lambda$  and  $\widehat{\lambda}$ , whenever they exist).

**Proposition 2.1**  $\widehat{\Lambda}$  is the dual predictable projection of  $\Lambda$  on  $\widehat{\mathbb{F}}$ , so

$$\widehat{\Lambda} = \Lambda^p. \quad (4)$$

Moreover, in case  $\widehat{\Lambda}$  and  $\Lambda$  are time-differentiable with related  $\widehat{\mathbb{F}}$ - and  $\mathbb{F}$ -intensity processes  $\widehat{\lambda}$  and  $\lambda$ , then  $\widehat{\lambda}$  is the optional projection of  $\lambda$  on  $\widehat{\mathbb{F}}$ , so

$$\widehat{\lambda} = {}^\circ\lambda. \quad (5)$$

## 3 Top, Top-Down and Bottom-Up Approaches: an Overview

This section provides an overview and a discussion about the so called top, top-down and bottom-up approaches in portfolio credit risk modeling. Some related discussion can also be found in Inglis et al. [31].

### 3.1 Top and Top-Down Approaches

The approach, that we dub the *pure top* approach takes as  $\mathbb{F}$  the filtration generated by the loss process alone. Thus, in the pure top approach we have that  $\mathbb{F} = \mathbb{F}^L$ . Examples are Laurent, Cousin and Fermanian [36], Cont and Minca [11], or (most of) Herbertsson [30].

The approach that we dub the *top* approach takes as  $\mathbb{F}$  the filtration generated by the loss process and by some additional relevant (preferably low dimensional) *auxiliary factor process*, say  $Y$ . Thus, in this case,  $\mathbb{F} = \mathbb{F}^L \vee \mathbb{F}^Y$ . Examples are Bennani [2], Schönbucher [41], Sidenius, Piterbarg and Andersen [42], Arnsdorf and Halperin [1] or Ehlers and Schönbucher [17].

The so-called *top-down* approach starts from *top*, that is, it starts with modeling of evolution of the portfolio loss process subject to information structure  $\mathbb{F}$ . Then, it attempts to ‘decompose’ the dynamics

of the portfolio loss process *down* on the individual constituent names of the portfolio, so to deduce the dynamics of processes  $H^i$  (for the purpose typically of hedging of credit portfolio derivatives by vanilla individual contracts such as default swaps). This decomposition is done by a method of *random thinning* formalized in Giesecke and Goldberg [24] (see also Halperin and Tomecek [27]).

### 3.2 Bottom-Up Approaches

The approach that we dub the *pure bottom-up* approach takes as  $\mathbb{F}$  the filtration generated by the state of the pool process  $H = (H^1, \dots, H^n)$ , i.e.,  $\mathbb{F} = \mathbb{F}^H$  (see, for instance, Herbertsson [29]).

The approach that we dub the *bottom-up* approach takes as  $\mathbb{F}$  the filtration generated by process  $H$  and by an *auxiliary factor process*  $Z$ . Thus, in this case,  $\mathbb{F} = \mathbb{F}^H \vee \mathbb{F}^Z$ . Examples are Duffie and Garleanu [16], Frey and Backhaus [20, 21], Bielecki, Crépey, Jeanblanc and Rutkowski [3], or Bielecki, Vidozzi and Vidozzi [8].

**Remark 3.1** A bottom-up model may be such that

$$\mathbb{F}^H \subseteq \mathbb{F}^Z. \quad (6)$$

For example, take  $n = 2$ , and take three positive random variables:  $\theta_j, j = 1, 2, 3$ . Next, define  $Z_t^j = \mathbb{1}_{\theta_j \leq t}$ ,  $j = 1, 2, 3$ . Finally, let  $\tau_1 = \theta_1 \wedge \theta_3$  and  $\tau_2 = \theta_2 \wedge \theta_3$  (so in this case there may be common jumps of  $H^1$  and  $H^2$ ). We can interpret  $\theta_1$  and  $\theta_2$  as ‘idiosyncratic default times,’ and we can interpret  $\theta_3$  as a ‘systemic default time.’ Letting  $Z = (Z^j)_{1 \leq j \leq 3}$ , we see that (6) holds.

#### 3.2.1 Interacting Particles Approaches

As an aside to bottom-up approaches, let us mention the interacting particles approaches (see Liggett [37] and [38] for a general reference, and Giesecke and Weber [25] or Frey and Backhaus [22] for applications to portfolio credit derivatives). Experience seems to show however that interacting particle models are not appropriate for risk management of portfolio credit derivatives. We can see two reasons for this:

- Firstly, interacting particle models ultimately rely on *homogeneity* assumptions which are obviously not satisfied in the case of credit portfolios, in general. Attempts to turn round this shortcoming by considering sub-group of homogeneous obligors face the difficulty that there is no way to determine such groups in a manner which would be consistent across time; for instance, economic sectors do not define groups of obligors which would be homogeneous in terms of credit risk, whereas homogeneous groups which would be defined by tranching the range of CDS spreads would vary over time (note however that a homogeneous groups set-up will be fruitfully used for numerical illustration purposes in Section 6);
- Secondly, the kind of contagion typically embedded in interacting particle systems (*nearest neighbor interaction* as of Liggett [37]) is not appropriate, neither quantitatively (not enough contagion and frailty) nor qualitatively, for portfolio credit derivatives management.

It is possible that interacting particle approaches might be of interest for *large portfolio credit value at risk assessment* (rather than credit derivatives management), however (see Dai Pra, Runggaldier, Sartori and Tolotti [13]).

**Remark 3.2** On a different note, interacting particles approaches also lead to generic importance sampling techniques that can fruitfully be applied to simulation in the context of dynamic Markovian models of portfolio credit risk (see Crépey and Carmona [12]).

### 3.3 Discussion

To discuss the previous approaches a prerequisite is to provide analysis criteria as for what a good credit basket model (or credit portfolio model) should be:

- Firstly, a good model should of course contain *the right inputs*, namely the inputs with respect to which the trader wishes to compute sensitivities or *Greeks* (typically sensitivities with respect to index and/or CDS

spreads in the case of CDO tranches);

- Secondly, a good model should be *calibrable* to the market consistently over time, since consistency or *robustness* of calibrated parameters over time effectively means that a model produces the right Greeks (this can be considered as a heuristic principle largely valid in practice: in any class of models achieving consistent calibration to the market, one gets essentially the same Greeks);
- Thirdly, pricing and calibration (the latter is of course the most demanding) should be doable *in real time*.

The *pure top* approach is undoubtedly the best suited for *fast calibration and fast valuation*, as it only refers to a single driver – the loss process itself.

However, it probably produces incorrect pricing results, as it is rather unlikely that financial market evaluates derivatives of the loss process based only on the history of evolution of the loss process alone. Note in particular that loss process is not a traded instrument.

Thus, it seems to be necessary to work with a larger amount of information than the one carried by filtration  $\mathbb{F}^L$  alone. This is quite likely the reason why several versions of the top approach have been developed. Enlarging filtration from  $\mathbb{F}^L$  to  $\mathbb{F}^L \vee \mathbb{F}^Y$  may lead to increased computational complexity, but at the same time it is rather sure to increase accuracy in calculation of important quantities, such as CDO tranche spreads and/or CDO prices.

From the *hedging* perspective both the pure top approach and the top approach appear to be inadequate. Since the loss process is not a traded security, a user of the top approaches is forced to hedge one derivative of the loss process, say  $\xi$ , with another loss process derivative, say  $\chi$ , which is available for (liquid) trading. This may not always be such a good idea since, for one, it is only possible to compute sensitivities of  $\xi$  with respect to  $\chi$  indirectly, via sensitivities of  $\xi$  and  $\chi$  with respect to  $L$  (*factor or correlation hedging*), so that hedging may not be quite precise, or even not possible. Moreover, this kind of hedging may be quite expensive (e.g., hedging a CDO tranche using iTraxx).

Secondly, operating on the top level definitely prohibits computing sensitivities of a loss process derivative with respect to constituents of the portfolio of credits generating the loss process in question. So, for example, when operating just on top level one can't compute sensitivities of CDO tranche prices with respect to prices of the CDS contracts underlying the portfolio. This is of course the problem that led to the idea of the *top-down approach*, that is the idea of *thinning*.

## 4 Thinning

Note that processes  $H^i$  and  $H^{(i)}$  are sub-martingales with respect to any filtration for which they are adapted, as non-decreasing processes, and therefore they can be compensated with respect to any filtration for which they are adapted (see section A.2). *Thinning* refers to the recovery of individual compensators of  $H^{(i)}$  and (in case  $\tau_i$  is an  $\mathbb{F}$ -stopping time, see section 4.5)  $H^i$ , starting from the loss compensator  $\Lambda$  as input data. Since the compensator is an information- (filtration-) dependent quantity, thinning of course depends on the filtration under consideration.

### 4.1 Motivation

A preliminary question regarding thinning is why would one wish to know the individual compensators. The answer depends on one's objectives.

#### 4.1.1 Pricing

Suppose that all one wants to do is to compute the expectation  $\mathbb{E}(\xi | \mathcal{F}_t)$  for  $0 \leq t < T$ , where the integrable random variable  $\xi = \pi(L_T)$  represents the stylized payoff of a portfolio loss derivative. In general, this is not an easy task. Sometimes, exact formulas may be available for  $\mathbb{E}(\xi | \mathcal{F}_t)$ . But in general, computation of such expectations will need to be done by simulation. Since the value of  $\xi$  does not depend on identities of defaulting names, computing of the expectation  $\mathbb{E}(\xi | \mathcal{F}_t)$  by simulation will only require simulation of the

process  $L$ , which is the same as simulation of the sequence of the  $\tau_{(i)}$ 's. If one additionally makes Markovian assumptions, or conditionally Markovian assumptions (assuming further factors  $Y$ ), about process  $L$  with respect to the filtration  $\mathbb{F}$ , then, in principle, the expectation  $\mathbb{E}(\xi | \mathcal{F}_t)$  can be computed (at least numerically). The point is that for computation of  $\mathbb{E}(\xi | \mathcal{F}_t)$ , one does not really need to know the individual compensators of the  $\tau_i$ 's (which do not even need to be assumed to be  $\mathbb{F}$ -stopping times in this regard). So, with regard to the problem of pricing of derivatives of the loss process, a top model may be fairly adequate. In particular, the filtration  $\mathbb{F}$  may not necessarily contain the pool filtration  $\mathbb{H}$ . Also, the representation  $L = \sum_{i=1}^n H^i$  (cf. (2)) need not be considered at all in this context.

#### 4.1.2 Hedging

But there is a fundamental reason why one may need to know the individual  $\mathbb{F}$ -compensators  $\Lambda^i$ 's of the  $\tau_i$ 's (assuming here the  $\tau_i$ 's are  $\mathbb{F}$  stopping times). Computing the price  $\mathbb{E}(\xi | \mathcal{F}_t)$  is just one task of interest, which of course is important in the context of valuation of derivatives written on credit portfolio. Yet the key task is *hedging*. From the mathematical point of view hedging relies on the derivation of a *martingale representation* of  $\mathbb{E}(\xi | \mathcal{F}_t)$ , which is useful in the context of computing sensitivities of the price of  $\xi$  with respect to changes in prices of (liquid) instruments, such as CDS contracts, corresponding to the credit names composing the credit pool underlying the loss process  $L$ . Sensitivities computed in this way account for both spread risk and jump-to-default risk.

Typically, one will seek a martingale representation in the form

$$\mathbb{E}(\xi | \mathcal{F}_t) = \mathbb{E}\xi + \sum_{i=1}^n \int_0^t \zeta_s^i dM_s^i + \sum_{j=1}^m \int_0^t \eta_s^j dN_s^j, \quad (7)$$

where the  $M^i$ 's are some fundamental martingales associated with the non-decreasing processes  $H^i$ 's, and the  $N^j$ 's are some fundamental martingales associated with all relevant auxiliary factors included in the model. The coefficients  $\zeta^i$ 's and  $\eta^j$ 's can, in principle, be computed given a particular model specification; now, for the practical computation of the  $\zeta^i$ 's and  $\eta^j$ 's, but also for the very definition of the  $M_i$ 's and  $N_j$ 's, one will typically need to know the compensators  $\Lambda^i$ 's (see Section 5 for illustrative examples).

#### 4.1.3 Multi-Name versus Single-Name Credit

Assume that  $\mathbb{F}$  can be decomposed as  $\tilde{\mathbb{F}} \vee \mathbb{H}$ , where  $\mathbb{H} = (\mathcal{H}_t)_{t \geq 0}$  is the filtration generated by the default indicator process  $H_t = \mathbb{1}_{t \geq \tau}$  of an  $\mathbb{F}$ -stopping time  $\tau$  (which will be taken below as one of the  $\tau_i$ 's, assumed to be an  $\mathbb{F}$ -stopping time), and  $\tilde{\mathbb{F}}$  is a reference filtration. Let further  $\xi$  stand for an  $\mathcal{F}_T$  measurable, integrable random variable. In problems of *single-name* credit risk, namely, when  $\tau$  is the only default time involved in the problem at hand, the knowledge of the compensator is very helpful in computing quantities like  $\mathbb{E}(J_T \xi | \mathcal{F}_t)$  (where  $J = 1 - H$ ).

To understand why (see, e.g., [6]), let us first denote  $G_t = \mathbb{P}(\tau > t | \tilde{\mathcal{F}}_t)$ . Assuming (with little loss of generality) that  $G$  is (strictly) positive, we define the corresponding  $\tilde{\mathbb{F}}$ -hazard process

$$\Gamma_t = -\ln G_t. \quad (8)$$

The importance of the hazard process comes, among other reasons, from the fact that using this process we can provide the following convenient representation:

$$\mathbb{E}(J_T \xi | \mathcal{F}_t) = J_t \mathbb{E}(e^{\Gamma_t - \Gamma_T} \xi | \tilde{\mathcal{F}}_t). \quad (9)$$

Moreover, under our positivity assumption on  $G$ , there exists a uniquely defined  $\tilde{\mathbb{F}}$ -adapted process  $\tilde{\Lambda}$ , called the  $\tilde{\mathbb{F}}$ -martingale hazard process, such that  $\Lambda = \tilde{\Lambda}_{\cdot \wedge \tau}$ , and additional assumptions discussed below imply that

$$\Gamma = \tilde{\Lambda}. \quad (10)$$

So one of the reasons why sometimes one may want to compute the process  $\Lambda$  (or, more precisely,  $\tilde{\Lambda} = \Gamma$ ) is to use it in (9).

Sufficient conditions ensuring (10) are that  $\Gamma$  is a continuous and non-decreasing process, where these requirements are typically met by postulating that  $\tau$  is an  $\tilde{\mathbb{F}}$ -pseudo-stopping time avoiding  $\tilde{\mathbb{F}}$ -stopping times (see, e.g., Coculescu and Nikeghbali [10]).

Recall that the  $\tilde{\mathbb{F}}$ -random time  $\tau$  being an  $\tilde{\mathbb{F}}$ -pseudo-stopping means that  $\tilde{\mathbb{F}}$ -martingales stopped at  $\tau$  are  $\mathbb{F}$ -martingales (see Nikeghbali and Yor [39]). This is of course satisfied when  $\tilde{\mathbb{F}}$ -martingales are  $\mathbb{F}$ -martingales, namely, when *immersion* (also referred to as *the  $(\mathcal{H})$  Hypothesis*) holds between  $\tilde{\mathbb{F}}$  and  $\mathbb{F}$ .

Now, in the case of multi-name credit risk with  $\tau$  given by one of the  $\tau_i$ 's (assumed to be an  $\mathbb{F}$ -stopping time) and  $\tilde{\mathbb{F}} = \tilde{\mathbb{F}}^i$ , the typical situation is that  $\mathbb{H}^j \subset \tilde{\mathbb{F}}^i$  for  $j \neq i$ . In this case immersion typically does *not* hold between  $\tilde{\mathbb{F}}^i$  and  $\mathbb{F}$  (unless we are in degenerate situations like the  $\tau_i$ 's being either ordered or independent, cf. Ehlers and Schönbucher [17]; see also Proposition 5.2 and the comments following it for a concrete illustration of this). Moreover,  $\tau_i$  is typically not an  $\tilde{\mathbb{F}}^i$ -pseudo-stopping time either. So the identity  $\Gamma^i = \tilde{\Lambda}^i$  may not hold and identity (9) (applied to  $\tau_i$ ) may not be exploited, fault of knowing  $\Gamma^i$  (even knowing  $\Lambda^i$  and  $\tilde{\Lambda}^i$ ).

## 4.2 Ordered Thinning of $\Lambda$

Let  $\Lambda^{(i)}$  denote the  $\mathbb{F}$ -compensator of  $\tau_{(i)}$  (recall that the  $\tau_{(i)}$  are  $\mathbb{F}$ -stopping times).

**Proposition 4.1** *We have, for  $t \geq 0$ ,*

$$\Lambda_t^{(i)} = \Lambda_{t \wedge \tau_{(i)}} - \Lambda_{t \wedge \tau_{(i-1)}}. \quad (11)$$

*So in particular  $\Lambda^{(i)} = 0$  on the set  $t \leq \tau_{(i-1)}$ .*

*Proof.* Note first that

$$L_{t \wedge \tau_{(i)}} - \Lambda_{t \wedge \tau_{(i)}} \quad (12)$$

is an  $\mathbb{F}$ -martingale, as it is equal to the  $\mathbb{F}$ -martingale  $L - \Lambda$  (cf. (62)) stopped at the  $\mathbb{F}$ -stopping time  $\tau_{(i)}$ . Taking the difference between expression in (12) for  $i$  and  $i - 1$  yields that  $H_t^{(i)} - \bar{\Lambda}_t^{(i)}$ , with  $\bar{\Lambda}_t^{(i)}$  defined as the RHS of (11), is an  $\mathbb{F}$ -martingale (stopped at  $\tau_{(i)}$ ). Hence (11) follows, due to uniqueness of compensators (recall  $\Lambda$  is continuous, so  $\bar{\Lambda}^{(i)}$  is continuous, hence predictable).  $\square$

Formula (11) represents the ‘ordered thinning’ of  $\Lambda$ . Note that Proposition 4.1 is true regardless of whether the  $\tau_i$ 's are  $\mathbb{F}$ -stopping times or not. This reflects the simple truth that modeling the loss process  $L$  is the same as modeling the ordered sequence of the  $\tau_{(i)}$ 's, no matter what is the informational context of the model otherwise.

In sections 4.3 and 4.4 we only consider the case that each random time  $\tau_i$  is an  $\mathbb{F}$ -stopping time, and we are interested in unordered thinning, that is computing the compensators of  $\tau_i$  relative to various sub-filtrations of  $\mathbb{F}$ , which respect to which  $\tau_i$  is a stopping time, starting from the process  $\Lambda$ .

## 4.3 $\mathbb{F}$ -Thinning of $\Lambda$

Let us first denote by  $\Lambda^i$  the  $\mathbb{F}$ -compensator of  $\tau_i$ . We of course have that

$$\Lambda = \sum_{i=1}^n \Lambda^i. \quad (13)$$

Moreover, the following is true (see also Giesecke and Goldberg [24]).

**Proposition 4.2** *There exists  $\mathbb{F}$ -predictable non-negative processes  $Z^i$ ,  $i \in \mathbb{N}_n^*$ , such that  $Z^1 + Z^2 + \dots + Z^n = 1$  and*

$$\Lambda^i = \int_0^\cdot Z_t^i d\Lambda_t, \quad i \in \mathbb{N}_n^*. \quad (14)$$

*Proof.* In view of (13), existence of  $Z^i (= \frac{d\Lambda^i}{d\Lambda})$  follows from Theorem VI 68, page 130, in Dellacherie and Meyer [15].  $\square$

In the special case where the random times  $\tau_i$ 's constitute an ordered sequence (so  $\tau_i = \tau_{(i)}$ ), then the ordered thinning formula (11) yields that  $Z_t^i = \mathbb{1}_{\tau_{i-1} < t \leq \tau_i}$ .

Proposition 4.2 tells us that, if one starts building a model from top, that is, if one starts building the model by first modeling the  $\mathbb{F}$ -compensator  $\Lambda$  of the loss process  $L$ , then the only way to go down relative to the information carried by  $\mathbb{F}$ , i.e., to obtain  $\mathbb{F}$ -compensators  $\Lambda^i$ , is to do thinning in the sense of equation (14). We shall refer to this as to  $\mathbb{F}$ -*thinning* of  $\Lambda$ .

**Remark 4.1 : When Top-Down becomes to Bottom-Up?** Given that all  $\tau_i$ 's are  $\mathbb{F}$ -stopping times, this thinning is of course equivalent to building the model from the bottom up. That is, modeling processes  $\Lambda$  and  $Z^i$ 's is equivalent to modeling the processes  $\Lambda^i$ 's. The interest of constructing top-down by thinning a model in a filtration containing all the  $\mathbb{H}^i$ 's (so, a bottom-up model, ultimately), might seem questionable. A justification might be that it gives a direct (thus better) control on the loss portfolio dynamics (which come first in the definition of the model, whereas the definition of the marginals' dynamics only come second). It is thus easier to impose realistic dynamics on the loss portfolio by this reverse-engineering construction (see Halperin and Tomecek [27]).

#### 4.4 $\widehat{\mathbb{F}}^i$ -Thinning of $\Lambda$

Now, suppose that  $\widehat{\mathbb{F}}^i$  is some sub-filtration of  $\mathbb{F}$  and that  $\tau_i$  is an  $\widehat{\mathbb{F}}^i$ -stopping time. We want to compute the  $\widehat{\mathbb{F}}^i$ -compensator  $\widehat{\Lambda}^i$  of  $\tau_i$ , starting with  $\Lambda$ .

The first step is to do the  $\mathbb{F}$ -thinning of  $\Lambda$ , that is, to obtain the  $\mathbb{F}$ -compensator  $\Lambda^i$  of  $\tau_i$  (cf. Section 4.3, formula (14)). The second step is to obtain the  $\widehat{\mathbb{F}}^i$ -compensator  $\widehat{\Lambda}^i$  of  $\tau_i$  from  $\Lambda^i$ . The following results follow by application of Proposition 2.1.

**Proposition 4.3**  *$\widehat{\Lambda}^i$  is the dual predictable projection of  $\Lambda^i$  on  $\widehat{\mathbb{F}}^i$ , so*

$$\widehat{\Lambda}^i = (\Lambda^i)^{p_i}. \quad (15)$$

*Moreover, in case  $\widehat{\Lambda}^i$  and  $\Lambda^i$  are time-differentiable with related  $\widehat{\mathbb{F}}^i$ - and  $\mathbb{F}$ - intensity processes  $\widehat{\lambda}^i$  and  $\lambda^i$ , then  $\widehat{\lambda}^i$  is the optional projection of  $\lambda^i$  on  $\widehat{\mathbb{F}}^i$ , so*

$$\widehat{\lambda}^i = o_i(\lambda^i). \quad (16)$$

Note that  $\widehat{\Lambda}^i$  is also the dual predictable projection of  $H^i$  on  $\widehat{\mathbb{F}}^i$ , so  $\widehat{\Lambda}^i = (H^i)^{p_i}$  (see section A.2).

##### 4.4.1 Calibration Issues

The above relations are important regarding the issue of *calibration* of a model to marginal data, one of the key issues in financial modeling.

For example, one may want to calibrate the credit portfolio model to spreads on individual CDS contracts. If the spread on the  $i^{th}$  CDS contract is computed using conditioning with respect to  $\widehat{\mathbb{F}}^i$ , then the  $\widehat{\mathbb{F}}^i$ -compensator  $\widehat{\Lambda}^i$  of  $\tau_i$  will typically be used in calibration, which is tantamount to solving (16) in  $\lambda^i$  with  $\widehat{\lambda}^i$  observed on the market. We refer the reader to the comments following Proposition 5.2 for an illustration in a pure bottom-up situation where the individual CDS spreads reflect only information relevant to the given obligor, so in this case  $\widehat{\mathbb{F}}^i = \mathbb{H}^i$ .

## 4.5 The case when $\tau_i$ 's are not $\mathbb{F}$ -stopping times

In case  $\tau_i$  is not an  $\mathbb{F}$ -stopping time, Giesecke and Goldberg [24] introduce a notion of (we call it *top-down*) intensity of  $\tau_i$ , defined as the time-derivative, assumed to exist, of the dual predictable projection  $(H^i)^p$  of  $H^i$  on  $\mathbb{F}$  (which is but the  $\mathbb{F}$ -predictable non-decreasing Doob–Meyer component of the optional projection  ${}^o(H^i)$  of  $H^i$  on  $\mathbb{F}$ , see, e.g., Brémaud and Yor [7]). However, our opinion is that such a top-down intensity does not make much sense. Indeed the *market* intensity of name  $i$  (intensity of name  $i$  as extracted from the marginal market data on name  $i$ , typically the CDS curve on  $i$ ) corresponds to an intensity in a filtration adapted to  $\tau_i$ , which in particular vanishes after  $\tau_i$  (contrarily to a top-down intensity, unless  $\tau_i$  is an  $\mathbb{F}$ -stopping time). A top-down intensity is thus not represented in the market, and it can therefore not be calibrated (unless, again,  $\tau_i$  is an  $\mathbb{F}$ -stopping time).

## 5 Toy Models

There are two major (and rather natural) messages in the previous section:

- The concept of thinning of the compensator of the loss process so to obtain compensators of the individual default times makes sense only if there is enough information to do so. However, the ‘enough information’ requirement renders the thinning really irrelevant;
- Insufficient information about the pool of credit names does not allow for hedging with respect to individual names.

It is also crucial to emphasize the following observations:

- Representation (7) is key to computing hedging ratios for  $\mathbb{E}(\pi(L_T) | \mathcal{F}_t)$  with respect to instruments derived from the sub-pools of the pool of given  $n$  credits (in particular, with respect to individual instruments, such as CDS contracts);
- Such a representation can’t be obtained if the model is not a bottom-up type model (model with a filtration including the full default filtration  $\mathbb{H}$ ), since in this case the fundamental martingales  $M^i$  are no more available, so they cannot be used in (7).

We shall now illustrate these points in simple set-ups of the pure bottom-up, bottom-up, pure top and top models, generally involving only two random times  $\tau_1$  and  $\tau_2$ , for simplicity of presentation. We shall also discuss the extension of the results to  $n$  random times.

In particular, we are going to provide various ways of shedding a dynamic perspective on two random times  $\tau_1$  and  $\tau_2$ , introducing in each case related  $\mathbb{F}$ -adapted point processes assumed to admit ( $\mathbb{F}$ -predictable, without loss of generality [9, 40])  $\mathbb{F}$ -intensity processes. The dynamic perspective is important if one is interested in hedging credit portfolio derivatives (e.g. CDOs), as well as if one is interested in pricing and hedging derivatives on credit portfolio derivatives (e.g. an option on a CDO tranche).

In every set-up (pure bottom-up, bottom-up, pure top and top) there will be basically two practical ways of ‘dynamizing’  $\tau_1$  and  $\tau_2$ : the *Markovian* approach, mainly, but also, under certain circumstances, a *distributional* approach (also exploited for various purposes in Jiao [34], El Karoui et al. [18] or Jeanblanc and Le Cam [33]).

The Markovian approach relies on the possibility to perform suitable Markovian change of measure, starting from models in which all the ingredients (default related point processes, auxiliary factor process if any) are independent. The model primitives are the *generator* of the related Markovian factor process, or equivalently (at least in pure top or bottom approaches), the  $\mathbb{F}$ -intensities of the point processes at hand.

In the distributional approach the model primitives are related marginal and/or joint *distributions* (in pure top or bottom approaches) or *conditional distributions* (when there are auxiliary factor processes involved), under suitable regularity assumptions on these distributions.

Incidentally, the connection between the two approaches is a delicate issue only partially addressed in this paper (we can only conclude from our examples that the two approaches are non-inclusive).

In order to discuss hedging, we introduce (Borel-measurable and bounded, say) *loss payoff functionals*  $\pi$ ,  $\phi$  and  $\psi$  (the exact nature of which will be made precise later). The basic idea is to hedge a claim with payoff  $\pi(L_T)$  at  $T$  by a claim with payoff  $\phi(L_T)$  at  $T$ , and, possibly also, a claim with payoff  $\psi(L_T)$  at  $T$ .

We denote the price process  $\Pi_t = \mathbb{E}(\pi(L_T) | \mathcal{F}_t)$ , and we introduce likewise  $\Phi$  and  $\Psi$ . The *tracking error* process  $e = e(\zeta)$  of the (self-financing) dynamic hedging strategy  $\zeta = (\zeta^1, \zeta^2)$  based on  $\Phi$  and  $\Psi$  (and the riskless, constant asset) satisfies, for  $t \in [0, T]$

$$de_t = d\Pi_t - \zeta_t d \begin{pmatrix} \Phi_t \\ \Psi_t \end{pmatrix} \quad (17)$$

(and  $e_0 = 0$ ). In particular, restricting oneself to single-instrument hedges, one can min-variance hedge the  $\pi$ -claim by the  $\phi$ -claim and the riskless asset (so  $\zeta^2 = 0$ , here) by using the strategy  $\zeta^1$  in  $\Phi$  defined by, for  $t \in [0, T]$ :

$$\zeta_t^1 = \frac{d\langle \Pi, \Phi \rangle_t}{d\langle \Phi \rangle_t}. \quad (18)$$

Here by *min-variance hedging strategy* we mean the strategy which minimizes the (risk-neutral) variance of the tracking error  $e_T$  among all the self-financing strategies in the  $\phi$ -claim (strategies  $\zeta$  with  $\zeta^2 = 0$ , in the above formalism). Under mild integrability assumptions this strategy corresponds to  $\zeta^1$  given by (17) (see, e.g., [5] page 85), which can often be given a more explicit form in specific set-ups, as we shall see at length in section 5.

Of course, the analysis of the tracking error will depend, in particular, on the information (filtration  $\mathbb{F}$ ) which is used.

## 5.1 Bottom-Up Approaches

In the bottom-up approaches,  $\tau_1$  and  $\tau_2$  are  $\mathbb{F}$ -stopping times, and  $H = (H_1, H_2)$  is therefore an  $\mathbb{F}$ -adapted process. We denote by  $M^1$  and  $M^2$  the  $\mathbb{F}$ -compensated martingales of  $H^1$  and  $H^2$ , so

$$M^1 = H^1 - \int_0^\cdot J_t^1 \tilde{\lambda}_t^1 dt, \quad M^2 = H^2 - \int_0^\cdot J_t^2 \tilde{\lambda}_t^2 dt \quad (19)$$

where  $\tilde{\lambda}^1$  and  $\tilde{\lambda}^2$  are the *pre-default  $\mathbb{F}$ -intensities* of  $\tau^1$  and  $\tau^2$ .

We denote by  $\iota = (i_1, i_2)$  a generic pair in  $\mathbb{N}_1^2$ . Moreover for every  $\iota = (i_1, i_2) \in \mathbb{N}_1^2$  we denote  $j = (j_1, j_2) = (1 - i_1, 1 - i_2)$ .

### 5.1.1 Pure Bottom-Up Approaches

Here we assume that available information is carried by the filtration  $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ . So in this case  $\mathbb{F} = \mathbb{F}^H = \mathbb{H}$ .

◇ **MARKOVIAN APPROACH** To cast the model in a Markovian framework, in the sense that the pair  $H = (H^1, H^2)$  is an  $\mathbb{H}$ -Markov process, one starts with the generator of  $H$ , given in the form of the following matrix,

$$\mathcal{A}_t = \begin{pmatrix} -(\lambda_{0,0}^1(t) + \lambda_{0,0}^2(t)) & \lambda_{0,0}^1(t) & \lambda_{0,0}^2(t) & 0 \\ 0 & -\lambda_{1,0}^2(t) & 0 & \lambda_{1,0}^2(t) \\ 0 & 0 & -\lambda_{0,1}^1(t) & \lambda_{0,1}^1(t) \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

In  $\lambda_i^l(t)$  the superscript  $l$  refers to ‘which obligors defaults’ and the subscript  $\iota = (i_1, i_2)$  to ‘from which (bivariate) state’. The  *$\mathbb{F}$ -intensity functions*  $\lambda_i^1(t)$  and  $\lambda_i^2(t)$ , sometimes also denoted  $\lambda^1(t, \iota)$  and  $\lambda^2(t, \iota)$ , are of the form, with  $(j_1, j_2) = (1 - i_1, 1 - i_2)$ :

$$\lambda_i^1(t) = j_1 \tilde{\lambda}_{i_2}^1(t), \quad \lambda_i^2(t) = j_2 \tilde{\lambda}_{i_1}^2(t) \quad (21)$$

for (non-negative) *pre-default intensity functions*  $\tilde{\lambda}_i^1(t)$  and  $\tilde{\lambda}_i^2(t)$ , or  $\tilde{\lambda}^1(t, i)$  and  $\tilde{\lambda}^2(t, i)$ , such that (cf. (19))

$$\tilde{\lambda}_i^1 = \tilde{\lambda}^1(t, H_t^i), \quad \tilde{\lambda}_i^2 = \tilde{\lambda}^2(t, H_t^i).$$

In other words, the  $\mathbb{H}$ -intensity process  $\lambda_i^l$  of  $\tau_l$  is given by, for  $l = 1, 2$  :

$$\lambda_i^l = J_t^l \tilde{\lambda}^l(t, H_t^{1-l}) = J_t^l J_t^{1-l} \tilde{\lambda}^l(t, 0) + J_t^l H_t^{1-l} \tilde{\lambda}^l(t, 1).$$

Note that, since there are no common jumps between processes  $H^l$ 's, the individual pre-default intensity functions  $\tilde{\lambda}^l$ 's are in one-to-one correspondence with the generator  $\mathcal{A}$ . The  $\tilde{\lambda}^l$ 's can thus be considered as the *model primitives* in this context.

In this paragraph we shall consider hedging of claims that are of the form  $\bar{\pi}(H_T)$  by means of trading of the claims of the form  $\bar{\psi}(H_T)$  and  $\bar{\phi}(H_T)$ . Taking  $\bar{\pi}(i) = \pi(i_1 + i_2)$ , and likewise for  $\bar{\psi}$  and  $\bar{\phi}$ , we can specialize these hedging results to claim depending solely on the loss process.

Since  $H$  is here a Markov process, we have that

$$\Pi_t = u(t, H_t), \quad (22)$$

where  $u(t, i)$  (or  $u_i(t)$ ), for  $t \in [0, T]$  and  $i \in \mathbb{N}_1^2$  is the *pricing function* (system of time-functionals  $u_i$ ). Using the Itô formula in conjunction with the martingale property of  $\Pi$ , the pricing function can then be characterized as the solution to the following *pricing equation* (system of ODEs):

$$(\partial_t + \mathcal{A}_t)u = 0 \text{ on } [0, T], \quad u_i(T) = \bar{\pi}(i). \quad (23)$$

Moreover we also get the following martingale representation, for  $t \in [0, T]$ :

$$\Pi_t = u(t, H_t) = \mathbb{E}(\bar{\pi}(H_T)) + \int_0^t \delta^1 u(s, H_{s-}) dM_s^1 + \int_0^t \delta^2 u(s, H_{s-}) dM_s^2, \quad (24)$$

where the delta functions  $\delta^1 u$  and  $\delta^2 u$  are defined by

$$\delta^1 u_i(t) = u_{1, i_2}(t) - u_{0, i_2}(t), \quad \delta^2 u_i(t) = u_{i_1, 1}(t) - u_{i_1, 0}(t),$$

or in a short-hand notation immediately extendable to the case of  $n$  obligors, for every  $l = 1$  or  $2$  :

$$\delta^l u_i(t) = u_{\bar{i}^l}(t) - u_{\underline{i}^l}(t)$$

where  $\underline{i}^l$  and  $\bar{i}^l$  denote the vector  $i$  with the  $l^{th}$  component replaced by 0 and 1, respectively.

Introducing likewise the pricing functions  $v$  and  $w$  of the  $\bar{\phi}$  and  $\bar{\psi}$ -claims, and plugging all this in (17)–(18), the following hedging results follow.

**Proposition 5.1 (i)** *One can replicate  $\bar{\pi}(H_T)$  at  $T$  by using the strategy  $\zeta = (\zeta^1, \zeta^2)$  based on  $\bar{\phi}$  and  $\bar{\psi}$  (and the riskless, constant asset) defined by, for  $t \in [0, T]$  (under the related matrix-invertibility assumption):*

$$\zeta_t = (\delta^1 u, \delta^2 u) \begin{pmatrix} \delta^1 v & \delta^2 v \\ \delta^1 w & \delta^2 w \end{pmatrix}^{-1} (t, H_{t-}).$$

**(ii)** *Alternatively, it is possible to min-variance hedge the  $\bar{\pi}$ -claim by the  $\bar{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :*

$$\zeta_t^1 = \frac{\lambda^1(\delta^1 u)(\delta^1 v) + \lambda^2(\delta^2 u)(\delta^2 v)}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2} (t, H_{t-}). \quad (25)$$

◇ **DISTRIBUTIONAL APPROACH** Let  $G^i$  (for  $i = 1, 2$ ) and  $G$  denote the marginal and joint *survival functions* of  $\tau_1, \tau_2$ , so for every  $u, v \in \mathbb{R}_+$ ,

$$G^i(u) = \mathbb{P}(\tau_i > u), \quad G(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v).$$

We assume the  $G^i$ 's of class  $\mathcal{C}^1$  and  $G$  of class  $\mathcal{C}^2$ . In particular there is therefore no common jump in the distributional model, consistently with our standing assumptions in this paper.

Let here and henceforth  $\partial_i^j f$  (or simply  $\partial_i f$  in case  $j = 1$ ) denote the partial derivative of order  $j$  of a function  $f$  with respect to its  $i^{\text{th}}$  argument.

**Remark 5.1** Since  $G$  is continuous, there exists a unique (*survival*) *copula function*  $C(\cdot, \cdot)$  such that

$$G(u, v) = C(G^1(u), G^2(v)).$$

So in particular, since  $G, G^1$  and  $G^2$  are differentiable:

$$\partial_1 G(u, v) = \partial_1 G^1(u) \partial_1 C(G^1(u), G^2(v))$$

and likewise for  $\partial_2 G$ .

The following proposition can be established by using standard conditioning techniques. Note that in the present approach, the *model primitive* is  $G$ , which determines the joint distribution of  $\tau_1$  and  $\tau_2$  under  $\mathbb{P}$ . The  $\mathbb{H}$ -intensities of  $\tau_1$  and  $\tau_2$  are then deduced as follows.

**Proposition 5.2 (i)** *Let*

$$\bar{\lambda}_t^1 = -\frac{\partial_1 G^1(t)}{G^1(t)}.$$

*Then the process  $N_t^1 = H_t^1 - \widehat{\Lambda}_t^1$ , with  $\widehat{\Lambda}^1 = \int_0^\cdot J_t^1 \bar{\lambda}_t^1 dt$ , is an  $\mathbb{H}^1$ -martingale.*

**(ii)** *Let*

$$\widetilde{\lambda}_t^1 = -J_t^2 \frac{\partial_1 G(t, t)}{G(t, t)} - H_t^2 \frac{\partial_1 \partial_2 G(t, \tau_2)}{\partial_2 G(t, \tau_2)} = -\frac{\partial_1 \partial_2^{H_t^2} G(t, t \wedge \tau_2)}{\partial_2^{H_t^2} G(t, t \wedge \tau_2)}. \quad (26)$$

*Then the process*

$$M_t^1 = H_t^1 - \Lambda_t^1, \quad (27)$$

*with  $\Lambda^1 = \int_0^\cdot J_t^1 \widetilde{\lambda}_t^1 dt$ , is an  $\mathbb{H}$ -martingale.*

*The (predictable versions of the)  $\mathbb{H}^1$ - and  $\mathbb{H}$ -intensities of  $\tau_1$  are thus given as, respectively,*

$$\widehat{\lambda}_t^1 = J_{t-}^1 \bar{\lambda}_{t-}^1, \quad \lambda_t^1 = J_{t-}^1 \widetilde{\lambda}_{t-}^1. \quad (28)$$

For comparison with the Markovian case discussed in the previous paragraph, observe that the process  $H = (H^1, H^2)$  is not a Markov process here, unless  $\frac{\partial_1 \partial_2 G(t, \tau_2)}{\partial_2 G(t, \tau_2)}$  does not depend on  $\tau_2$  in (26).

Proposition 5.2 shows explicitly how the pre-default intensity of  $\tau_1$  depends on the underlying filtration. In particular, since  $\widehat{\lambda}^1$  and  $\lambda^1$  obviously differ in (28), thus the  $\mathbb{H}^1$ -martingale  $N^1$  of Proposition 5.2(i) is therefore *not* an  $\mathbb{H}$ -martingale, and we recover the fact that immersion (of  $\mathbb{H}^1$  into  $\mathbb{H} = \mathbb{H}^1 \vee \mathbb{H}^2$ , here) typically does *not* hold in multi-name credit (cf. the discussion at the end of section 4.1.3).

Proposition 5.2 is also interesting with respect to the calibration issue risen in Section 4.4.1. We know by application of Proposition 4.3 that

$$(\Lambda^1)^{P^1} = \widehat{\Lambda}^1, \quad \circ_1(\lambda^1) = \widehat{\lambda}^1. \quad (29)$$

This can indeed be verified directly using the forms of  $\lambda^1$  and  $\widehat{\lambda}^1$  derived in Proposition 5.2 and the definitions of the optional and dual predictable projections (see Appendix A).

**Remark 5.2** In a practical situation, relation (29) would be used in the reverse-engineering fashion. By this we mean that (29) would provide a calibration constraint for the model, so that  $(\Lambda^1)^{P_1}$  computed from the model, meets  $\widehat{\Lambda}^1$ , which is extracted from the CDS market on name one (cf. section 4.4.1).

In this paragraph we shall consider hedging of claims that are of the form  $\widehat{\pi}(\tau_1, \tau_2)$  by means of trading of the claims of the form  $\widehat{\phi}(\tau_1, \tau_2)$  and  $\widehat{\psi}(\tau_1, \tau_2)$ . Taking

$$\widehat{\pi}(\tau_1, \tau_2) = \pi(\mathbb{1}_{\tau_1 \leq T} + \mathbb{1}_{\tau_2 \leq T}) = \pi(H_T^1 + H_T^2) = \pi(L_T), \quad (30)$$

and likewise for  $\widehat{\psi}$  and  $\widehat{\phi}$ , we can specialize these hedging results to claim depending solely on the loss process.

Let us introduce the following notation:

$$\begin{aligned} \Pi_{1,1}(s, t) &= \widehat{\pi}(s, t), \quad \Pi_{1,0}(s, t) = -\frac{\int_t^\infty \widehat{\pi}(s, v) \partial_1 \partial_2 G(s, v) dv}{\partial_1 G(s, t)} \\ \Pi_{0,1}(t, s) &= -\frac{\int_t^\infty \widehat{\pi}(u, s) \partial_1 \partial_2 G(u, s) du}{\partial_2 G(t, s)}, \quad \Pi_{0,0}(t, t) = \frac{\int_t^\infty \int_t^\infty \widehat{\pi}(u, v) \partial_1 \partial_2 G(u, v) dudv}{G(t, t)}, \end{aligned}$$

so in a symbolic short-hand notation immediately extendable to the case of  $n$  default times, for every  $\theta = (\theta_1, \theta_2) \in \mathbb{R}_+^2$  and  $\iota = (i_1, i_2) \in \mathbb{N}_1^2$ , with  $j = (j_1, j_2) = (1 - i_1, 1 - i_2)$ :

$$\Pi_\iota(\theta) = \frac{\left( \int_{\theta_1}^\infty \right)^{j_1} \left( \int_{\theta_2}^\infty \right)^{j_2} \widehat{\pi} \partial_1 \partial_2 G (u_1^{j_1} \theta_1^{i_1}, u_2^{j_2} \theta_2^{i_2}) (du_1)^{j_1} (du_2)^{j_2}}{\left( \int_{\theta_1}^\infty \right)^{j_1} \left( \int_{\theta_2}^\infty \right)^{j_2} \partial_1 \partial_2 G (u_1^{j_1} \theta_1^{i_1}, u_2^{j_2} \theta_2^{i_2}) (du_1)^{j_1} (du_2)^{j_2}}. \quad (31)$$

**Lemma 5.3** *The following decomposition holds true, for every  $t \geq 0$ :*

$$\begin{aligned} \Pi_t &= \Pi_{1,1}(\tau_1, \tau_2) H_t^1 H_t^2 + \Pi_{1,0}(\tau_1, t) H_t^1 J_t^2 \\ &\quad + \Pi_{0,1}(t, \tau_2) J_t^1 H_t^2 + \Pi_{0,0}(t, t) J_t^1 J_t^2 \\ &= \Pi_{H_t}(t \wedge \tau_1, t \wedge \tau_2). \end{aligned}$$

We are now ready to derive the following martingale representation.

**Proposition 5.4** *One has,*

$$\Pi_t = \mathbb{E}(\widehat{\pi}(\tau_1, \tau_2)) + \int_0^t \delta^1 \Pi_s dM_s^1 + \int_0^t \delta^2 \Pi_s dM_s^2, \quad (32)$$

where

$$\begin{aligned} \delta^1 \Pi_t &= \Pi_{1, H_t^2}(t, t \wedge \tau_2) - \Pi_{0, H_t^2}(t, t \wedge \tau_2) \\ \delta^2 \Pi_t &= \Pi_{H_t^1, 1}(t \wedge \tau_1, t) - \Pi_{H_t^1, 0}(t \wedge \tau_1, t), \end{aligned}$$

or in short-hand notation, for every  $l = 1, 2$ :

$$\delta^l \Pi_t = \Pi_{\overline{H}_t^l}(\theta_t^l) - \Pi_{\underline{H}_t^l}(\theta_t^l)$$

where  $\underline{H}^l$  and  $\overline{H}^l$  denote the vector  $H$  with the  $l^{\text{th}}$  component replaced by 0 and 1, respectively, and where  $\theta_t^l$  denotes the vector with entries  $t \wedge \tau_k$  for  $k \neq l$  and  $t$  for  $k = l$ .

**Remark 5.3** A direct proof of this result may be derived by using Proposition 5.2 in combination with Lemma 5.3. Note that the existence of a martingale representation of the general form (32) for  $\Pi$  is well known by standard results (see, e.g., Brémaud [9] or Last and Brandt [35]). Under conditions on  $G$  stated above, the expression for the coefficient  $\delta^1 \Pi$  (and likewise for  $\delta^2 \Pi$ ) is natural in view of Lemma 5.3, noting that:

- in case  $\tau_1 < \tau_2$ , the process  $\Pi$  has a jump at time  $\tau_1$  equal to  $\Pi_{1,0}(\tau_1, t) - \Pi_{0,0}(t, t)$  at time  $t = \tau_1$ ;
- in case  $\tau_1 > \tau_2$ , the process  $\Pi$  has a jump at time  $\tau_1$  equal to  $\Pi_{1,1}(\tau_1, \tau_2) - \Pi_{0,1}(t, \tau_2)$  at time  $t = \tau_1$ .

Using similar decompositions for processes  $\Phi$  and  $\Psi$ , the following analog to Proposition 5.1 may then be formulated, by application in the present set-up of (17)–(18).

**Proposition 5.5 (i)** *One can replicate  $\widehat{\pi}(\tau_1, \tau_2)$  at  $T$  by using the hedging strategy  $\zeta$  in the  $\widehat{\phi}$ - and  $\widehat{\psi}$ -claims (and the riskless asset) defined by, for  $t \in [0, T]$  (under the related matrix-invertibility assumption):*

$$\zeta_t = (\delta^1 \Pi_t, \delta^2 \Pi_t) \begin{pmatrix} \delta^1 \Phi_t & \delta^2 \Phi_t \\ \delta^1 \Psi_t & \delta^2 \Psi_t \end{pmatrix}^{-1}.$$

**(ii)** *Alternatively, it is possible to min-variance hedge the  $\widehat{\pi}$ -claim by the  $\widehat{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :*

$$\zeta_t^1 = \frac{\lambda_t^1 (\delta^1 \Pi_t) (\delta^1 \Phi_t) + \lambda_t^2 (\delta^2 \Pi_t) (\delta^2 \Phi_t)}{\lambda_t^1 (\delta^1 \Phi_t)^2 + \lambda_t^2 (\delta^2 \Phi_t)^2}. \quad (33)$$

It is worth stressing that the explicit formulas of this paragraph are derived in a dynamic *non-Markovian* model of credit risk.

### 5.1.2 Adding a Reference Filtration

Let us now assume, more generally, that  $\mathbb{F} = \mathbb{F}^H \vee \mathbb{F}^Z$ , where  $Z$  is a suitable factor process (to be specified later). We call the filtration  $\mathbb{F}^Z$  the *reference filtration*.

◇ **MARKOVIAN APPROACH** To cast the model in a Markovian framework, in the sense that the pair  $(H, Z)$  is an  $\mathbb{F}$  – Markov process, one starts with a generator, which we take of the following form:

$$\mathcal{A} = \mathcal{A}^H + \mathcal{A}^Z,$$

where  $\mathcal{A}^H$  corresponds to  $H$  and  $\mathcal{A}^Z$  corresponds to  $Z$ . Since we assume that there are no common jumps between processes  $H^i$ , so individual pre-default intensities are in one-to-one correspondence with  $\mathcal{A}^H$  (cf. (20)–(21)). To determine  $\mathcal{A}^H$  it thus suffices to specify pre-default individual intensities, say

$$\widetilde{\lambda}_t^1 = \widetilde{\lambda}^1(t, H_t^2, Z_t), \quad \widetilde{\lambda}_t^2 = \widetilde{\lambda}^2(t, H_t^1, Z_t), \quad (34)$$

so for  $l = 1$  or  $2$ :

$$\lambda_t^l = J_t^l \widetilde{\lambda}_t^l =: \lambda^l(t, H_t, Z_t). \quad (35)$$

The construction of a Markovian model  $(H, Z)$  of stopping times  $\tau_1$  and  $\tau_2$  with  $\mathbb{F}$ -intensity processes  $\lambda_t^1$  and  $\lambda_t^2$  satisfying (35) can for example be realized by Markovian change of probability measure, starting from a model with independent default times and factor process (see [3]).

Setting  $\mathcal{Z}_t = (t, H_t, Z_t)$ , one may then define the pricing functions  $u, v, w = u, v, w_\iota(t, z)$  for any claims  $\bar{\pi}, \bar{\phi}, \bar{\psi}(\mathcal{Z}_T)$  (in particular,  $\pi, \phi, \psi(L_T)$ ), characterized as the solutions to the related pricing equations with generator  $\mathcal{A}$ . The delta functions  $\delta^1 u$  and  $\delta^2 u$  are defined as in Section 5.1.1, except for the fact that they involve an additional argument  $z$ .

Moreover we have the following hedging results, by application in the present set-up of (17)–(18) (we refer to [4], Section 3.3, and [32], page 109, for mathematical details behind these results),

**Proposition 5.6 (i)** *Assume  $Z$  satisfies the following  $d$ -dimensional SDE*

$$dZ_t = b(Z_t)dt + \sigma(Z_t)dB_t,$$

for suitable coefficients  $b$  and  $\sigma$  and a  $d$ -dimensional standard  $\mathbb{F}$  – Brownian motion  $B$ . Then, denoting by  $\partial$  the row-gradient of a function with respect to the argument  $z$ , we have:

$$de_t = \left( \delta^1 u - \zeta_t \begin{pmatrix} \delta^1 v \\ \delta^1 w \end{pmatrix} \right) (\mathcal{Z}_{t-}) dM_t^1 + \left( \delta^2 u - \zeta_t \begin{pmatrix} \delta^2 v \\ \delta^2 w \end{pmatrix} \right) (\mathcal{Z}_{t-}) dM_t^2 \\ + \left( \left( \partial u - \zeta_t \begin{pmatrix} \partial v \\ \partial w \end{pmatrix} \right) \sigma \right) (\mathcal{Z}_t) dB_t.$$

In particular one can min-variance hedge the  $\bar{\pi}$ -claim by the  $\bar{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :

$$\zeta_t^1 = \frac{\lambda^1(\delta^1 u)(\delta^1 v) + \lambda^2(\delta^2 u)(\delta^2 v) + (\partial u)\sigma\sigma^\top(\partial v)}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2 + (\partial v)\sigma\sigma^\top(\partial v)} (\mathcal{Z}_{t-}). \quad (36)$$

(ii) Assume  $Z$  is given as a pure jump process with finite state space  $E$  of cardinality  $d$ , jump times disjoint from  $\tau_1$  and  $\tau_2$ , jump intensity vector-process  $\bar{\lambda}(\mathcal{Z}_{t-}, z)_{z \in E}$  and compensated jump vector-martingale  $(M_t(z), z \in E)$ . Then, denoting by  $\Delta u_i(t, z)$  the  $d$ -dimensional row-vector  $(u_i(t, z') - u_i(t, z))_{z' \in E}$  and likewise for  $\Delta v$  and  $\Delta w$ , we have:

$$de_t = \left( \delta^1 u - \zeta_t \begin{pmatrix} \delta^1 v \\ \delta^1 w \end{pmatrix} \right) (\mathcal{Z}_{t-}) dM_t^1 + \left( \delta^2 u - \zeta_t \begin{pmatrix} \delta^2 v \\ \delta^2 w \end{pmatrix} \right) (\mathcal{Z}_{t-}) dM_t^2 \\ + \left( \Delta u - \zeta_t \begin{pmatrix} \Delta v \\ \Delta w \end{pmatrix} \right) (\mathcal{Z}_{t-}) dM_t$$

In particular one can min-variance hedge the  $\bar{\pi}$ -claim by the  $\bar{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :

$$\zeta_t^1 = \frac{\lambda^1(\delta^1 u)(\delta^1 v)(\mathcal{Z}_{t-}) + \lambda^2(\delta^2 u)(\delta^2 v)(\mathcal{Z}_{t-}) + \sum_{z \in E} \bar{\lambda}(\mathcal{Z}_{t-}, z) \Delta u(\mathcal{Z}_{t-}, z) \Delta v(\mathcal{Z}_{t-}, z)}{\lambda^1(\delta^1 v)^2(\mathcal{Z}_{t-}) + \lambda^2(\delta^2 v)^2(\mathcal{Z}_{t-}) + \sum_{z \in E} \bar{\lambda}(\mathcal{Z}_{t-}, z) \Delta v(\mathcal{Z}_{t-}, z) \Delta v(\mathcal{Z}_{t-}, z)}. \quad (37)$$

**Remark 5.4** Of course the auxiliary factor process  $Z$  introduces potentially several additional sources of randomness that need to be hedged. This can be dealt with by taking as a hedging instrument (on top of the bank account) a possibly multidimensional claim  $\phi$ . The results of Proposition 5.6 can then be easily extended to the case of multidimensional claim  $\phi$  by formulating appropriate systems of linear equations.

◇ DISTRIBUTIONAL APPROACH Let  $G_t^i$  (for  $i = 1, 2$ ) and  $G_t$  denote the marginal and joint conditional survival functions of  $\tau_1$  and  $\tau_2$ , so for every  $t, u, v \geq 0$ ,

$$G_t^i(u) = \mathbb{P}(\tau_i > u \mid \mathcal{F}_t^Z), \quad G_t(u, v) = \mathbb{P}(\tau_1 > u, \tau_2 > v \mid \mathcal{F}_t^Z).$$

In particular  $G_0^i$  and  $G_0$  reduces to the (unconditional) marginal and joint survival function  $G$ , for  $\mathcal{F}_0^Z$  trivial (case of a pure bottom-up distributional approach). Assuming the  $G_t^i$ 's of class  $\mathcal{C}^1$  and  $G_t$  of class  $\mathcal{C}^2$  with respect to  $u$  and  $v$ , we may then easily derive formal extensions of the initial times approach of the pure bottom case of Section 5.1.1. Then, we have (see, for instance, Zargari [43]),

**Proposition 5.7** The pre-default  $\mathbb{H}^1 \vee \mathbb{F}^Z$ - and  $\mathbb{H} \vee \mathbb{F}^Z$ - intensities of  $\tau^1$  are given by, respectively:

$$\bar{\lambda}_t^1 = -\frac{\partial_1 G_t^1(t)}{G_t^1(t)}, \quad \tilde{\lambda}_t^1 = -\frac{\partial_1 \partial_2^{H_t^2} G_t(t, t \wedge \tau_2)}{\partial_2^{H_t^2} G_t(t, t \wedge \tau_2)}. \quad (38)$$

Moreover (see [43]) formulas (38) admit the following straightforward generalization to the case of  $n$  default times,  $\tilde{\lambda}_t^1$  in (39) denoting the pre-default  $\mathbb{H} \vee \mathbb{F}$  intensity of  $\tau^1$  (in which  $\mathbb{H}$  stands as usual for the defaults filtration):

$$\tilde{\lambda}_t^1 = -\frac{\partial_1 \partial_2^{H_t^2} \cdots \partial_n^{H_t^n} G_t(t, t \wedge \tau_2, \dots, t \wedge \tau_n)}{\partial_2^{H_t^2} \cdots \partial_n^{H_t^n} G_t(t, t \wedge \tau_2, \dots, t \wedge \tau_n)}. \quad (39)$$

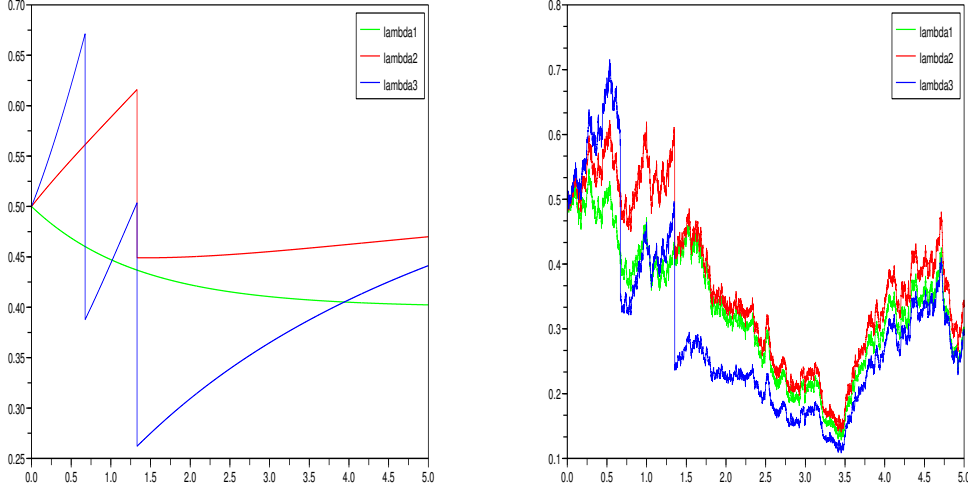


Figure 1: Simulated sample path of the pre-default intensity of  $\tau_1$  with respect to  $\mathbb{H}^1 \vee \mathbb{F}^Z$ ,  $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^Z$  or  $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{H}^3 \vee \mathbb{F}^Z$ , with  $Z$  trivial (constant) in the left pane versus  $Z = W$  in the right pane.

Figure 1 provides an illustration of formula (39) in a simple model with  $n = 3$  stopping times<sup>2</sup>. The figure shows a trajectory over the time interval  $(0, 5yr)$  of the (pre-default) intensity of  $\tau_1$  with respect to  $\mathbb{H}^1 \vee \mathbb{F}^Z$ ,  $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{F}^Z$  and  $\mathbb{H}^1 \vee \mathbb{H}^2 \vee \mathbb{H}^3 \vee \mathbb{F}^Z$  (respectively labelled `lambda1`, `lambda2` and `lambda3` in the Figure), where the  $\mathbb{H}^i$ 's correspond to the filtrations generated by the  $H^i$ 's, and:

- the reference filtration  $\mathbb{F}^Z$  is trivial in the left pane,
- it is given as a (scalar) Brownian filtration  $\mathbb{F}^Z = \mathbb{F}^W$  in the right pane.

We refer the reader to Zargari [43] for every detail about these simulations. On this example we have  $\tau_2 = 1.354$ ,  $\tau_3 = 0.669$  in the case where  $\mathbb{F}^Z$  is trivial and  $\tau_2 = 1.3305$ ,  $\tau_3 = 0.676$  in the case where  $\mathbb{F}^Z = \mathbb{F}^W$  (the same random numbers were used in the two experiments). Observe that:

- `lambda2` and `lambda3` jump at  $\tau_2$  ( $= 1.354$  in the left pane and  $1.3305$  in the right pane),
- only `lambda3` jumps at  $\tau_3$  ( $= 0.669$  in the left pane and  $0.676$  in the right pane), and
- `lambda1` does not jump at all.

Also note the effect of adding a reference filtration (noisy pre-default intensities in the right-pane, versus pre-default intensities ‘deterministic between default times’ in the left pane).

As for hedging, it seems difficult to derive explicit and constructive martingale representations in the set-up of this paragraph (so hedging cannot be implemented either), unless we are in the case of an auxiliary factor process  $Z$  given as a pure jump process with finite state space. In this case it is possible to derive an elementary martingale representation and a suitable analog to Proposition 5.6(ii), valid for payoffs  $\widehat{\pi}, \widehat{\phi}, \widehat{\psi}(\tau_1, \tau_2)$ . The detail is left to the reader.

## 5.2 Top Approaches

We now work with a top filtration  $\mathbb{F}$ . We directly consider  $n$  default times. Since we work with a top filtration  $\mathbb{F}$ , the  $\tau_i$ 's are not  $\mathbb{F}$ -stopping times, as opposed to the *ordered* default times  $t_i$ 's. The loss process is therefore

<sup>2</sup>We thank Behnaz Zargari from the Mathematics Departments at University of Evry, France, and University of Sharif, Iran, for these simulations.

an  $\mathbb{F}$ -adapted, non-decreasing process. We denote by  $M$  its  $\mathbb{F}$ -compensated martingale, so

$$M = L - \int_0^\cdot \lambda_t dt$$

where  $\lambda$  is the (predictable version of the)  $\mathbb{F}$ -intensity, assumed to exist, of  $L$ .

### 5.2.1 Pure Top Approaches

Here  $\mathbb{F} = \mathbb{F}^L$ .

◇ **MARKOVIAN APPROACH** In the Markovian case  $L$  is a pure birth-and-death process (*local intensity process* as of Laurent, Cousin and Fermanian [36] or Cont and Minca [11]), with  $\mathbb{F}$ -intensity  $\lambda_t = \lambda(t, L_t)$ , for a suitable  $\mathbb{F}$ -intensity function  $\lambda(t, i)$  (vanishing for  $i \geq n$ , consistently with the fact that the loss process  $L$  is stopped at level  $n$ ).

In this set-up, we obtain that, for  $t \in [0, T]$

$$\Pi_t = \mathbb{E}(\pi(L_T) | \mathcal{F}_t) = u(t, L_t), \quad (40)$$

where  $u(t, i)$  or  $u_i(t)$  for  $(t, i) \in [0, T] \times \mathbb{N}_n$ , is the *pricing function* (system of time-functionals  $u_i$ ), solution to the related system of backward Kolmogorov differential equations. Moreover we have the following martingale representation, for  $t \in [0, T]$ :

$$\Pi_t = u(t, L_t) = \mathbb{E}(\pi(L_T)) + \int_0^t \delta u(s, L_{s-}) dM_s \quad (41)$$

where the *delta function*  $\delta u$  is defined by, for  $t \in [0, T]$  and  $i \in \mathbb{N}_{n-1}$  :

$$\delta u_i(t) = u_{i+1}(t) - u_i(t). \quad (42)$$

It is rather clear that in this present case it is enough to use just one claim, say  $\phi$  (and the riskless account) so to replicate claim  $\pi$ . So, using the analogous martingale representation for the  $\phi$  claim, the following result follows (in view of (17)),

**Proposition 5.8** *One can replicate  $\pi(L_T)$  at  $T$  by using the strategy  $\zeta$  based on the  $\phi$ -claim (and on the riskless asset) defined by, for  $t \in [0, T]$  (assuming  $\delta v \neq 0$ ):*

$$\zeta_t = \frac{\delta u}{\delta v}(t, L_{t-}).$$

◇ **DISTRIBUTIONAL APPROACH** We denote  $G_t^i(u) = \mathbb{P}(L_u = i | \mathcal{F}_t)$ , for  $i \in \mathbb{N}_n$ .

As an  $\mathbb{F}^L$ -martingale, the process  $G_t^i(u)$  admits a representation of the form

$$G_t^i(u) = G_0^i(u) + \int_0^t \delta G_s^i(u) dM_s$$

for some integrand  $\delta G_s^i(u)$ . We are now ready to write the representation for  $\Pi$  in terms of  $\delta G$ . We thus have, for  $t \in [0, T]$  :

$$\Pi_t = \sum_{0 \leq i \leq n} \pi(i) G_t^i(T) = \mathbb{E}(\pi(L_T)) + \int_0^t \delta \Pi_s dM_s, \quad (43)$$

where we set

$$\delta \Pi_t = \sum_{0 \leq i \leq n} \pi(i) \delta G_t^i(T).$$

Using the analogous representation regarding the  $\phi$  claim, and plugging all these expressions in (17), one gets the following,

**Proposition 5.9** *One can replicate  $\pi$  at  $T$  by using the strategy  $\zeta$  based on the  $\phi$ -claim (and on the riskless asset) defined by, for  $t \in [0, T]$  (assuming  $\delta\Phi_t \neq 0$ ):*

$$\zeta_t = \frac{\delta\Pi_t}{\delta\Phi_t}. \quad (44)$$

To illustrate the feasibility (and the limits) of the approach of this paragraph we need to provide specific examples in which  $G_t^i(u)$  and  $\delta G_t^i(u)$  are computable. For this it is enough that the joint cumulative distribution of the consecutive default times  $t_k$ 's of  $L$  be computable and continuous. Indeed we have, since  $\mathbb{F} = \mathbb{F}^L$ :

$$G_t^i(u) = \mathbb{P}(L_u = i \mid \mathcal{F}_t) = \mathbb{P}(L_u = i \mid \mathcal{F}_t \vee \sigma(L_t)), \quad (45)$$

which on the random time interval  $t \in [t_l, t_{l+1})$  (or, equivalently, on the event  $\{L_t = l\}$ ), is easily seen to coincide with  $\mathbb{P}(t_i \leq u < t_{i+1} \mid t_1, \dots, t_l; t_i \leq t < t_{i+1})$ . Now, by the Bayes rule, the latter quantity is determined by the joint law of the  $t_k$ 's. Moreover, if the cumulative distribution of  $(t_k)_{1 \leq k \leq n}$  is continuous, then  $G_t^i(u)$  is thus continuous on every interval  $[t_l, t_{l+1})$ , and the only possible jumps of  $G_t^i(u)$  occur at the  $t_l$ 's, where they are given by

$$G_{t_l}^i(u) - G_{t_l-}^i(u) = \mathbb{P}(t_i \leq u < t_{i+1} \mid t_1, \dots, t_l) - \mathbb{P}(t_i \leq u < t_{i+1} \mid t_1, \dots, t_{l-1}), \quad (46)$$

which can also be evaluated by the Bayes rule given the joint cumulative distribution of the  $t_k$ 's. So  $\delta G_t^i(u)$  is computable too.

**Example 5.5** We consider two ordered times  $t_1$  and  $t_2$  defined by, given IID unit exponential random variables  $\mathcal{E}_1$  and  $\mathcal{E}_2$ :

$$t_1 = \inf\{t > 0 \mid \int_0^t \mu_1(s) ds > \mathcal{E}_1\}, \quad t_2 = \inf\{t > t_1 \mid \int_{t_1}^t \mu_2(s, t_1) ds > \mathcal{E}_2\}$$

with  $\mu_1(t) = 1$ ,  $\mu_2(t, t_1) = t_1$ . So

$$t_1 = \mathcal{E}_1, \quad t_2 = t_1 + \frac{\mathcal{E}_2}{t_1}.$$

One can show that the related loss process  $L$  is a non-Markovian *Hawkes process* (see Errais, Giesecke and Goldberg [19], Hawkes [28]). Moreover, we have in this case

$$\mathbb{P}(t_1 > u_1, t_2 > u_2) = \mathbb{P}(u_1 < \mathcal{E}_1, u_2 < \mathcal{E}_1 + \frac{\mathcal{E}_2}{\mathcal{E}_1}),$$

which is explicitly given by  $\int_{x>u_1} \int_{y>x(u_2-x)^+} e^{-(x+y)} dx dy$ .  $G_t^i(u)$  and  $\delta G_t^i(u)$  can thus be computed, and so can in turn be  $\delta\Pi$ ,  $\delta\Phi$  and  $\zeta$  in (44).

## 5.2.2 Adding a Reference Filtration

We now assume  $\mathbb{F} = \mathbb{F}^L \vee \mathbb{F}^Y$ , for a suitable factor process  $Y$ .

◇ **MARKOVIAN APPROACH** We suppose that the pair  $(L, Y)$  is an  $\mathbb{F}$ -Markov process with generator  $\mathcal{A}$ , assuming more specifically that the  $\mathbb{F}$ -intensity of  $L$  satisfies

$$\lambda_t = \lambda(t, L_t, Y_t) \quad (47)$$

for a given *intensity function*  $\lambda_i(t, y)$  (vanishing for  $i \geq n$ ). The construction of such a model  $(L, Y)$  can be realized by a Markovian change of probability measure, starting from an auxiliary model with independent loss and factor processes.

Setting  $\mathcal{Y}_t = (t, L_t, Y_t)$ , one may then define the pricing function  $u, v, w = u, v, w_i(t, z)$  for any  $\tilde{\pi}, \tilde{\phi}, \tilde{\psi}(\mathcal{Y}_T)$ -claims (such as any  $\pi, \phi, \psi(L_T)$ -claims), characterized as the solutions to the related pricing equations with generator  $\mathcal{A}$ .

The loss delta function  $\delta u$  is defined by (cf. (42))

$$\delta u_i(t, y) = u_{i+1}(t, y) - u_i(t, y) .$$

Moreover, we have the following hedging result, which is an analog to Proposition 5.6 (and the analog of Remark 5.4 also holds),

**Proposition 5.10 (i)** *Assume  $Y$  satisfies the following  $d$ -dimensional SDE*

$$dY_t = b(\mathcal{Y}_t)dt + \sigma(\mathcal{Y}_t)dB_t ,$$

for suitable coefficients  $b$  and  $\sigma$  and a  $d$ -dimensional standard  $\mathbb{F}$  – Brownian motion  $B$ . Then, denoting by  $\partial$  the row-gradient of a function with respect to the argument  $y$ , we have:<sup>3</sup>

$$de_t = \left( \delta u - \zeta_t \delta v \right) (\mathcal{Y}_{t-}) dM_t + \left( \partial u \sigma - \zeta_t \partial v \sigma \right) (\mathcal{Y}_t) dB_t .$$

In particular one can min-variance hedge the  $\tilde{\pi}$ -claim by the  $\tilde{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :

$$\zeta_t^1 = \frac{\lambda(\delta u)(\delta v) + (\partial u)\sigma\sigma^\top(\partial v)}{\lambda(\delta v)^2 + (\partial v)\sigma\sigma^\top(\partial v)} (\mathcal{Y}_{t-}) . \quad (48)$$

**(ii)** *Assume  $Y$  given as a pure jump process with finite state space  $E$  of cardinality  $d$ , jump times disjoint from  $L$ , jump intensity vector-process  $\check{\lambda}(\mathcal{Y}_{t-}, y)_{y \in E}$  and compensated jump vector-martingale  $(N_t(y), y \in E)$ . Then, denoting by  $\Delta u_i(t, y)$  the  $d$ -dimensional row-vector  $(u_i(t, y') - u_i(t, y))_{y' \in E}$  and likewise for  $\Delta v$ , we have:*

$$de_t = \left( \delta u - \zeta_t \delta v \right) (\mathcal{Y}_{t-}) dM_t + \left( \Delta u - \zeta_t \Delta v \right) (\mathcal{Y}_{t-}) dN_t$$

In particular one can min-variance hedge the  $\tilde{\pi}$ -claim by the  $\tilde{\phi}$ -claim and the riskless asset using the strategy  $\zeta$  such that  $\zeta^2 = 0$  and, for  $t \in [0, T]$ :

$$\zeta_t^1 = \frac{\lambda(\delta u)(\delta v)(\mathcal{Y}_{t-}) + \sum_{y \in E} \check{\lambda}(\mathcal{Y}_{t-}, y) \Delta u(\mathcal{Y}_{t-}, y) \Delta v(\mathcal{Y}_{t-}, y)}{\lambda(\delta v)^2(\mathcal{Y}_{t-}) + \sum_{y \in E} \check{\lambda}(\mathcal{Y}_{t-}, y) (\Delta v(\mathcal{Y}_{t-}, y))^2} . \quad (49)$$

◇ DISTRIBUTIONAL APPROACH As in the bottom-up case of section 5.1.2, there is little hope to obtain a constructive martingale representation (with computable integrands) in a distributional top approach with an auxiliary factor process  $Y$ , unless maybe we consider a very simple factor process  $Y$  taking a finite number of values.

## 6 Sufficient Statistics

For credit derivatives with stylized payoff given as  $\xi = \pi(L_T)$  at maturity time  $T$ , it is tempting to adopt a Black–Scholes like approach, modeling  $L$  as a Markov point process and performing factor hedging of one derivative by another as in Proposition 5.8, balancing the related sensitivities computed by the Itô–Markov formula. However, since the loss process  $L$  is far from being Markovian in the market (unless maybe additional factors are considered to form a Markovian vector state-process), it is quite likely that  $L$  a not a

<sup>3</sup>with  $\delta u(\mathcal{Y}_{t-}) = u_{L_{t-}+1}(t, Y_t) - u_{L_{t-}}(t, Y_t)$ , and likewise for  $\delta v$ .

sufficient statistics for the purpose of valuation and hedging of portfolio credit risk. In other words, ignoring the potentially non-Markovian dynamics of  $L$  for pricing and/or hedging may cause significant model risk, even though the payoffs of the products at hand are given as functions of  $L_T$ .

In this section we want to illustrate this point by means of numerical hedging simulations. For more realism in these numerical experiments we introduce a non-zero recovery  $R$ , taken as a constant  $R = 40\%$ . We thus need to distinguish the *cumulative default process*  $N_t = \sum_{i=1}^n H_t^i$  and the *cumulative loss process*  $L_t = (1 - R)N_t$ .

We shall consider the benchmark problem of pricing and hedging a stylized loss derivative. Specifically, for simplicity, we only consider protection legs of *equity tranches*, resp. *super-senior tranches* (i.e. detachment of 100%), with stylized payoffs

$$\pi(N_T) = \frac{L_T}{n} \wedge k, \text{ resp. } \left(\frac{L_T}{n} - k\right)^+$$

at a maturity time  $T$ . The ‘strike’ (detachment, resp. attachment point)  $k$  belongs to  $[0, 1]$ . In this formalism the (stylized) *credit index* corresponds to the equity tranche with  $k = 100\%$  (or senior tranche with  $k = 0$ ). With a slight abuse of terminology, we shall refer to our stylized loss derivatives as to *tranches* and *index*, respectively.

We shall now consider the problem of hedging the tranches with the index, using a simplified market model of credit risk.

## 6.1 Homogeneous Groups Model

We consider a Markov chain model of credit risk as of Frey and Backhaus [21] (see also Bielecki et al. [3]). Namely, the  $n$  names of a pool are grouped in  $d$  classes of  $\nu - 1 = \frac{n}{d}$  homogeneous obligors (assuming  $\frac{n}{d}$  integer), and the cumulative default processes  $N^l$ ,  $l \in \mathbb{N}_d^*$ , of the different groups (so  $N = \sum N^l$ ) are jointly modeled as a  $d$ -variate Markov point process  $\mathcal{N}$ , with  $\mathbb{F}^{\mathcal{N}}$ -intensity of  $N^l$  given as

$$\lambda_t^l = (\nu - 1 - N_t^l) \tilde{\lambda}^l(t, \mathcal{N}_t), \quad (50)$$

for some *pre-default individual intensity functions*  $\tilde{\lambda}^l = \tilde{\lambda}^l(t, \iota)$ , where  $\iota = (i_1, \dots, i_d) \in \mathbb{N}_{\nu-1}^d$ . The related generator (spatial generator at time  $t$ ) may then be written in the form of a  $\nu^d$ -dimensional (very sparse) matrix  $\mathcal{A}_t$ .

For  $d = 1$ , we recover the well-known *local intensity model* already considered in the first paragraph of Section 5.2.1 (pure top approach with  $N$  modeled as a Markov birth point process stopped at level  $n$ ; see, for instance, Laurent, Cousin and Fermanian [36] or Cont and Minca [11]). At the other extreme, for  $d = n$ , we are in effect modeling the vector of the default indicator processes of the pool names. As  $d$  varies between 1 and  $n$ , we thus get a variety of models of credit risk, ranging from pure top models for  $d = 1$  to pure bottom-up models for  $d = n$ .

**Remark 6.1** Observe that in the *homogeneous case* where  $\tilde{\lambda}^l(t, \iota) = \hat{\lambda}(t, \sum_j i_j)$  for some function  $\hat{\lambda} = \hat{\lambda}(t, i)$  (independent of  $l$ ), the model (whatever the nominal value of  $d$  / structure of the matrix generator used for encoding the model) effectively reduces to a local intensity model (with  $d = 1$  and pre-default individual intensity  $\hat{\lambda}(t, i)$  therein).

Further specifying the model to  $\hat{\lambda}$  independent of  $i$  corresponds to the situation of homogeneous and independent obligors.

In general, introducing parsimonious parameterizations of the intensities allows one to account for inhomogeneity between groups and/or defaults contagion. It is also possible to extend this set-up to more general credit migrations models, or to generic bottom-up models of credit migrations influenced by macro-economic factors (see Bielecki et al. [3, 8] or Frey and Backhaus [22]).

Note that we only use this model for its flexibility and ease of implementation in view of illustrative purposes. We are not claiming here that this would be a good model for dealing with credit derivatives in actual practice (cf. in particular the reservation made in Section 3.2.1).

### 6.1.1 Pricing in the Homogeneous Groups Model

Since  $\mathcal{N}$  is a Markov process and  $N$  is a function of  $\mathcal{N}$ , the related tranche price process writes, for  $t \in [0, T]$  (assuming  $\pi(N_T)$  integrable):

$$\Pi_t = \mathbb{E}(\pi(N_T) | \mathcal{F}_t^{\mathcal{N}}) = u(t, \mathcal{N}_t), \quad (51)$$

where  $u(t, \iota)$  or  $u_\iota(t)$  for  $t \in [0, T]$  and  $\iota \in \mathbb{N}_{\nu-1}^d$ , is the *pricing function* (system of time-functions  $u_\iota$ ). Using the Itô formula in conjunction with the martingale property of  $\Pi$ , the pricing function can then be characterized as the solution to the following *pricing equation* (system of ODEs):

$$(\partial_t + \mathcal{A}_t)u = 0 \text{ on } [0, T] \quad (52)$$

with terminal condition  $u_\iota(T) = \pi(\iota)$ , for  $\iota \in \mathbb{N}_{\nu-1}^d$ . In particular, in the case of a time-homogeneous generator  $\mathcal{A}$  (independent of  $t$ ), one has the semi-closed formula,

$$u(t) = \exp[(T - t)\mathcal{A}]\pi. \quad (53)$$

Pricing in this model can be achieved by various means, like numerical resolution of the ODE system (52), numerical matrix exponentiation based on (53) (in the time-homogeneous case) or Monte Carlo simulation. However resolution of (52) or computation of (53) by deterministic numerical schemes is typically precluded by the curse of dimensionality for  $d$  greater than a few units (depending on  $\nu$ ). So for high  $d$  simulation methods appear to be the only viable computational alternative. Appropriate variance reduction methods may help in this regard (see, for instance, Carmona and Crépey [12]).

The distribution of the vector of time- $t$  losses (for each group), that is,  $q_\iota(t) = \mathbb{P}(\mathcal{N}_t = \iota)$  for  $t \in [0, T]$  and  $\iota \in \mathbb{N}_{\nu-1}^d$ , and the portfolio cumulative loss distribution,  $p = p_i(t) = \mathbb{P}(N_t = i)$  for  $t \in [0, T]$  and  $i \in \mathbb{N}_n$ , can be computed likewise by numerical solution of the associated forward Kolmogorov equations (for more detail, see, e.g., [12]).

### 6.1.2 Hedging in the Homogeneous Groups Model

In general, in the Markovian model described above, it is possible to replicate dynamically in continuous time any payoff provided  $d$  non-redundant hedging instruments are available (see Frey and Backhaus [20] or Bielecki, Vidozzi and Vidozzi [8]; see also Laurent, Cousin and Fermanian [36] for results in the special case where  $d = 1$ ). From the mathematical side this corresponds to the fact that the model is of *multiplicity*  $d$  (see, e.g., Davis and Varaiya[14]), in general. So, in general, it is not possible to replicate a payoff, such as tranche, *by the index alone* in this model, unless the model dimension  $d$  is equal to one (or reducible to one, cf. Remark 6.1). Now our point is that this potential lack of replicability is not purely speculative, but can be very significant in practice.

Since delta-hedging in continuous time is expensive in terms of transaction costs, and because main changes occur at default times in this model (in fact, default times are the only events in this model, if not for time flow and the induced time-decay effects), we shall focus on *semi-static hedging* in what follows, only updating at default times the composition of the hedging portfolio. More specifically, denoting by  $t_1$  the first default time of a reference obligor, we shall examine the result at  $t_1$  of a static hedging strategy on the random time interval  $[0, t_1]$ .

Let  $\Pi$  and  $P$  denote the tranche and index model price processes, respectively. Using a constant hedge ratio  $\widehat{\delta}_0$  over the time interval  $[0, t_1]$ , the *tracking error* or *profit-and-loss* of a delta-hedged tranche at  $t_1$  writes:

$$e_{t_1} = (\Pi_{t_1} - \Pi_0) - \widehat{\delta}_0(P_{t_1} - P_0). \quad (54)$$

The question we want to consider is whether it is possible to make this quantity ‘small’, in terms, say, of (risk-neutral) variance, relative to the variance of  $\Pi_{t_1} - \Pi_0$  (which corresponds to the risk without hedging), by a suitable choice of  $\widehat{\delta}_0$ . It is expected that this should depend:

- First, on the characteristics of the tranche, and in particular on the value of the strike  $k$ : A high strike equity

tranche or low strike senior tranche (*in-the-money* tranche) is quite close to the index in terms of cashflows, and should therefore exhibit a higher degree of correlation and be easier to hedge with the index, than a low strike equity tranche or high strike senior tranche (*out-of-the money* tranche);

- Second, on the ‘degree of Markovianity’ of the loss process  $L$ , which in the case of the homogeneous groups model depends both on the model nominal dimension  $d$  and on the specification of the intensities (see, e.g., Remark 6.1).

Moreover, it is intuitively clear that for too large values of  $t_1$  time-decay effects matter and the hedge should be rebalanced at some intermediate points of the time interval  $[0, t_1]$  (even though no default occurred yet). To keep it as simple as possible we shall merely apply a cutoff and restrict our attention to the random set  $\{\omega : t_1(\omega) < T_1\}$  for some fixed  $T_1 \in [0, T]$ .

## 6.2 Numerical Results

We work with the above model for  $d = 2$  and  $\nu = 5$ . We thus consider a two-dimensional model of a stylized credit portfolio of  $n = 8$  obligors. The model generator is a  $\nu^d \otimes \nu^d$  – (sparse) matrix with  $\nu^{2d} = 5^4 = 625$ . Recall that the computation time for exact pricing (using matrix exponentiation based on (53)) in such model grows as  $\nu^{2d}$ , which motivated the previous modest choices for  $d$  and  $\nu$ .

Moreover we take the  $\tilde{\lambda}^l$ 's given by, (cf. (50)):

$$\tilde{\lambda}^1(t, \iota) = \frac{2(1+i_l)}{9n}, \quad \tilde{\lambda}^2(t, \iota) = \frac{16(1+i_l)}{9n}. \quad (55)$$

So in this case (which is an admittedly extreme case of inhomogeneity between two independent groups of obligors), the individual intensities of the obligors of group 1 and 2 are given as  $\frac{1+i_1}{36}$  and  $\frac{8(1+i_2)}{36}$ , where  $i_1$  and  $i_2$  represent the number of currently defaulted obligors in groups 1 and 2, respectively.

For instance, at time 0 with  $\mathcal{N}_0 = (0, 0)$ , the individual intensities of obligors of group 1 and 2 are equal to  $1/36$  and  $8/36$ , respectively; the average individual intensity at time 0 is thus equal to  $1/8 = 0.125 = 1/n$ .

We set the maturity  $T$  equal to 5 years and the cutoff  $T_1$  equal to 1 year. We thus make a focus on the random set of trajectories for which  $t_1 < 1$ , meaning that a default occurred during the first year of hedging.

### 6.2.1 Model Simulation

In this toy model the simulation takes the following very simple form (see also [20] or [3] for more details in more general set-ups):

Compute  $\Pi_0$  (for the tranche) and  $P_0$  (for the index) by numerical matrix exponentiation based on (53), and then for every  $j = 1, \dots, m$ :

- Draw a pair  $(\tilde{t}_1^j, \hat{t}_1^j)$  of independent exponential random variables with parameter (cf. (50)–(55))

$$(\lambda_0^1, \lambda_0^2) = 4 \times \left( \frac{1}{36}, \frac{8}{36} \right) = \left( \frac{1}{9}, \frac{8}{9} \right); \quad (56)$$

- Set  $t_1^j = \min(\tilde{t}_1^j, \hat{t}_1^j)$  and  $\mathcal{N}_{t_1} = (1, 0)$  or  $(0, 1)$  depending on whether  $t_1^j = \tilde{t}_1^j$  or  $\hat{t}_1^j$ ;
- Compute  $\Pi_{t_1^j}$  (for the tranche) and  $P_{t_1^j}$  (for the index) by (53).

Doing this for  $m = 10^4$ , we got 9930 draws with  $t_1 < T = 5yr$ , among which 6299 ones with  $t_1 < T_1 = 1yr$ , subdividing themselves into 699 defaults in the first group of obligors and 5600 defaults in the second one.

### 6.2.2 Pricing Results

We consider two  $T = 5yr$ -tranches in the above model: an ‘equity tranche’ with  $k = 30\%$ , corresponding to a payoff  $\frac{(1-R)N_T}{n} \wedge k = \left(\frac{60N_T}{8} \wedge 30\right)\%$  (of a unit nominal amount), and a ‘senior tranche’ defined simply as the complement of the equity tranche to the index, thus with payoff  $\left(\frac{(1-R)N_T}{n} - k\right)^+ = \left(\frac{60N_T}{8} - 30\right)^+\%$ .

We also computed the portfolio loss distribution at maturity by numerical matrix exponentiation corresponding to explicit solution of the associated forward Kolmogorov equations (see, e.g., [12]).

Note that there is virtually no error involved in the previous computations, in the sense that our simulation is exact (without simulation bias), and the prices and loss probabilities are computed by (quasi-exact) matrix exponentiation.

The left pane of Figure 2 represents the histogram of the loss distribution at the time horizon  $T$ ; we indicate by a vertical line the loss level  $x$  beyond which the equity tranche is wiped out, and the senior tranche starts being hit (so  $\frac{(1-R)x}{n} = k$ , e.g.  $x = 4$ ).

The right pane of Figure 2 displays the equity (labeled by  $+$ ), senior ( $\times$ ) and index ( $\circ$ ) tranche prices at  $t_1$  (in ordinate) versus  $t_1$  (in abscissa), for all the points in the simulated data with  $t_1 < 5$  (9930 points). Blue and red points correspond to defaults in the first ( $\mathcal{N}_{t_1} = (1, 0)$ ) and in the second ( $\mathcal{N}_{t_1} = (0, 1)$ ) group of obligors, respectively. We also represented in black the points  $(0, \Pi_0)$  (for the tranches) and  $(0, P_0)$  (for the index).

Note that in the case of the senior tranche and of the index, there is a clear difference between prices at  $t_1$  depending on whether  $t_1$  corresponds to a default in the first or in the second group of obligors, whereas in the case of the equity tranche there seems to be little difference in this regard.

In view of the portfolio loss distribution in the left pane, this can be explained by the fact that in the case of the equity tranche, the probability conditional on  $t_1$  that the tranche will be wiped out at maturity is important unless  $t_1$  is rather large. Therefore the equity tranche price at  $t_1$  is close to  $k = 30\%$  for  $t_1$  close to 0. Moreover for  $t_1$  close to  $T$  the intrinsic value of the tranche at  $t_1$  constitutes the major part of the equity tranche price at  $t_1$  (since the tranche has low time-value close to maturity). In conclusion the state of  $\mathcal{N}$  at  $t_1$  has a low impact on  $\Pi_{t_1}$ , unless  $t_1$  is in the middle of the time-domain.

On the other hand, in the case of the senior tranche or in case of the index, the state of  $\mathcal{N}$  at  $t_1$  has a high impact on the corresponding price, unless  $t_1$  is close to  $T$  (in which case intrinsic value effects are dominant). This explains the ‘two-track’ pictures seen for the senior tranche and for the index on the right pane of Figure 2, except close to  $T$  (whereas the two-tracks are superimposed close to 0 and  $T$  in the case of the equity tranche).

Looking at these results in terms of price changes  $\Pi_0 - \Pi_{t_1}$  of a tranche versus the corresponding index price changes  $P_0 - P_{t_1}$ , we obtain the graphs of Figure 3 for the equity tranche and 4 for the senior tranche. We consider all points with  $t_1 < T$  on the left panes and focus on the points with  $t_1 < T_1$  on the right ones. We use the same blue/red color code as above, and we further highlight in green on the left panes the points with  $t_1 < 1$ , which are focused upon on the right panes.

Figure 3 gives a further graphical illustration of the low level of correlation between price changes of the equity tranche and of the index. Indeed the cloud of points on the right pane is obviously “far from a straight line”, due to the partitioning of points between blue points / defaults in group one on one segment versus red points / defaults in group two on a different segment.

On the opposite (Figure 4), at least for  $t_1$  not too far from 0 (right pane), there is an evidence of linear correlation between price changes of the senior tranche and of the index, since in this case the blue and the red segments are not far from being on a common line.

### 6.2.3 Hedging Results

We then computed the (empirical, risk-neutral) variance of  $\Pi_{t_1} - \Pi_0$  and of the profit-and-loss  $e_{t_1}$  in (54) (restricting attention to the subset  $t_1 < T_1 = 1$ ), using for  $\hat{\delta}_0$  the empirical regression delta of the tranche with respect to the index at time 0, so

$$\hat{\delta}_0 = \frac{\widehat{\text{Cov}}(\Pi_{t_1} - \Pi_0, P_{t_1} - P_0)}{\widehat{\text{Var}}(P_{t_1} - P_0)}. \quad (57)$$

Moreover, we also did these computations restricting further attention to the subsets of  $t_1 < 1$  corresponding to defaults in the first and in the second group of obligors (blue and red points on the figures), respectively. The

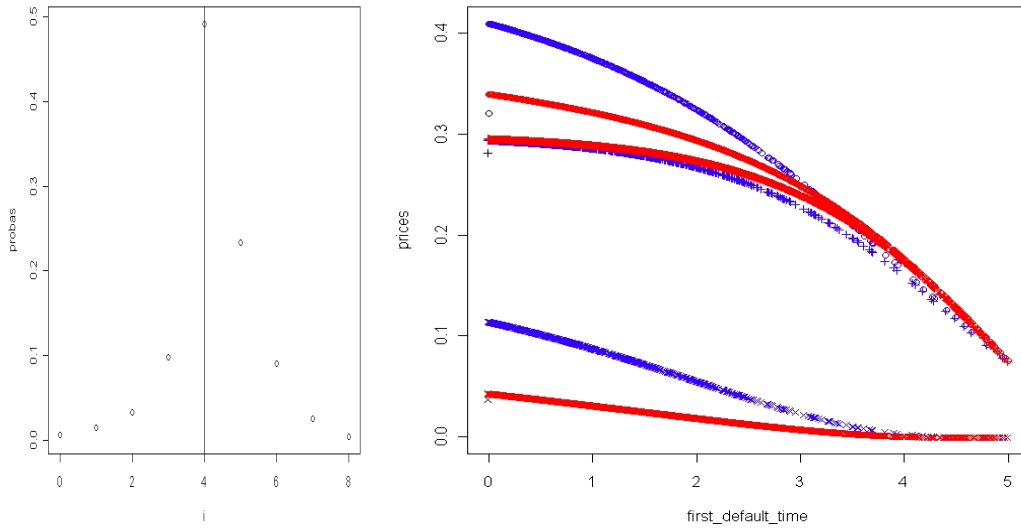


Figure 2: (Left) Portfolio loss distribution at maturity  $T = 5yr$ ; (Right) Tranche Prices at  $t_1$  for  $t_1 < T = 5$  (equity tranche (+), senior tranche ( $\times$ ) and index ( $\circ$ )). On this and the following figures, blue and red points correspond to defaults in the first and in the second group of obligors, respectively.

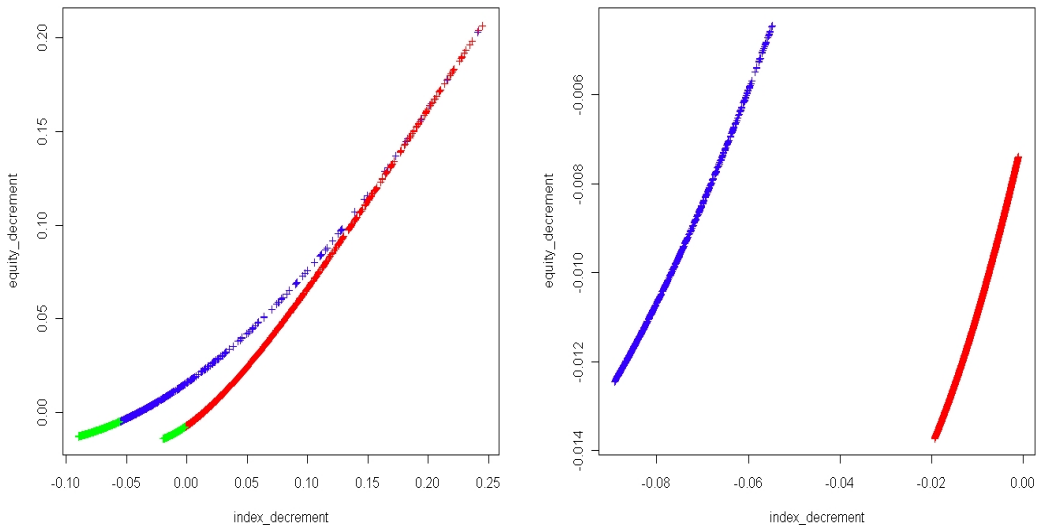


Figure 3: Equity tranche vs Index Price Changes between 0 and  $t_1$  ( $t_1 < T = 5$ , left pane; zoom on  $t_1 < T_1 = 1$ , right pane).

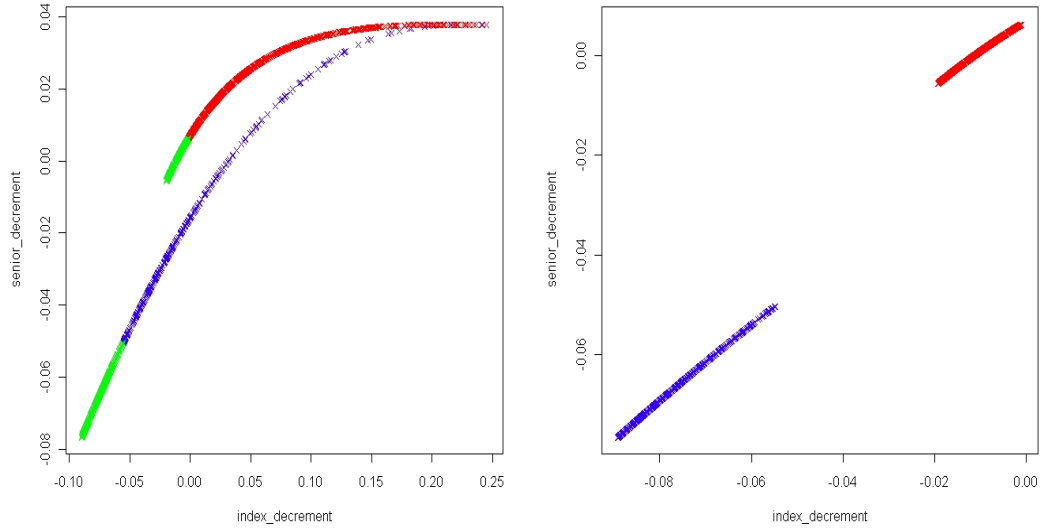


Figure 4: *Senior tranche vs Index Price Changes between 0 and  $t_1$  ( $t_1 < T = 5$ , last pane; zoom on  $t_1 < T_1 = 1$ , right pane).*

latter results are to be understood as giving proxies of the situation that would prevail in a one-dimensional complete model of credit risk (local intensity model for  $N$ ).

The results are displayed in Tables 1 and 2. Note that the prices and deltas of the equity and senior tranche of same strike  $k$  respectively sum up to  $P$  and to one, by construction. So the results for the senior tranche could in a sense be deduced from those for the equity tranche and conversely. However we present detailed results for the equity and senior tranche, for the reader's convenience.

In Table 1 we denote by:

- $\Sigma_0 = \frac{10^4}{kT} \Pi_0$  or  $\frac{10^4}{(1-R-k)T} \Pi_0$  (for the equity or senior tranche) or  $S_0 = \frac{10^4}{(1-R)T} P_0$  (for the index), stylized 'bp spreads' corresponding to the time zero prices  $\Pi_0$  and  $P_0$  of the equity or senior tranche and of the index;
- $\delta_0^1$ ,  $\delta_0^2$  and  $\delta_0$ , the functions  $\frac{\delta^1 u}{\delta^1 v}$ ,  $\frac{\delta^2 u}{\delta^2 v}$  and the *continuous time min-variance delta function* (as it follows easily by application of (18))

$$\frac{\lambda^1(\delta^1 u)(\delta^1 v) + \lambda^2(\delta^2 u)(\delta^2 v)}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2} = \frac{\lambda^1(\delta^1 v)^2}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2} \left( \frac{\delta^1 u}{\delta^1 v} \right) + \frac{\lambda^2(\delta^2 v)^2}{\lambda^1(\delta^1 v)^2 + \lambda^2(\delta^2 v)^2} \left( \frac{\delta^2 u}{\delta^2 v} \right)$$

evaluated at  $t = 0$  and  $v = \mathcal{N}_{0-} = (0, 0)$ , so

$$\delta_0^1 = \frac{u_{1,0} - u_{0,0}}{v_{1,0} - v_{0,0}}(0), \quad \delta_0^2 = \frac{u_{0,1} - u_{0,0}}{v_{0,1} - v_{0,0}}(0) \quad (58)$$

$$\delta_0 = \frac{\lambda_0^1(u_{1,0} - u_{0,0})(v_{1,0} - v_{0,0}) + \lambda_0^2(u_{0,1} - u_{0,0})(v_{0,1} - v_{0,0})}{\lambda_0^1(v_{1,0} - v_{0,0})^2 + \lambda_0^2(v_{0,1} - v_{0,0})^2} \quad (59)$$

where we recall from (56) that  $(\lambda_0^1, \lambda_0^2) = (\frac{1}{9}, \frac{8}{9})$ .

The three deltas  $\delta_0^1$ ,  $\delta_0^2$  and  $\delta_0$  were thus computed by matrix exponentiation based on (53) for the various terms  $u, v_i(0)$  involved in formulas (58), (59).

**Remark 6.2** The instantaneous min-variance delta  $\delta_0$  (which is a suitably weighted average of  $\delta_0^1$  and  $\delta_0^2$ ) can be considered as a measure of the *moneyness* of a tranche: *out-of-the-money* low strike equity tranche or high strike senior tranche with  $\delta_0$  less than 0.5, versus *in-the-money* tranche high strike equity tranche or low strike senior tranche with  $\delta_0$  greater than 0.5. The further out-of-the money a tranche and/or 'the less Markovian' a portfolio loss process  $L$ , the poorer the hedge by the index (cf. end of section 6.1.2).

	$\Pi_0$ or $P_0$	$\Sigma_0$ or $S_0$	$\hat{\delta}_0^1$	$\hat{\delta}_0^2$	$\delta_0$
Eq	0.2821814	1881.209	0.1396623	0.7157741	0.2951399
Sen	0.03817907	254.5271	0.8603377	0.2842259	0.7048601

Table 1: Time  $t = 0$  – Prices, Spreads and Instantaneous Deltas in the Semi-Homogeneous Model.

In Table 2 (cf. also (57)):

- $\rho$  in column two is the empirical correlation of the tranche price increments  $\Pi_{t_1} - \Pi_0$  versus the index price increments  $P_{t_1} - P_0$ ,
- $R2 = \rho^2$  in column three is the *coefficient of determination* of the regression,
- Dev in column 4 stands for  $\widehat{\text{Stdev}}(\Pi_{t_1} - \Pi_0)/\Pi_0$ ,
- The *hedging variance reduction factor*  $\text{RedVar} = \frac{\widehat{\text{Var}}(\Pi_{t_1} - \Pi_0)}{\widehat{\text{Var}}(e_{t_1})}$  in the last column is equal to  $\frac{1}{1-\rho^2}$ .

**Remark 6.3** It is expected that  $\hat{\delta}_0$  should converge to  $\delta_0$  in the limit where the cutoff  $T_1$  would tend to zero, provided the number of simulations  $m$  jointly goes to infinity. For  $T_1 = 1yr$  and  $m = 10^4$  simulations however, we shall see below that there is a clear discrepancy between  $\delta_0$  and  $\hat{\delta}_0$ , and all the more so that we are in a non-homogeneous model with low correlation between the tranche and index price changes between times 0 and  $t_1$ . The reason is that the coefficient of determination ( $R2$ ) of the linear regression with slope  $\hat{\delta}_0$  is given by  $R2 = \rho^2$ . In case  $\rho$  is small,  $R2$  is even smaller, and the significance of the estimator (for low  $T_1$ 's)  $\hat{\delta}_0$  of  $\delta_0$  is low too. In other words, in case  $\rho$  is small, we recover mainly noise through  $\hat{\delta}_0$ ; this however does not weaken our statements below regarding the ability or not to hedge the tranche by the index, since the variance reduction factor  $\text{RedVar} = \frac{\widehat{\text{Var}}(\Pi_{t_1} - \Pi_0)}{\widehat{\text{Var}}(e_{t_1})}$  is equal to  $\frac{1}{1-\rho^2}$ , which only depends on  $\rho$  and not on  $\hat{\delta}_0$ .

	$\hat{\delta}_0$	$\rho$	$R2$	Dev	RedVar
Eq	-0.00275974	-0.03099014	0.0009603885	0.006612626	<b>1.000961</b>
Eq1	0.2269367	0.9980242	0.9960522	0.007576104	253.306
Eq2	0.3391563	0.997375	0.994757	0.006134385	190.7276
Sen	1.002760	0.9960836	0.9921825	0.07475331	<b>127.9176</b>
Sen1	0.7730633	0.9998293	0.9996586	0.02576152	2928.847
Sen2	0.6608437	0.9993066	0.9986137	0.01192970	721.3244

Table 2: Hedging Tranches by the Index in the Semi-Homogeneous Model.

Recall that qualitatively the senior tranche's dynamics is rather close to that of the index (at least for  $t_1$  close to 0, see Section 6.2.2, right pane of Figure 4). Accordingly, we find that hedging the senior tranche with the index is possible (variance reduction factor of about 128 in bold blue in the last column). This case thus seems to be supportive of the claim according to which one could use the index for hedging a loss derivative, even in a non Markovian model of portfolio loss process  $L$ .

But in the case of the equity tranche we get the opposite message: the index is useless for hedging the equity tranche (variance reduction factor essentially equal to 1 in bold red in the table, so *no variance reduction* in this case).

Moreover, the equity tranche variance reduction factors conditional on defaults in the first and in the second group of obligors (in purple in the table) amount to 253 and 190, respectively. This supports the interpretation that the unhedgeability of the equity tranche by the index really comes from the fact that the full model dynamics is not represented in the loss process.

Incidentally this also means that hedging the senior tranche by the equity tranche, or vice versa, is not possible either.

We conclude that in general, at least for certain ranges of the model parameters and tranche characteristics (strong to strong non-Markovianity of  $L$  and out-of-the money to far out-of-the money tranche), hedging tranches with the index may not be possible in a non Markovian model of portfolio loss process  $L$ .

Since the equity and the senior tranche sum-up to the index, therefore a perfect static replication of the equity tranche (say) is provided by a long position in the index and a short position in the senior tranche. As a reality-check of this statement, we performed a bilinear regression of the equity price increments versus the index and the senior tranche price increments, in order to minimize over  $(\widehat{\delta}_0^{ind}, \widehat{\delta}_0^{sen})$  the (risk-neutral) variance of

$$\tilde{e}_{t_1} = (\Pi_{t_1}^{eq} - \Pi_0^{eq}) - \widehat{\delta}_0^{ind}(P_{t_1} - P_0) - \widehat{\delta}_0^{sen}(\Pi_{t_1}^{sen} - \Pi_0^{sen}). \quad (60)$$

The results are displayed in Tables 3. We recover numerically the perfect two-instruments replication strategy mentioned above, whereas a single-instrument hedge using only the index was essentially useless in this case (cf. bold red entry in Table 2).

$\widehat{\delta}_0^{ind}$	$\widehat{\delta}_0^{sen}$	RedVar
1	-1	2.566254e+29

Table 3: *Replicating the equity tranche by the index and the senior tranche in the Semi-Homogeneous Model.*

#### 6.2.4 Fully Homogeneous Case

For confirmation of the previous analysis and interpretation of the results, we redid the computations using the same values as before for all the model, products and simulation parameters, except for the fact that the following pre-default individual intensities were used, for  $l = 1, 2$  :

$$\tilde{\lambda}^l(i) = \frac{1}{n} + \frac{\sum_{1 \leq \ell \leq d} i_\ell}{nd} =: \widehat{\lambda}(\sum_{1 \leq \ell \leq d} i_\ell). \quad (61)$$

For instance, at time 0 with  $\mathcal{N}_0 = 0$ , the individual intensities of the obligors are all equal to  $1/8 = 0.125 = 1/n$ .

We are thus in a case of homogeneous obligors, reducible to a local intensity model (with  $d = 1$  and pre-default individual intensity  $\widehat{\lambda}(i)$  therein, see Remark 6.1). So in this case we expect that hedging tranches by the index should work, including in the case of the out-of-the-money equity tranche.

This is what happens numerically, as it is evident from the following Figures and Tables (which are the analogs of those in previous sections, using the same notation everywhere). Note that all red and blue curves are superimposed, which is consistent with the fact that the group of a defaulted name has no bearing in this case, given the present specification of the identities.

Out of new  $10^4$  draws using the intensities given in (61), we got 9922 draws with  $t_1 < 5$ , among which 6267 ones with  $t_1 < 1$ , subdividing themselves into 3186 defaults in the first group of obligors and 3081 defaults in the second one.

Looking at Table 5, we find as in the semi-homogeneous case that hedging the senior tranche with the index works very well (still better than before, variance reduction factor of 11645 in bold blue in the last column; yet this may be partly due to a moneyness effect: the senior tranche is further in-the-money than before, with a senior tranche  $\delta_0$  of about 0.7 in Table 1 versus 0.8 in Table 4).

But as opposed to the situation in the semi-homogeneous case, hedging the equity tranche with the index also works very well (variance reduction factor of about 123 in bold purple in the last column), and this holds even though the equity tranche is further out-of-the-money now than it was before, with an equity tranche  $\delta_0$  of about 0.3 in Table 1 versus 0.2 in Table 4 (cf. Remark 6.2). So the degradation of the hedge when we pass from the homogeneous model to the semi-homogeneous model is really due to the non-Markovianity of  $L$ ,

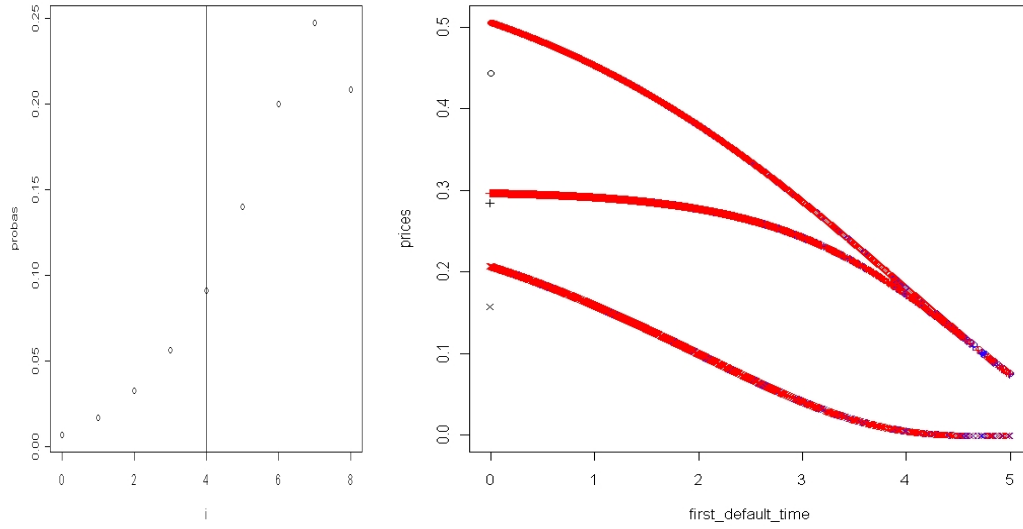


Figure 5: (Left) Portfolio loss distribution at maturity  $T = 5y$  (Right) Tranche Prices at  $t_1$  (for  $t_1 < T$ ).

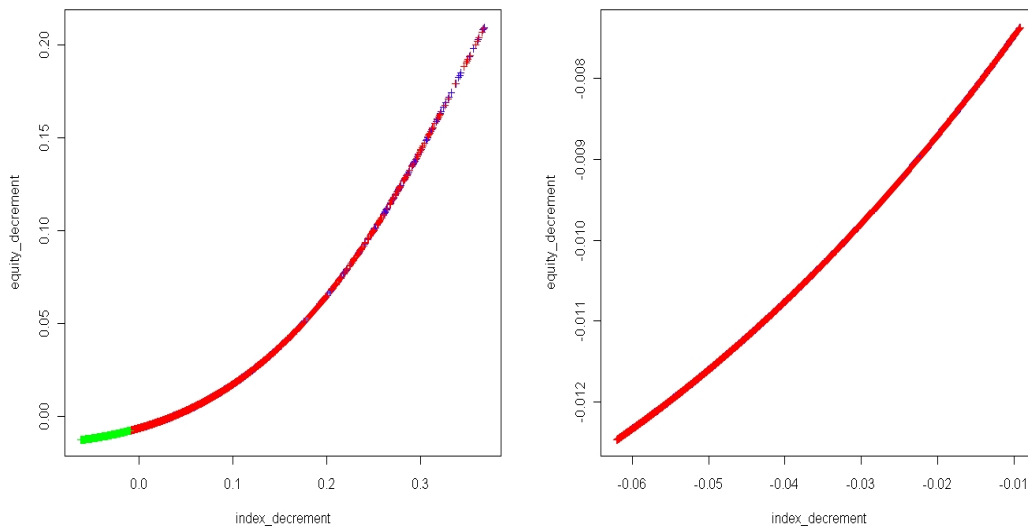


Figure 6: Equity tranche vs Index Price Decrements between 0 and  $t_1$  ( $t_1 < T$ , left pane; zoom on  $t_1 < 1$ , right pane).

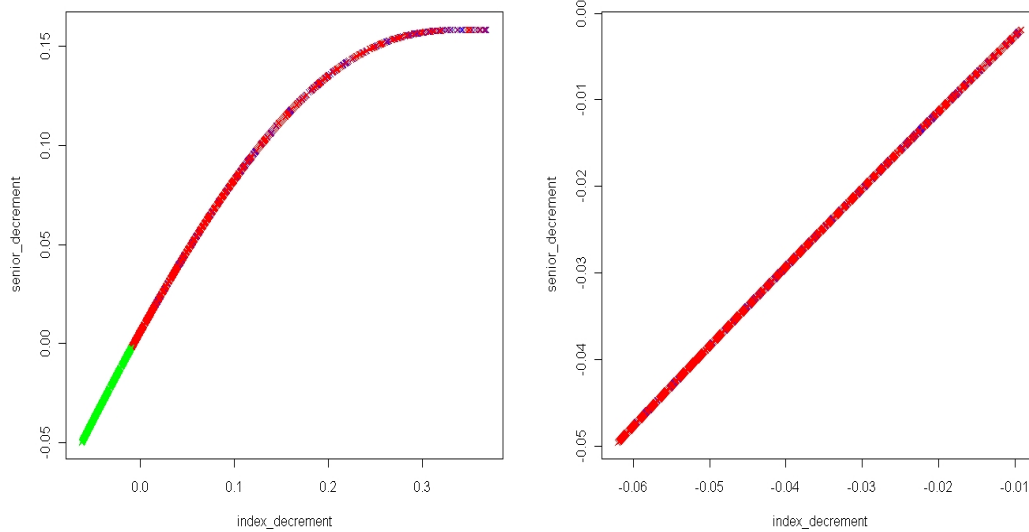


Figure 7: Senior tranche vs Index Price Decrements between 0 and  $t_1$  ( $t_1 < T$ , left pane; zoom on  $t_1 < 1$ , right pane).

and not to an effect of moneyness (cf. end of section 6.1.2).

Moreover the unconditional variance reduction factor and variance reduction factor conditional on defaults in the first and in the second group of obligors are now essentially the same (for the equity tranche as for the senior tranche).

This also means that hedging the equity tranche by the senior tranche, or vice versa, is quite effective in this case.

These results support our previous analysis that the impossibility of hedging the equity tranche by the index in the semi-homogeneous model was due to the non-Markovianity of the loss process  $L$ .

Note incidentally that  $\widehat{\delta}_0$  and  $\delta_0$  are closer now (in Tables 4–5) than they were previously (in Tables 1–2). This is consistent with the fact that  $R2$  is now larger than before ( $\widehat{\delta}_0$  and  $\delta_0$  would be even closer if the cutoff  $T_1$  was less than  $1yr$ , provided of course the number of simulations  $m$  is large enough; see Remark 6.3).

	$\Pi_0$ or $P_0$	$\Sigma_0$ or $S_0$	$\delta_0^1$	$\delta_0^2$	$\delta_0$
Eq	0.2850154	1900.103	0.2011043	0.2011043	0.2011043
Sen	0.1587075	1058.050	0.7988957	0.7988957	0.7988957

Table 4: Time  $t = 0$  – Prices, Spreads and Instantaneous Deltas in the Fully-Homogeneous Model.

	$\widehat{\delta}_0$	$\rho$	$R2$	Dev	RedVar
Eq	0.0929529	0.9959361	0.9918887	0.004754811	<b>123.2852</b>
Eq1	0.09307564	0.995929	0.9918745	0.004794916	123.0695
Eq2	0.09282067	0.995946	0.9919084	0.004713333	123.5853
Sen	0.9070471	0.999957	0.9999141	0.04621152	<b>11645.15</b>
Sen1	0.9069244	0.9999569	0.9999137	0.04653322	11590.83
Sen2	0.9071793	0.9999573	0.9999146	0.0458808	11710.42

Table 5: Hedging Tranches by the Index in the Fully-Homogeneous Model.

## 7 Conclusions

In case of a non ( $\mathbb{F}^L$ -)Markovian portfolio process  $L$ , factor hedging of a loss derivative by another one may not work, and hedging by individuals is necessary.

Bottom-up models are the only ones suited for this purpose (possibly constructed top-down by thinning, admittedly).

This effectively means that the prices of credit derivatives depend on factors others than  $L$ , like the identity (and not only the number) of the defaulted names, ratings or implied ratings (and not only identities) of survivors, etc. This makes of course perfect sense since it is rather clear that the default of a major name in the index does not bear the same informational content as that of an arbitrary firm, and, moreover, pricing is done by agents with regard to the quality of the remaining names in the portfolio rather than with regard to the defaulted names.

At this point one may raise the issue of the so called *curse of dimensionality* that is commonly associated with the bottom-up approaches. However, recent developments in the bottom up modeling enable one to efficiently cope with this curse of dimensionality. It is thus possible to specify high-dimensional ('bottom-up') dynamic Markovian models of portfolio credit risk *with automatically calibrated model marginals* (to the individual CDS curves, say), see Bielecki, Vidozzi and Vidozzi [8].

Much like in the standard static copula framework, this effectively reduces the main computational cost issue, that relative to model calibration, to calibration of the few *dependence parameters* in the model at hand. This calibration can thus be achieved in a very reasonable time, including by pure simulation procedures if need be (without using any closed pricing formulae, if there aren't any in the model under consideration). Appropriate variance reduction methods (see, e.g., [12]) may help in this regard.

## A A glimpse of General Theory

In this Appendix we recall definitions and (essentially well known) results from the theory of processes used in this paper (see, e.g., Dellacherie and Meyer [15]).

### A.1 Optional Projections

Let  $X$  be an integrable process (not necessarily adapted). Then there exists a unique adapted process  $({}^oX_t)_{t \geq 0}$ , called the *optional projection* of  $X$ , such that, for any stopping time  $\tau$ ,

$$\mathbb{E} \left( X_\tau \mathbb{1}_{\{\tau < +\infty\}} \mid \mathcal{F}_\tau \right) = {}^oX_\tau \mathbb{1}_{\{\tau < +\infty\}} .$$

In case  $X$  is non-decreasing, then  ${}^oX$  a submartingale.

### A.2 Dual Predictable Projections and Compensators

Let  $K$  be a non-decreasing and bounded process (not necessarily adapted; typically in the context of this paper,  $K$  corresponds to marginal or portfolio loss processes  $H^i$  or  $L$ ). Then there exists a unique predictable non-decreasing process  $(K_t^p)_{t \geq 0}$ , called the *dual predictable projection* of  $K$ , such that, for any positive predictable process  $H$ :

$$\mathbb{E} \left( \int_0^\infty H_s dK_s \right) = \mathbb{E} \left( \int_0^\infty H_s dK_s^p \right) .$$

In case  $K$  is adapted, it is a sub-martingale, and it admits as such a unique Doob-Meyer decomposition

$$K_t = M_t + \Lambda_t \tag{62}$$

where  $M$  is a uniformly integrable martingale (since  $K$  is bounded) and the *compensator*  $\Lambda$  of  $K$  is a predictable finite variation process. So  $K^p = \Lambda$ , by identification in the Doob-Meyer decomposition.

If, moreover,  $K$  is stopped at some stopping time  $\tau$ , and if  $K^p = \Lambda$  is continuous, then  $K^p = \Lambda$  is also stopped at  $\tau$ , by uniqueness of the Doob-Meyer decomposition of  $K = K_{\cdot \wedge \tau}$ . In case  $\Lambda$  is time-differentiable, so  $\Lambda = \int_0^\cdot \lambda_t dt$  for some *intensity process*  $\lambda$  of  $K$  (also called *intensity of  $\tau$* , when  $K = \mathbb{1}_{\tau \geq t}$  for some stopping time  $\tau$ ), the intensity process  $\lambda$  vanishes after  $\tau$ .

**Remark A.1** If  $K$  is a point-process (like a marginal or cumulative default process  $H^i$  or  $L$  in this paper), the continuity of  $\Lambda$  is equivalent to the ordered jump times of  $K$  being *totally inaccessible* stopping times (see, e.g., Dellacherie and Meyer [15]).

Moreover (see Dellacherie and Meyer [15] or Last and Brandt [35], Brémaud [9]), these definitions and results admit straightforward extensions to *integer-valued random measures* (rather than non-decreasing processes)  $K$ , viewing a random measure  $K$  as a family, parameterized by  $\kappa$ , of non-decreasing processes  $K_t(\omega, \kappa)$ , counting the jumps with mark  $\kappa$  in the mark space  $\mathcal{K}$  of an underlying marked point process.

### A.3 Proof of Proposition 2.1

Let  $\bar{\Lambda}$  denote the  $\widehat{\mathbb{F}}$ -predictable non-decreasing component of the  $\widehat{\mathbb{F}}$ -submartingale  ${}^o\Lambda$  (see section A.1). The tower property of iterated conditional expectations yields,

$$\begin{aligned} \mathbb{E}\left(\int_t^T dK_u - d\bar{\Lambda}_u \mid \widehat{\mathcal{F}}_t\right) &= \mathbb{E}\left(\int_t^T dK_u - d({}^o\Lambda)_u \mid \widehat{\mathcal{F}}_t\right) \\ &= \mathbb{E}\left(\int_t^T dK_u - d\Lambda_u \mid \widehat{\mathcal{F}}_t\right) = \mathbb{E}\left(\mathbb{E}\left(\int_t^T dK_u - d\Lambda_u \mid \mathcal{F}_t\right) \mid \widehat{\mathcal{F}}_t\right) = 0, \end{aligned}$$

since  $K - \Lambda$  is an  $\mathbb{F}$ -martingale. This proves that

$$\bar{\Lambda} = \widehat{\Lambda}. \quad (63)$$

Moreover, one has  $\bar{\Lambda} = \Lambda^p$ , (see, e.g., Proposition 3 of Brémaud and Yor [7]), hence (4) follows.

Now, in case  $\Lambda$  and  $\widehat{\Lambda}$  are time-differentiable with related intensity processes  $\lambda$  and  $\widehat{\lambda}$ , (63) means that

$$\int_0^t \widehat{\lambda}_s ds - \mathbb{E}\left(\int_0^t \lambda_s ds \mid \widehat{\mathcal{F}}_t\right) \quad (64)$$

is an  $\widehat{\mathbb{F}}$ -martingale. Moreover it is immediate to check, using the tower property of iterated conditional expectations, that

$$\mathbb{E}\left(\int_0^t \lambda_s ds \mid \widehat{\mathcal{F}}_t\right) - \int_0^t \mathbb{E}\left(\lambda_s \mid \widehat{\mathcal{F}}_s\right) ds \quad (65)$$

is an  $\widehat{\mathbb{F}}$ -martingale as well. By addition between (64) and (65),

$$\int_0^t \widehat{\lambda}_s ds - \int_0^t \mathbb{E}(\lambda_s \mid \widehat{\mathcal{F}}_s) ds$$

is in turn an  $\widehat{\mathbb{F}}$ -martingale. Since it is also a predictable (as continuous) finite variation process, it is thus in fact identically equal to 0, so for  $t \geq 0$ ,

$$\widehat{\lambda}_t = \mathbb{E}(\lambda_t \mid \widehat{\mathcal{F}}_t),$$

which proves (5).

## References

- [1] ARNSDORF, M. AND HALPERIN, I. BSLP: Markovian Bivariate Spread-Loss Model for Portfolio Credit. *Working Paper*, 2007.
- [2] BENNANI, N.: The Forward Loss Model: A Dynamic Term Structure Pricing of Portfolio Credit Derivatives. *Working Paper*, 2006.
- [3] BIELECKI, T.R., CRÉPEY, S., JEANBLANC, M. AND RUTKOWSKI, M.: Valuation of basket credit derivatives in the credit migrations environment. *Handbook of Financial Engineering*, J. Birge and V. Linetsky eds., Elsevier, 2007.
- [4] BIELECKI, T.R., CRÉPEY, S., JEANBLANC, M. AND RUTKOWSKI, M.: Defaultable options in a Markovian intensity model of credit risk. *Mathematical Finance*, vol. 18, pp. 493-518, 2008 (updated version available online at [www.defaultrisk.com](http://www.defaultrisk.com)).
- [5] BIELECKI, T.R., JEANBLANC, M. AND RUTKOWSKI, M.: *Hedging of Defaultable Claims*, in Paris-Princeton Lectures on Mathematical Finance 2003, eds. R.A. Carmona, E. Çinlar, I. Ekeland, E. Jouini, J.A. Scheinkman, N. Touzi, Springer 2004.
- [6] BIELECKI, T.R. AND RUTKOWSKI, M.: *Credit Risk: Modeling, Valuation and Hedging*. Springer-Verlag, Berlin, 2002.
- [7] BRÉMAUD, P. AND YOR, M.: Changes of filtrations and of probability measures. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, 45, 269–295, 1978.
- [8] BIELECKI, T.R., VIDOZZI, A. AND VIDOZZI, L.: A Markov Copulae Approach to Pricing and Hedging of Credit Index Derivatives and Ratings Triggered Step-Up Bonds, *J. of Credit Risk*, 2008.
- [9] BRÉMAUD, P.: *Point processes and queues, martingale dynamics*, Springer, 1981.
- [10] COCULESCU, D. NIKEGHBALI, A.: Hazard Process and Martingale Hazard Process, *Working Paper*, 2008.
- [11] CONT, R., MINCA, A.: Recovering Portfolio Default Intensities Implied by CDO Quotes, *Working Paper*, 2008.
- [12] CRÉPEY, S. AND CARMONA, R. : Importance Sampling and Interacting Particle Systems for the Estimation of Markovian Credit Portfolios Loss Distribution, *Submitted*, 2008.
- [13] DAI PRA, P., RUNGALDIER, W., SARTORI, E., TOLOTTI, M.: Large portfolio losses; A dynamic contagion model. *Working Paper*, 2007.
- [14] DAVIS, M.H. AND VARAIYA, P.: The multiplicity of an increasing family of  $\sigma$ -fields, *Annals of Probab.*, 2(1974), 958–963.
- [15] DELLACHERIE, C. AND MEYER, P.-A.: *Probabilities and Potential*, North-Holland, 1982.
- [16] DUFFIE, D. AND GARLEANU: Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, 7 (1), 4159, 2001.
- [17] EHLERS, P., SCHÖNBUCHER, P.: Background Filtrations and Canonical Loss Processes for Top-Down Models of Portfolio Credit Risk. *Working Paper*, 2006.
- [18] EL KAROUI, N., JEANBLANC, M. AND JIAO, Y.: Modelling of Successive Defaults Events with density. *Working Paper*.
- [19] ERRAIS, E., GIESECKE, K., GOLDBERG, L.: Pricing Credit from the Top Down with Affine Point Processes. *Working Paper*, 2007.
- [20] FREY, R., BACKHAUS, J.: Dynamic Hedging of Synthetic CDO Tranches with Spread Risk and Default Contagion. *Working Paper*, 2007.

- [21] FREY, R., BACKHAUS, J.: Pricing and Hedging of Portfolio Credit Derivatives with Interacting Default Intensities. *Working Paper*, 2007.
- [22] FREY, R., BACKHAUS, J.: Portfolio credit risk models with interacting default intensities: a Markovian approach. *Working Paper*, 2004.
- [23] GASPAR, R., SCHMIDT, T.: Quadratic Models for Portfolio Credit Risk with Shot-Noise Effects. *Working Paper*, 2008.
- [24] GIESECKE, K., GOLDBERG, L.: A Top Down Approach to Multi-Name Credit. *Working Paper*, 2007.
- [25] GIESECKE, K., WEBER, S.: Credit Contagion and Aggregate Losses. *Journal of Economic Dynamics and Control*, 30, Issue 5, pp. 741-767, 2006 .
- [26] HALPERIN, I.: The “Top” and the “Down” in Credit Top Down Models. Slides of Presentation at *Credit Risk Summit*, New York, November 2, 2007.
- [27] HALPERIN, I. AND TOMECEK, P.: Climbing Down from the Top: Single Name Dynamics in Credit Top Down Models. Working paper, Quantitative Research JP Morgan, September 2008.
- [28] HAWKES, A.: Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58 (1971), pp. 8390.
- [29] HERBERTSSON, A.: Pricing kth-to-default swaps under default contagion: the matrix-analytic approach. *The Journal of Computational Finance*, (1-30) Volume 12/Number 1, Fall 2008.
- [30] HERBERTSSON, A.: Pricing portfolio credit derivatives. *PhD Thesis*, Göteborg University, 2007.
- [31] INGLIS, S., LIPTON, A., SAVESCU, I., SEPP, A.: Dynamic credit models *Statistics and Its Interface*, Forthcoming.
- [32] JACOD, J.: Calcul Stochastique et Problèmes de Martingales. *Lecture Notes in Mathematics*, vol. 714, Springer-Verlag, 1979.
- [33] JEANBLANC, M. AND LE CAM, Y.: *Progressive enlargement of filtrations with initial times*. Working paper.
- [34] JIAO, Y.: Le risque de crédit: modélisation et simulation numérique. *PhD Thesis* (in English), École polytechnique.
- [35] LAST, G., BRANDT, A.: *Marked Point Processes on the Real Line: The Dynamical Approach*, Springer, 1995.
- [36] LAURENT, J.-P., COUSIN, A. , FERMANIAN, J-D.: Hedging default risks of CDOs in Markovian contagion models. *Working Paper*, 2007.
- [37] LIGGET, T.: *Interacting particle systems*. Springer, 1995.
- [38] LIGGET, T.: *Stochastic Interacting Systems: Contact, Voter, and Exclusion Processes*. Springer-Verlag, 1999.
- [39] NIKEGHBALI, A. AND YOR, M.: A definition and some characteristic properties of pseudo-stopping times. *Annals of Probability* 33 (2005), 1804–1824.
- [40] PROTTER, P.E.: *Stochastic Integration and Differential Equations, Second Edition, Version 2.1* Springer, Berlin Heidelberg New York, 2005.
- [41] SCHÖNBUCHER, P.: Portfolio Losses And The Term Structure Of Loss Transition Rates: A New Methodology For The Pricing Of Portfolio Credit Derivatives. *Working Paper*, 2006.
- [42] SIDENIUS, J., PITERBARG, V. , ANDERSEN, L.: A New Framework For Dynamic Credit Portfolio Loss Modelling. *Working Paper*, 2006.

- [43] ZARGARI, B.: Dependence Modelling in Multi-Name Credit Risk, *PhD Thesis University of Evry–University of Sharif*, In preparation.