INDIFFERENCE PRICING OF DEFAULTABLE CLAIMS

Tomasz R. Bielecki [1]
Department of Applied Mathematics
Illinois Institute of Technology
Chicago, USA
bielecki@iit.edu

Monique Jeanblanc
Equipe d'Analyse et Probabilités
Université d'Évry-Val-d'Essonne
Évry, France
monique.jeanblanc@maths.univ-evry.fr

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Contents

Chapter 1

INDIFFERENCE PRICING OF DEFAULTABLE CLAIMS

The goal of this chapter is to give an application of the theory of indifference prices in the context of defaultable claims within the *reduced-form approach*. In this approach the defaultable market is incomplete and there does not exist a (perfect) hedging strategy for claims which depend on the occurrence of the default. An important issue is the issue of choice of relevant information.

The chapter is organized as follows. Section 1.1 contains a brief description of the basic concepts of default risk that are used in the sequel. The second section is devoted to indifference pricing in the filtration of default-free assets. The following section studies the case where the investor additionally uses the information on the default in the choice of the portfolio and is endowed with an exponential utility function. In a last section, we present the quadratic hedging problem.

For details on credit risk, the reader can refer to the books of Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Schönbucher (2003) and to the survey papers of Bielecki et al (2004a, 2004b) where many references are given.

1.1 Preliminaries

In this section, we introduce the basic notions that will be used in what follows. First, we define a default-free market model. Then, we examine the concept of a default time and we present the associated hazard process. We make precise the choice of the filtration, which is an important aspect of our presentation.

1.1.1 Default-Free Market

Consider an economy in continuous time, with the time parameter $t \in I\!R_+$. A probability space $(\Omega, \mathcal{G}, \mathbf{P})$ endowed with a one-dimensional standard Brownian motion $(W_t, t \geq 0)$ is given. We assume that *the reference filtration* $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ is the $\mathbf{P}$-augmented and right-continuous version of the natural filtration generated by W. We have $\mathcal{F}_t \subset \mathcal{G}$, for any $t \in I\!R_+$, however we do not assume that $\mathcal{G} = \mathcal{F}_\infty$.

In the first step, we introduce a Black and Scholes arbitrage-free *default-free market.* In this market, we have the following primary assets:

- A *money market account* B satisfying

$$dB_t = rB_t\,dt, \quad B_0 = 1,$$

  or, equivalently, $B_t = \exp(rt)$, where the interest rate r is assumed to be constant.

- A *default-free asset* whose price $(S_t, t \geq 0)$ follows a geometric Brownian motion dynamics

$$dS_t = S_t(\nu dt + \sigma dW_t)\,,$$

  where ν and σ are two constants, with $\sigma \neq 0$.

It is not difficult to extend the study to the case where r, ν and σ are $\mathbf{F}$-adapted process as soon as some regularity is assumed in order that the default-free market is arbitrage free. In the last part of the chapter, we shall turn to a more general model of the primary market.

As it is well known, the Black and Scholes default-free market is arbitrage-free and complete, and the (unique) risk-neutral probability $\mathbf{Q}$ is obtained via its Radon-Nikodym density, i.e.

$$d\mathbf{Q}|_{\mathcal{F}_t} = \eta_t d\mathbf{P}|_{\mathcal{F}_t},$$

where $(\eta_t, t \geq 0)$ is the $(\mathbf{P}, \mathbf{F})$-martingale given as

$$\eta_t = \exp(-\theta W_t - \frac{1}{2}\theta^2 t)\,,$$

where $\theta = (\nu - r)\sigma^{-1}$ is the risk premium. From Girsanov's theorem, the process $W_t^{\mathbf{Q}} = W_t + \theta t$, is a $(\mathbf{Q}, \mathbf{F})$-Brownian motion.

1.1.2 Default Time

The default time τ is defined as a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbf{P})$. We introduce the default process $H_t = 1\!\!1_{\{\tau \leq t\}}$ and we denote by $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$ the filtration generated by this process (this filtration is right-continuous, and, as usual, we take the completion of this filtration). Note that $\mathcal{H}_t = \sigma(t \wedge \tau)$, hence, any $\mathcal{H}_t$-measurable random variable if a deterministic function of the random variable $\tau \wedge t$.

Hazard process. It is generally assumed that the investor knows when the default takes place, that is the observation of the investor includes the filtration $\mathbf{H}$. At time t, the investor knows whether or not the default has occurred. If the default has not occurred in the past, the investor has no information on the date when the default will appear. Therefore, we consider the filtration of information which takes into account the information on the asset price and of the occurrence of default: $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t \cup \mathcal{H}_t)$ for every $t \in I\!R_+$. The filtration $\mathbf{G}$ is referred to as to the *full filtration*. It is clear that τ is an $\mathbf{H}$-stopping time, as well as a $\mathbf{G}$-stopping time (but not necessarily an $\mathbf{F}$-stopping time). The concept of the hazard process of a random time τ is closely related to the process $(F_t, t \geq 0)$ which is defined as follows:

$$F_t = \mathbf{P}\{\tau \leq t \,|\, \mathcal{F}_t\}, \quad \forall\, t \in I\!R_+.$$

Let us denote $G_t = 1 - F_t = \mathbf{P}\{\tau > t \,|\, \mathcal{F}_t\}$ and let us assume that $G_t > 0$ for every $t \in I\!R_+$ (hence, we exclude the case where τ is an $\mathbf{F}$-stopping time – a case that corresponds to the so-called structural approach). Then the process $(\Gamma_t, t \geq 0)$, given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t, \quad \forall\, t \geq 0,$$

is well defined. It is termed the *hazard process* of the random time τ with respect to the reference filtration $\mathbf{F}$. We postulate that $F_\infty = 1$ (i.e. τ is finite with probability one).

We now formulate an important

Hypothesis: We assume in this chapter that the Brownian motion $(W_t, t \geq 0)$ is a $(\mathbf{P}, \mathbf{G})$-Brownian motion.

In other words, we assume that the so-called (H) hypothesis is satisfied and, as a consequence, the process F (hence Γ) is increasing. We do not comment here on that hypothesis, we simply mention that this hypothesis is necessary in order that there is no-arbitrage in the default-free market using $\mathbf{G}$-adapted strategies. See Elliott et al. (2000) or Bielecki et al. (2004b) for comments.

Note that, due to (H) hypothesis, the process $(\eta_t, t \geq 0)$ is a $(\mathbf{P}, \mathbf{G})$-martingale.This allows us to define the probability $\mathbf{Q}^*$ whose the restriction to $\mathcal{G}_t$ is

$$d\mathbf{Q}^*|_{\mathcal{G}_t} = \eta_t d\mathbf{P}|_{\mathcal{G}_t}.$$

Obviously, the restriction of $\mathbf{Q}^*$ to $\mathbf{F}$ is equal to $\mathbf{Q}$. We shall omit the superscript $*$ in what follows.

Moreover, for simplicity, we assume that the process $(F_t, t \geq 0)$ is absolutely continuous, that is,

$$F_t = \int_0^t f_u \, du$$

for some density process $f : I\!R_+ \to I\!R_+$. Then we have

$$F_t = 1 - e^{-\Gamma_t} = 1 - \exp\left(- \int_0^t \gamma_u \, du \right), \quad \forall \, t \geq 0$$

where

$$\gamma_t = \frac{f_t}{1 - F_t}, \quad \forall \, t \geq 0.$$

The process γ is non-negative and satisfies $\int_0^\infty \gamma_u \, du = \infty$. It is called the *stochastic intensity* of τ (or the *hazard rate*). It can be checked by direct calculations that the process

$$M_t = H_t - \int_0^{t \wedge \tau} \gamma_u \, du = H_t - \int_0^t (1 - H_{u-}) \gamma_u \, du \tag{1.1}$$

is a (purely discontinuous) $(\mathbf{P}, \mathbf{G})$ -martingale. This implies that the random time τ is totally inaccessible in the filtration $\mathbf{G}$. We emphasize that, in our setting, the intensity process is uniquely defined up to infinity and is $\mathbf{F}$-adapted. Moreover, from the definition of $\mathbf{Q}$ (relative to the full filtration $\mathbf{G}$) the process M is a $(\mathbf{Q}, \mathbf{G})$ -martingale. Indeed, the change of probability has an effect only on the Brownian motion W and no effect on the martingale M, which is orthogonal to W. In the particular case where the random time τ is independent of the filtration $\mathbf{F}$, the hazard process is deterministic.

Furthermore, note that, for any $\mathbf{G}$-predictable process ψ such that $\psi_s > -1, \forall s,\ a.s.$, and $\int_0^t (1 + \psi_s)(1 - H_s)\gamma_s ds < \infty$, the process $M_t^\psi = M_t - \int_0^t (1 - H_s)\psi_s \gamma_s ds$ is a $\mathbf{Q}^\psi$-martingale, where

$$d\mathbf{Q}^\psi|_{\mathcal{G}_t} = \eta_t \mathcal{E}(\psi \bullet M)_t d\mathbf{P}|_{\mathcal{G}_t}.$$

Here, the process $\mathcal{E}(\psi \bullet M)_t$ – the Doléans-Dade exponential of $\psi \bullet M$ – is the unique process Y, which is the solution of $dY_t = Y_{t-}\psi_t dM_t$. The restriction of $\mathbf{Q}^\psi$ to the σ-algebra $\mathcal{F}_t$ is equal to $\mathbf{Q}$, and $\mathbf{Q}$ equals to $\mathbf{Q}^0$ on $\mathcal{G}_t$.

1.1.3 Defaultable Claims

A *defaultable* claim (X_1, X_2, τ) with maturity date T consists of:

- The *default time* τ specifying the random time of default and thus also the default events $\{\tau \leq t\}$ for every $t \in [0, T]$. It is always assumed that τ is strictly positive with probability 1.
- The *promised payoff* X_1, which represents the random payoff received by the owner of the claim at time T, if there was no default prior to or at time T. The actual payoff at time T associated with X_1 thus equals $X_1 1\!\!1_{\{\tau > T\}}$. We assume that X_1 is an $\mathcal{F}_T$-measurable random variable.
- The *recovery payoff* X_2, where X_2 is an $\mathcal{F}_T$-measurable random variable which is received by the owner of the claim at maturity, provided that the default occurs prior to or at maturity date T.

In what follows, we shall denote by $X = X_1 1\!\!1_{T<\tau} + X_2 1\!\!1_{\tau \leq T}$ the value of the defaultable contingent claim at maturity.

1.1.4 Hodges Indifference Price

In this section we discuss the concept of Hodges indifference price in our setup. The difference between our approach and the approach of Barrieu and El Karoui (see the corresponding chapter in the present volume) is that we study two different problems, corresponding to the choice of two different filtrations (i.e. the reference filtration and the full filtration). When considering Hodges indifference prices one starts with a given utility function, say u. Typically, u is assumed to be strictly increasing and strictly concave. We shall also apply a similar methodology in the case where u is assumed to be strictly convex (namely $u(x) = x^2$) for quadratic hedging. In this case howevere one can not use the term indifference price and one solves a minimization problem.

Problem $(\mathcal{P})$: Optimization in the default-free market.

The agent invests his initial wealth $v > 0$ in the default-free financial market using a self-financing strategy. The associated optimization problem is,

$$(\mathcal{P})\ :\ \mathcal{V}(v) := \sup_{\phi\in\Phi(F)} \mathbf{E_P}\big\{u\big(V_T^v(\phi)\big)\big\},$$

where the wealth process $(V_t = V_t^v(\phi),\, t \le T)$, is solution of

$$dV_t = rV_t\,dt + \phi_t(dS_t - rS_t dt), \quad V_0 = v. \tag{1.2}$$

Here $\Phi(F)$ is the class of all **F**-adapted, self-financing trading strategies.

Problem $(\mathcal{P}^X_{\mathbf{F}})$: Optimization in the default-free market using F-adapted strategies and buying the defaultable claim.

The agent buys the defaultable claim X at price p, and invests his remaining wealth $v - p$ in the default-free financial market, using a trading strategy $\phi \in \Phi(F)$. The resulting *global terminal wealth* will be

$$V_T^{v-p,X}(\phi) = V_T^{v-p}(\phi) + X.$$

The associated optimization problem is

$$(\mathcal{P}^X_{\mathbf{F}})\ :\ \mathcal{V}^{\mathbf{F}}_X(v-p) := \sup_{\phi\in\Phi(F)} \mathbf{E_P}\big\{u\big(V_T^{v-p}(\phi) + X\big)\big\},$$

where the process $V^{v-p}(\phi)$ is solution of (1.2) with the initial condition $V_0^{v-p}(\phi) = v - p$. We emphasize that the class $\Phi(F)$ of admissible strategies is the same as in the problem $(\mathcal{P})$, that is, we restrict here our attention to trading strategies that are adapted to the reference filtration **F**.

Problem $(\mathcal{P}^X_{\mathbf{G}})$: Optimization in the default-free market using G-adapted strategies and buying the defaultable claim.

The agent buys the defaultable contingent claim X at price p, and invests the remaining wealth $v - p$ in the financial market, using a strategy adapted to the enlarged filtration **G**. The associated optimization problem is

$$(\mathcal{P}^X_{\mathbf{G}})\ :\ \mathcal{V}^{\mathbf{G}}_X(v-p) := \sup_{\phi\in\Phi(G)} \mathbf{E_P}\big\{u\big(V_T^{v-p}(\phi) + X\big)\big\},$$

where $\Phi(G)$ is the class of all $\mathbf{G}$-admissible trading strategies.

Remark. It is easy to check that the solution of

$$(\mathcal{P}_{\mathbf{G}}) \; : \; \sup_{\phi\in\Phi(G)} \mathbf{E}_{\mathbf{P}}\big\{u\big(V_T^v(\phi)\big)\big\},$$

is the same as the solution of $(\mathcal{P})$.

Definition 1.1 For a given initial endowment v, the $\mathbf{F}$-*Hodges buying price* of the defaultable claim X is the real number $p^*_{\mathbf{F}}(v)$ such that

$$\mathcal{V}(v) = \mathcal{V}_X^{\mathbf{F}}\big(v - p^*_{\mathbf{F}}(v)\big)\,.$$

Similarly, the $\mathbf{G}$-*Hodges buying price* of X is the real number $p^*_{\mathbf{G}}(v)$ such that $\mathcal{V}(v) = \mathcal{V}_X^{\mathbf{G}}\big(v - p^*_{\mathbf{G}}(v)\big)$.

Remark. We can define the $\mathbf{F}$-*Hodges selling price* $p_*^{\mathbf{F}}(v)$ of X by considering $-p$, where p is the buying price of $-X$, as specified in Definition 1.1.

If the contingent claim X is $\mathcal{F}_T$-measurable, then (See Rouge and ElKaroui (2000)) the $\mathbf{F}$- and the $\mathbf{G}$-Hodges selling and buying prices coincide with the hedging price of X, i.e.,

$$p^*_{\mathbf{F}}(v) = p^*_{\mathbf{G}}(v) = \mathbf{E}_{\mathbf{P}}(\zeta_T X) = \mathbf{E}_{\mathbf{Q}}(X) = p_*^{\mathbf{G}}(v) = p_*^{\mathbf{F}}(v)\,,$$

where we denote by ζ the deflator process $\zeta_t = \eta_t e^{-rt}$.

1.2 Hodges prices relative to the reference filtration

In this section, we study the problem $(\mathcal{P}^X_{\mathbf{F}})$ (i.e., we use strategies adapted to the reference filtration). First, we compute the value function, i.e., $\mathcal{V}_X^{\mathbf{F}}(v-p)$. Next, we establish a quasi-explicit representation for the Hodges price of X in the case of exponential utility. Finally, we compare the spread obtained via the risk-neutral valuation with the spread determined by the Hodges price of a defaultable zero-coupon bond.

1.2.1 Solution of Problem $(\mathcal{P}^X_{\mathbf{F}})$

In view of the particular form of the defaultable claim X it follows that

$$V_T^{v-p,X}(\phi) = 1\!\!1_{\{\tau>T\}}(V_T^{v-p}(\phi) + X_1) + 1\!\!1_{\{\tau\le T\}}(V_T^{v-p}(\phi) + X_2).$$

Since the trading strategies are $\mathbf{F}$-adapted, the terminal wealth $V_T^{v-p}(\phi)$ is an $\mathcal{F}_T$-measurable random variable. Consequently, it holds that

$$\mathbf{E}_{\mathbf{P}}\Big[u\big(V_T^{v-p,X}(\phi)\big)\Big] =$$

$$
\begin{aligned}
&= \mathbf{E_P}\left(u\big(V_T^{v-p}(\phi)+X_1\big)1\!\!1_{\{\tau>T\}} + u\big(V_T^{v-p}(\phi)+X_2\big)1\!\!1_{\{\tau\le T\}}\right) \\
&= \mathbf{E_P}\left(\mathbf{E_P}\left[u\left(V_T^{v-p}(\phi)+X_1\right)1\!\!1_{\{\tau>T\}} + u\left(V_T^{v-p}(\phi)+X_2\right)1\!\!1_{\{\tau\le T\}}|\mathcal{F}_T\right]\right) \\
&= \mathbf{E_P}\left[u\big(V_T^{v-p}(\phi)+X_1\big)(1-F_T) + u\big(V_T^{v-p}(\phi)+X_2\big)F_T\right],
\end{aligned}
$$

where $F_T = \mathbf{P}\{\tau \le T \,|\, \mathcal{F}_T\}$. Thus, problem $(\mathcal{P}_{\mathbf{F}}^X)$ is equivalent to the following problem:

$$
(\mathcal{P}_{\mathbf{F}}^X) \;:\; \mathcal{V}_X^{\mathbf{F}}(v-p) := \sup_{\phi\in\Phi(F)} \mathbf{E_P}\left(J_X\big(V_T^{v-p}(\phi),\cdot\big)\right),
$$

where

$$
J_X(y,\omega) = u(y+X_1(\omega))(1-F_T(\omega)) + u(y+X_2(\omega))F_T(\omega),
$$

for every $\omega \in \Omega$ and $y \in I\!\!R$. The real-valued mapping $J_X(\cdot,\omega)$ is strictly concave and increasing. Consequently, for any $\omega \in \Omega$, we can define the mapping $I_X(z,\omega)$ by setting $I_X(z,\omega) = \big(J'_X(\cdot,\omega)\big)^{-1}(z)$ for $z \in I\!\!R$, where $(J'_X(\cdot,\omega))^{-1}$ denotes the inverse mapping of the derivative of J_X with respect to the first variable. To simplify the notation, we shall usually suppress the second variable, and we shall write $I_X(\cdot)$ in place of $I_X(\cdot,\omega)$.

The following lemma provides the form of the optimal solution for the problem $(\mathcal{P}_{\mathbf{F}}^X)$,

Lemma 1.1 *The optimal terminal wealth for the problem* $(\mathcal{P}_{\mathbf{F}}^X)$ *is given by* $V_T^{v-p,*} = I_X(\lambda^*\zeta_T)$, $\mathbf{P}$*-a.s., for some* λ^* *such that*

$$
v-p = \mathbf{E_P}\big(\zeta_T V_T^{v-p,*}\big). \tag{1.3}
$$

Thus the optimal global wealth equals $V_T^{v-p,X,*} = V_T^{v-p,*} + X = I_X(\lambda^*\zeta_T) + X$ *and the value function of the objective criterion for the problem* $(\mathcal{P}_{\mathbf{F}}^X)$ *is*

$$
\mathcal{V}_X^{\mathbf{F}}(v-p) = \mathbf{E_P}(u(V_T^{v-p,X,*})) = \mathbf{E_P}(u(I_X(\lambda^*\zeta_T)+X)). \tag{1.4}
$$

Proof. It is well known (see, e.g., Karatzas and Shreve (1998)) that, in order to find the optimal wealth it is enough to maximize $u(\Delta)$ over the set of square-integrable and $\mathcal{F}_T$-measurable random variables Δ, subject to the budget constraint, given by

$$
\mathbf{E_P}(\zeta_T\Delta) \le v-p.
$$

The mapping $J_X(\cdot)$ is strictly concave (for all ω). Hence, for every pair of $\mathcal{F}_T$-measurable random variables (Δ,Δ^*) subject to the budget constraint, by tangent inequality, we have

$$
\mathbf{E_P}\big\{J_X(\Delta) - J_X(\Delta^*)\big\} \;\le\; \mathbf{E_P}\big\{(\Delta-\Delta^*)J'_X(\Delta^*)\big\}.
$$

For $\Delta^* = V_T^{v-p,*}$ given in the formulation of the Lemma we obtain

$$
\mathbf{E_P}\big\{J_X(\Delta) - J_X(V_T^{v-p,*})\big\} \le \lambda^*\mathbf{E_P}\big\{\zeta_T(\Delta - V_T^{v-p,*})\big\} \le 0,
$$

where the last inequality follows from the budget constraint and the choice of λ^*. Hence, for any $\phi \in \Phi(F)$,

$$\mathbf{E_P}\big\{J_X(V_T^{v-p}(\phi)) - J_X(V_T^{v-p,*})\big\} \le 0\,.$$

To end the proof, it remains to observe that the first order conditions are also sufficient in the case of a concave criterion. Moreover, by virtue of strict concavity of the function J_X, the optimal strategy is unique. □

1.2.2 Exponential Utility: Explicit Computation of the Hodges Price

For the sake of simplicity, we assume here that $r = 0$.

Proposition 1.1 *Let $u(x) = 1 - \exp(-\varrho x)$ for some $\varrho > 0$. Assume that the random variables $\zeta_T e^{-\varrho X_i}$, $i = 1, 2$ are $\mathbf{P}$-integrable. Then the $\mathbf{F}$-Hodges buying price is given by*

$$p^*_{\mathbf{F}}(v) = -\frac{1}{\varrho}\,\mathbf{E_P}\left(\zeta_T \ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big)\right) = \mathbf{E_P}(\zeta_T \Psi),$$

where the $\mathcal{F}_T$-measurable random variable Ψ equals

$$\Psi = -\frac{1}{\varrho}\ln\big((1-F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}\big)\,. \tag{1.5}$$

Thus, the $\mathbf{F}$-Hodges buying price $p^*_{\mathbf{F}}(v)$ is the arbitrage price of the associated claim Ψ. In addition, the claim Ψ enjoys the following meaningful property

$$\mathbf{E_P}\big\{u\big(X - \Psi\big)\,\big|\,\mathcal{F}_T\big\} = 0. \tag{1.6}$$

Proof. In view of the form of the solution to the problem $(\mathcal{P})$, we obtain

$$V_T^{v,*} = -\frac{1}{\varrho}\ln\left(\frac{\mu^*\zeta_T}{\varrho}\right).$$

The budget constraint $\mathbf{E_P}(\zeta_T V_T^{v,*}) = v$ implies that the Lagrange multiplier μ^* satisfies

$$\frac{1}{\varrho}\ln\left(\frac{\mu^*}{\varrho}\right) = -\frac{1}{\varrho}\mathbf{E_P}\big(\zeta_T \ln\zeta_T\big) - v. \tag{1.7}$$

The solution to the problem $(\mathcal{P}^X_{\mathbf{F}})$ is obtained in a general setting in Lemma 1.1. In the case of an exponential utility, we have (recall that the variable ω is suppressed)

$$J_X(y) = (1 - e^{-\varrho(y+X_1)})(1-F_T) + (1 - e^{-\varrho(y+X_2)})F_T,$$

so that

$$J'_X(y) = \varrho\, e^{-\varrho y}(e^{-\varrho X_1}(1-F_T) + e^{-\varrho X_2}F_T).$$

Thus, setting

$$A = e^{-\varrho X_1}(1 - F_T) + e^{-\varrho X_2} F_T = e^{-\varrho \Psi},$$

we obtain

$$I_X(z) = -\frac{1}{\varrho} \ln\left(\frac{z}{A\varrho}\right) = -\frac{1}{\varrho} \ln\left(\frac{z}{\varrho}\right) - \Psi.$$

It follows that the optimal terminal wealth for the initial endowment $v - p$ is

$$V_T^{v-p,*} = -\frac{1}{\varrho} \ln\left(\frac{\lambda^* \zeta_T}{A\varrho}\right) = -\frac{1}{\varrho} \ln\left(\frac{\lambda^*}{\varrho}\right) - \frac{1}{\varrho} \ln \zeta_T - \Psi,$$

where the Lagrange multiplier λ^* is chosen to satisfy the budget constraint $\mathbf{E_P}(\zeta_T V_T^{v-p,*}) = v - p$, that is,

$$\frac{1}{\varrho} \ln\left(\frac{\lambda^*}{\varrho}\right) = -\frac{1}{\varrho} \mathbf{E_P}\big(\zeta_T \ln \zeta_T\big) - \mathbf{E_P}\big(\zeta_T \Psi\big) - v + p. \tag{1.8}$$

¿From definition, the $\mathbf{F}$-Hodges buying price is a real number $p^* = p^*_{\mathbf{F}}(v)$ such that

$$\mathbf{E_P}\big(\exp(-\varrho V_T^{v,*})\big) = \mathbf{E_P}\big(\exp(-\varrho(V_T^{v-p^*,*} + X))\big),$$

where μ^* and λ^* are given by (1.7) and (1.8), respectively. After substitution and simplifications, we arrive at the following equality

$$\mathbf{E_P}\Big\{\exp\Big(-\varrho\big(\mathbf{E_P}(\zeta_T \Psi) - p^* + X - \Psi\big)\Big)\Big\} = 1. \tag{1.9}$$

It is easy to check that

$$\mathbf{E_P}\big(e^{-\varrho(X-\Psi)} \,\big|\, \mathcal{F}_T\big) = 1 \tag{1.10}$$

so that equality (1.6) holds, and $\mathbf{E_P}\big(e^{-\varrho(X-\Psi)}\big) = 1$. Combining (1.9) and (1.10), we conclude that $p^*_{\mathbf{F}}(v) = \mathbf{E_P}(\zeta_T \Psi)$. △

We briefly provide the analog of (1.5) for the $\mathbf{F}$-Hodges selling price of X . We have $p_*^{\mathbf{F}}(v) = \mathbf{E_P}(\zeta_T \widetilde{\Psi})$, where

$$\widetilde{\Psi} = \frac{1}{\varrho} \ln\big((1 - F_T)e^{\varrho X_1} + F_T e^{\varrho X_2}\big). \tag{1.11}$$

Remark. It is important to notice that the $\mathbf{F}$-Hodges prices $p^*_{\mathbf{F}}(v)$ and $p_*^{\mathbf{F}}(v)$ do not depend on the initial endowment v. This is an interesting property of the exponential utility function. In view of (1.6), the random variable Ψ will be called the *indifference conditional hedge.*

From concavity of the logarithm function we obtain

$$\ln((1 - F_T)e^{-\varrho X_1} + F_T e^{-\varrho X_2}) \geq (1 - F_T)(-\varrho X_1) + F_T(-\varrho X_2).$$

Hence, using that ζ_T is $\mathbf{F}_T$-measurable,

$$p^*_{\mathbf{F}}(v) \leq \mathbf{E_P}(\zeta_T((1 - F_T)X_1 + F_T X_2)) = \mathbf{E_Q}(X).$$

Comparison with the Davis price. Let us present the results derived from the marginal utility pricing approach. The *Davis price* (see Davis (1997)) is given by

$$d^*(v) = \frac{\mathbf{E}_{\mathbf{P}}\big\{u'\big(V_T^{v,*}\big)X\big\}}{\mathcal{V}'(v)}\,.$$

In our context, this yields

$$d^*(v) = \mathbf{E}_{\mathbf{P}}\big\{\zeta_T\big(X_1F_T + X_2(1-F_T)\big)\big\}.$$

In this case, the risk aversion ϱ has no influence on the pricing of the contingent claim. In particular, when F is deterministic, the Davis price reduces to the arbitrage price of each (default-free) financial asset X^i, $i=1,2$, weighted by the corresponding probabilities F_T and $1-F_T$.

1.2.3 Risk-Neutral Spread Versus Hodges Spreads

In our setting the price process of the T-maturity unit discount Treasury (default-free) bond is $B(t,T) = e^{-r(T-t)}$. Let us consider the case of a defaultable bond with zero recovery, i.e., $X_1 = 1$ and $X_2 = 0$. It follows from (1.11) that the **F**-Hodges buying and selling prices of the bond are (it will be convenient here to indicate the dependence of the Hodges price on maturity T)

$$D^*_{\mathbf{F}}(0,T) = -\frac{1}{\varrho}\,\mathbf{E}_{\mathbf{P}}\big\{\zeta_T \ln(e^{-\varrho}(1-F_T)+F_T)\big\}$$

and

$$D^{\mathbf{F}}_*(0,T) = \frac{1}{\varrho}\,\mathbf{E}_{\mathbf{P}}\big\{\zeta_T \ln(e^{\varrho}(1-F_T)+F_T)\big\},$$

respectively.

Let $\tilde{\mathbf{Q}}$ be a risk-neutral probability for the filtration $\mathbf{G}$, that is, for the enlarged market. The "market" price at time $t=0$ of defaultable bond, denoted as $D^0(0,T)$, is thus equal to the expectation under $\tilde{\mathbf{Q}}$ of its discounted pay-off, that is,

$$D^0(0,T) = \mathbf{E}_{\tilde{\mathbf{Q}}}\big(\mathbb{1}_{\{\tau>T\}}R_T\big) = \mathbf{E}_{\tilde{\mathbf{Q}}}\big((1-\widetilde{F}_T)R_T\big),$$

where $\widetilde{F}_t = \tilde{\mathbf{Q}}\,\{\tau \le t\,|\,\mathcal{F}_t\}$ for every $t\in[0,T]$. Let us emphasize that the risk-neutral probability $\tilde{\mathbf{Q}}$ is chosen by the market, via the price of the defaultable asset. The Hodges buying and selling spreads at time $t=0$ are defined as

$$S^*(0,T) = -\frac{1}{T}\ln\frac{D^*_{\mathbf{F}}(0,T)}{B(0,T)}$$

and

$$S_*(0,T) = -\frac{1}{T}\ln\frac{D^{\mathbf{F}}_*(0,T)}{B(0,T)}\,,$$

respectively. Likewise, the *risk-neutral spread* at time $t = 0$ is given as

$$S^0(0,T) = -\frac{1}{T} \ln \frac{D^0(0,T)}{B(0,T)}\,.$$

Since $D^*_{\mathbf{F}}(0,0) = D^{\mathbf{F}}_*(0,0) = D^0(0,0) = 1$, the respective *backward short spreads* at time $t = 0$ are given by the following limits (provided the limits exist)

$$s^*(0) = \lim_{T\downarrow 0} S^*(0,T) = -\frac{d^+ \ln D^*_{\mathbf{F}}(0,T)}{dT}\Big|_{T=0} - r$$

and

$$s_*(0) = \lim_{T\downarrow 0} S_*(0,T) = -\frac{d^+ \ln D^{\mathbf{F}}_*(0,T)}{dT}\Big|_{T=0} - r,$$

respectively. We also set

$$s^0(0) = \lim_{T\downarrow 0} S^0(0,T) = -\frac{d^+ \ln D^0(0,T)}{dT}\Big|_{T=0} - r.$$

Assuming, as we do, that the processes $\widetilde{F}_T$ and F_T are absolutely continuous with respect to the Lebesgue measure, and using the observation that the restriction of $\tilde{\mathbf{Q}}$ to $\mathcal{F}_T$ is equal to $\mathbf{Q}$, we find out that

$$\begin{aligned}\frac{D^*_{\mathbf{F}}(0,T)}{B(0,T)} &= -\frac{1}{\varrho}\,\mathbf{E}_{\mathbf{Q}}\big\{\ln\big(e^{-\varrho}(1-F_T)+F_T\big)\big\}\\ &= -\frac{1}{\varrho}\,\mathbf{E}_{\mathbf{Q}}\Big\{\ln\Big(e^{-\varrho}\Big(1-\int_0^T f_t\,dt\Big)+\int_0^T f_t\,dt\Big)\Big\},\end{aligned}$$

and

$$\begin{aligned}\frac{D^{\mathbf{F}}_*(0,T)}{B(0,T)} &= \frac{1}{\varrho}\,\mathbf{E}_{\mathbf{Q}}\big\{\ln\big(e^{\varrho}(1-F_T)+F_T\big)\big\}\\ &= \frac{1}{\varrho}\,\mathbf{E}_{\mathbf{Q}}\Big\{\ln\Big(e^{\varrho}\Big(1-\int_0^T f_t\,dt\Big)+\int_0^T f_t\,dt\Big)\Big\}.\end{aligned}$$

Furthermore,

$$\frac{D^0(0,T)}{B(0,T)} = \mathbf{E}_{\mathbf{Q}}(1-\widetilde{F}_T) = \mathbf{E}_{\mathbf{Q}}\Big(1-\int_0^T \widetilde{f}_t\,dt\Big).$$

Consequently,

$$s^*(0) = \frac{1}{\varrho}\,\big(e^{\varrho}-1\big)f_0, \quad s_*(0) = \frac{1}{\varrho}\,\big(1-e^{-\varrho}\big)f_0,$$

and $s^0(0) = \widetilde{f}_0$. Now, if we postulate, for instance, that $s_*(0) = s^0(0)$ (it would be the case if the market price is the selling Hodges price), then we must have

$$\widetilde{f}_0 = \frac{1}{\varrho}\,\big(1-e^{-\varrho}\big)f_0 = \frac{1}{\varrho}\,\big(1-e^{-\varrho}\big)\gamma_0$$

so that $\widetilde{\gamma}_0 < \gamma_0$. Similar calculations can be made for any $t \in [0, T[$. It can be noticed that, if the market price is the selling Hodges price, $\widetilde{f}_0$ corresponds to the risk-neutral intensity at time 0 whereas γ_0 is the historical intensity. The reader may refer to Bernis and Jeanblanc (2002) for other comments.

1.2.4 Recovery paid at time of default

Assume now that the recovery payment is made at time τ, if $\tau \leq T$. More precisely, let $(X^3_t, t \geq 0)$ be some $\mathbf{F}$-adapted process. If $\tau < T$, the payoff X^3_t is paid at time $t = \tau$ and re-invested in the riskless asset. The terminal global wealth is now

$$(V_T^{v-p}(\pi) + X_1)\mathbb{1}_{T<\tau} + (V_T^{v-p}(\pi) + Z_\tau)\mathbb{1}_{\tau \leq T}$$

where $Z_t = X^3_t e^{r(T-t)}$, and we are still interested in optimization of wealth at time T.

The corresponding optimization problem is

$$(\widehat{\mathcal{P}}^Z_{\mathbf{F}}) : \mathcal{V}(v-p) := \sup_{\phi \in \Phi(F)} \mathbf{E_P}\left(U(V_T^{v-p}(\phi) + X_1)\mathbb{1}_{T<\tau} + U(V_T^{v-p}(\phi) + Z_\tau)\mathbb{1}_{\tau \leq T}\right).$$

The supremum part above can be written as

$$\sup_{\phi \in \Phi(F)} \mathbf{E_P}\left\{ \widetilde{J}\big(V_T^{v-p}(\phi)\big)\right\},$$

where, for $\mathbf{P}$-a.e. $\omega \in \Omega$,

$$\widetilde{J}(y, \omega) = U(y + X_1(\omega))(1 - F_T(\omega)) + \int_0^T U(y + Z_t(\omega)) f_t dt.$$

Let us introduce the conditional indifference hedge:

$$\Phi := -\frac{1}{\varrho}\ln\Big(\int_0^T \exp(-\varrho Z_t) f_t dt + \exp(-\varrho X_1)(1 - F_T)\Big). \tag{1.12}$$

We have the following result,

Theorem 1.2.1 *Assume that* $\sup_{0\leq t\leq T} \exp(-\varrho Z_t)$ *and* $\exp(-\varrho X^1)$ *are* $\mathbf{Q}$*-integrable. The Hodges price of* $(X^1, X^3_\cdot)$ *is the arbitrage price of the indifference conditional hedge*Φ*, the pay-off of which is given by (1.12).*

Proof. Observe first that problem $(\widehat{\mathcal{P}}^Z_{\mathbf{F}})$ can be written as

$$\mathcal{V}(x - p) = \sup_{\phi \in \Phi(F)} \mathbf{E_P}\left\{\exp\big(-\varrho\,[V_T^{v-p}(\phi) + \Phi]\big)\right\}.$$

Thus, problem $(\widehat{\mathcal{P}}^Z_{\mathbf{F}})$ is the same as problem $(\mathcal{P}^X_{\mathbf{F}})$ with $X = \Phi$, so that finding the Hodges price of $(X^1, X^3_\cdot)$ amounts to finding the Hodges price of Φ. But now, the claim Φ is a $\mathcal{F}_T$-measurable random variable. Thus, its Hodges price must coincide with its arbitrage price.

□

Observe that Φ is a pay-off at time T. However, at time of default selling the derivative Φ yields enough money to obtain the utility needed.

1.3 Optimization Problems and BSDEs

We now consider strategies ϕ that are predictable with respect to the full filtration $\mathbf{G}$. The dynamics of the risky asset $(S_t, t \geq 0)$ are

$$dS_t = S_t(\nu dt + \sigma dW_t). \tag{1.13}$$

In order to simplify notation, we denote by $(\xi_t, t \geq 0)$ the $\mathbf{G}$-predictable process such that $dM_t = dH_t - \xi_t\, dt$ is a $\mathbf{G}$-martingale, i.e., $\xi_t = \gamma_t(1 - H_{t-})$. (See equation (1.1).)

We assume for simplicity that $r = 0$, so that now $\theta = \nu/\sigma$, and we change the definition of admissible portfolios to one that will be more suitable for problems considered here: instead of using the number of shares ϕ as before, we set $\pi = \phi S$, so that π represents the value invested in the risky asset. In addition, we adopt here the following relaxed definition of admissibility of trading strategies.

Definition 1.2 The class $\Pi(\mathbf{F})$ ($\Pi(\mathbf{G})$, respectively) of $\mathbf{F}$-*admissible* ($\mathbf{G}$-*admissible*, respectively) trading strategies is the set of all $\mathbf{F}$-adapted ($\mathbf{G}$-predictable, respectively) processes π such that $\int_0^T \pi_t^2\, dt < \infty$, $\mathbf{P}$-a.s.

The wealth process of a strategy π satisfies

$$dV_t(\pi) = \pi_t\big(\nu dt + \sigma dW_t\big). \tag{1.14}$$

Let X be a given contingent claim, represented by a $\mathcal{G}_T$-measurable random variable. We shall study the following problem:

$$\sup_{\pi \in \Pi(\mathbf{G})} \mathbf{E_P}\big\{u\big(V_T^v(\pi) + X\big)\big\}.$$

in the case of the exponential utility. In a last step, for the determination of Hodges' price, we shall change v into $v - p$.

1.3.1 Optimization Problem

Our first goal is to solve an optimization problem for an agent who sells a claim X. To this end, it suffices to find a strategy $\pi \in \Pi(\mathbf{G})$ that maximizes $\mathbf{E_P}(u(V_T^v(\pi)+X))$, where the wealth process $(V_t = V_t^v(\pi), t \geq 0)$ (for simplicity, we shall frequently skip v and π from the notation) satisfies

$$dV_t = \phi_t\, dS_t = \pi_t(\nu dt + \sigma dW_t),\ V_0 = v.$$

We consider the exponential utility function $u(x) = 1 - e^{-\varrho x}$, with $\varrho > 0$. Therefore,

$$\sup_{\pi \in \Pi(\mathbf{G})} \mathbf{E_P}\big\{u(V_T^v(\pi) + X)\big\} = 1 - \inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E_P}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big).$$

We shall give three different methods to solve $\inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E_P}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big)$.

Direct method

We describe the idea of a solution; the idea follows the dynamic programming principle.

Suppose that we can find a $\mathbf{G}$-adapted process $(Z_t, t \geq 0)$ with $Z_T = e^{-\varrho X}$, which depends only on the claim X and parameters ϱ, σ, ν, and such that the process $(e^{-\varrho V_t^v(\pi)} Z_t, t \geq 0)$ is a $(\mathbf{P}, \mathbf{G})$-submartingale for any admissible strategy π, and is a martingale under $\mathbf{P}$ for some admissible strategy $\pi^* \in \Pi(\mathbf{G})$. Then, we would have

$$\mathbf{E}_{\mathbf{P}}(e^{-\varrho V_T^v(\pi)} Z_T) \geq e^{-\varrho V_0^v(\pi)} Z_0 = e^{-\varrho v} Z_0$$

for any $\pi \in \Pi(\mathbf{G})$, with equality for some strategy $\pi^* \in \Pi(\mathbf{G})$. Consequently, we would obtain

$$\inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E}_{\mathbf{P}}\big(e^{-\varrho V_T^v(\pi)} e^{-\varrho X}\big) = \mathbf{E}_{\mathbf{P}}\big(e^{-\varrho V_T^v(\pi^*)} e^{-\varrho X}\big) = e^{-\varrho v} Z_0, \tag{1.15}$$

and thus we would be in the position to conclude that π^* is an optimal strategy. In fact, it will turn out that in order to implement the above idea we shall need to restrict further the class of $\mathbf{G}$-admissible trading strategies to such strategies that the "martingale part" in (1.17) determines a true martingale rather than a local-martingale.

In what follows, we shall use the BSDE framework. We refer the reader to the chapter by ElKaroui and Hamadéne in this volume and to the papers of Barles (1997), Rong (1997) and the thesis of Royer (2002) for BSDE with jumps.

We shall search the process Z in the class of all processes satisfying the following BSDE

$$dZ_t = z_t\, dt + \widehat{z}_t\, dW_t + \widetilde{z}_t\, dM_t,\ t \in [0, T[,\ Z_T = e^{-\varrho X}, \tag{1.16}$$

where the process $z = (z_t, t \geq 0)$ will be determined later (see equation (1.19) below). By applying Itô's formula, we obtain

$$d(e^{-\varrho V_t}) = e^{-\varrho V_t} \left(\left(\tfrac{1}{2}\, \varrho^2 \pi_t^2 \sigma^2 - \varrho \pi_t \nu \right) dt - \varrho \pi_t \sigma\, dW_t \right),$$

so that

$$\begin{aligned} d(e^{-\varrho V_t} Z_t) &= e^{-\varrho V_t}\big(z_t + Z_t(\tfrac{1}{2}\varrho^2 \pi_t^2 \sigma^2 - \varrho \pi_t \nu) - \varrho \pi_t \sigma \widehat{z}_t\big) dt \\ &\quad + e^{-\varrho V_t}\big((\widehat{z}_t - \varrho \pi_t \sigma Z_t)\, dW_t + \widetilde{z}_t\, dM_t\big). \end{aligned} \tag{1.17}$$

Let us choose $\pi^* = (\pi_t^*, t \geq 0)$ such that it minimizes, for every t, the following expression

$$Z_t\big(\tfrac{1}{2}\, \varrho^2 \pi_t^2 \sigma^2 - \varrho \pi_t \nu\big) - \varrho \pi_t \sigma \widehat{z}_t = -\varrho \pi_t (\nu Z_t + \sigma \widehat{z}_t) + \tfrac{1}{2}\, \varrho^2 \pi_t^2 \sigma^2 Z_t.$$

It is easily seen that, assuming that the process Z is strictly positive, we have

$$\pi_t^* = \frac{\nu Z_t + \sigma \widehat{z}_t}{\varrho \sigma^2 Z_t} = \frac{1}{\varrho \sigma} \left(\theta + \frac{\widehat{z}_t}{Z_t} \right). \tag{1.18}$$

Now, let us choose the process z as follows

$$\begin{aligned} z_t &= Z_t \left(\varrho\pi_t^*\nu - \tfrac{1}{2}\,\varrho^2(\pi_t^*)^2\sigma^2\right) + \varrho\pi_t^*\sigma\widehat{z}_t \\ &= \varrho\pi_t^*(Z_t\nu + \sigma\widehat{z}_t) - \tfrac{1}{2}\,\varrho^2(\pi_t^*)^2\sigma^2 Z_t = \frac{(\nu Z_t + \sigma\widehat{z}_t)^2}{2\sigma^2 Z_t} \\ &= \frac{1}{2}\theta^2 Z_t + \theta\widehat{z}_t + \frac{1}{2Z_t}\widehat{z}_t^2\,. \end{aligned} \tag{1.19}$$

Note that with the above choice of the process z the drift term in (1.17) is positive for any admissible strategy π, and it is zero for $\pi = \pi^*$.

Given the above, it appears that we have reduced our problem to the problem of solving the BSDE (1.16) with the process z given by (1.19), i.e.,

$$\begin{cases} dZ_t = (\frac{1}{2}\theta^2 Z_t + \theta\widehat{z}_t + \frac{1}{2Z_t}\widehat{z}_t^2)dt + \widehat{z}_t dW_t + \widetilde{z}_t dM_t,\ t \in [0,T), \\ Z_T = e^{-\varrho X}\,. \end{cases} \tag{1.20}$$

In fact, assuming that (1.20) admits a solution $(Z, \widehat{z}, \widetilde{z})$, so that with $\pi = \pi^*$ the "martingale part" in (1.17) is a true martingale part rather than a local-martingale part, then the process

$$\pi_t^* = \frac{1}{\varrho\sigma}\left(\theta + \frac{\widehat{z}_t}{Z_t}\right),$$

will be an optimal portfolio, i.e.,

$$\inf_{\pi\in\Pi(\mathbf{G})} \mathbf{E_P}\left(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\right) = \mathbf{E_P}\left(e^{-\varrho V_T^v(\pi^*)}e^{-\varrho X}\right).$$

However, this BSDE is not of standard. This is a BSDE with jumps, and existence theorems and comparison theorems are known only if the driver is Lipschitz. Hence, we shall establish the existence using another approach, an approach due to Mania and Tevzadze.

Mania and Tevzadze approach

In a very general setting, when the underlying asset is of the form

$$dS_t = d\mu_t + \lambda_t d\langle\mu\rangle_t$$

where μ is a continuous local martingale, Mania and Tevzadze (2003a) study the family of processes

$$\mathcal{V}_t(v) = \max_\phi \mathbf{E_P}(U(v + \int_t^T \phi_s dS_s)|\mathcal{G}_t)$$

where v is a real-valued deterministic parameter. They establish that the process $(\mathcal{V}(t,v) = \mathcal{V}_t(v), t \geq 0)$ (which depends on the parameter v) is solution of a BSDE

$$\begin{aligned} d\mathcal{V}(t,v) &= \frac{1}{2}\frac{1}{\mathcal{V}_{vv}(t,v)}(\varphi_v(t,v) + \lambda_t\mathcal{V}_v(t,v))^2 d\langle\mu\rangle_t + \varphi(t,v)d\mu_t + dN_t(v), \\ \mathcal{V}(T,v) &= U(v), \end{aligned} \tag{1.21}$$

where N is a martingale orthogonal to μ, and the optimal portfolio is proved to be

$$\phi_t^* = -S_t \frac{\varphi_v(t, V_t^*) - \lambda_t \mathcal{V}_v(t, V_t^*)}{\mathcal{V}_{vv}(t, V_t^*)}.$$

Analysis of the proof of the equation (1.4) in Mania and Tevzadze (2003a) reveals that their results carry to the case when

$$\mathcal{V}_t(v) = \max_{\phi} E(U(v + \int_t^T \phi_s dS_s + X)|\mathcal{G}_t)$$

for a claim X satisfying appropriate integrability conditions, in which case the process $(\mathcal{V}_t(v), t \geq 0)$ satisfies the BSDE (1.21) with terminal condition $\mathcal{V}(T, v) = U(v + X)$. We note however that there are several technical conditions postulated in Mania and Tevzadze (2003a) that need to be verified before their results can be adopted.

In the particular case when the dynamics of the underlying asset follows

$$dS_t = S_t(\nu dt + \sigma dW_t)$$

we have $d\mu_t = S_t \sigma dW_t$ and $\lambda_t = \nu/(S_t \sigma^2)$, and the BSDE (1.21) reads

$$\begin{aligned} d\mathcal{V}(t, v) &= \frac{S_t^2 \sigma^2}{2\mathcal{V}_{vv}(t, v)} (\varphi(t, v) + \frac{\nu}{\sigma^2 S_t} \mathcal{V}_v(t, v))^2 dt + \varphi(t, v) S_t \sigma dW_t + dN_t \\ &= \frac{1}{2\sigma^2 \mathcal{V}_{vv}(t, v)} (\varphi(t, v)\sigma^2 S_t + \nu \mathcal{V}_v(t, v))^2 dt + \varphi(t, v) S_t \sigma dW_t + dN_t \end{aligned}$$

where N is a martingale orthogonal to W (hence, in our setting a martingale of the form $\int_0^t \psi_s dM_s$). The terminal condition is

$$\mathcal{V}(T, v) = U(v + X).$$

and the optimal portfolio is

$$\phi_t^* = -S_t \frac{\varphi_v + \mathcal{V}_v \nu/(\sigma^2 S_t)}{\mathcal{V}_{vv}}.$$

Here, U is an exponential function. Thus, it is convenient to factorize process $\mathcal{V}$ as $\mathcal{V}(t, v) = e^{-\varrho v} Z_t$, and to factorize process φ as $\varphi(t, v) = \widehat{\varphi}(t) e^{-\varrho v}$. It follows that Z satisfies

$$dZ_t = \frac{(\widehat{\varphi}(t) + \frac{\nu}{\sigma^2 S_t} Z_t)^2}{2Z_t} S_t^2 \sigma^2 dt + \widehat{\varphi}(t) S_t \sigma dW_t + dN_t, \quad Z_T = e^{-\varrho X}.$$

Setting $\widehat{z}_t = \widehat{\varphi}(t) \sigma S_t$, we get

$$dZ_t = \frac{1}{2Z_t} (\widehat{z}_t + \frac{\nu}{\sigma} Z_t)^2 dt + \widehat{z}_t dW_t + dN_t, \quad Z_T = e^{-\varrho X},$$

which is exactly equation (1.19), where N is a stochastic integral w.r.t. the martingale M, orthogonal to W. Thus, it appears that a solution to equation (1.19) is given as

$$Z_t = e^{\varrho v}\mathcal{V}(t,v), \quad \widehat{z}_t = \widehat{\varphi}(t)\sigma S_t, \quad \text{and} \quad \widetilde{z}_t = \frac{dN_t}{dM_t}.$$

The optimal portfolio is

$$\frac{\sigma\widehat{z}_t + Z_t\nu}{\varrho\sigma^2 Z_t}$$

which is exactly (1.18).

Remark. Analogous results follow from by Mania and Tevzadze (2003b) where a more general case of utility function is studied.

Duality Approach

We present now the duality approach (See for example Delbaen et al. (2002), or Mania and Tevzadze (2003b)). In the case $dS_t = S_t(\nu dt + \sigma dW_t)$, the set of equivalent martingale measure (emm) is the set of probability measures $\mathbf{Q}^\psi$ defined as

$$d\mathbf{Q}^\psi|_{\mathcal{G}_t} = L_t d\mathbf{P}|_{\mathcal{G}_t}$$

where

$$dL_t = L_{t-}(-\theta dW_t + \psi_t dM_t)$$

where ψ is a $\mathbf{G}$-predictable process, with $\psi > -1$ and θ is the risk premium $\theta = \nu/\sigma$. Indeed, using Kusuoka representation theorem (1999), we know that any strictly positive martingale can be written of the form

$$dL_t = L_{t-}(\ell_t dW_t + \psi_t dM_t)\,.$$

The discounted price of the default-free asset is a martingale under the change of probability, hence, it is easy to check that $\ell_t = -\theta$. (We have already noticed that the restriction of any emm to the filtration $\mathbf{F}$ is equal to $\mathbf{Q}$.) Let us denote by $W_t^{\mathbf{Q}} = W_t + \theta t$ and $\widehat{M}_t = M_t - \int_0^t \psi_s\xi_s ds$. The processes $W^{\mathbf{Q}}$ and $\widehat{M}$ are $\mathbf{Q}^\psi$ martingales. Then,

$$\begin{aligned} L_t &= \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t + \int_0^t \ln(1+\psi_s)dH_s - \int_0^t \psi_s\xi_s ds\right) \\ &= \exp\left(-\theta W_t^{\mathbf{Q}} + \frac{\theta^2 t}{2} + \int_0^t \ln(1+\psi_s)d\widehat{M}_s + \int_0^t [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right) \end{aligned}$$

Hence, the relative entropy of $\mathbf{Q}^\psi$ with respect to $\mathbf{P}$ is

$$H(\mathbf{Q}^\psi|\mathbf{P}) = E_{\mathbf{Q}^\psi}(\ln L_T) = E_{\mathbf{Q}^\psi}\left(\frac{1}{2}\theta^2 T + \int_0^T [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right).$$

From duality theory, the optimization problem

$$\inf_{\pi\in\Pi(\mathbf{G})} \mathbf{E_P}\left(e^{-\varrho V_T^v(\pi)}e^{-\varrho X}\right)$$

reduces to maximization over ψ of

$$E_{\mathbf{Q}^\psi}(X - \frac{1}{\varrho}H(\mathbf{Q}^\psi|\mathbf{P})),$$

that is, maximization over ψ of

$$E_{\mathbf{Q}^\psi}\left(X - \frac{1}{2\varrho}\theta^2 T - \frac{1}{\varrho}\int_0^T [(1+\psi_s)\ln(1+\psi_s) - \psi_s]\xi_s ds\right).$$

We solve this latter problem by operating

$$\begin{aligned} dU_t &= \left(\frac{1}{\varrho}[(1+\psi_t)\ln(1+\psi_t) - \psi_t]\xi_t\right)dt + \widehat{u}_t dW_t^{\mathbf{Q}} + \widetilde{u}_t d\widehat{M}_t, \\ U_T &= X - \frac{1}{2\varrho}\theta^2 T. \end{aligned}$$

Setting $Y_t = \varrho U_t$ we obtain

$$\begin{aligned} dY_t &= ([(1+\psi_t)\ln(1+\psi_t) - \psi_t]\xi_t)\,dt + \widehat{y}_t dW_t^{\mathbf{Q}} + \widetilde{y}_t d\widehat{M}_t, \\ Y_T &= \varrho X - \frac{1}{2}\theta^2 T. \end{aligned}$$

In terms of the martingale M, we get

$$dY_t = ([(1+\psi_t)\ln(1+\psi_t) - \psi_t(1+\widetilde{y}_t)]\xi_t)\,dt + \widehat{y}_t dW_t^{\mathbf{Q}} + \widetilde{y}_t dM_t,$$

The solution is obtained by maximization of the drift in the above equation w.r.t. ψ, which leads to $1+\psi_s = \widetilde{y}_s$. Consequently, the BSDE reads

$$dY_t = -\left(e^{\widetilde{y}_t} - 1 - \widetilde{y}_t\right)\xi_t dt + \widehat{y}_t dW_t^{\mathbf{Q}} + \widetilde{y}_t dM_t, \quad Y_T = \varrho X - \frac{1}{2}\theta^2 T,$$

and setting $Z_t^* = \exp(-Y_t)$ we conclude that

$$dZ_t^* = \frac{1}{2}Z_t^*\widehat{y}_t^2 dt - Z_t^*\widehat{y}_t dW_t^{\mathbf{Q}} + Z_{t-}^*(e^{\widehat{y}_t} - 1)dM_t, \quad Z_T^* = \exp(-\varrho X + \frac{1}{2}\theta^2 T),$$

or, denoting $\widehat{z}_t = -Z_t^*\widehat{y}_t$, $\widetilde{z}_t = Z_{t-}^*(e^{\widehat{y}_t} - 1)$

$$dZ_t^* = \frac{1}{2Z_t^*}\widehat{z}_t^2 dt + \widehat{z}_t dW_t^{\mathbf{Q}} + \widehat{z}_t dM_t, \quad Z_T^* = \exp(-\varrho X + \frac{1}{2}\theta^2 T),$$

which is equivalent to (1.20). (Note that $Z_t = Z_t^* e^{-\frac{1}{2}\theta^2(T-t)}$.)

1.3.2 Hodges Buying and Selling Prices

Particular case: attainable claims

Assume, as before, that $r = 0$ and let us check that the Hodges buying price is the hedging price in case of attainable claims. Assume that a claim X is $\mathcal{F}_T$-measurable. By virtue of the predictable representation theorem, there exists a pair $(x, \widehat{x})$, where x is a constant and $\widehat{x}_t$ is an $\mathbf{F}$-adapted process, such that $X = x + \int_0^T \widehat{x}_u \, dW_u^{\mathbf{Q}}$, where $W_t^{\mathbf{Q}} = W_t + \theta t$. Here $x = \mathbf{E}_{\mathbf{Q}} X$ is the arbitrage price of X and the replicating portfolio is obtained through $\widehat{x}$. Hence, the time t value of X is $X_t = x + \int_0^t \widehat{x}_u \, dW_u^{\mathbf{Q}}$. Then $dX_t = \widehat{x}_t \, dW_t^{\mathbf{Q}}$ and the process

$$Z_t = e^{-\theta^2 (T-t)/2} e^{-\varrho X_t}$$

satisfies

$$\begin{aligned} dZ_t &= Z_t \Big(\big(\tfrac{1}{2}\theta^2 + \tfrac{1}{2}\varrho^2 \widehat{x}_t^2 \big) \, dt + \varrho \widehat{x}_t \, dW_t^{\mathbf{Q}} \Big) \\ &= \frac{1}{2\sigma^2 Z_t} (\nu Z_t + \sigma \varrho Z_t \widehat{x}_t)^2 \, dt + \varrho Z_t \widehat{x}_t \, dW_t, \\ Z_T &= e^{-\varrho X}. \end{aligned}$$

Hence $(Z_t, \varrho Z_t \widehat{x}_t, 0)$ is the solution of (1.20) with the terminal condition $e^{-\varrho X}$, and

$$Z_0 = e^{-\theta^2 T/2} e^{-\varrho x} .$$

Note that, for $X = 0$, we get $Z_0 = e^{-\theta^2 T/2}$, therefore

$$\inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E}_{\mathbf{P}} (e^{-\varrho V_T^v(\pi)}) = e^{-\varrho v} e^{-\theta^2 T/2}.$$

The $\mathbf{G}$-Hodges buying price of X is the value of p such that

$$\inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E}_{\mathbf{P}} \big(e^{-\varrho V_T^v(\pi)} \big) = \inf_{\pi \in \Pi(\mathbf{G})} \mathbf{E}_{\mathbf{P}} \big(e^{-\varrho (V_T^{v-p}(\pi) + X)} \big),$$

that is,

$$e^{-\varrho v} e^{-\theta^2 T/2} = e^{-\varrho (v - p + \mathbf{E}_{\mathbf{Q}} X)} e^{-\theta^2 T/2}.$$

We conclude easily that $p_*^{\mathbf{G}}(X) = \mathbf{E}_{\mathbf{Q}} X$. Similar arguments show that $p_{\mathbf{G}}^*(X) = \mathbf{E}_{\mathbf{Q}} X$.

General case

Assume now that a claim X is $\mathcal{G}_T$-measurable. Assuming that the process Z introduced in (1.20) is strictly positive, we can use its logarithm. Let us denote $\widehat{\psi}_t = Z_t / \widehat{z}_t =$, $\widetilde{\psi}_t = Z_t / \widetilde{z}_t =$ and

$$\kappa_t = \frac{\widetilde{\psi}_t}{\ln(1 + \widetilde{\psi}_t)} \geq 0.$$

Then we get

$$d(\ln Z_t) = \tfrac{1}{2}\,\theta^2 dt + \widehat{\psi}_t\, dW_t^{\mathbf{Q}} + \ln(1+\widetilde{\psi}_t)\big(dM_t + \xi_t(1-\kappa_t)\,dt\big),$$

and thus

$$d(\ln Z_t) = \tfrac{1}{2}\,\theta^2 dt + \widehat{\psi}_t\, dW_t^{\mathbf{Q}} + \ln(1+\widetilde{\psi}_t)\, d\widehat{M}_t,$$

where

$$d\widehat{M}_t = dM_t + \xi_t(1-\kappa_t)\,dt = dH_t - \xi_t\kappa_t\,dt.$$

The process $\widehat{M}$ is a martingale under the probability measure $\widehat{\mathbf{Q}}$ defined as $d\widehat{\mathbf{Q}}|_{\mathcal{G}_t} = \widehat{\eta}_t\, d\mathbf{P}|_{\mathcal{G}_t}$, where $\widehat{\eta}$ satisfies

$$d\widehat{\eta}_t = -\widehat{\eta}_{t-}\big(\theta\, dW_t + \xi_t(1-\kappa_t)\,dM_t\big)$$

with $\widehat{\eta}_0 = 1$.

Proposition 1.2 *The* $\mathbf{G}$*-Hodges buying price of* X *with respect to the exponential utility is the real number* p *such that* $e^{-\varrho(v-p)}Z_0^X = e^{-\varrho v}Z_0^0$*, that is,* $p^*_{\mathbf{G}}(X) = \varrho^{-1}\ln(Z_0^0/Z_0^X)$ *or, equivalently,* $p^*_{\mathbf{G}}(X) = \mathbf{E}_{\widehat{\mathbf{Q}}}X$*.*

Our previous study establishes that the dynamic hedging price of a claim X is the process $X_t = \mathbf{E}_{\widehat{\mathbf{Q}}}(X\,|\,\mathcal{G}_t)$. This price is the expectation of the payoff, under some martingale measure, as is any price in the range of no-arbitrage prices.

Remark All the results presented in this section remain valid if ν and σ are adapted processes.

1.4 Quadratic Hedging

We work under the same hypothesis as before; in particular, the wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t), \quad V_0^v(\pi) = v\,.$$

In the last part of this section we shall study a more general case.

The objective of this section is to examine the issue of quadratic pricing and hedging. Specifically, for a given $\mathbf{P}$-square-integrable claim $X \in \mathcal{G}_T$, we study the following problems:

- For a given initial endowment v, solve the minimization problem:

$$\min_{\pi} \mathbf{E}_{\mathbf{P}}\big((V_T^v(\pi) - X)^2\big)\,.$$

A solution to this problem provides the portfolio which, among the portfolios with a *given initial wealth*, has the closest terminal wealth to a given claim X, in the sense of L^2-norm under the historical probability $\mathbf{P}$. The solution of this problem exists, since the set of stochastic integrals of the form $\int_0^T \phi_s dS_s$ is closed in L^2.

• Solve the minimization problem:

$$\min_{\pi,v} \mathbf{E_P}((V_T^v(\pi) - X)^2).$$

The optimal value of v is called the *quadratic hedging price* and the optimal π the *quadratic hedging strategy.*

The quadratic hedging problem was examined in a fairly general framework of incomplete markets by means of BSDEs in several papers; see, for example, Mania (2000), Mania and Tevzadze (2003a), Bobrovnytska and Schweizer (2004), Hu and Zhou (2004) or Lim (2004). Since this list is by no means exhaustive, the interested reader is referred to the references quoted in the above-mentioned papers. The reader may refer to Bielecki et al. (2004b) for a study of the same problem under a constraint on the expectation. Also, some additional references can be found in that paper.

1.4.1 Quadratic Hedging with F-Adapted Strategies

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbf{F})} \mathbf{E_P}((V_T^v(\pi) - X)^2),$$

where X is given as

$$X = X_1 \mathbb{1}_{\{\tau > T\}} + X_2 \mathbb{1}_{\{\tau \le T\}}$$

for some $\mathcal{F}_T$-measurable, $\mathbf{P}$-square-integrable random variables X_1 and X_2. Using the same approach as in Section 1.2.1, we define

$$J_X(y) = (y - X_1)^2(1 - F_T) + (y - X_2)^2 F_T$$

and its derivative

$$J'_X(y) = 2\left[(y - X_1)(1 - F_T) + (y - X_2)F_T\right] = 2\left[y - X_1(1 - F_T) - X_2 F_T\right].$$

Hence, the inverse of $J'_X(y)$ is

$$I_X(z) = \frac{1}{2}z + X_1(1 - F_T) + X_2 F_T$$

and thus the optimal terminal wealth equals

$$V_T^{v,*} = \tfrac{1}{2}\lambda^* \zeta_T + X_1(1 - F_T) + X_2 F_T,$$

where λ^* is specified through the budget constraint:

$$\mathbf{E_P}(\zeta_T V_T^{v,*}) = \tfrac{1}{2}\lambda^* \mathbf{E_P}(\zeta_T^2) + \mathbf{E_P}(\zeta_T X_1(1 - F_T)) + \mathbf{E_P}(\zeta_T X_2 F_T) = v.$$

The optimal strategy is the one, which hedges the $\mathcal{F}_T$-measurable contingent claim

$$\lambda^* \zeta_T + X_1(1 - F_T) + X_2 F_T = 2e^{-\theta_2 T}(v - \mathbf{E_Q}(X))\zeta_T + X_1(1 - F_T) + X_2 F_T.$$

We deduce that

$$\begin{aligned}
&\min_{\pi} \mathbf{E_P}((V_T^v - X)^2) \\
&= \mathbf{E_P}\left[\left(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1-F_T) + X_2F_T - X_1\right)^2(1-F_T)\right] \\
&\quad + \mathbf{E_P}\left[\left(\tfrac{1}{2}\lambda^*\zeta_T + X_1(1-F_T) + X_2F_T) - X_2\right)^2 F_T\right] \\
&= \tfrac{1}{4}(\lambda^*)^2\,\mathbf{E_P}(\zeta_T^2) + \mathbf{E_P}\left((X_1-X_2)^2F_T(1-F_T)\right) \\
&= \frac{1}{2\mathbf{E_P}(\zeta_T^2)}\Big(v - \mathbf{E_P}(\zeta_T(X_1 + F_T(X_2 - X_1))\Big)^2 \\
&\quad + \mathbf{E_P}((X_1-X_2)^2F_T(1-F_T)).
\end{aligned}$$

It remains to minimize over v the right-hand side, which is now simple. Therefore, we obtain the following result.

Proposition 1.3 *If we restrict our attention to* $\mathbf{F}$*-adapted strategies, the quadratic hedging price of the claim* $X = X_1 1\!\!1_{\{\tau>T\}} + X_2 1\!\!1_{\{\tau\le T\}}$ *equals*

$$\mathbf{E_P}(\zeta_T(X_1 + F_T(X_2 - X_1)) = \mathbf{E_Q}(X_1(1-F_T) + F_TX_2)\,.$$

The optimal quadratic hedging of X *is the strategy which replicates the* $\mathcal{F}_T$*-measurable contingent claim* $X_1(1-F_T) + F_TX_2$*.*

Let us now examine the case of a generic $\mathcal{G}_T$-measurable random variable X. Here, we shall only examine the solution of the second problem introduced above, that is,

$$\min_{v,\pi} \mathbf{E_P}((V_T^v(\pi) - X)^2)\,.$$

As explained in Bielecki et al. (2004b), this problem is essentially equivalent to a problem where we restrict our attention to the terminal wealth so that we may reduce the problem to $\min_{V\in\mathcal{F}_T}\mathbf{E_P}((V-X)^2)$. From the properties of conditional expectations, we have

$$\min_{V\in\mathcal{F}_T}\mathbf{E_P}((V-X)^2) = \mathbf{E_P}((\mathbf{E_P}(X\,|\,\mathcal{F}_T) - X)^2)$$

and the initial value of the strategy with terminal value $\mathbf{E_P}(X\,|\,\mathcal{F}_T)$ is

$$\mathbf{E_P}(\zeta_T\mathbf{E_P}(X\,|\,\mathcal{F}_T)) = \mathbf{E_P}(\zeta_TX).$$

In essence, the latter statement is a consequence of the completeness of the default-free market model. Indeed, the fact that the conditional expectation $\mathbf{E_P}(X\,|\,\mathcal{F}_T)$ can be written as a stochastic integral w.r.t. S follows directly from the completeness of the default-free market. In conclusion, the quadratic hedging price equals $\mathbf{E_P}(\zeta_TX) = \mathbf{E_Q}X$ and the quadratic hedging strategy is the replicating strategy of the attainable claim $\mathbf{E_P}(X\,|\,\mathcal{F}_T)$ associated with X.

1.4.2 Quadratic Hedging with G-Adapted Strategies

Similarly as in the previous subsection we assume here that the price process of the underlying asset obeys

$$dS_t = S_t(\nu dt + \sigma dW_t).$$

The wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t), \quad V_0^v(\pi) = v\,.$$

We shall first solve, for a given initial endowment v, the following minimization problem

$$\min_{\pi \in \Pi(\mathbf{G})} \mathbf{E}_{\mathbf{P}}((V_T^v(\pi) - X)^2).$$

As discussed in Bielecki et al. (2004b) one way of solving this problem is to project the random variable X on the closed set of stochastic integrals of the form $\int_0^T \varphi_s dS_s$. Here, we present an alternative approach. We are looking for $\mathbf{G}$-adapted processes X, Θ and Ψ such that the process

$$J_t(\pi, v) = \big(V_t^v(\pi) - X_t\big)^2 \Theta_t + \Psi_t, \quad \forall\, t \in [0, T], \tag{1.22}$$

is a $\mathbf{G}$-submartingale for any $\mathbf{G}$-adapted trading strategy π and a $\mathbf{G}$-martingale for some strategy π^*. In addition, we require that $X_T = X$, $\Theta_T = 1$, $\Phi_T = 0$ so that $J_T(\pi, v) = (V_T^v(\pi) - X)^2$. Let us assume that the dynamics of these processes are of the form

$$dX_t = x_t\, dt + \widehat{x}_t\, dW_t + \widetilde{x}_t\, dM_t, \tag{1.23}$$

$$d\Theta_t = \Theta_{t-}\big(\vartheta_t\, dt + \widehat{\vartheta}_t\, dW_t + \widetilde{\vartheta}_t\, dM_t\big), \tag{1.24}$$

$$d\Psi_t = \psi_t\, dt + \widehat{\psi}_t\, dW_t + \widetilde{\psi}_t\, dM_t, \tag{1.25}$$

where the drifts x_t, ϑ_t and ψ_t are yet to be determined. From Itô's formula, we obtain (recall that $\xi_t = \gamma_t \mathbb{1}_{\{\tau > t\}}$)

$$\begin{aligned} d(V_t - X_t)^2 &= 2(V_t - X_t)(\pi_t\sigma - \widehat{x}_t)\, dW_t - 2(V_t - X_{t-})\widetilde{x}_t\, dM_t \\ &+ \big[(V_t - X_{t-} - \widetilde{x}_t)^2 - (V_t - X_{t-})^2\big] dM_t \\ &+ \Big(2(V_t - X_t)(\pi_t\nu - x_t) + (\pi_t\sigma - \widehat{x}_t)^2 \\ &+ \xi_t\big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big) dt, \end{aligned}$$

where we denote $V_t = V_t^v(\pi)$. Then, using integration by parts formula, we obtain by straightforward calculations

$$J_t(\pi) = k(t, \pi_t, \vartheta_t, x_t, \psi_t)dt + martingale$$

where

$$\begin{aligned} k(t, \pi_t, \vartheta_t, x_t, \psi_t) &= \psi_t + \Theta_t \Big[\vartheta_t (V_t - X_t)^2 \\ &+ 2(V_t - X_t)\big[(\pi_t\nu - x_t) + \widehat{\vartheta}_t(\pi_t\sigma - \widehat{x}_t) + \xi_t \widetilde{x}_t\big] \\ &+ (\pi_t\sigma - \widehat{x}_t)^2 + \xi_t(\widetilde{\vartheta}_t + 1)\big[(V_t - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\big]\Big]\,. \end{aligned}$$

The process $J(\pi)$ is a (local) martingale if and only if its drift term $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ equals 0 for every $t\in[0,T]$.

In the first step, for any $t\in[0,T]$ we shall find π^*_t such that the minimum of $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is attained. Subsequently, we shall choose the processes $x=x^*,\vartheta=\vartheta^*$ and $\psi=\psi^*$ in such a way that $k(t,\pi^*_t,x^*_t,\vartheta^*_t,\psi^*_t)=0$. This choice will imply that $k(t,\pi_t,x^*_t,\vartheta^*_t,\psi^*_t)\geq 0$ for any trading strategy π and any $t\in[0,T]$.

The strategy π^* which minimizes $k(t,\pi_t,x_t,\vartheta_t,\psi_t)$ is the solution of the following equation:

$$(V^v_t(\pi)-X_t)(\nu+\widehat{\vartheta}_t\sigma)+\sigma(\pi_t\sigma-\widehat{x}_t)=0,\quad \forall\, t\in[0,T].$$

Hence, the strategy π^* is implicitly given by

$$\pi^*_t=\sigma^{-1}\widehat{x}_t-\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)(V^v_t(\pi^*)-X_t)=A_t-B_t(V^v_t(\pi^*)-X_t),$$

where we denote

$$A_t=\sigma^{-1}\widehat{x}_t,\quad B_t=\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma).$$

After some computations, we see that the drift term of the process $J(\pi^*)$ admits the following representation:

$$\begin{aligned}k(t,\pi_t,\vartheta_t,x_t,\psi_t)&=\psi_t+\Theta_t(V_t-X_t)^2(\vartheta_t-\sigma^2B^2_t)\\&\quad+2\Theta_t(V_t-X_t)\big(\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t-x_t\big)+\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}^2_t.\end{aligned}$$

From now on, we shall assume that the auxiliary processes ϑ,x and ψ are chosen as follows:

$$\begin{aligned}\vartheta_t&=\vartheta^*_t=\sigma^2B^2_t,\\x_t&=x^*_t=\sigma^2A_tB_t-\widehat{\vartheta}_t\widehat{x}_t-\widetilde{\vartheta}_t\widetilde{x}_t\xi_t,\\\psi_t&=\psi^*_t=-\Theta_t\xi_t(\widetilde{\vartheta}_t+1)\widetilde{x}^2_t.\end{aligned}$$

Straightforward computation verifies that if the drift coefficients ϑ,x,ψ in (1.23)-(1.25) are chosen as above, then the drift term in dynamics of J is always non-negative, and it is equal to 0 for $\pi^*_t=A_t-B_t(V^v_t(\pi^*)-X_t)$.

Our next goal is to solve equations (1.23)-(1.25). Since $\vartheta_t=\sigma^2B^2_t$, the three-dimensional process $(\Theta,\widehat{\vartheta},\widetilde{\vartheta})$ is the unique solution to the linear BSDE (1.24)

$$d\Theta_t=\Theta_t\big(\sigma^{-2}(\nu+\widehat{\vartheta}_t\sigma)^2\,dt+\widehat{\vartheta}_t\,dW_t+\widetilde{\vartheta}_t\,dM_t\big),\ \Theta_T=1.$$

It is obvious that a solution is

$$\widehat{\vartheta}_t=0,\quad \widetilde{\vartheta}_t=0,\quad \Theta_t=\exp(-\theta^2(T-t)),\quad \forall\, t\in[0,T]. \tag{1.26}$$

The three-dimensional process $(X,\widehat{x},\widetilde{x})$ solves equation (1.23) with $x_t=x^*_t=\sigma^2A_t(\nu/\sigma^2)=\theta\widehat{x}_t$. This means that $(X,\widehat{x},\widetilde{x})$ is the unique solution to the linear BSDE

$$dX_t=\theta\widehat{x}_t\,dt+\widehat{x}_t\,dW_t+\widetilde{x}_t\,dM_t,\ X_T=X.$$

The unique solution to the last equation is $X_t = \mathbf{E_Q}(X \,|\, \mathcal{G}_t)$. The components $\widehat{x}$ and $\widetilde{x}$ are given by the integral representation of the $\mathbf{G}$-martingale $(X_t, t \geq 0)$ with respect to $W^{\mathbf{Q}}$ and M, where $W_t^{\mathbf{Q}} = W_t + \theta t$. Notice also that since $\widehat{\vartheta} = 0$, the optimal portfolio π^* is given by the feedback formula

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t^v(\pi^*) - X_t)\big).$$

Finally, since $\widetilde{\vartheta} = 0$, we have $\psi_t = -\xi_t \widetilde{x}_t^2 \Theta_t$. Therefore, we can solve explicitly the BSDE (1.25) for the process Ψ. Indeed, we are now looking for a three-dimensional process $(\Psi, \widehat{\psi}, \widetilde{\psi})$, which is the unique solution of the BSDE

$$d\Psi_t = -\Theta_t \xi_t \widetilde{x}_t^2 \, dt + \widehat{\psi}_t \, dW_t + \widetilde{\psi}_t \, dM_t, \;\; \Psi_T = 0.$$

Noting that the process

$$\Psi_t + \int_0^t \Theta_s \xi_s \widetilde{x}_s^2 \, ds$$

is a $\mathbf{G}$-martingale under $\mathbf{P}$ with terminal value $\int_0^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds$, we obtain the value of Ψ in a closed form:

$$\begin{aligned} \Psi_t &= \mathbf{E_P}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \,\Big|\, \mathcal{G}_t \Big) \\ &= \int_t^T e^{-\theta^2(T-s)} \, \mathbf{E_P}\big(\gamma_s \widetilde{x}_s^2 \mathbb{1}_{\{\tau > s\}} \,\big|\, \mathcal{G}_t \big) \, ds \\ &= \int_t^T e^{-\theta^2(T-s)} \, \mathbf{E_P}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \,\big|\, \mathcal{F}_t \big) \, ds \end{aligned} \tag{1.27}$$

where we have identified the process $\widetilde{x}$ with its $\mathbf{F}$-adapted version (recall that any $\mathbf{G}$-predictable process is equal, prior to default, to an $\mathbf{F}$-predictable process).

Substituting (1.26) and (1.27) in (1.22), we conclude that for a fixed v the value function for our problem is $J_t^*(v) = J_t(\pi^*, v)$, where in turn

$$J_t(\pi^*, v) = (V_t^v(\pi^*) - X_t)^2 e^{-\theta^2(T-t)} + \mathbb{1}_{\{\tau > t\}} \int_t^T e^{-\theta^2(T-s)} \, \mathbf{E_P}\big(\gamma_s \widetilde{x}_s^2 e^{\Gamma_t - \Gamma_s} \,\big|\, \mathcal{F}_t \big) \, ds \,.$$

In particular,

$$J_0^*(v) = e^{-\theta^2 T} \Big((v - X_0)^2 + \mathbf{E_P}\Big(\int_0^T e^{\theta^2 s} \gamma_s \widetilde{x}_s^2 e^{-\Gamma_s} \, ds \Big) \Big).$$

The quadratic hedging price, say v^*, is obtained by minimizing $J_0^*(v)$ with respect to v. From the last formula, it is obvious that the quadratic hedging price is $v^* = X_0 = \mathbf{E_Q} X$. We are in the position to formulate the main result of this section. A corresponding theorem for a default-free financial model was established by Kohlmann and Zhou (2000).

Proposition 1.4 *Let a claim X be $\mathcal{G}_T$-measurable and square-integrable under* $\mathbf{P}$. *The optimal trading strategy π^*, which solves the quadratic problem*

$$\min_{\pi \in \Pi(\mathbf{G})} \mathbf{E_P}((V_T^v(\pi) - X)^2),$$

is given by the feedback formula

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t^v(\pi^*) - X_t)\big),$$

where $X_t = \mathbf{E_Q}(X \,|\, \mathcal{G}_t)$ for every $t \in [0,T]$, and the process $\widehat{x}_t$ is specified by

$$dX_t = \widehat{x}_t \, dW_t^{\mathbf{Q}} + \widetilde{x}_t \, dM_t.$$

The quadratic hedging price of X is $\mathbf{E_Q}X$.

Example: Survival Claim

Let us consider a simple survival claim $X = 1\!\!1_{\{\tau > T\}}$, and let us assume that Γ is deterministic, specifically, $\Gamma(t) = \int_0^t \gamma(s)\, ds$. In that case, from the representation theorem (see Bielecki and Rutkowski (2002), Page 159), we have $dX_t = \widetilde{x}_t \, dM_t$ with $\widetilde{x}_t = -e^{\Gamma(t)-\Gamma(T)}$. Hence

$$\begin{aligned}
\Psi_t &= \mathbf{E_P}\Big(\int_t^T \Theta_s \xi_s \widetilde{x}_s^2 \, ds \,\Big|\, \mathcal{G}_t \Big) \\
&= \mathbf{E_P}\Big(\int_t^T \Theta_s \gamma(s) 1\!\!1_{\{\tau > s\}} e^{2\Gamma(s) - 2\Gamma(T)} \, ds \,\Big|\, \mathcal{G}_t \Big) \\
&= 1\!\!1_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \, \mathbf{E_P}\Big(\int_t^T e^{-\theta^2 (T-s)} \gamma(s) e^{\Gamma(s)} \, ds \,\Big|\, \mathcal{F}_t \Big) \\
&= 1\!\!1_{\{\tau > t\}} \, e^{\Gamma(t) - 2\Gamma(T)} \int_t^T e^{-\theta^2 (T-s)} \gamma(s) e^{\Gamma(s)} \, ds.
\end{aligned}$$

One can check that, at time 0, the value function is indeed smaller that the one obtained with $\mathbf{F}$-adapted portfolios.

Case of an Attainable Claim

Assume now that a claim X is $\mathcal{F}_T$-measurable. Then $X_t = \mathbf{E_Q}(X \,|\, \mathcal{G}_t)$ is the price of X, and it satisfies $dX_t = \widehat{x}_t \, dW_t^{\mathbf{Q}}$. The optimal strategy is, in a feedback form,

$$\pi_t^* = \sigma^{-1}\big(\widehat{x}_t - \theta(V_t - X_t)\big)$$

and the associated wealth process satisfies

$$dV_t = \pi_t^*(\nu dt + \sigma dW_t) = \pi_t^* \sigma \, dW_t^{\mathbf{Q}} = \sigma^{-1}\big(\sigma \widehat{x}_t - \nu(V_t - X_t)\big) \, dW_t^{\mathbf{Q}}.$$

Therefore,

$$d(V_t - X_t) = -\theta(V_t - X_t) \, dW_t^{\mathbf{Q}}.$$

Hence, if we start with an initial wealth equal to the arbitrage price $\mathbf{E_Q}X$ of X, then we that $V_t = X_t$ for every $t \in [0,T]$, as expected.

Hodges Price

Let us emphasize that the Hodges price has no real meaning here, since the problem $\min \mathbf{E_P}((V_T^v)^2)$ has no financial interpretation. We have studied in Bielecki et al. (2004b) a more pertinent problem, with a constraint on the expected value of V_T^v under $\mathbf{P}$. Nevertheless, from a mathematical point of view, the Hodges price would be the value of p such that

$$(v^2 - (v-p)^2) = \int_0^T e^{\theta^2 s} \mathbf{E_P}(\gamma_s \widetilde{x}_s^2 e^{-\Gamma_s}) \mathbb{1}_{\{\tau > t\}} ds$$

In the case of the example studied in Section 1.4.2, the Hodges price would be the non-negative value of p such that

$$2vp - p^2 = e^{-2\Gamma_T} \int_0^T e^{\theta^2 s} \gamma_s e^{\Gamma_s}\, ds.$$

Let us also mention that our results are different from results of Lim (2004). Indeed, Lim studies a model with Poisson component, and thus in his approach the intensity of this process does not vanish after the first jump.

1.4.3 Jump-Dynamics of Price

We assume here that the price process follows

$$dS_t = S_{t-}(\nu dt + \sigma dW_t + \varphi dM_t), \quad S_0 > 0$$

where the constant φ satisfy $\varphi > -1$ so that the price S_t is strictly positive. Hence, the primary market, where the savings account and the asset S are traded is arbitrage free, but incomplete (in general). It follows that the wealth process follows

$$dV_t^v(\pi) = \pi_t(\nu dt + \sigma dW_t + \varphi dM_t), \quad V_0^v(\pi) = v.$$

As in the previous subsection, our aim is, for a given initial endowment v, solve the minimization problem:

$$\min_\pi \mathbf{E_P}((V_T^v(\pi) - X)^2).$$

In order to characterize the value function we proceed analogously as before. That is, we are looking for processes X, Θ and Ψ such that the process (for simplicity we write V_t in place of $V_t^v(\pi)$)

$$J(t, V_t) = (V_t - X_t)^2 \Theta_t + \Psi_t$$

is a submartingale for any π and a martingale for some π^*, and such that $\Psi_T = 0$, $X_T = X$, $\Theta_T = 1$. (Note that Mania and Tevzadze (2003a) did a similar approach for continuous processes, with a value function of the form

$J_t = \Phi_0(t) + \Phi_1(t)V_t + \Phi_2(t)V_t^2$.) Let us assume that the dynamics of these processes are of the form

$$\begin{aligned} dX_t &= f_t dt + \widehat{x}_t dW_t + \widetilde{x}_t dM_t, & (1.28)\\ d\Theta_t &= \Theta_t(\vartheta_t dt + \widehat{\vartheta}_t dW_t + \widetilde{\vartheta}_t dM_t) & (1.29)\\ d\Psi_t &= \psi_t dt + \widehat{\psi}_t dW_t + \widetilde{\psi}_t dM_t & (1.30)\end{aligned}$$

where the drifts f_t, ϑ_t and ψ_t have to be determined.
From Itô's formula we obtain

$$\begin{aligned} d(V_t - X_t)^2 &= 2(V_t - X_t)(\pi_t\sigma - \widehat{x}_t)dW_t\\ &+ \left[(V_t + \pi_t\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\right] dM_t\\ &+ \Big(2(V_t - X_t)(\pi_t\mu - f_t) + (\pi_t\sigma - \widehat{x}_t)^2\\ &+ \xi_t\left[(V_t + \pi_t\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2 - 2(V_t - X_t)(\pi_t\varphi - \widetilde{x}_t)\right]\Big) dt.\end{aligned}$$

Process $\Theta_t(V_t - X_t)^2 + \Psi_t$ is a (local) martingale iff $k(\pi_t, f_t, \vartheta_t, \psi_t) = 0$ for all t, where

$$\begin{aligned} k(\pi, \vartheta, f, \psi) &= \psi + \Theta_t\Big[\vartheta_t(V_t - X_t)^2\\ &+ 2(V_t - X_t)\Big((\pi\mu - f) + \widehat{\vartheta}_t(\pi\sigma - \widehat{x}_t) - \xi_t(\pi\varphi - \widetilde{x}_t)\Big)\\ &+ (\pi\sigma - \widehat{x}_t)^2\\ &+ \xi_t(\widetilde{\vartheta}_t + 1)\Big((V_t + \pi\varphi - X_t - \widetilde{x}_t)^2 - (V_t - X_t)^2\Big)\Big]\,.\end{aligned}$$

In the first step, we find $\pi^\sharp$ such that the maximum of $k(\pi)$ is obtained. Then, one defines $(f^*, \vartheta^*, \psi^*)$ such that $k(\pi^\sharp, f^*, \vartheta^*, \psi^*) = 0$. This implies that, for any π, $k(\pi, f^*, \vartheta^*, \psi^*) \le 0$, and that $k(\pi^\sharp, f^*, \vartheta^*, \psi^*) = 0$.

The optimal $\pi^\sharp$ is the solution of

$$\begin{aligned} &(V_t - X_t)(\mu - \xi_t\varphi + \widehat{\vartheta}_t\sigma) + \sigma(\pi\sigma - \widehat{x}_t)\\ &+ \xi_t(\widetilde{\vartheta}_t + 1)\varphi(V_t + \pi\varphi - X_t - \widetilde{x}_t) = 0\,.\end{aligned}$$

hence

$$\begin{aligned} \pi_t^\sharp &= \frac{1}{\sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)}\Big((\sigma\widehat{x}_t + \xi_t\varphi(\widetilde{\vartheta}_t + 1)\widetilde{x}_t) - (\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t)(V_t - X_t)\Big)\\ &= A_t - B_t(V_t - X_t)\end{aligned}$$

with

$$\begin{aligned} A_t &= \Big(\sigma\widehat{x}_t + \xi_t\varphi(\widetilde{\vartheta}_t + 1)\widetilde{x}_t\Big)\Delta_t^{-1}\\ B_t &= \Big(\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t\Big)\Delta_t^{-1}\\ \Delta_t &= \sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)\,.\end{aligned}$$

After some computations the drift term of $\Theta_t(V_t - X_t) + \Psi_t$ is found to be

$$\begin{aligned}&\Theta_t(V_t - X_t)^2(\vartheta_t - B_t^2\Delta_t) + 2\Theta_t(V_t - X_t)\left(A_tB_t\Delta_t - \widehat{\vartheta}_t\widehat{x}_t - \xi_t\widetilde{\vartheta}_t\widetilde{x}_t - f_t\right)\\ + \quad &\Theta_t\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 + \Theta_t(A_t\sigma - \widehat{x}_t)^2 + \psi_t\,.\end{aligned}$$

Then, we choose

$$\begin{aligned}\vartheta_t^* &= B_t^2\Delta_t\\ f_t^* &= A_tB_t\Delta_t - \widehat{\varphi}_t\widehat{x}_t - \xi_t\widetilde{\vartheta}_t\widetilde{x}_t\\ \psi_t^* &= -\Theta_t\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 - \Theta_t(A_t\sigma - \widehat{x}_t)^2.\end{aligned}$$

Let us suppose that with this choice of drifts equations (1.29)–(1.30) admit solutions (we shall discuss this issue below). Next, let us denote these solutions as $(\Theta^*, \widehat{\vartheta}^*, \widetilde{\vartheta}^*)$, $(X^*, \widehat{x}^*, \widetilde{x}^*)$ and $(\Psi^*, \widehat{\psi}^*, \widetilde{\psi}^*)$; the corresponding processes A, B and Δ will be denoted as A^*, B^* and Δ^*. Consequently, the drift term of $\Theta_t^*(V_t^*(\pi) - X_t^*) + \Psi_t^*$ is non-positive for any admissible π and it is equal to 0 for $\pi^* = A_t^* - B_t^*(V_t^{v,*}(\pi^*) - X_t^*)$.

The three dimensional process $(\Theta^*, \widehat{\vartheta}^*, \widetilde{\vartheta}^*)$ is supposed to satisfy the BSDE

$$\begin{aligned}d\Theta_t &= \Theta_t\left(\frac{(\mu + \widehat{\vartheta}_t\sigma + \xi_t\varphi\widetilde{\vartheta}_t)^2}{\sigma^2 + \varphi^2\xi_t(\widetilde{\vartheta}_t + 1)}dt + \widehat{\vartheta}_t dW_t + \widetilde{\vartheta}_t dM_t\right) \qquad (1.31)\\ \Theta_T &= 1.\end{aligned}$$

We shall discuss this equation later.

The three dimensional process $(X^*, \widehat{x}^*, \widetilde{x}^*)$ is a solution of the **linear BSDE**

$$\begin{aligned}dX_t &= \frac{1}{\Delta_t}\left(\kappa_{1,t}\widehat{x}_t + \kappa_{2,t}\widetilde{x}_t\right)dt + \widehat{x}_t dW_t + \widetilde{x}_t dM_t\\ X_T &= X\end{aligned}$$

where

$$\kappa_{1,t} = \sigma\mu + \sigma\varphi\xi_t\widetilde{\vartheta}_t - \varphi^2\widehat{\vartheta}_t\xi_t(1 + \widetilde{\vartheta}_t),\ \kappa_{2,t} = \varphi\xi_t(1 + \widetilde{\vartheta}_t)(\mu + \sigma\widehat{\vartheta}_t) - \sigma^2\xi_t\widetilde{\vartheta}_t\,.$$

Thus,

$$X_t^* = \mathbf{E}_{\mathbf{Q}^\kappa}(X|\mathcal{G}_t),$$

where $d\mathbf{Q}^\kappa|_{\mathcal{G}_t} = L_t^{(\kappa)}d\mathbf{P}|_{\mathcal{G}_t}$ and

$$dL_t^{(\kappa)} = -L_{t-}^{(\kappa)}\left(\frac{\kappa_{1,t}}{\Delta_t}dW_t + \frac{\kappa_{2,t}}{\xi\Delta_t}dM_t\right).$$

The three dimensional process $(\Psi^*, \widehat{\psi}^*, \widetilde{\psi}^*)$ is solution of

$$\begin{aligned}d\Psi_t &= -\Theta_t\left(\xi_t(\widetilde{\vartheta}_t + 1)(A_t\varphi - \widetilde{x}_t)^2 + (A_t\sigma - \widehat{x}_t)^2\right)dt + \widehat{\psi}_t dW_t + \widetilde{\psi}_t dM_t\\ \Psi_T &= 0.\end{aligned}$$

Thus, noting that

$$\Psi_t^* + \int_0^t \Theta_s \left(\xi_s(\widetilde{\vartheta}_s + 1)(A_s\varphi - \widetilde{x}_s)^2 + (A_s\sigma - \widehat{x}_s)^2 \right) ds$$

is a **G**-martingale, we obtain that

$$\Psi_t^* = E\left(\int_t^T \Theta_s \left(\xi_s(\widetilde{\vartheta}_s + 1)(A_s\varphi - \widetilde{x}_s)^2 + (A_s\sigma - \widehat{x}_s)^2 \right) ds|\mathcal{G}_t \right). \tag{1.32}$$

Discussion of equation (1.31): Duality approach

Our aim is here to prove that the BSDE (1.31) has a solution. We take the opportunuity to correct a mistake in Bielecki et al (2004b) where we claim that, in the particular case where the intensity γ_t is constant, we get a solution of the form $\widetilde{\theta}_t$ constant. The solution that appear in Bielecki et al. is valid only in the case $P(\tau < T) = 1$. We proceed using duality approach.

The set of equivalent martingale measure is determined by the set of densities. From Kusuoka (1999) representation theorem, it follows that any strictly positive martingale in the filtration **G** can be written as

$$dL_t = L_{t-}(\ell_t dW_t + \chi_t dM_t) \tag{1.33}$$

for a **G**-predictable process χ satisfying $\chi_t > -1$. In order that L corresponds to the Radon-Nikodym density of an emm, a relation between ℓ and χ has to be satisfied in order to imply that process $L_t S_t$ is a **P** (local) martingale. (Recall that $r = 0$.) Straightforward application of integration by parts formula proves that the drift term of LS vanishes iff

$$\varphi\chi_t\xi_t + \sigma\ell_t + \nu = 0$$

Recall that by definition the variance optimal measure for L is a probability measure $\mathbf{Q}^*$ such that it minimizes $\mathbf{E}_{\mathbf{Q}^*}(L_T^2)$. At this moment we are unable to verify existence/uniqueness of such a measure in the context of our model. We thus assume that the measure exists,

Hypothesis: We assume that the variance optimal measure exists.

In what follows we shall use the same argument as in Bobrovnytska and Schweizer (2004). Towards this end we denote by L^* the Radon-Nikodym density of the variance optimal martingale measure. Let Z be the martingale $Z_t = \mathbf{E}_{\mathbf{Q}^*}(L_T^*|\mathcal{G}_t)$ and $U = L^*/Z$. It is proved in Delbaen and Shachermayer (Lemma 2.2) that, if the variance optimal martingale measure exists, then there exists a predictable process $\widehat{z}$ such that

$$dZ_t/Z_{t-} = \widehat{z}_t dS_t = z_t(\sigma dW_t + \varphi dM_t + \nu dt)$$

where $z_t = \widehat{z}_t S_{t-}$ (in the proof of lemma 2.2, the hypothesis of continuity of the asset is not required). The process L^* is a $(\mathbf{P}, \mathbf{G})$ martingale, hence there exist ℓ and χ such that

$$dL_t^* = L_{t-}^*(\ell_t dW_t + \chi_t dM_t)$$

From Itô's calculus, setting $U = L^*/Z$, we obtain

$$dU_t = U_{t-}\left(A_t dt + (\ell_t - z_t\sigma)dW_t + \left(\frac{1}{1+z_t\varphi}(\chi_t+1)-1\right)dM_t\right), \quad U_T = 1,$$

where

$$\begin{aligned} A_t &= z_t^2\sigma^2 + \xi_t(1+\chi_t)(z_t\varphi + \frac{1}{1+z_t\varphi} - 1) \\ &= z_t^2\sigma^2 + \xi_t(1+\chi_t)\frac{z_t^2\varphi^2}{1+z_t\varphi} \\ &= z_t^2\left(\sigma^2 + \xi_t(1+\chi_t)\frac{\varphi^2}{1+z_t\varphi}\right). \end{aligned}$$

We recall that $\varphi\chi_t\xi_t + \sigma\ell_t + \nu = 0$. Hence, letting

$$\begin{aligned} \widehat{u}_t &= \ell_t - z_t\sigma \\ \widetilde{u}_t &= \frac{1}{1+z_t\varphi}(\chi_t+1) - 1, \end{aligned}$$

we get

$$z_t = -\frac{\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)}.$$

It follows that

$$\begin{aligned} A_t &= z_t^2\left(\sigma^2 + \xi_t(1+\chi_t)\frac{\varphi^2}{1+z_t\varphi}\right) \\ &= z_t^2\left(\sigma^2 + \xi_t(1+z_t\varphi)(1+\widetilde{u}_t)\frac{\varphi^2}{1+z_t\varphi}\right) \\ &= z_t^2\left(\sigma^2 + \xi_t(1+\widetilde{u}_t)\varphi^2\right) \\ &= \frac{(\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t)^2}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)} \end{aligned}$$

so that process U is a solution of

$$dU_t = U_{t-}\left(\frac{(\nu + \sigma\widehat{u}_t + \varphi\xi_t\widetilde{u}_t)^2}{\sigma^2 + \varphi^2\xi_t(1+\widetilde{u}_t)}dt + \widehat{u}_t dW_t + \widetilde{u}_t dM_t\right), \quad U_T = 1,$$

which establishes that the BSDE (1.31) has a solution as long as the variance optimal martingale measure exists in our set-up.

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