HEDGING OF CREDIT DEFAULT SWAPTIONS IN A HAZARD PROCESS MODEL

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December 14, 2008

*The research of T.R. Bielecki was supported by NSF Grant 0604789.
†This research benefited from the support of the "Chaire Risque de Crédit", Fédération Bancaire Française.
‡The research of M. Rutkowski was supported by the ARC Discovery Project DP0881460.

Introduction

An important issue arising in the context of credit default swap (CDS) rates is a construction of an appropriate model in which a family of options written on credit default swaps, referred to hereafter as credit default swaptions, can be valued and hedged. Some previous efforts in this direction were largely motivated by the market practice of using a suitable version of the Black swaption formula. For an option written on a single-name forward credit default swap, such pricing formula was derived by Schönbucher [32]–[33] and Jamshidian [24], who formally used the risk-neutral valuation formula in an intensity-based credit risk model, which was not fully specified. The derivations of a version of the Black formula for credit default swaptions presented in these papers are based on rather abstract approximation arguments for a positive martingale, as opposed to an explicit construction of a (lognormal) model for a family of CDS rates associated with a given tenor structure, in which the pricing formula for a credit default swaption could be supported by strict replication arguments. Such a construction was provided, albeit under some simplifying assumptions, in recent papers by Brigo [8]–[9], who analyzed the joint dynamics of certain families of forward CDS rates under judiciously chosen martingale measures. He showed that in some cases (most notably, for a family of one-period forward CDS rates), it is possible to develop a change of a numéraire approach, which is analogous to arbitrage-free modeling of forward LIBOR rates. He also emphasized the difficulties that arise in the context of modeling of a family of co-terminal forward CDS rates.

In an alternative approach, Ben-Ameur et al. [2] and Brigo and Cousot [12] dealt with the valuation of European and Bermudan credit default swaptions within the framework of the intensity-based SSRD (Shifted Square-Root Diffusion) model, which was introduced by Brigo and Alfonsi [11]. More recently, Brigo and El-Bachir [13] provided a complete study of valuation of credit default swaptions in the SSRJD (Shifted Square-Root Jump-Diffusion) model. It is worth stressing that none of the above-mentioned papers addresses the issue of dynamic hedging of credit default swaptions. The only work, that we are aware of, that discusses hedging of CDS option is Zhang et al. [35]. The methodology used in this paper is driven by the idea of simultaneous derivation of the price of the swaption and of the hedging strategy by solving a sequence of some constrained optimization problems. It appears that the hedging strategy discussed in [35] is not necessarily self-financing. It is thus not quite clear what is the meaning of the calculated "price" of the swaption. In addition, the relationship between the par CDS rate and the risk-neutral intensity in [35] (see formula (9) therein) appears to be questionable. Such relationship is known to be valid in the case of constant default intensity and recovery rate, in which case the par CDS spread is constant as well, but it fails to hold in general. Consequently, there appears to be a mismatch between the postulated evolution of the par CDS rate and the stochastic evolution of the hazard rate.

In our previous papers Bielecki et al. [4]–[6] (see also Schmidt [34]), we provided some general results on hedging of general defaultable claims in copula-based and intensity-based frameworks. The goal of this work is to exemplify the usefulness of these results in the context of valuation and hedging of credit default swaptions in hazard process models. In Section 1, we recall some results from [6], which will prove useful in the sequel. Section 2, deals with the preliminary results on the valuation and hedging of a credit default swaption in a general hazard process model. We also present there the market convention for pricing of a credit default swaption. In Section 3, the modeling of default time and dynamics of prices are examined in some detail. Finally, Section 4 is devoted to the study of hedging strategies for a credit default swaption in the CIR intensity model.

Let us set some notation used throughout this work. A *random time* is a strictly positive random variable τ defined on an underlying probability space $(\Omega, \mathcal{G}, \mathbb{Q})$. In view of its financial interpretation, we will refer to it as a *default time*. We define the *default indicator process* $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process. We assume that we are given, in addition, some auxiliary filtration $\mathbb{F}$ and we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$, meaning that $\mathcal{G}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. The filtration $\mathbb{G}$ is referred to as to the *full filtration*. It is clear that τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time, but not necessarily an $\mathbb{F}$-stopping time. All processes are defined on the space $(\Omega, \mathbb{G}, \mathbb{Q})$. In what follows, T is a finite horizon and, unless otherwise stated, all processes considered are assumed to be $\mathbb{G}$-adapted and with càdlàg (i.e., RCLL) sample paths.

1 Dynamics of Asset Prices in a Hazard Process Model

We assume that the underlying market model is arbitrage-free, meaning that it admits a *spot martingale measure* $\mathbb{Q}^*$ (not necessarily unique) equivalent to $\mathbb{Q}$. A spot martingale measure is associated with the choice of the savings account B as a numéraire, in the sense that the price process of any tradeable security, which pays no coupons or dividends, is a $(\mathbb{Q}^*, \mathbb{G})$-martingale when discounted by the *savings account* B, which is given by

$$B_t = \exp\Big(\int_0^t r_u\,du\Big), \quad \forall\, t \in \mathbb{R}_+, \tag{1}$$

where the short-term r is assumed to follow an $\mathbb{F}$-progressively measurable stochastic process. The choice of a suitable term structure model is arbitrary and it is beyond the scope of the present work.

Let us denote by $G_t = \mathbb{Q}^*(\tau > t \,|\, \mathcal{F}_t)$ the *survival process* of τ with respect to the reference filtration $\mathbb{F}$. We assume throughout that $\mathcal{F}_0$ is trivial. We postulate that $G_0 = 1$ and $G_t > 0$ for every $t \in \mathbb{R}_+$, so that the *hazard process* $\Gamma = -\ln G$ of τ with respect to the filtration $\mathbb{F}$ is well defined. This means, in particular, that the case where τ is an $\mathbb{F}$-stopping time is not covered by the foregoing results.

For any $\mathbb{Q}^*$-integrable and $\mathcal{F}_T$-measurable random variable Y, the following classic formula holds

$$\mathbb{E}_{\mathbb{Q}^*}(\mathbb{1}_{\{T<\tau\}}Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\, G_t^{-1}\mathbb{E}_{\mathbb{Q}^*}(G_T Y \,|\, \mathcal{F}_t).$$

Clearly, the process G is a bounded $(\mathbb{Q}^*, \mathbb{G})$-supermartingale and thus it admits the unique Doob-Meyer decomposition $G = \mu - \nu$, where μ is a martingale part and ν is a predictable increasing process. Note that, if G is continuous, then the processes μ and ν are continuous as well.

In this section, we work under the following standing assumption.

Assumption 1.1 The survival process G is continuous and the increasing process ν in the Doob-Meyer decomposition $G = \mu - \nu$ is absolutely continuous with respect to the Lebesgue measure, so that $d\nu_t = \upsilon_t\,dt$ for some $\mathbb{F}$-progressively measurable, non-negative process υ. We denote by λ the $\mathbb{F}$-progressively measurable process defined as $\lambda_t = G_t^{-1}\upsilon_t$. The process λ is called the $\mathbb{F}$-*intensity* of default time.

Assumption 1.1 implies that $dG_t = d\mu_t - \lambda_t G_t\,dt$, where the $(\mathbb{Q}^*, \mathbb{F})$-martingale μ is continuous. Moreover, continuity of G implies that $\mathbb{Q}^*(\tau = t) = 0$ for any $t \in \mathbb{R}_+$. Finally, it is known (see, e.g., [17] or [18]) that under Assumption 1.1 the process M, which is given by the formula

$$M_t = H_t - \int_0^{t\wedge\tau} \lambda_u\,du = H_t - \int_0^t (1 - H_u)\lambda_u\,du, \tag{2}$$

is a $(\mathbb{Q}^*, \mathbb{G})$-martingale.

We will now recall some valuation results for a generic defaultable claim.

Definition 1.1 A *defaultable claim maturing at* T is a quadruple (X, A, Z, τ), where X is an $\mathcal{F}_T$-measurable random variable, $A = (A_t)_{t\in[0,T]}$ is an $\mathbb{F}$-adapted, continuous process of finite variation with $A_0 = 0$, $Z = (Z_t)_{t\in[0,T]}$ is an $\mathbb{F}$-predictable process and τ is a default time. The *dividend process* $D = (D_t)_{t\in\mathbb{R}_+}$ of a defaultable claim maturing at T equals, for every $t \in \mathbb{R}_+$,

$$D_t = X\mathbb{1}_{\{T<\tau\}}\mathbb{1}_{[T,\infty[}(t) + \int_{]0,t\wedge T]} (1 - H_u)\,dA_u + \int_{]0,t\wedge T]} Z_u\,dH_u.$$

The financial interpretation of this definition is the following: X is the *promised payoff*, A represents the process of *promised dividends* and the process Z, termed the *recovery process*, specifies the recovery payoff at default.

Definition 1.2 The *ex-dividend price* process S associated with the dividend process D equals, for every $t \in [0,T]$,

$$S_t = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big).$$

The *cumulative price* process S^c associated with the dividend process D is given by the following expression, for every $t \in [0,T]$,

$$S_t^c = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]0,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big) = S_t + B_t \int_{]0,t]} B_u^{-1} \, dD_u.$$

It can be shown (see [6]) that the ex-dividend price of the defaultable claim (X, A, Z, τ) satisfies, for every $t \in [0,T]$,

$$S_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \, \mathbb{E}_{\mathbb{Q}^*} \bigg(B_T^{-1} G_T X \mathbb{1}_{\{t<T\}} + \int_t^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_t^T B_u^{-1} G_u \, dA_u \, \bigg| \, \mathcal{F}_t \bigg). \tag{3}$$

This means, in particular, that for any $t \in [0,T]$ the ex-dividend price S equals $S_t = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t$ for the $\mathbb{F}$-adapted process $\widetilde{S}$, which is termed the *ex-dividend pre-default price* of a defaultable claim.

Under the assumption that all $(\mathbb{Q}^*, \mathbb{F})$-martingales are continuous (see Assumption 1.2 in [6]), the following result is valid (we refer to [6] for the proof).

Proposition 1.1 *Let μ be the $(\mathbb{Q}^*, \mathbb{F})$-martingale part of the Doob-Meyer decomposition of G and let m be the $(\mathbb{Q}^*, \mathbb{F})$-martingale given by the formula*

$$m_t = \mathbb{E}_{\mathbb{Q}^*} \bigg(B_T^{-1} G_T X + \int_0^T B_u^{-1} G_u Z_u \lambda_u \, du + \int_0^T B_u^{-1} G_u \, dA_u \, \bigg| \, \mathcal{F}_t \bigg).$$

(i) The dynamics of the ex-dividend price S on $[0,T]$ are

$$\begin{aligned} dS_t &= -S_{t-} \, dM_t + (1-H_t)\big((r_t S_t - \lambda_t Z_t)\, dt + dA_t\big) \\ &\quad + (1-H_t) G_t^{-1} \big(B_t \, dm_t - S_t \, d\mu_t\big) + (1-H_t) G_t^{-2} \big(S_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big). \end{aligned}$$

(ii) The dynamics of the pre-default price $\widetilde{S}$ on $[0,T]$ are

$$\begin{aligned} d\widetilde{S}_t &= \big((\lambda_t + r_t)\widetilde{S}_t - \lambda_t Z_t\big)\, dt + dA_t + G_t^{-1}\big(B_t \, dm_t - \widetilde{S}_t \, d\mu_t\big) \\ &\quad + G_t^{-2}\big(\widetilde{S}_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big). \end{aligned}$$

(iii) The dynamics of the cumulative price S^c on $[0,T]$ are

$$\begin{aligned} dS_t^c &= r_t S_t^c \, dt + (Z_t - S_{t-}) \, dM_t \\ &\quad + (1-H_t) G_t^{-1} \big(B_t \, dm_t - S_t \, d\mu_t\big) + (1-H_t) G_t^{-2} \big(S_t \, d\langle \mu \rangle_t - B_t \, d\langle \mu, m \rangle_t\big). \end{aligned}$$

2 Credit Default Swaptions

We are in a position to analyze credit default swaps and related options in a hazard process model introduced in Section 1. Throughout this section, we maintain the standing Assumption 1.1.

2.1 Forward CDS

A forward CDS can be issued at any time $s \in [0,U]$ and it gives default protection over the future time interval $[U,T]$. If the reference entity defaults prior to the start date U the contract is terminated and no payments are made. In what follows, both the recovery process δ and the time period $[U,T]$ are fixed.

Definition 2.1 A *forward CDS* issued at time s, with start date U and maturity T, a constant rate κ and recovery at default is a defaultable claim $(0, A, Z, \tau)$ where $dA_t = -\kappa \mathbb{1}_{]U,T]}(t)\, dL_t$ and $Z_t = \delta_t \mathbb{1}_{[U,T]}(t)$ for every $t \in [0, T]$. An $\mathbb{F}$-predictable process $\delta : [U, T] \to \mathbb{R}$ represents the *default protection* and a constant κ is the *CDS rate.*

The increasing, $\mathbb{F}$-adapted process L, such that $L_T - L_U > 0$, represents the tenor structure of the fee leg of a forward CDS (for more information on specific market conventions, we refer to [8]–[9] or [31]). For a stylized forward CDS, one may wish to set $L_t = t$, as in [5]–[6]. This convention is by no means necessary for our further developments, however. Note also that κ is constant over time, but it is an $\mathcal{F}_s$-measurable random variable for a forward CDS issued at time s.

Since a forward CDS does not have any cash flows prior to its start date U and, under our assumptions, $\mathbb{Q}^*(\tau = U) = 0$, its price $S_t(\kappa)$ for any $t \in [s, U]$ can be considered as either the cum-dividend price or the ex-dividend price. Therefore, the price of a forward CDS equals, for every $t \in [s, U]$,

$$S_t(\kappa) = S^c_t(\kappa) = B_t \, \mathbb{E}_{\mathbb{Q}^*} \Big(\mathbb{1}_{\{U \le \tau \le T\}} B^{-1}_\tau \delta_\tau - \kappa \int_{]\tau \wedge U, \tau \wedge T]} B^{-1}_u \, dL_u \,\Big|\, \mathcal{G}_t \Big) \tag{4}$$

and formula (3) yields

$$\begin{aligned} S_t(\kappa) &= \mathbb{1}_{\{t<\tau\}} \Big\{ \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \Big(-\int_U^T B^{-1}_u \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big) - \kappa \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big) \Big\} \\ &= \mathbb{1}_{\{t<\tau\}} \big(\widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T) \big) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa), \end{aligned}$$

where $\widetilde{P}(t, U, T)$ represents the pre-default value at time t of the protection leg and the *CDS annuity* $\widetilde{A}(t, U, T)$ represents the pre-default value at time t of the survival annuity stream per unit of the rate κ. It is worth noting that neither $\widetilde{P}(t, U, T)$ nor $\widetilde{A}(t, U, T)$ depend on s and the CDS annuity $(\widetilde{A}(t, U, T),\, t \in [0, U])$ follows a strictly positive process.

By the *forward market CDS* issued at time $t \in [0, U]$ we mean a forward CDS in which the $\mathcal{F}_t$-measurable rate κ is chosen in such a way that the contract is valueless at time t. The corresponding (pre-default) *forward CDS rate* at time t is the unique $\mathcal{F}_t$-measurable random variable $\kappa(t, U, T)$ solving the equation $\widetilde{S}_t(\kappa(t, U, T)) = 0$. It is easily seen that, for every $t \in [0, U]$,

$$\kappa(t, U, T) = \frac{\widetilde{P}(t, U, T)}{\widetilde{A}(t, U, T)} = -\frac{\mathbb{E}_{\mathbb{Q}^*} \Big(\int_U^T B^{-1}_u \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big)}{\mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big)} = \frac{M^P_t}{M^A_t}, \tag{5}$$

where the $(\mathbb{Q}^*, \mathbb{F})$-martingales $(M^P_t,\, t \in [0, U])$ and $(M^A_t,\, t \in [0, U])$ are given by the following expressions

$$M^P_t = -\mathbb{E}_{\mathbb{Q}^*} \Big(\int_U^T B^{-1}_u \delta_u \, dG_u \,\Big|\, \mathcal{F}_t \Big), \quad M^A_t = \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]U,T]} B^{-1}_u G_u \, dL_u \,\Big|\, \mathcal{F}_t \Big). \tag{6}$$

Lemma 2.1 *For a forward CDS issued at time s with an $\mathcal{F}_s$-measurable rate κ we have, for every $t \in [s, U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)(\kappa(t, U, T) - \kappa). \tag{7}$$

Proof. It suffices to observe that

$$\begin{aligned} S_t(\kappa) &= S_t(\kappa) - S_t(\kappa(t, U, T)) \\ &= \mathbb{1}_{\{t<\tau\}} \big(\widetilde{P}(t, U, T) - \kappa \widetilde{A}(t, U, T) - \widetilde{P}(t, U, T) + \kappa(t, U, T) \widetilde{A}(t, U, T) \big) \\ &= \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t, U, T)(\kappa(t, U, T) - \kappa) \end{aligned}$$

as required. □

2.2 Credit Default Swaption

Let us consider a European call option with expiry date R (where $0 < R \le U$) and zero strike written on the value of a forward start CDS issued at time s (with $0 \le s < R$) with start date U, maturity T and an $\mathcal{F}_s$-measurable rate κ. The swaption's payoff C_R at its expiry is thus given by the equality $C_R = (S_R(\kappa))^+$. Using Lemma 2.1, we obtain

$$C_R = \mathbb{1}_{\{R<\tau\}} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+.$$

The formula above shows that a call option with zero strike on the value of the forward CDS with rate κ is formally equivalent to a call option on the forward CDS rate with strike κ. Note that this option is knocked out if default occurs prior to R. For $R < U$, we find it convenient to refer to this contract as the forward credit default swaption.

Remark. It is worth stressing that the credit default swaption is considered here as a derivative asset in a market model in which CDSs with maturities U and T are assumed to be traded assets. The corresponding forward CDS can be easily synthetized by static positions in traded CDSs.

In the first step, we are interested in the value C_t of this claim at time $t \in [s,R]$.

Lemma 2.2 *Assume that the claim C_R is attainable. Then the price at time $t \in [s,R]$ of the credit default swaption equals*

$$C_t = \mathbb{1}_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}^*}\left(\frac{G_R}{B_R} \widetilde{A}(R,U,T)\big(\kappa(R,U,T) - \kappa\big)^+ \,\Big|\, \mathcal{F}_t \right). \tag{8}$$

Proof. It suffices to use the risk-neutral valuation formula. □

In order to simplify formula (8), we will define an equivalent probability measure $\widehat{\mathbb{Q}}$ on $(\Omega, \mathcal{F}_R)$. Towards this end, we recall that the process $(M^A_t,\, t \in [0,T])$ is a strictly positive $(\mathbb{Q}^*, \mathbb{F})$-martingale. We postulate that the Radon-Nikodým density of $\widehat{\mathbb{Q}}$ with respect to $\mathbb{Q}^*$ is given as

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}^*} = \frac{M^A_R}{M^A_0}, \quad \mathbb{Q}^*\text{-a.s.} \tag{9}$$

Therefore, for every $t \in [0,R]$,

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}^*}\Big|\mathcal{F}_t = \frac{M^A_t}{M^A_0}, \quad \mathbb{Q}^*\text{-a.s.}$$

It is advantageous to work under $\widehat{\mathbb{Q}}$ since the process $(M^P_t = M^A_t \kappa(t,U,T),\, t \in [0,R])$, defined in (6), is manifestly a $(\mathbb{Q}^*, \mathbb{F})$-martingale, which in turn implies that the forward CDS rate $(\kappa(t,U,T),\, t \in [0,R])$ is a $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale. In addition, the pricing formula (8) for the credit default swaption takes a more convenient shape under $\widehat{\mathbb{Q}}$, as can be seen from the following result (cf. [14] or [31]).

Proposition 2.1 *The price of the credit default swaption is given by the following expression, for every $t \in [s,R]$,*

$$C_t = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa(R,U,T) - \kappa)^+ \,\big|\, \mathcal{F}_t \big) = \mathbb{1}_{\{t<\tau\}} \widetilde{C}_t, \tag{10}$$

where $\widetilde{C}_t$ is the pre-default value of the credit default swaption at time t.

Proof. Using (8), we obtain

$$\begin{aligned}
C_t &= \mathbb{1}_{\{t<\tau\}} B_t G_t^{-1}\, \mathbb{E}_{\mathbb{Q}^*}\left(G_R B_R^{-1} \widetilde{A}(R,U,T)(\kappa(R,U,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)(M^A_t)^{-1}\, \mathbb{E}_{\mathbb{Q}^*}\left(M^A_R (\kappa(R,U,T) - \kappa)^+ \,\Big|\, \mathcal{F}_t \right) \\
&= \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big((\kappa(R,U,T) - \kappa)^+ \,\big|\, \mathcal{F}_t \big),
\end{aligned}$$

where the last equality follows from the Bayes formula. □

To proceed further, we assume that the filtration $\mathbb{F}$ is generated by a (possibly multidimensional) Brownian motion W under $\mathbb{Q}^*$. Since M^P and M^A are strictly positive $(\mathbb{Q}^*, \mathbb{F})$-martingales, we deduce from the predictable representation theorem for a Brownian filtration that

$$dM_t^P = M_t^P \sigma_t^P \, dW_t, \quad dM_t^A = M_t^A \sigma_t^A \, dW_t,$$

for some $\mathbb{F}$-predictable processes σ^P and σ^A. To establish the next result, it suffices to apply the Itô formula to (5) and use the Girsanov theorem.

Lemma 2.3 *Let the filtration $\mathbb{F}$ be generated by a Brownian motion W under $\mathbb{Q}^*$. Then the forward CDS rate $(\kappa(t,U,T),\, t \in [0,R])$ is $(\widehat{\mathbb{Q}}, \mathbb{F})$-martingale and*

$$d\kappa(t,U,T) = \kappa(t,U,T)\sigma_t^\kappa \, d\widehat{W}_t, \tag{11}$$

where the $\mathbb{F}$-predictable process σ^κ satisfies $\sigma^\kappa = \sigma^P - \sigma^A$ and the $(\widehat{\mathbb{Q}}, \mathbb{F})$-Brownian motion $\widehat{W}$ is given by the formula

$$\widehat{W}_t = W_t - \int_0^t \sigma_u^A \, du, \quad \forall\, t \in [0,R].$$

To a large extent, this result is model independent. In particular, if the process σ^κ is deterministic so that Black formula can be applied for pricing and hedging. Such an assumption is quite constraining, as explained, for example, in Brigo and El-Bachir [13]. In Section 4, we shall present pricing and hedging results for the model in which σ^κ is not deterministic, but in which hedging and pricing results are still numerically feasible due to special structure of the hazard process.

2.3 Hedging with Forward CDS and Swap Portfolio

In order to get the simplest form of a hedging strategy for a credit default swaption, we assume that a forward CDS with an $\mathcal{F}_s$-measurable rate κ is traded. As a second traded instrument, we take the defaultable bond portfolio corresponding to the CDS annuity. Note that $\widetilde{A}(t,U,T)$ can be seen as the pre-default value at time $t \in [s,U]$ of a particular portfolio of defaultable bonds with zero recovery, referred to hereafter as the *swap portfolio*. If default occurs at some date $t \in [s,U]$, the wealth of this portfolio will necessarily fall to zero. Of course, the same property holds for the forward CDS as well as for the credit default swaption. This means that in what follows we will only need to focus on the dynamics of the pre-default value of the swaption and the pre-default wealth of a hedging portfolio.

Let $A(t,U,T)$ be the price process of the swap portfolio at time $t \in [s,U]$. Formally, we set $A(t,U,T) = \mathbb{1}_{\{t<\tau\}} \widetilde{A}(t,U,T)$. Recall also that $S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \widetilde{S}_t(\kappa)$.

Let $\varphi = (\varphi^1, \varphi^2)$ be a *trading strategy*, where φ^1 and φ^2 are $\mathbb{G}$-predictable processes. The wealth of φ equals, for any $t \in [s,R]$,

$$V_t(\varphi) = \varphi_t^1 S_t(\kappa) + \varphi_t^2 A(t,U,T),$$

and thus the pre-default wealth satisfies, for any $t \in [s,R]$,

$$\widetilde{V}_t(\varphi) = \varphi_t^1 \widetilde{S}_t(\kappa) + \varphi_t^2 \widetilde{A}(t,U,T).$$

Of course, the equality $V_t(\varphi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\varphi)$ holds for any $t \in [s,R]$ and thus it suffices to examine a replicating strategy on the interval $[s, s \vee \tau \wedge R]$. Therefore, it is enough to search for $\mathbb{F}$-predictable processes $\widetilde{\varphi}^i$, $i = 1,2$ such that for every $t \in [s,R]$ we have that $\mathbb{1}_{\{t<\tau\}} \varphi_t^i = \widetilde{\varphi}_t^i$ for $i = 1,2$. We then say that φ replicates a credit default swaption if $\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{C}_t$ for every $t \in [s,R]$ or, equivalently, if $V_t(\varphi) = C_t$ for every $t \in [s,R]$.

A replicating strategy φ is required to be *self-financing*, in the sense that

$$d\widetilde{V}_t(\varphi) = d\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{\varphi}_t^1 \, d\widetilde{S}_t(\kappa) + \widetilde{\varphi}_t^2 \, d\widetilde{A}(t,U,T).$$

It can be easily shown by Itô's formula that the relative pre-default wealth satisfies

$$d(\widetilde{V}_t(\widetilde{\varphi})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d(\widetilde{S}_t(\kappa)/\widetilde{A}(t,U,T)). \tag{12}$$

Proposition 2.2 *Assume that the Brownian motion W is one-dimensional. Then the replicating strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ for the credit default swaption is given by, for every $t \in [s,R]$,*

$$\widetilde{\varphi}^1_t = \frac{\widetilde{\xi}_t}{\kappa(t,U,T)\sigma^\kappa_t}, \quad \widetilde{\varphi}^2_t = \frac{\widetilde{C}_t - \widetilde{\varphi}^1_t \widetilde{S}_t(\kappa)}{\widetilde{A}(t,U,T)}, \tag{13}$$

where $\widetilde{\xi}$ is the process satisfying

$$\frac{\widetilde{C}_R}{\widetilde{A}(R,U,T)} = \frac{\widetilde{C}_0}{\widetilde{A}(0,U,T)} + \int_0^R \widetilde{\xi}_t \, d\widehat{W}_t. \tag{14}$$

Proof. The existence of $\widetilde{\xi}$ is a consequence of the predictable representation theorem for the Brownian filtration. In view of (7), (11) and (12), we have that

$$d(\widetilde{V}_t(\widetilde{\varphi})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d(\widetilde{S}_t(\widetilde{\kappa})/\widetilde{A}(t,U,T)) = \widetilde{\varphi}^1_t \, d\kappa(t,U,T) = \widetilde{\varphi}^1_t \kappa(t,U,T)\sigma^\kappa_t \, d\widehat{W}_t. \tag{15}$$

A comparison of (14) and (15) yields the desired expression for the hedge ratio $\widetilde{\varphi}^1$. Standard arguments show that the strategy $\widetilde{\varphi}$ given by (13) is self-financing and its pre-default wealth satisfies $\widetilde{V}_t(\widetilde{\varphi}) = \widetilde{C}_t$ for every $t \in [s,R]$. As already mentioned above, it default occurs prior to or at expiration date R of a swaption then the wealth of the portfolio φ falls to zero and the same property is satisfied by the price process of the credit default swaption. □

In view of the last result, the problem of replication of the credit default swaption is reduced to the computation of the quantities $\widetilde{\xi}_t, \kappa(t,U,T)$ and σ^κ_t that appear in formula (13). Of course, such computations are far from trivial, in general, and they rarely lead to closed-form analytical solution. In the next subsection, we will consider the classic case where they are fairly standard. In Section 4, we will address the same issue in the framework of the CIR default intensity model.

2.4 Black Formula for Credit Default Swaptions

The goal of this section is to examine the case of a deterministic volatility of a forward CDS rate. For a more detailed discussion of applicability of some variant of the Black formula to valuation of credit default (index) swaptions, the interested reader is referred to Brigo and Morini [14], Morini and Brigo [29], and Rutkowski and Armstrong [31]. Let us only mention here that although an arbitrage-free model of a family of forward CDS rates underpinning Proposition 2.3 is rather difficult to construct, the pricing formula (16) was nevertheless adopted as the market convention for quotation of credit default swaptions, and thus we find it is natural to consider it here as a benchmark approach.

Proposition 2.3 *Assume that the Brownian motion W is one-dimensional and the volatility σ^κ of the forward CDS rate $(\kappa(t,U,T),\, t \in [0,U])$ is deterministic for $t \in [s,R]$. Then the pre-default value of the (forward) credit default swaption with $\mathcal{F}_s$-measurable strike κ and expiry date $R \leq U$ equals, for every $t \in [s,R]$,*

$$\widetilde{C}_t = \widetilde{A}(t,U,T)\Big(\kappa_t N\big(d_+(\kappa_t,t,R)\big) - \kappa N\big(d_-(\kappa_t,t,R)\big)\Big), \tag{16}$$

where we write $\kappa_t = \kappa(t,U,T)$ and where

$$d_\pm(\kappa_t,t,R) = \frac{\ln(\kappa_t/\kappa) \pm \frac{1}{2}\int_t^R (\sigma^\kappa(u))^2\,du}{\sqrt{\int_t^R (\sigma^\kappa(u))^2\,du}}.$$

Equivalently,

$$\widetilde{C}_t = \widetilde{P}(t,U,T)\, N\big(d_+(\kappa_t,t,R)\big) - \kappa \widetilde{A}(t,U,T)\, N\big(d_-(\kappa_t,t,R)\big). \tag{17}$$

The replicating strategy $\widetilde{\varphi}$ is given by $\widetilde{\varphi}^1_t = N\big(d_+(\kappa_t,t,R)\big)$ and $\widetilde{\varphi}^2_t$ given by (13)

Proof. Derivation of the pricing formula (16) relies on standard arguments and thus it is omitted. To find the hedging strategy, it suffices to observe that, by an application of the Itô formula to expression (16), the volatility of the process $\widetilde{C}$ equals, for every $t \in [0,T]$,

$$\sigma^C_t = (\widetilde{C}_t)^{-1} \kappa_t \sigma^\kappa_t\, N\big(d_+(\kappa(t,U,T),t,R)\big).$$

The result is now a straightforward consequence of Proposition 2.2. □

In practice, hedging should rather be done by taking positions at any date t in the *market CDS*, that is, the just-issued CDS with the fixed spread $\kappa(t,U,T)$. An explicit representation for this strategy in continuous-time is more cumbersome, however, since one needs to deal with an uncountable family of traded assets (see Section 1.1.3 in [6]).

3 Modeling of Default Time

The goal of this section is to discuss a particular method of modeling the default time, which, under some assumptions, is consistent with the hazard process model introduced in Section 1. One of our main goals will be to establish explicit formulae for the volatilities of $\mathbb{F}$-martingales M^P and M^A introduced in Section 2 in terms of some fundamental quantities associated with $\mathcal{F}_t$-conditional distributions of default time.

To proceed, we need first to introduce suitable definitions and assumptions underpinning the above-mentioned modeling approach. Let then τ be a random time defined on a probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$.

Definition 3.1 For any fixed $u \in \mathbb{R}_+$, we define the $\mathbb{F}$-martingale $G^u_t = \mathbb{Q}^*(\tau > u \,|\, \mathcal{F}_t)$ for $t \in [0,T]$.

For conciseness, we shall frequently write G_t instead of G^t_t (of course, this convention is consistent with notation introduced in Section 1). Recall that the process $(G_t,\, t \in [0,T])$ is an $\mathbb{F}$-supermartingale. It is assumed throughout that G is a strictly positive and continuous process.

We will work throughout under the following standing assumption, which was also postulated in related papers by El Karoui et al. [18] and Jeanblanc and Le Cam [25].

Assumption 3.1 There exists a family of $\mathbb{F}$-adapted processes $(f^x_t,\, t \in [0,T])$, where $x \in \mathbb{R}_+$, such that for any t the map $(\omega, x) \to f^x_t(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$ measurable and, for any $u \in \mathbb{R}_+$,

$$G^u_t = \int_u^\infty f^x_t\, dx, \quad \forall\, t \in [0,T].$$

Assumption 3.1 implies, in particular, that the probability distribution of the random variable τ has the probability density function $f^{\cdot}_0$ with respect to Lebesgue measure, so that $\mathbb{Q}^*(\tau \in dx) = f^x_0\, dx$. More generally, for any $t \in [0,T]$, the random variable $f^{\cdot}_t$ represents the conditional density of τ with respect to the σ-field $\mathcal{F}_t$, that is, $f^x_t\, dx = \mathbb{Q}^*(\tau \in dx \,|\, \mathcal{F}_t)$. To alleviate notation, we denote $f^t_t = f_t$ and $\widehat{\lambda}_t = G^{-1}_t f_t$. Let us observe that, for any fixed x and every $t \in [0,T]$,

$$f^x_t = \lim_{h \downarrow 0} \frac{1}{h}\, \mathbb{Q}^*(\tau \in [x, x+h] \,|\, \mathcal{F}_t).$$

This leads to the following simple lemma, which will prove useful in the sequel.

Lemma 3.1 *The process $(f_t^x,\, t \in [0,T])$ is a non-negative $(\mathbb{Q}^*,\mathbb{F})$-martingale. In particular, for any fixed $x \in \mathbb{R}_+$, it holds that $\mathbb{E}_{\mathbb{Q}^*}(f_x \,|\, \mathcal{F}_t) = f_t^x$ for every $t \in [0,x]$.*

Note that Assumption 3.1 implies that any $(\mathbb{Q}^*,\mathbb{F})$-martingale is a $(\mathbb{Q}^*,\mathbb{G})$-semimartingale (see Jacod [21] or Jeanblanc and Le Cam [25]). Moreover, we have the following well known result.

Lemma 3.2 *Under Assumption 3.1, the process $(\widehat{M}_t,\, t \in [0,T])$, given by the formula*

$$\widehat{M}_t = H_t - \int_0^{t\wedge\tau} \widehat{\lambda}_u \, du,$$

is a $(\mathbb{Q}^*,\mathbb{G})$-martingale.

Proof. For the sake of completeness, we provide the proof of the lemma (it can be found in [18] and [25]). Let us fix the dates $0 \le s < t \le T$. We have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) &= \mathbb{1}_{\{s<\tau\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{s<\tau\le t\}} - \int_s^{t\wedge\tau} \widehat{\lambda}_u \, du \,\Big|\, \mathcal{G}_s\Big)\\ &= \mathbb{1}_{\{s<\tau\}}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{s<\tau\le t\}} - \mathbb{1}_{\{t<\tau\}}\int_s^{t} \widehat{\lambda}_u \, du - \mathbb{1}_{\{\tau\le t\}}\int_s^t \mathbb{1}_{\{u<\tau\le t\}}\widehat{\lambda}_u\, du \,\Big|\, \mathcal{G}_s\Big).\end{aligned}$$

For any $0 \le s \le t \le T$, the following equalities hold, on the event $\{s < \tau\}$,

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{1}_{\{s<\tau\le t\}} \,|\, \mathcal{G}_s\big) &= \frac{1}{G_s}(G_s - G_s^t),\\ \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{t<\tau\}}\int_s^t \widehat{\lambda}_u\, du \,\Big|\, \mathcal{G}_s\Big) &= \frac{1}{G_s}\, \mathbb{E}_{\mathbb{Q}^*}\Big(G_t \int_s^t \widehat{\lambda}_u\, du \,\Big|\, \mathcal{F}_s\Big),\\ \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{\tau\le t\}}\int_s^t \mathbb{1}_{\{u<\tau\le t\}}\widehat{\lambda}_u\, du \,\Big|\, \mathcal{G}_s\Big) &= \frac{1}{G_s}\, \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u (G_t^u - G_t)\, du \,\Big|\, \mathcal{F}_s\Big),\end{aligned}$$

where two last equalities are obtained by conditioning with respect to $\mathcal{F}_t$. Consequently,

$$\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) = \mathbb{1}_{\{s<\tau\}} \frac{1}{G_s}\left(G_s - G_s^t - \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u G_t^u\, du \,\Big|\, \mathcal{F}_s\Big)\right).$$

To conclude that $\mathbb{E}_{\mathbb{Q}^*}(\widehat{M}_t - \widehat{M}_s \,|\, \mathcal{G}_s) = 0$ for any $0 \le s \le t \le T$, it suffices to note that

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t \widehat{\lambda}_u G_t^u\, du \,\Big|\, \mathcal{F}_s\Big) &= \int_s^t \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{E}_{\mathbb{Q}^*}\big(\widehat{\lambda}_u G_t^u \,|\, \mathcal{F}_u\big) \,\Big|\, \mathcal{F}_s\Big)\, du\\ &= \int_s^t \mathbb{E}_{\mathbb{Q}^*}\Big(\widehat{\lambda}_u\, \mathbb{E}_{\mathbb{Q}^*}\big(G_t^u \,|\, \mathcal{F}_u\big) \,\Big|\, \mathcal{F}_s\Big)\, du = \int_s^t \mathbb{E}_{\mathbb{Q}^*}(\widehat{\lambda}_u G_u \,|\, \mathcal{F}_s)\, du\\ &= \mathbb{E}_{\mathbb{Q}^*}\Big(\int_s^t f_u\, du \,\Big|\, \mathcal{F}_s\Big) = \int_s^t f_s^u\, du = G_s - G_s^t,\end{aligned}$$

since $\mathbb{E}_{\mathbb{Q}^*}(G_t^u \,|\, \mathcal{F}_u) = \mathbb{Q}^*(\tau > u \,|\, \mathcal{F}_u) = G_u$ for arbitrary $0 \le u \le t \le T$ and, by Lemma 3.1, $\mathbb{E}_{\mathbb{Q}^*}(f_u \,|\, \mathcal{F}_s) = f_s^u$ for $s \le u$. □

The next standing assumption will allow us to make use of the predictable representation property of the Brownian filtration.

Assumption 3.2 The filtration $\mathbb{F}$ is generated by a one-dimensional Brownian motion $(W_t,\, t \in [0,T])$ under $\mathbb{Q}^*$.

Under Assumption 3.2, for any fixed $u \in \mathbb{R}_+$, the $(\mathbb{Q}^*,\mathbb{F})$-martingale G^u admits the following integral representation, for every $t \in [0,T]$,

$$G_t^u = G_0^u + \int_0^t g_s^u\, dW_s \tag{18}$$

for some $\mathbb{F}$-predictable, real-valued process $(g_t^u,\, t \in [0,T])$.

Similarly, for any fixed $x \in \mathbb{R}_+$, the process $(f_t^x,\, t \in [0,T])$ is a non-negative $(\mathbb{Q}^*, \mathbb{F})$-martingale by virtue of Lemma 3.1. Therefore, for any fixed $x \in \mathbb{R}_+$ there exists an $\mathbb{F}$-predictable process $(\sigma_t^x,\, t \in [0,T])$ such that, for every $t \in [0,T]$,

$$f_t^x = f_0^x + \int_0^t \sigma_s^x \, dW_s. \tag{19}$$

Since the stochastic Fubini theorem yields

$$G_t^u = \int_u^\infty f_t^x \, dx = \int_u^\infty \Big(f_0^x + \int_0^t \sigma_s^x \, dW_s \Big) dx = G_0^u + \int_0^t dW_s \int_u^\infty \sigma_s^x \, dx,$$

we conclude that the following relationship is valid, for every $u \in \mathbb{R}_+$ and $t \in [0,T]$,

$$g_t^u = \int_u^\infty \sigma_t^x \, dx.$$

By applying the Itô-Wentzell-Kunita formula (see Theorem 3.3.1 in Kunita [27] or Section 5 below), we obtain the following auxiliary result, in which we denote $g_s^s = g_s$ and $f_s^s = f_s$.

Lemma 3.3 *Under Assumptions 3.1–3.2, the Doob-Meyer decomposition of the survival process G reads, for every $t \in [0,T]$,*

$$G_t = G_0 + \int_0^t g_u \, dW_u - \int_0^t f_u \, du. \tag{20}$$

In particular, the survival process G of τ with respect to $\mathbb{F}$ is continuous.

The following result shows that under Assumptions 3.1–3.2 we can make use of all results of Section 1.

Lemma 3.4 *Assumptions 3.1–3.2 imply that Assumption 1.1 is satisfied. Moreover, the equality $\widehat{\lambda} = \lambda$ is valid and thus the process $\widehat{M} = M$ is a $(\mathbb{Q}^*, \mathbb{G})$-martingale.*

Proof. In view of Lemma 3.3, for the process ν in the Doob-Meyer decomposition of G we have that $d\nu_t = f_t \, dt = \widehat{\lambda}_t G_t \, dt$. Therefore, Assumption 1.1 is satisfied and the equality $\widehat{\lambda} = \lambda$ holds. This in turn implies that, for every $t \in [0,T]$,

$$\widehat{M}_t = H_t - \int_0^{t\wedge\tau} \widehat{\lambda}_u \, du = H_t - \int_0^{t\wedge\tau} \lambda_u \, du = M_t$$

and thus $\widehat{M}$ is a $(\mathbb{Q}^*, \mathbb{G})$-martingale since, under Assumption 1.1, the process M given by (2) is known to be a $(\mathbb{Q}^*, \mathbb{G})$-martingale. □

It is shown in [25] that, under Assumptions 3.1–3.2, the process

$$\widetilde{W}_t = W_t - \int_0^{t\wedge\tau} \frac{g_u}{G_u} \, du + \int_{t\wedge\tau}^t \frac{\sigma_u^\tau}{f_u^\tau} \, du \tag{21}$$

is a $(\mathbb{Q}^*, \mathbb{G})$-Brownian motion. We take this result for granted and we refer to [25] for its proof. Let us note that on the event $\{t < \tau\}$ the process $\widetilde{W}$ satisfies

$$d\widetilde{W}_t = dW_t - g_t G_t^{-1} \, dt.$$

3.1 Price Dynamics of a Defaultable Claim

The following assumption will allow us to establish more explicit representations for (pre-default) prices.

Assumption 3.3 The quantities Z, X, A and B are deterministic. To emphasize this property, we will write $Z(t), A(t), B(t)$ and $\beta(t) = B^{-1}(t)$, rather than Z_t, A_t, B_t and $\beta_t = B_t^{-1}$.

The main goal of the next result is to derive a more explicit representation for the volatility term appearing in the price dynamics.

Proposition 3.1 *Under Assumptions 3.1–3.3, the dynamics of the pre-default price* $\widetilde{S}$ *are*

$$d\widetilde{S}_t = \big((r(t) + \lambda_t)\widetilde{S}_t - \lambda_t Z(t)\big)\,dt + dA(t) + \zeta_t\big(dW_t - g_t G_t^{-1}\,dt\big),$$

whereas the cumulative price S^c *satisfies*

$$dS_t^c = r(t)S_t^c\,dt + (Z(t) - \widetilde{S}_t)\,dM_t + (1 - H_t)\zeta_t\,d\widetilde{W}_t,$$

where the $(\mathbb{Q}^*, \mathbb{G})$*-Brownian motion* $\widetilde{W}$ *is given by (21) and the process* ζ *equals, for every* $t \in [0, T]$,

$$\zeta_t = G_t^{-1}\big(B(t)\nu_t - \widetilde{S}_t g_t\big) \tag{22}$$

with

$$\nu_t = \beta(T)XG_t^T + \int_t^T \beta(u)Z(u)\sigma_t^u\,du + \int_t^T \beta(u)g_t^u\,dA(u). \tag{23}$$

Proof. Under Assumptions 3.1–3.3 and using the martingale properties of the processes G^u and f^u, we obtain from (3)

$$S_t = \mathbb{1}_{\{t<\tau\}}\frac{B(t)}{G_t}\left(\beta(T)XG_t^T + \int_t^T \beta(u)\big(Z(u)f_t^u\,du + G_t^u\,dA(u)\big)\right) \tag{24}$$

and

$$\begin{aligned} m_t &= \beta(T)XG_t^T + \int_0^t \beta(u)Z(u)f_u\,du + \int_t^T \beta(u)Z(u)f_t^u\,du \\ &\quad + \int_0^t \beta(u)G_u\,du + \int_t^T \beta(u)G_t^u\,du. \end{aligned}$$

It then follows that

$$dm_t = \left(\beta(T)XG_t^T + \int_t^T \beta(u)Z(u)\sigma_t^u\,du + \int_t^T \beta(u)g_t^u\,dA(u)\right)dW_t = \nu_t\,dW_t,$$

where the process ν is given by formula (23). Hence, using the equality $\mu_t = \int_0^t g_u\,dW_u$, we obtain $d\langle\mu\rangle_t = g_t^2\,dt$ and $d\langle\mu, m\rangle_t = g_t\nu_t\,dt$. In view of (22), the asserted formulae now follow directly from Proposition 1.1. □

3.2 Price Dynamics of the Forward CDS

In the sequel, we shall work under Assumptions 3.1–3.3. Let $S(\kappa)$ be the price of the forward CDS with the protection payment $\delta(\tau)$ at time τ on the event $\{U \le \tau \le T\}$, where δ is some function, κ is the constant spread and L is an increasing function such that $L(T) - L(U) > 0$. By the risk-neutral valuation formula (4), the price $S_t(\kappa)$ of a forward CDS equals, for every $t \in [0, U]$,

$$S_t(\kappa) = B(t)\,\mathbb{E}_{\mathbb{Q}^*}\left(\mathbb{1}_{\{U\le\tau\le T\}}\beta(\tau)\delta(\tau) - \kappa\int_{]\tau\wedge U,\tau\wedge T]}\beta(u)\,dL(u)\,\Big|\,\mathcal{G}_t\right) \tag{25}$$

for an increasing function L, which specifies the tenor structure of the fee leg of the forward CDS.

The following result is a rather straightforward consequence of formula (25) and Proposition 3.1.

Corollary 3.1 *Under Assumptions 3.1–3.3, the price $S(\kappa)$ satisfies, for every $t \in [0,U]$,*

$$S_t(\kappa) = \mathbb{1}_{\{t<\tau\}} \frac{B(t)}{G_t} \Big(\int_U^T \beta(u)\delta(u) f_t^u \, du - \kappa \int_{]U,T]} \beta(u) G_t^u \, dL(u) \Big). \tag{26}$$

The dynamics of the process $\widetilde{S}(\kappa)$ are, for $t \in [0,U]$,

$$d\widetilde{S}_t(\kappa) = (r(t) + \lambda_t)\widetilde{S}_t(\kappa)\, dt + \zeta_t \Big(dW_t - \frac{g_t}{G_t}\, dt \Big),$$

where

$$\zeta_t = \frac{B(t)}{G_t} \Big(\int_U^T \beta(u)\delta(u)\sigma_t^u \, du - \kappa \int_{]U,T]} \beta(u) g_t^u \, dL(u) \Big) - \widetilde{S}_t(\kappa) \frac{g_t}{G_t}.$$

Consequently, the ex-dividend prices satisfies, for $t \in [0,U]$,

$$dS_t(\kappa) = r(t) S_t(\kappa)\, dt - S_{t-}(\kappa)\, dM_t + (1-H_t)\zeta_t \Big(dW_t - \frac{g_t}{G_t}\, dt \Big)$$

and the cumulative price satisfies, for $t \in [0,U]$,

$$dS_t^c(\kappa) = r_t S_t^c(\kappa)\, dt + \big(\delta(t) - S_{t-}(\kappa)\big)\, dM_t + (1-H_t)\zeta_t \Big(dW_t - \frac{g_t}{G_t}\, dt \Big).$$

The final step is the computation of the volatilities appearing in dynamics (11) of the forward CDS rate. Of course, the usefulness of Proposition 3.2 depends on the possibility of explicit computations of these volatilities. In Section 4, we will argue that such computations can be performed in the framework of the CIR default intensity model under deterministic interest rates. Let us recall that the positive $(\mathbb{Q}^*, \mathbb{F})$-martingales M^P and M^A were defined in Section 2.1 (see formula (6)).

Proposition 3.2 *Under Assumptions 3.1–3.3, the volatility processes σ^P and σ^A of strictly positive $(\mathbb{Q}^*, \mathbb{F})$-martingales M^P and M^A are given by the following expressions*

$$\sigma_t^P = \Big(\int_U^T \beta(u)\delta(u)\sigma_t^u \, du \Big) \Big(\int_U^T \beta(u)\delta(u) f_t^u \, du \Big)^{-1},$$
$$\sigma_t^A = \Big(\int_{]U,T]} \beta(u) g_t^u \, dL(u) \Big) \Big(\int_{]U,T]} \beta(u) G_t^u \, dL(u) \Big)^{-1}.$$

Proof. Using the martingale properties of processes G^u and f^u, we obtain

$$\begin{aligned} M_t^P &= -\mathbb{E}_{\mathbb{Q}^*} \Big(\int_U^T \beta(u)\delta(u)\, dG_u \,\Big|\, \mathcal{F}_t \Big) = \mathbb{E}_{\mathbb{Q}^*} \Big(\int_U^T \beta(u)\delta(u) f_u \, du \,\Big|\, \mathcal{F}_t \Big) \\ &= \int_U^T \beta(u)\delta(u)\, \mathbb{E}_{\mathbb{Q}^*}(f_u|\mathcal{F}_t)\, du = \int_U^T \beta(u)\delta(u) f_t^u \, du, \\ M_t^A &= \mathbb{E}_{\mathbb{Q}^*} \Big(\int_{]U,T]} \beta(u) G_u \, dL(u) \,\Big|\, \mathcal{F}_t \Big) = \int_{]U,T]} \beta(u) \mathbb{E}_{\mathbb{Q}^*}(G_u|\mathcal{F}_t)\, dL(u) = \int_{]U,T]} \beta(u) G_t^u \, dL(u). \end{aligned}$$

Therefore,

$$dM_t^P = \int_U^T \beta(u)\delta(u)\, d_t f_t^u du = \Big(\int_U^T \beta(u)\delta(u)\sigma_t^u \, du \Big) dW_t,$$
$$dM_t^A = \int_{]U,T]} \beta(u)\, d_t G_t^u dL(u) = \Big(\int_{]U,T]} \beta(u) g_t^u \, dL(u) \Big) dW_t.$$

We conclude that

$$dM_t^P = M_t^P \sigma_t^P \, dW_t, \quad dM_t^A = M_t^A \sigma_t^A \, dW_t$$

with the volatility processes σ^P and σ^A given in the statement of the proposition. □

3.3 Immersion Property

It is fairly common to construct a default time τ in such a way that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ under a martingale measure. Recall that a filtration $\mathbb{F}$ is said to be *immersed* in a filtration $\mathbb{G}$ under $\mathbb{Q}^*$, where $\mathbb{F} \subset \mathbb{G}$, if any $(\mathbb{Q}^*, \mathbb{F})$-martingale is a $(\mathbb{Q}^*, \mathbb{G})$-martingale; this condition is also frequently referred to as the hypothesis (H). It is worth noting that the immersion property is not preserved under an equivalent change of a probability measure, in general.

Assumption 3.4 The filtration $\mathbb{F}$ is immersed in the full filtration $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ under $\mathbb{Q}^*$.

In our setting, Assumption 3.4 implies that G is an increasing process and, for any $u \in [0, T]$, the $\mathbb{F}$-martingale G^u is stopped at time u. The following lemma is thus easy to establish.

Lemma 3.5 *Under Assumptions 3.1–3.4, we have that $g_t = 0$ for every $t \in [0, T]$. Moreover, $f^u_t = f^u_u = f_u$ for every $0 \le u < t \le T$ and thus $\sigma^u_t = 0$ for every $0 \le u < t \le T$. Consequently, the equality $W = \widetilde{W}$ holds and thus a $(\mathbb{Q}^*, \mathbb{F})$-Brownian motion W is also a $(\mathbb{Q}^*, \mathbb{G})$-Brownian motion.*

By combining Proposition 3.1 with Lemma 3.3, we obtain the following result. Of course, this result can be also applied to a forward CDS.

Corollary 3.2 *Under Assumptions 3.1–3.4, we have that, for every $t \in [0, T]$,*

$$d\widetilde{S}_t = \Big((r(t) + \lambda_t) \widetilde{S}_t - \lambda_t Z(t) \Big)\, dt + dA(t) + \zeta_t \, dW_t,$$

and the dynamics of the cumulative price are

$$dS^c_t = r(t) S^c_t \, dt + (Z(t) - \widetilde{S}_t) \, dM_t + (1 - H_t) \zeta_t \, dW_t$$

with $\zeta_t = G_t^{-1} B(t) \nu^T_t$.

3.4 Modeling of Conditional Default Distributions

Before examining in Section 4 the standard case of an intensity-based credit risk model, let us study a more general set-up that was introduced in Definition 3.1 and Assumption 3.1. Formally, we will search for a family of non-negative $(\mathbb{Q}^*, \mathbb{F})$-martingales $(f^x_t, \, t \in [0, T], x \in \mathbb{R}_+)$ such that $\int_0^\infty f^x_t \, dx = 1$ for every $t \in \mathbb{R}_+$, so that, as a function of x for any fixed t, they can be interpreted as the $\mathcal{F}_t$-conditional probability densities of the default time.

3.4.1 Backward Method

Let us fix some $T \le \infty$ and let us assume that we are given a family $(f^x_T, \, x \in \mathbb{R}_+)$ of $\mathcal{F}_T$-measurable and non-negative random variables that satisfy

$$\int_0^\infty f^x_T \, dx = 1.$$

By setting, for every $t \in [0, T]$,

$$f^x_t = \mathbb{E}_{\mathbb{Q}^*}(f^x_T \,|\, \mathcal{F}_t) \tag{27}$$

we obtain a family $(f^x_t, \, t \in [0, T], x \in \mathbb{R}_+)$ of conditional probability densities. As expected, a family of f^x_T can be constructed in several alternative ways. We present below two examples of such constructions.

In the first method, we start with a parametrized family $(\varphi(x,\alpha),\, x \in \mathbb{R}_+,\, \alpha \in A)$ of densities, where $A \subset \mathbb{R}^d$. Hence we have that, for every $\alpha \in A$,

$$\int_0^\infty \varphi(x,\alpha)\, dx = 1.$$

Let X be an $\mathcal{F}_T$-measurable random variable taking values in A. We define $f_T^x := \varphi(x,X)$. For instance, we may take $\varphi(x,\alpha) = \alpha e^{-\alpha x}$. In that case, $f_t^x = \mathbb{E}_{\mathbb{Q}^*}(Xe^{-xX} \,|\, \mathcal{F}_t)$ can be computed as $f_t^x = -\partial_x \mathbb{E}_{\mathbb{Q}^*}(e^{-xX} \,|\, \mathcal{F}_t)$ for every $t \in [0,T]$.

In the second method, a family $(f_T^x,\, x \in \mathbb{R}_+)$ is defined by the expression

$$f_T^x = \lambda_T^x \exp\Big(- \int_0^x \lambda_T^u \, du \Big),$$

where $(\lambda_T^u,\, u \in \mathbb{R}_+)$ is an arbitrary family of non-negative, $\mathcal{F}_T$-measurable random variables such that $\int_0^\infty \lambda_T^u \, du = \infty$.

3.4.2 Forward Method

In the *forward method*, we define the random field $(f_t^x,\, t \in [0,T], x \in \mathbb{R}_+)$ by setting

$$f_t^x = \lambda_t^x \exp\Big(- \int_0^x \lambda_t^u \, du \Big) \tag{28}$$

for a family of non-negative, $\mathbb{F}$-adapted processes $(\lambda^u,\, u \in \mathbb{R}_+)$ that satisfy

$$d\lambda_t^u = \lambda_t^u \big(a_t^u \, dt + \sigma_t^u \, dW_t \big), \tag{29}$$

where W is a $(\mathbb{Q}^*, \mathbb{F})$-Brownian motion. For this construction to be valid, we need to impose additional restrictions on the model coefficients a_t^u and σ_t^u, since we need to ensure that for each $x \in \mathbb{R}_+$ the process $(f_t^x,\, t \in [0,T])$ follows a non-negative $(\mathbb{Q}^*, \mathbb{F})$-martingale. The necessary and sufficient condition for this property to hold is provided by the following result.

Proposition 3.3 *The martingale property of processes f^x holds if and only if the following condition is satisfied, for every $t \in [0,T]$ and $x \in \mathbb{R}_+$,*

$$a_t^x - \int_0^x a_t^u \lambda_t^u \, du - \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du + \frac{1}{2} \Big(\int_0^x \sigma_t^u \lambda_t^u \, du \Big)^2 = 0. \tag{30}$$

Proof. By applying Itô's lemma to (28) and (28), we obtain

$$df_t^x = \exp\Big(-\int_0^x \lambda_t^u \, du\Big) \Big\{ d_t\lambda_t^x - \lambda_t^x \Big(\int_0^x d_t \lambda_t^u \, du + \frac{1}{2} \Big(\int_0^x \lambda_t^u \sigma_t^u du \Big)^2 \Big) dt - \sigma_t^x \Big(\lambda_t^x \int_0^x \lambda_t^u \sigma_t^u du \Big) dt \Big\}$$

and thus

$$df_t^x = \exp\Big(- \int_0^x \lambda_t^u \, du \Big) \lambda_t^x \big(\mu_t^x \, dt + \Sigma_t^x \, dW_t \big), \tag{31}$$

where we denote

$$\mu_t^x = a_t^x - \int_0^x a_t^u \lambda_t^u \, du - \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du + \frac{1}{2} \Big(\int_0^x \sigma_t^u \lambda_t^u \, du \Big)^2.$$

Hence the drift term in (31) vanishes whenever (30) holds. □

It is not difficult to check that condition (30) is satisfied if, for example, we take the coefficient a_t^x to be given by the following expression

$$a_t^x = \sigma_t^x \int_0^x \lambda_t^u \sigma_t^u \, du.$$

In that case, equation (29) becomes

$$d\lambda_t^x = \lambda_t^x \sigma_t^x \Big(\int_0^t \lambda_t^u \sigma_t^u \, du \Big) \, dt + \lambda_t^x \sigma_t^x \, dW_t. \tag{32}$$

It is worth noting an interesting analogy between the above equation and the no-arbitrage condition in the HJM approach to modeling of instantaneous forward interest rates. Let us also remark that it is not guaranteed that the solution of the equation (32) is non-negative.

Let us now provide a general template for models yielding non-negative families $(\lambda^u,\, u \in \mathbb{R}_+)$ of solutions to (32). Towards this end, we first rewrite equation (32) as follows

$$\lambda_t^x = \lambda_0^x + \int_0^t \psi_s^x \Psi_s^x \, ds + \int_0^t \psi_s^x \, dW_s,$$

where we denote $\psi_t^x = \lambda_t^x \sigma_t^x$ and $\Psi_t^x = \int_0^x \psi_t^u \, du$.

The non-negativity of λ^x is satisfied under the following assumptions:

- the process ψ^x is non-negative and thus the process $\psi^x \Psi^x$ is non-negative as well,
- the process $Z_t^x := \lambda_0^x + \int_0^t \psi_s^x \, dW_s$ is the Doléans-Dade exponential with the initial value $\lambda_0^x > 0$, meaning that Z^x satisfies, for every $t \in [0,T]$,

  $$Z_t^x = \lambda_0^x + \int_0^t Z_s^x b_s^x \, dW_s$$

  for some $\mathbb{F}$-progressively measurable process b^x. A sufficient condition for the latter condition to be satisfied is that ψ^x is such that the equality

  $$\int_0^t \psi_s^x \, dW_s = \int_0^t b_s^x Z_s^x \, dW_s$$

  holds for every $t \in [0,T]$. This means that ψ^x can be represented as follows

  $$\psi_t^x = b_t^x Z_t = b_t^x \lambda_0^x \exp\Big(\int_0^t b_s^x \, dW_s - \frac{1}{2} \int_0^t (b_s^x)^2 \, ds \Big). \tag{33}$$

Let us summarize the above considerations by stating the following result.

Proposition 3.4 *Let $\lambda_0^x > 0$ and let $(b^x,\, x \in \mathbb{R}_+)$ be a family non-negative $\mathbb{F}$-progressively measurable processes. For every $t \in [0,T]$ and $x \in \mathbb{R}_+$, we define*

$$\psi_t^x = b_t^x \lambda_0^x \, \exp\Big(\int_0^t b_s^x \, dW_s - \frac{1}{2} \int_0^t (b_s^x)^2 \, ds \Big) \tag{34}$$

and we set

$$f_t^x = \lambda_t^x \exp\Big(- \int_0^x \lambda_t^u \, du \Big)$$

where

$$\lambda_t^x = \lambda_0^x + \int_0^x \psi_s^x \, dW_s + \int_0^t \psi_s^x \Psi_s^x \, ds.$$

If the family $(\lambda^x,\, x \in \mathbb{R}_+)$ satisfies $\int_0^\infty \lambda_t^x \, dx = \infty$ for every $t \in [0,T]$ then $(f_t^x,\, t \in [0,T], x \in \mathbb{R}_+)$ is a family of non-negative $(\mathbb{Q}^*, \mathbb{F})$-martingales such that $\int_0^\infty f_t^x \, dx = 1$ for every $t \in [0,T]$.

3.4.3 Cox Process Approach

Let us now present a method motivated by construction of the first jump of a Cox processes. Assume that a non-negative, $\mathbb{F}$-adapted process λ is given and set $\Lambda_t = \int_0^t \lambda_u \, du$. Let Θ be a random variable independent of $\mathcal{F}_\infty$ with unit exponential law and let V be an $\mathcal{F}_\infty$-measurable non-negative random variable. We define

$$\tau = \inf \big\{ t \in \mathbb{R}_+ : \, \Lambda_t \geq \Theta / V \big\}.$$

Then we have, for any $x \in \mathbb{R}_+$ and $t \in \mathbb{R}_+$,

$$\mathbb{Q}^*(\tau > x \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}^*}\big(\mathbb{Q}^*(\Lambda_x V < \Theta \,|\, \mathcal{F}_\infty) \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}^*}\big(e^{-V\Lambda_x} \,|\, \mathcal{F}_t\big) = \int_x^\infty f_t^u \, du$$

with

$$f_t^x = -\frac{d}{dx} \, \mathbb{E}_{\mathbb{Q}^*}\big(e^{-V\Lambda_x} \,|\, \mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}^*}\big(V \lambda_x \, e^{-V\Lambda_x} \,|\, \mathcal{F}_t\big). \tag{35}$$

It is straightforward to check that the family $(f_t^x, \, t \in [0,T], x \in \mathbb{R}_+)$ defined by (35) satisfies the required conditions stated at the beginning of this section.

3.4.4 Functional Approach

Let us assume that $(f_t^x, \, t \in [0,T], x \in \mathbb{R}_+)$ is a strictly positive random field. Then we have, for every $x \in \mathbb{R}_+$,

$$f_t^x = f_0^x + \int_0^t f_u^x \Sigma_u^x \, dW_u$$

for some predictable process Σ^x. The normalization condition $\int_0^\infty f_t^x \, dx = 1$, which needs to be satisfied for every $t \in [0,T]$, implies that the equality

$$\int_0^\infty f_t^x \Sigma_t^x \, dx = 0$$

holds for every $t \in [0,T]$. A sufficient condition for the random field $(\Sigma_t^x, \, t \in [0,T], x \in \mathbb{R}_+)$ to satisfy the above condition is that

$$\Sigma_t^x = \Psi_t^x - \int_0^\infty f_t^y \Psi_t^y \, dy$$

for some family of $\mathbb{F}$-predictable processes Ψ^y. In that case, we obtain the following representation for f_t^x

$$f_t^x = f_0^x + \int_0^t f_u^x \Big(\Psi_u^x - \int_0^\infty f_u^y \Psi_u^y \, dy \Big) \, dW_u. \tag{36}$$

This suggests a possible method for modeling of f_t^x in terms of f_0^x and Ψ_u^x starting from (36). Any family f^x satisfying (36) is obviously a family of local martingales fulfilling the normalization condition. The non-negativity and martingale properties of such a family are not obvious, however. At this time, we were unable to construct any example of family f^x satisfying the above equation and fulfilling the assumptions specified in the beginning of this section. Let us finally note that equation of this type has been recently studied by Macrina et al. [19].

4 Intensity-Based Modeling of Default Time

The goal of this section is to provide an example of a model in which our standing assumptions are satisfied and to show that this model is amenable for quasi-explicit computations of the price of a credit default swaption and its replicating strategy in term of the underlying forward CDS and the corresponding swap portfolio process.

It is not uncommon to start modeling by specifying the dynamics of the default intensity process. Following this approach, we postulate that we are given a non-negative and $\mathbb{F}$-predictable process λ defined on some probability space $(\Omega, \mathcal{G}, \mathbb{Q}^*)$ endowed with a filtration $\mathbb{F}$ and we set $\Lambda_t = \int_0^t \lambda_u \, du$. The default time is defined by the formula

$$\tau = \inf \big\{ t \in \mathbb{R}_+ : \Lambda_t \geq \Theta \big\}, \tag{37}$$

where Θ is a random variable with unit exponential distribution, independent of the filtration $\mathbb{F}$. Note that τ can be here seen as the moment of the first jump of a Cox process with the intensity process λ. It is easy to check that the $(\mathbb{Q}^*, \mathbb{F})$-martingale $(G_t^u,\, t \in [0,T])$ satisfies

$$G_t^u = \mathbb{E}_{\mathbb{Q}^*}\big(e^{-\Lambda_u} \,\big|\, \mathcal{F}_t\big) \ \text{ for } t \in [0,u[, \quad G_t^u = e^{-\Lambda_u} \ \text{ for } t \in [u,T],$$

so that $G_t = e^{-\Lambda_t}$ for every $t \in [0,T]$. Moreover, the $(\mathbb{Q}^*, \mathbb{F})$-martingale $(f_t^x,\, t \in [0,T])$ satisfies

$$f_t^x = \mathbb{E}_{\mathbb{Q}^*}\big(\lambda_x e^{-\Lambda_x} \,\big|\, \mathcal{F}_t\big) \ \text{ for } t \in [0,x[, \quad f_t^x = \lambda_x e^{-\Lambda_x} \ \text{ for } t \in [x,T].$$

It is well known that the immersion property holds between $\mathbb{F}$ and $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$. If, in addition, $\mathbb{F}$ is the Brownian filtration then Assumptions 3.1–3.4 are satisfied.

4.1 CIR Default Intensity Model

For the sake of concreteness, we examine a special case of the Cox process model in which the default intensity λ is governed by the CIR dynamics (see Cox et al. [16])

$$d\lambda_t = \mu(\lambda_t)\, dt + \nu(\lambda_t)\, dW_t, \quad \lambda_0 > 0, \tag{38}$$

where $\mu(\lambda) = a - b\lambda$, $\nu(\lambda) = c\sqrt{\lambda}$ and W is a one-dimensional Brownian motion generating the filtration $\mathbb{F}$. It is well known that under the assumption that $2a > c^2$ the unique solution to this SDE is strictly positive. We postulate that the default time τ is given by formula (37), so that $G_t = e^{-\Lambda_t}$. Let us denote, for arbitrary $0 \leq t \leq u \leq T$,

$$H_t^u = \mathbb{E}_{\mathbb{Q}^*}\big(e^{-(\Lambda_u - \Lambda_t)} \,\big|\, \mathcal{F}_t\big) = \frac{G_t^u}{G_t}. \tag{39}$$

It is known (see, e.g., Section 3.2.3 in [15], Section 6.3.4 in [26], or Page 357 in [30]) that

$$H_t^u = e^{m(t,u) - n(t,u)\lambda_t} = \widehat{H}(\lambda_t, t, u), \tag{40}$$

where

$$\widehat{H}(y,t,u) = e^{m(t,u) - n(t,u)y} \tag{41}$$

and the functions m and n are given by the following expressions

$$m(t,u) = \frac{2a}{c^2} \ln \left\{ \frac{\gamma e^{b(u-t)/2}}{\gamma \cosh \gamma(u-t) + \frac{1}{2} b \sinh \gamma(u-t)} \right\}$$

and

$$n(t,u) = \frac{\sinh \gamma(u-t)}{\gamma \cosh \gamma(u-t) + \frac{1}{2} b \sinh \gamma(u-t)}\,,$$

where in turn $2\gamma = (b^2 + 2c^2)^{1/2}$. It is important to notice that, for any fixed $t \in \mathbb{R}_+$, the function $n(t,u)$, $u \geq t$, is strictly increasing. Moreover, the function n is strictly positive and thus, for any fixed u and t, the auxiliary function $\widehat{H}(y,t,u)$ is decreasing and continuous in $y \in \mathbb{R}_+$.

Now, let $D^0(t,u)$ be the price at time t of a unit defaultable zero-coupon bond with zero recovery maturing at $u \geq t$, and let $B(t,u)$ be the price at time t of a (default-free) unit discount bond maturing at $u \geq t$. It is well known that if the interest rate process r is independent of the default intensity λ then $D^0(t,u)$ is given by the following formula

$$D^0(t,u) = \mathbb{1}_{\{t<\tau\}} B(t,u) H_t^u = \mathbb{1}_{\{t<\tau\}} \widetilde{D}^0(t,u), \tag{42}$$

where $\widetilde{D}^0(t,u)$ represents the *pre-default value* of the bond at time t.

4.2 Volatility of the Forward CDS Rate

We will now analyze the volatility of the forward CDS rate in the CIR default intensity model. To this end, we need to find the integral representations of $\mathbb{F}$-martingales $(G_t^u,\, t \in [0,u])$ and $(f_t^x,\, t \in [0,x])$.

Lemma 4.1 *For any fixed $0 < u \le T$ and $t \in [0,u]$, we have that $dG_t^u = g_t^u \, dW_t$, where*

$$g_t^u = -e^{-\Lambda_t} H_t^u \nu(\lambda_t) n(t,u). \tag{43}$$

For any fixed $0 < x \le T$ and every $t \in [0,x]$, we have that

$$f_t^x = -e^{-\Lambda_t} \partial_x H_t^x = e^{-\Lambda_t} \alpha_t^x H_t^x, \tag{44}$$

where $\alpha_t^x = \lambda_t \partial_x n(t,x) - \partial_x m(t,x)$. Moreover, for $t \in [0,x]$, the equality $df_t^x = \sigma_t^x \, dW_t$ holds with

$$\sigma_t^x = e^{-\Lambda_t} H_t^x \nu(\lambda_t) \big(n_x(t,x) - \alpha_t^x n(t,x) \big). \tag{45}$$

Proof. Let us first establish (44). To this end, we note that

$$f_t^x = \mathbb{E}_{\mathbb{Q}^*}(\lambda_x e^{-\Lambda_x} \,|\, \mathcal{F}_t) = e^{-\Lambda_t} \, \mathbb{E}_{\mathbb{Q}^*}(\lambda_x e^{-(\Lambda_x - \Lambda_t)} \,|\, \mathcal{F}_t) = -e^{-\Lambda_t} \partial_x H_t^x.$$

Using (41), we obtain

$$\partial_x H_t^x = (\partial_x m(t,x) - \lambda_t \partial_x n(t,x)) H_t^x = -\alpha_t^x H_t^x, \tag{46}$$

so that equality (44) is valid. Since G^u and f^x are $\mathbb{F}$-martingales, to derive (43) and (45), it suffices to focus on martingale terms in their differentials. Noting that $G_t^u = e^{\Lambda_t} H_t^u$ and applying the Itô formula to (39), we obtain (43). Similarly, to establish (45), it suffices to apply the Itô integration by parts formula to $\alpha_t^x H_t^x$. □

As in Section 3.2, we consider a forward CDS with the protection payment $\delta(\tau)$ at time τ on the event $\{U \le \tau \le T\}$, where δ is some function, κ is the constant (over time) spread and L is an increasing function such that $L(T) - L(U) > 0$. Also, the short-term interest rate r is deterministic.

Recall that the volatility of the forward CDS rate is denoted by σ^κ. In view of Proposition 4.1, it is natural to conjecture that σ^κ is not deterministic, thereby precluding the possibility to justify the use of the Black formula in the CIR default intensity model.

Proposition 4.1 *Assume that $\delta > 0$ is constant. Then the volatility of the forward CDS rate satisfies $\sigma^\kappa = \sigma^P - \sigma^A$, where*

$$\sigma_t^P = \nu(\lambda_t) \, \frac{\beta(T) H_t^T n(t,T) - \beta(U) H_t^U n(t,U) + \int_U^T r(u)\beta(u) H_t^u n(t,u)\, du}{\beta(U) H_t^U - \beta(T) H_t^T - \int_U^T r(u)\beta(u) H_t^u \, du} \tag{47}$$

and

$$\sigma_t^A = \nu(\lambda_t) \frac{\int_{]U,T]} \beta(u) H_t^u n(t,u)\, dL(u)}{\int_{]U,T]} \beta(u) H_t^u \, dL(u)} \, . \tag{48}$$

Proof. Equality $\sigma^\kappa = \sigma^P - \sigma^A$ is an immediate consequence of Lemma 2.3. Under the present assumptions, from Proposition 3.2 we obtain

$$\sigma_t^P = \Big(\int_U^T \beta(u) \sigma_t^u \, du \Big) \Big(\int_U^T \beta(u) f_t^u \, du \Big)^{-1}$$

and

$$\sigma_t^A = \Big(\int_{]U,T]} \beta(u) g_t^u \, dL(u) \Big) \Big(\int_{]U,T]} \beta(u) G_t^u \, dL(u) \Big)^{-1}.$$

In view of (44) and (45), the process σ^P can be represented as follows

$$\sigma_t^P = \frac{e^{-\Lambda_t}\nu(\lambda_t)\int_U^T \beta(u)H_t^u\big(n_u(t,u)-\alpha_t^u n(t,u)\big)\,du}{e^{-\Lambda_t}\int_U^T \beta(u)H_t^u\alpha_t^u\,du}.$$

Equality (46) and the integration by parts formula yield

$$\begin{aligned}\int_U^T \beta(u)H_t^u\big(n_u(t,u)-\alpha_t^u n(t,u)\big)\,du &= \int_U^T \beta(u)\partial_u(H_t^u n(t,u))\,du\\ &= \beta(T)H_t^T n(t,T)-\beta(U)H_t^U n(t,U)+\int_U^T r(u)\beta(u)H_t^u n(t,u)\,du\end{aligned}$$

and

$$-\int_U^T \beta(u)H_t^u\alpha_t^u\,du = \beta(T)H_t^T-\beta(U)H_t^U+\int_U^T r(u)\beta(u)H_t^u\,du,$$

and thus we conclude that (47) is valid. For the process σ^A, using (39) and (43), we obtain

$$\sigma_t^A = \frac{-e^{-\Lambda_t}\nu(\lambda_t)\int_{]U,T]}\beta(u)H_t^u n(t,u)\,dL(u)}{e^{-\Lambda_t}\int_{]U,T]}\beta(u)H_t^u\,dL(u)}.$$

This completes the proof. □

4.3 Credit Default Swaption

We continue working within the setup of Section 3.2, and we shall build upon the ideas borrowed from [13] (see also [10] and [22]). In the remaining part of this paper, we will work under the following standing assumption.

Assumption 4.1 We postulate that:
(i) the default time τ is given by (37) with the intensity λ governed by the CIR dynamics (38),
(ii) $\delta(t)=\delta$ for some positive constant δ,
(iii) the short-term interest rate r is a non-negative deterministic function of time.

The interested reader is referred to [13] for a discussion of assumption (iii) in the context of valuation of credit default swaptions.

Recall that a credit default swaption maturing at R is formally equivalent to a defaultable claim $(C_R,0,0,\tau)$, where $C_R=\big(S_R(\kappa)\big)^+$ and $S_R(\kappa)$ is the value at time R of a forward CDS over the time period $[U,T]$. For simplicity of presentation, we will assume that the underlying forward CDS was issued at time 0. Equivalently, the rate κ is assumed to be $\mathcal{F}_0$-measurable (and thus constant since $\mathcal{F}_0$ is trivial) and thus we may examine the price of the credit default swaption for any $t\in[0,R]$.

Under the present assumptions, the price at time t of a unit discount bond maturing at time u satisfies $B(t,u)=B(t)B^{-1}(u)=B(t)\beta(u)$. Let us write $\lambda_t^u=f_t^u G_t^{-1}$ so that $\lambda_t^u=\hat{h}(\lambda_t,t,u)$, where

$$\hat{h}(y,t,u)=-\partial_u\widehat{H}(y,t,u)$$

with $\widehat{H}$ given by (41). In view of (26), we thus obtain

$$C_R=\mathbb{1}_{\{R<\tau\}}\left(\delta\int_U^T B(R,u)\lambda_R^u\,du-\kappa\int_{]U,T]}B(R,u)H_R^u\,dL(u)\right)^+.$$

Straightforward computations lead to the following representation

$$C_R=\mathbb{1}_{\{R<\tau\}}\left(\delta B(R,U)H_R^U-\int_{]U,T]}B(R,u)H_R^u\,d\chi(u)\right)^+, \tag{49}$$

where the function $\chi : \mathbb{R}_+ \to \mathbb{R}$ satisfies

$$d\chi(u) = -\delta \frac{\partial \ln B(R,u)}{\partial u}\, du + \kappa\, dL(u) + \delta\, d\mathbb{1}_{[T,\infty[}(u). \tag{50}$$

Let us define auxiliary functions $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : \mathbb{R} \to \mathbb{R}_+$ by setting

$$\zeta(x) = \delta B(R,U)\widehat{H}(x,R,U), \quad \psi(y) = \int_{]U,T]} B(R,u)\widehat{H}(y,R,u)\, d\chi(u). \tag{51}$$

We note that χ generates a finite, non-negative measure μ_χ on $(]U,T], \mathcal{B}(]U,T]))$ and $\mu_\chi(]U,T]) > 0$. Therefore, the strictly positive function ψ is strictly decreasing. Moreover, since for any $u > R$ we have that $\lim_{y\to-\infty} \widehat{H}(y,R,u) = \infty$ and $\lim_{y\to\infty} \widehat{H}(y,R,u) = 0$, we see that $\lim_{y\to-\infty} \psi(y) = \infty$ and $\lim_{y\to\infty} \psi(y) = 0$. Moreover, since ψ is continuous, its inverse ψ^{-1} is continuous as well.

Formula (49) can be rewritten as follows

$$C_R = \mathbb{1}_{\{R<\tau\}} \big(\zeta(\lambda_R) - \psi(\lambda_R)\big)^+. \tag{52}$$

Our next goal is to examine representation (52) in more detail. Towards this end, we shall analyze the existence of a solution to the equation $\zeta(\lambda_R(\omega)) = \psi(\lambda^*_R(\omega))$ for any fixed $\omega \in \Omega$. Put another way, we deal with the following random equation

$$\delta B(R,U)\widehat{H}(\lambda_R,R,U) = \int_{]U,T]} B(R,u)\widehat{H}(\lambda^*_R,R,u)\, d\chi(u), \tag{53}$$

in which we search for a solution λ^*_R. In view of the properties of ζ and ψ, the following result is obvious (note that the second statement in the lemma follows from the strict positivity of λ_R).

Lemma 4.2 *There exists a unique $\mathcal{F}_R$-measurable random variable $\lambda^*_R = \psi^{-1}(\zeta(\lambda_R))$ such that*

$$\zeta(\lambda_R) = \delta B(R,U)\widehat{H}(\lambda_R,R,U) = \int_{]U,T]} B(R,u)\widehat{H}(\lambda^*_R,R,u)\, d\chi(u) = \psi(\lambda^*_R). \tag{54}$$

On the set $\{\zeta(\lambda_R) \ge \psi(0)\}$, we have $\lambda^*_R \le 0$ and thus the inequality $\zeta(\lambda_R) = \psi(\lambda^*_R) > \psi(\lambda_R)$ holds.

Proposition 4.2 *The payoff of the credit default swaption admits the following representation*

$$C_R = \mathbb{1}_{\{R<\tau\}} \int_{]U,T]} B(R,u)\big(\widehat{H}(\lambda^*_R,R,u) - \widehat{H}(\lambda_R,R,u)\big)^+\, d\chi(u). \tag{55}$$

Proof. In view of (49) and (54), we obtain

$$\begin{aligned} C_R &= \mathbb{1}_{\{R<\tau\}} \left(\int_{]U,T]} B(R,u)\big(\widehat{H}(\lambda^*_R,R,u) - \widehat{H}(\lambda_R,R,u)\big)\, d\chi(u) \right)^+ \\ &= \mathbb{1}_{\{R<\tau\}} \left(\int_{]U,T]} B(R,u)\big(\widehat{H}(\lambda^*_R,R,u) - \widehat{H}(\lambda_R,R,u)\big)\, \mu_\chi(du) \right)^+. \end{aligned}$$

Since μ_χ is a non-negative measure and the sign of the expression $\widehat{H}(\lambda^*_R,R,u) - \widehat{H}(\lambda_R,R,u)$ is constant with respect to u, the validity of equality (55) is clear. □

We are in a position to state the corollary, which gives a convenient representation for the payoff of a credit default swaption in the present setup. Observe that equality (56) is an immediate consequence of (55) combined with (42).

Corollary 4.1 *The payoff of the credit default swaption equals*

$$C_R = \int_{]U,T]} \big(K(u)D^0(R,R) - D^0(R,u)\big)^+\, d\chi(u), \tag{56}$$

where the random variable $K(u)$ satisfies $K(u) = B(R,u)\widehat{H}(\lambda^*_R,R,u)$ and χ is given by (50).

4.4 Hedging Strategies for the Credit Default Swaption

Throughout Section 4.4, we work under the standing assumption that $R = U$. In that case, the function ζ is constant, specifically, the equality $\zeta(x) = \delta$ holds for every $x \in \mathbb{R}_+$. Therefore, equation (53) simplifies to

$$\zeta(\lambda_U) = \delta = \int_{]U,T]} B(U,u)\widehat{H}(\lambda^*_U, U, u)\, d\chi(u).$$

The unique solution λ^*_U is deterministic and it is strictly positive whenever $\delta < \psi(0)$. If $\delta \geq \psi(0)$ then $\lambda^*_U \leq 0$ and thus $\zeta(\lambda_U) = \psi(\lambda^*_U) > \psi(\lambda_U)$. Consequently,

$$C_U = \mathbb{1}_{\{U<\tau\}} \big(\zeta(\lambda_U) - \psi(\lambda_U)\big) = \int_{]U,T]} \big(K(u)D^0(U,U) - D^0(U,u)\big)\, d\chi(u),$$

where $K(u) = B(U,u)\widehat{H}(\lambda^*_U, U, u)$ is deterministic. This shows that the credit default swaption can be seen here as a particular portfolio of defaultable bonds with zero recovery, and thus the hedging problem is of a lesser interest. For this reason, we will focus in what follows on the case when $\delta < \psi(0)$. From Corollary 4.1, we obtain

$$C_U = \int_{]U,T]} \big(K(u)D^0(U,U) - D^0(U,u)\big)^+\, d\chi(u) = \int_{]U,T]} C^u_U\, d\chi(u).$$

Formula (4.4) shows that the credit default swaption is formally equivalent to a weighted portfolio of *survival claims* $(C^u_U, 0, 0, \tau)$ maturing at U and indexed by $u \in]U,T]$, where C^u_U equals

$$C^u_U = \big(K(u)D^0(U,U) - D^0(U,u)\big)^+ = \mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+, \tag{57}$$

where $K :]U,T] \to \mathbb{R}_+$ is a deterministic function and $\widetilde{D}^0(U,u)$ stands for the pre-default value at time U of the defaultable bond with zero recovery maturing at time u.

4.4.1 Hedging with Defaultable Bonds

As discussed above, the problem of hedging of a credit default swaption can be formally reduced to a problem of hedging of a portfolio of options on zero-coupon defaultable bonds with zero recovery. To address the latter problem, we can employ techniques developed in Section 4.2.2 of [4]. Specifically, we take as the hedging instruments two zero-coupon defaultable bonds, with maturities $T_1, T_2 \geq U$ (for instance, $T_1 = U$ and $T_2 = T$), which are sensitive to the same default time τ.

Let $Y^1_t = D^0(t,T_1) =: \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^1_t$ and $Y^2_t = D^0(t,T_2) =: \mathbb{1}_{\{\tau>t\}}\widetilde{Y}^2_t$ be their prices. We fix $u \in [U,T]$ and we consider the claim C^u_U maturing at U. As shown in [4], if we can find a constant $x(u)$ and a predictable process $(\varphi^2_t(u),\, t \in [0,U])$ such that

$$x(u) + \int_0^U \varphi^2_s(u)\, d\widetilde{Y}^{2,1}_s = (K(u) - \widetilde{D}^0(R,u))^+,$$

where $\widetilde{Y}^{2,1}_t = \widetilde{Y}^2_t / \widetilde{Y}^1_t$, then we can find a predictable process $(\varphi^1_t(u),\, t \in [0,U])$ such that the pair $\varphi(u) := (\varphi^1(u), \varphi^2(u))$ is a self-financing portfolio replicating the claim $\mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+$. Assuming that $x(u)$ and $\varphi(u)$ are sufficiently regular functions of u, we can apply stochastic Fubini's theorem (see, e.g., [26]) to conclude that $\varphi = (\varphi^1, \varphi^2)$, where $\varphi^i_t = \int_{]U,T]} \varphi^i(u)\, d\chi(u)$, is a self-financing portfolio replicating C_U with the initial wealth equal to $\int_{]U,T]} x(u)\, d\chi(u)$. Let us finally observe that one can synthesize a zero-coupon corporate bond with a portfolio of corporate bonds and the bank account. We conclude that, at least in principle, the payoff C_U could be replicated by a self-financing portfolio of coupon-bearing corporate bonds and the savings account.

4.4.2 Hedging with Forward CDS and Swap Portfolio

The main goal of this section is the adaptation to the present setup of general hedging results established in Section 2.3. Recall that the hedge ratio $\widetilde{\varphi}^1$ was shown to be given by the generic expression (cf. formula (13))

$$\widetilde{\varphi}^1_t = \frac{\widetilde{\xi}_t}{\kappa(t,U,T)\sigma^\kappa_t}\,, \tag{58}$$

where $\widetilde{\xi}$ is implicitly defined by (14) with $R = U$ and $\sigma^\kappa = \sigma^P - \sigma^A$ is the volatility of the forward CDS rate (see Section 4.2). To make formula (58) operational, we need to compute explicitly the process $\widetilde{\xi}$.

Our strategy is to first use equality (56) in order to derive an explicit pricing formula for the swaption in terms of the intensity process λ and subsequently to use this formula for the computation the process $\widetilde{\xi}$.

Proposition 4.3 *The pre-default price of the credit default swaption equals, for every $t \in [0, U]$*

$$\widetilde{C}_t = \int_{]U,T]} B(t,u)P(\lambda_t, U, u, \widehat{K}(u))\,d\chi(u), \tag{59}$$

where $\widehat{K}(u) := K(u)/B(U,u) = \widehat{H}(\lambda^*_U, U, u)$ and where $P(\lambda_t, U, u, \widehat{K}(u))$ stands for the price at time t of a put bond option with strike $\widehat{K}(u)$ and expiry U written on a zero-coupon bond maturing at u computed in the CIR model with the short-term interest rate modeled by the process λ.

Proof. In view of (4.4) and (57), for the price C_t of the swaption we obtain

$$C_t = B(t,U)\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]U,T]} C^u_U\,d\chi(u)\,\Big|\,\mathcal{G}_t\Big) = B(t,U)\,\mathbb{E}_{\mathbb{Q}^*}\Big(\int_{]U,T]} \mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+\,d\chi(u)\,\Big|\,\mathcal{G}_t\Big)$$

and thus

$$C_t = B(t,U)\int_{]U,T]} \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+\,\Big|\,\mathcal{G}_t\Big)\,d\chi(u) = B(t,U)\int_{]U,T]} D(u)\,d\chi(u),$$

where we denote

$$D(u) = \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+\,\Big|\,\mathcal{G}_t\Big).$$

We observe that, for every $t \in [0, U]$,

$$\begin{aligned}
D(u) &= \mathbb{E}_{\mathbb{Q}^*}\Big(\mathbb{1}_{\{U<\tau\}}\big(K(u) - \widetilde{D}^0(U,u)\big)^+\,\Big|\,\mathcal{G}_t\Big)\\
&= \mathbb{1}_{\{t<\tau\}}\,\mathbb{E}_{\mathbb{Q}^*}\Big(e^{\Lambda_t-\Lambda_U}\big(K(u) - B(U,u)\widehat{H}(\lambda_U,U,u)\big)^+\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbb{1}_{\{t<\tau\}}\,\mathbb{E}_{\mathbb{Q}^*}\Big(e^{\Lambda_t-\Lambda_U}\big(K(u) - B(U,u)\widehat{H}(\lambda_U,U,u)\big)^+\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbb{1}_{\{t<\tau\}}\,B(U,u)\,\mathbb{E}_{\mathbb{Q}^*}\Big(e^{\Lambda_t-\Lambda_U}\big(\widehat{K}(u) - \widehat{H}(\lambda_U,U,u)\big)^+\,\Big|\,\mathcal{F}_t\Big)\\
&= \mathbb{1}_{\{t<\tau\}}B(U,u)P(\lambda_t, U, u, \widehat{K}(u)),
\end{aligned}$$

where we write $\widehat{K}(u) = K(u)/B(U,u)$ and where $P(\lambda_t, U, u, \widehat{K}(u))$ stands for the price at time t of a put bond option with strike $\widehat{K}(u)$ and expiry U written on a zero-coupon bond maturing at u in the CIR model with the short-term interest rate modeled by the process λ. We conclude that equality (59) is valid. □

The price $P^u_t := P(\lambda_t, U, u, \widehat{K}(u))$ of the put bond option in the CIR model is known to be given by a closed-form solution (see, for instance, Proposition 10.3.4 in [30]). Without going into details of its derivation, let us recall that P^u_t equals, using the notation of the present work,

$$P^u_t = \widehat{K}(u)\widehat{H}(\lambda_t,t,U)\,\mathbb{P}_U(\widehat{H}(\lambda_U,U,u) \le \widehat{K}(u)\,|\,\lambda_t) - \widehat{H}(\lambda_t,t,u)\,\mathbb{P}_u(\widehat{H}(\lambda_U,U,u) \le \widehat{K}(u)\,|\,\lambda_t)$$

or, equivalently,

$$P_t^u = \widehat{K}(u)\widehat{H}(\lambda_t, t, U)\,\mathbb{P}_U(\lambda_U \geq \bar{K}(U,u)\,|\,\lambda_t) - \widehat{H}(\lambda_t, t, u)\,\mathbb{P}_u(\lambda_U \geq \bar{K}(U,u)\,|\,\lambda_t), \tag{60}$$

where $\widehat{H}(\lambda_t, t, u)$ can be interpreted as the price at time t of the zero-coupon bond maturing at u in the CIR model with the short-term rate λ and the constant $\bar{K}(U,u)$ can be easily computed from (41). The conditional probability density functions of λ_U under the forward martingale measures $\mathbb{P}_U$ and $\mathbb{P}_u$ can be expressed in terms of the transition probability density function of the ν-dimensional squared Bessel process with $\nu = 4a/c^2$ (i.e., the non-central χ^2-distribution).

To alleviate notation, we will represent formula (60) as follows

$$P_t^u = \widehat{K}(u)H_t^U p_U(t,\lambda_t) - H_t^u p_u(t,\lambda_t) = \widehat{K}(u)H_t^U P_U(t,Z_t) - H_t^u P_u(t,Z_t), \tag{61}$$

where we set $Z_t = H_t^u/H_t^U$ and where the functions P_U and P_u are obtained by expressing λ_t in terms of Z_t. To justify this step, it suffices to note that the ratio H_t^u/H_t^U is a strictly decreasing function of λ_t for any fixed $u > U$ since $n(t,u) > n(t,U)$.

To complete the computation of the hedging strategy, it remains to find the process $\widetilde{\zeta}$ arising in (58). This is feasible in the present setup as the following result shows. Recall that we are searching for the process $\widetilde{\xi}$, which is implicitly defined by the equality (cf. (14))

$$d(\widetilde{C}_t/\widetilde{A}(t,U,T)) = \widetilde{\xi}_t\, d\widehat{W}_t. \tag{62}$$

Proposition 4.4 *We have that, for every $t \in [0,U]$,*

$$\widetilde{\xi}_t = \frac{1}{\widetilde{A}_t}\bigg(\int_{]U,T]} B(t,u)\Big(\vartheta_t H_t^u(b_t^u - b_t^U) - P_t^u b_t^U\Big)\,d\chi(u) - \widetilde{C}_t\sigma_t^A\bigg), \tag{63}$$

where $\widetilde{A}_t = \widetilde{A}(t,U,T)$, $H_t^u = \widehat{H}(\lambda_t,t,u)$, $b_t^u = cn(t,u)\sqrt{\lambda_t}$, $P_t^u = P(\lambda_t, U, u, \widehat{K}(u))$ and

$$\vartheta_t = \widehat{K}(u)\,\frac{\partial P_U}{\partial z}(t,Z_t) - P_u(t,Z_t) - Z_t\,\frac{\partial P_u}{\partial z}(t,Z_t). \tag{64}$$

Proof. In view of (59), it is clear that it suffices to examine the martingale part of the process $(P_t^u,\, t \in [0,U])$ for a fixed $u \in]U,T]$. To this end, we first note that an application of the Itô formula to the right-hand side in the equality (cf. (61))

$$\frac{P_t^u}{H_t^U} = \widehat{K}(u)P_U(t,Z_t) - Z_t P_u(t,Z_t),$$

yields (observe that P_t^u/H_t^U and $Z_t = H_t^u/H_t^U$ are $\mathbb{P}_U$-martingales)

$$d\bigg(\frac{P_t^u}{H_t^U}\bigg) = \vartheta_t\,dZ_t = \vartheta_t\,d\bigg(\frac{H_t^u}{H_t^U}\bigg) \tag{65}$$

with ϑ given by (64). Let the symbol $\overset{mart}{=}$ stand for the equality of martingale parts of two continuous semimartingales. In view of (38) and (40), the process $(H_t^u,\, t \in [0,U])$ satisfies under $\widehat{\mathbb{Q}}$

$$dH_t^u \overset{mart}{=} -cn(t,u)\sqrt{\lambda_t}H_t^u\,d\widehat{W}_t = -H_t^u b_t^u\,d\widehat{W}_t,$$

where we denote $b_t^u = cn(t,u)\sqrt{\lambda_t}$. This in turn implies that

$$d\bigg(\frac{H_t^u}{H_t^U}\bigg) \overset{mart}{=} \frac{H_t^u}{H_t^U}\,(b_t^U - b_t^u)\,d\widehat{W}_t$$

and thus also, in view of (65),

$$d\bigg(\frac{P_t^u}{H_t^U}\bigg) \overset{mart}{=} \vartheta_t\frac{H_t^u}{H_t^U}\,(b_t^u - b_t^U)\,d\widehat{W}_t.$$

Consequently, we obtain

$$d(B(t,u)P_t^u) \overset{mart}{=} B(t,u)\Big(\vartheta_t H_t^u\big(b_t^u - b_t^U\big) - P_t^u b_t^U\Big)\, d\widehat{W}_t.$$

Using (59) and stochastic Fubini's theorem, we arrive at the following equality

$$d\widetilde{C}_t \overset{mart}{=} \int_{]U,T]} B(t,u)\Big(\vartheta_t H_t^u\big(b_t^u - b_t^U\big) - P_t^u b_t^U\Big)\, d\chi(u)\, d\widehat{W}_t.$$

We also note that

$$d\widetilde{A}_t = d(B_t G_t^{-1} a_t) \overset{mart}{=} B_t G_t^{-1} a_t \sigma_t^A\, d\widehat{W}_t = \widetilde{A}_t \sigma_t^A\, d\widehat{W}_t,$$

which in turn yields

$$d\bigg(\frac{\widetilde{C}_t}{\widetilde{A}_t}\bigg) = \frac{d\widetilde{C}_t - \widetilde{C}_t \sigma_t^A\, d\widehat{W}_t}{\widetilde{A}_t}\,.$$

Upon substitution, this completes the derivation of formula (63). □

Remark. Note that the process ϑ is simply the hedge ratio for the put bond option in the CIR model. We are not aware whether this hedge ratio was examined in detail in the existing literature.

We conclude this section by stating the following corollary to Propositions 2.2, 4.1 and 4.4. Admittedly, despite the fact that the hedging strategy for the credit default swaption is given here in the closed form, its implementation is not straightforward due to obvious numerical difficulties.

Corollary 4.2 *Let us consider the CIR default intensity model with a deterministic short-term interest rate, as specified in Assumption 4.1. The replicating strategy $\widetilde{\varphi} = (\widetilde{\varphi}^1, \widetilde{\varphi}^2)$ for the credit default swaption maturing at $R = U$ equals, for any $t \in [0, U]$,*

$$\widetilde{\varphi}_t^1 = \frac{\widetilde{\xi}_t}{\kappa(t,U,T)\sigma_t^\kappa}\,, \quad \widetilde{\varphi}_t^2 = \frac{\widetilde{C}_t - \widetilde{\varphi}_t^1 \widetilde{S}_t(\kappa)}{\widetilde{A}(t,U,T)}\,,$$

where the processes $\sigma^\kappa, \widetilde{C}$ and $\widetilde{\xi}$ are given in Propositions 4.1, 4.3 and 4.4, respectively.

4.5 Further Research

As shown in the previous section, the valuation and hedging problems the CIR default intensity model for the credit default swaption maturing at $R = U$ are amenable to quasi-explicit solutions. Unfortunately, the case of the credit default swaption maturing at $R < U$ is more complicated. It seems to us that it is rather difficult to make use of representation (55) for analytical computations of the volatility of the swaption's price when $R < U$. That is why one may want to consider an alternative approach involving another possible representation of the swaption's payoff C_R. To obtain this representation, we recall that $\zeta(x) = \delta B(R,U)\widehat{H}(x,R,U)$ and we observe that $\zeta : \mathbb{R} \to \mathbb{R}_+$ is strictly positive and strictly decreasing (recall that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ enjoys the same properties). Let $x^* > 0$ denote a strictly positive solution (if it exists) to the deterministic equation

$$\zeta(x) = \psi(x). \tag{66}$$

Lemma 4.3 *If the inequality $\zeta(0) < \psi(0)$ holds then there exists a unique solution $x^* > 0$ to equation (66) and $\zeta(x) > \psi(x)$ whenever $x > x^*$. If $\zeta(0) \geq \psi(0)$ then $\zeta(x) \geq \psi(x)$ for every $x \in \mathbb{R}_+$.*

Proof. Recall that the inequality $n(R,U) < n(R,u)$ holds for every $u \in]U,T]$. It is easy to deduce from (41) and (51) that $\zeta'(x) = -n(R,U)\zeta(x)$ whereas $\psi'(x) < -n(R,U)\psi(x)$ for every $x > 0$. This implies that a unique strictly positive solution x to equation (66) exists if $\zeta(0) < \psi(0)$. It is also clear that there is no strictly positive solution to this equation if $\zeta(0) \geq \psi(0)$ since in that case $\zeta(x) \geq \psi(x)$ for every $x \in \mathbb{R}_+$. $\square$

Assume first that $\zeta(0) \geq \psi(0)$. Then (52) becomes

$$C_R = \mathbb{1}_{\{R<\tau\}} \Big(\delta B(R,U)\widehat{H}(\lambda_R, R, U) - \psi(\lambda_R)\Big) = \mathbb{1}_{\{R<\tau\}} \big(\zeta(\lambda_R) - \psi(\lambda_R)\big).$$

If $\zeta(0) < \psi(0)$ then (52) can be represented as follows

$$C_R = \mathbb{1}_{\{R<\tau\}}\mathbb{1}_{\{\lambda_R>x^*\}} \Big(\delta B(R,U)\widehat{H}(\lambda_R, R, U) - \psi(\lambda_R)\Big) = \mathbb{1}_{\{R<\tau\}}\mathbb{1}_{\{\lambda_R>x^*\}} \big(\zeta(\lambda_R) - \psi(\lambda_R)\big).$$

It is left as an open problem whether these representations can be helpful in finding hedging strategies for the credit default swaption maturing at $R < U$.

5 Appendix: Itô-Kunita-Wentzell Formula

For the reader's convenience, we recall here the Itô-Kunita-Wentzell formula used in the proof of Lemma 3.3. Let $F_t(x)$ be a family of stochastic processes, continuous in $(t,x) \in (\mathbb{R}_+ \times \mathbb{R}^d)$ a.s., and satisfying the following conditions:
(i) for each $t > 0$, $x \to F_t(x)$ is C^2 from $\mathbb{R}^d$ to $\mathbb{R}$,
(ii) for each x, $(F_t(x), t \geq 0)$ is a continuous semimartingale

$$dF_t(x) = \sum_{j=1}^n f_t^j(x)\, dM_t^j,$$

where M^j are continuous semimartingales, and $f^j(x)$ are stochastic processes continuous in (t,x), such that for every $s > 0$, the map $x \to f_s^j(x)$ is C^1, and for every x, $f^j(x)$ is an adapted process. Let $X = (X^1, \cdots, X^d)$ be a continuous semimartingale. Then

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^n \int_0^t f_s^j(X_s)\, dM_s^j + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)\, dX_s^i \\ &+ \sum_{i=1}^d \sum_{j=1}^n \int_0^t \frac{\partial f_s}{\partial x_i}(X_s)\, d\langle M^j, X^i\rangle_s + \frac{1}{2} \sum_{i,k=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k}\, d\langle X^k, X^i\rangle_s. \end{aligned}$$

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