

# *Pricing American currency options in an exponential Lévy model*

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Received 26 September 2002 and in revised form 28 April 2004

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In this article the problem of the American option valuation in a Lévy process setting is analysed. The perpetual case is first considered. Without possible discontinuities (i.e. with negative jumps in the call case), known results concerning the currency option value as well as the exercise boundary are obtained with a martingale approach. With possible discontinuities of the underlying process at the exercise boundary (i.e. with positive jumps in the call case), original results are derived by relying on first passage time and overshoot associated with a Lévy process. For finite life American currency calls, the formula derived by Bates or Zhang, in the context of a negative jump size, is tested. It is basically an extension of the one developed by Mac Millan and extended by Barone-Adesi and Whaley. It is shown that Bates' model generates pretty good results only when the process is continuous at the exercise boundary.

**Keywords:** American options, perpetual options, exercise boundary, incomplete markets, jump diffusion model, Laplace transform, stopping times, Lévy exponent, overshoot

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## 1. Introduction

Several articles have focused on the valuation of European options when the underlying value follows a jump diffusion process. Merton (1976) was the first author to obtain a closed form solution. The problem of the American option valuation is more complex. It was also tackled by several authors. Bates (1991) derived the early exercise premium, by relying on an extension of the Mac Millan (1987) and Barone-Adesi and Whaley (1987) approaches in a jump diffusion setting.

Pham (1997) considered the American put option valuation in a jump-diffusion model (Merton's assumptions) and related this optimal stopping problem to a parabolic integro-differential free boundary problem. By extending the Riesz decomposition obtained by Carr *et al.* (1992) in a diffusion model, he derived a decomposition of the American put price as the sum of its corresponding European price and the early exercise premium. The latter rests on the identification of the exercise boundary. In the same context, Zhang (1994) relies on variational inequalities and shows how to use numerical methods (finite difference methods) in order to price the American put. Zhang (1995) sets this pricing problem into a free boundary problem, and by using the Mac Millan's (1987) approximation, she obtains a price for the perpetual put and an approximation of the finite maturity put price. These

results are obtained only when jumps are positive, i.e. without discontinuities of the underlying process at the exercise boundary. Mastroeni and Matzeu (1995 and 1996) obtain an extension of Zhang (1994) results in a multidimensional state space. Boyarchenko and Levendorskii (2002), Mordecki (1999) and (2002), Gerber and Shui (1999) also consider the American option pricing problem. They obtain solutions which are explicit only when the distribution of the jump size is exponential or when the jump size is non-positive for a call (resp. non-negative for a put). Mordecki establishes the value of the boundary for a perpetual option in terms of the law of the extrema of the Lévy process. Michaud (1997) considers the case of a constant jump size and of a zero volatility and obtains explicit solutions. Gerber and Landry (1998) obtain a formula in a compound process when the dividend rate is null.

By relying on a martingale approach, this study also obtains original results.

First considered is the valuation of a perpetual currency call option, and obtain the exercise boundary in two cases: only negative jumps and then only positive ones. In the first case, known results are obtained. The exercise boundary is derived by using the stopping theorem and the Lévy exponent. In the second case, the valuation problem is more difficult to tackle, the process being possibly discontinuous at the critical boundary. The overshoot at the exercise boundary is introduced. Our original results are derived by relying on Bertoin's book (1996) on Lévy's processes. Indeed, Laplace transforms involving the first passage time and the overshoot at the exercise boundary are used.

As in Zhang (1997) or Bates (1991), it is then shown how an approximation of the finite maturity option price can be obtained by relying on the perpetual maturity case. As shown in the simulations, this approach generates good results only when jumps are negative in the call case, i.e. without discontinuities of the underlying process at the exercise boundary.

This article is organized as follows. In Section 2, our model is defined. In Section 3 (resp. 4) the perpetual American call (resp. put) is considered. In Section 5, decompositions of American options prices are obtained. In Section 6, an approximation of the finite maturity American option value is derived. In Section 7, the accuracy of the approximation formula is tested, by comparing it to a numerical approach. The article ends with an appendix, where some proofs are given.

## 2. Notation and assumptions

Suppose that, under the chosen risk neutral probability, the price's dynamics of the risky asset (a currency) are

$$S_t = S_0 e^{X_t} \quad (1)$$

Here  $X$  is a Lévy process of parameters  $(m, \sigma^2, \nu)$ , i.e. a process with stationary and independent increments with characteristic function

$$E(e^{iyX_1}) = \exp\left(iym - \frac{\sigma^2}{2}y^2 + \int (e^{iyx} - 1 - iyx\mathbb{1}_{|x|\leq 1})\nu(dx)\right)$$

where  $\nu$  is a Lévy measure. It is assumed that, for any  $u \in \mathbb{R}$ ,  $E(e^{uX_1}) < \infty$  so that

$$E(e^{uX_1}) = \exp\left(um + \frac{\sigma^2}{2}u^2 + \int (e^{ux} - 1 - ux\mathbb{1}_{|x|\leq 1})\nu(dx)\right)$$

Let us denote by  $r$  (resp.  $\delta$ ) the constant domestic (resp. foreign) interest rate. Since the dynamics are given under the risk-neutral probability, the process  $e^{-(r-\delta)t}e^{X_t}$  is a martingale. This last condition is equivalent to

$$-(r-\delta) + m + \frac{1}{2}\sigma^2 + \int (e^x - 1 - x\mathbb{1}_{|x|\leq 1})v(dx) = 0 \quad (2)$$

We denote by  $\Psi(u)$  the Lévy exponent of  $X$  defined as

$$E(e^{uX_t}) = e^{t\Psi(u)}$$

In an explicit form,

$$\Psi(u) = um + u^2 \frac{\sigma^2}{2} + \int (e^{ux} - 1 - ux\mathbb{1}_{|x|\leq 1})v(dx)$$

The function  $\Psi$  is convex,  $\Psi(0) = 0$ ,  $\Psi(1) = r - \delta$ . (This last property is a consequence of the martingale property.) We denote by  $\Psi^\#(r)$  the non-negative solution of  $\Psi(u) = r$ .

As examples, the case where the Lévy measure is proportional to a Dirac measure at point  $\varphi$ :  $v(dx) = \lambda\delta_\varphi(dx)$ , with  $\lambda > 0$  is presented. In that case, the dynamics of  $S$  can be written in the form

$$dS_t = S_{t-}((r-\delta)dt + \sigma dW_t + \phi dM_t)$$

where  $M_t = N_t - \lambda t$  is the compensated martingale associated with a Poisson process  $N$  with parameter  $\lambda$  and  $\phi = e^\varphi - 1$ . The Laplace exponent of the Lévy process  $X$  is

$$\begin{aligned} \Psi(u) &= u\mu + u^2 \frac{\sigma^2}{2} + \lambda(e^{u\varphi} - 1) \\ &= u\left(r - \delta - \frac{\sigma^2}{2}\right) + u^2 \frac{\sigma^2}{2} + \lambda((1 + \phi)^u - 1 - u\phi) \end{aligned}$$

with  $\mu = r - \delta - \lambda\phi - \frac{\sigma^2}{2}$ .

A second example is the case where  $v$  is integrable. This case corresponds to a compound Poisson process and was studied in Gerber and Landry when the dividend rate is equal to 0. In the case where the jump part of  $X$  is a compound Poisson process  $\sum_{k=1}^{N_t} Y_k$ , where  $N$  is a Poisson process with intensity  $\lambda$  and the r.v.s ( $Y_k, k \geq 1$ ) are iid with common law  $P(Y_k \in dx) = F(dy)$ , then the Lévy measure is  $v(dx) = \lambda F(dx)$ .

### 3. The perpetual American currency call

The value of a perpetual American call is  $C_A(S_0) = \sup_\tau E((S_\tau - K)e^{-r\tau})$ . Mordecki (1999) proves that one can reduce attention to the stopping times which are hitting times, i.e. to stopping times of the form

$$T^S(L) = \inf\{t \geq 0 : S_t \geq L\}$$

and

$$C_A(S_0) = \sup_L E\left((S_{T(L)} - K)e^{-rT(L)}\right)$$

Hence, introducing the function  $f$  defined for  $S_0 < L$  as  $f(S_0, L) = E(e^{-rT(L)}(S_0 \exp(X_{T(L)}) - K))$ , the value of the American call is

$$C_A(x) = \sup_{L \geq x} f(x, L)$$

Let  $T^X(\ell) = \inf\{t \geq 0, X_t \geq \ell\}$  be the hitting time of  $\ell$  for the process  $X$ , and  $K(\ell)$  be the overshoot defined in terms of  $X$  by  $X_{T(\ell)} = \ell + K(\ell)$ . (Obviously,  $T^X(\ell) = T^S(L)$ .) We introduce the function

$$g(x, \ell) = xe^\ell E\left(e^{-rT(\ell) + K(\ell)}\right) - KE\left(e^{-rT(\ell)}\right) = f(x, xe^\ell)$$

Then,

$$C_A(x) = \sup_{\ell \geq 0} E\left(e^{-rT(\ell)}(xe^{X_{T(\ell)}} - K)\right) = \sup_{\ell \geq 0} g(x, \ell)$$

The levels  $\ell$  and  $L$  are related by  $\ell = \ln(L/x)$ .

### 3.1. General formula

Introduce now the function  $\bar{g}$

$$\bar{g}(x, \ell) = g(xe^{-\ell}, \ell) = xE\left(e^{-rT(\ell) + K(\ell)}\right) - KE\left(e^{-rT(\ell)}\right) = f(xe^{-\ell}, x)$$

The Laplace transform of  $\bar{g}$  with respect to  $\ell$  is

$$\varphi(q, x) = \int_0^\infty e^{-q\ell} \bar{g}(x, \ell) d\ell = \int_0^x e^{-q \ln(x/y)} \bar{g}(x, \ln(x/y)) \frac{1}{y} dy \quad (3)$$

$$= \int_0^x \frac{1}{y} f(y, x) e^{-q \ln(x/y)} dy \quad (4)$$

On the one hand,

$$\varphi(q, x) = x\alpha(q, r) - K\beta(q, r)$$

with:

$$\alpha(q, r) = \int_0^{+\infty} e^{-q\ell} E\left(e^{-rT(\ell) + K(\ell)}\right) d\ell \quad (5)$$

$$\beta(q, r) = \int_0^{+\infty} e^{-q\ell} E\left(e^{-rT(\ell)}\right) d\ell \quad (6)$$

On the other hand, as in Gerber and Landry, by definition of the perpetual call exercise boundary  $b_c$ :

$$\text{for } x < b_c \quad f(x, b_c) = \sup_{L \geq x} f(x, L)$$

hence, assuming that  $f$  is differentiable,

$$\frac{\partial f}{\partial L}(x, b_c) = 0, \quad x < b_c \quad (7)$$

Therefore, by differentiation of (4) with respect to  $x$  at point  $b_c$

$$\frac{\partial \varphi}{\partial x}(q, b_c) = \frac{f(b_c, b_c)}{b_c} - \frac{q}{b_c} \varphi(q, b_c)$$

hence:

$$\alpha(q, r) = \frac{b_c - K}{b_c} - \frac{q}{b_c} (b_c \alpha(q, r) - K \beta(q, r))$$

Now, due to Pecherski–Rogozin (1969) identity (See Appendix A), the functions  $\alpha$  and  $\beta$  are known in terms of the ladder exponent  $\kappa$ :

$$\alpha(q, r) = \frac{\kappa(r, q) - \kappa(r, -1)}{(q+1)\kappa(r, q)}, \quad \beta(q, r) = \frac{\kappa(r, q) - \kappa(r, 0)}{q\kappa(r, q)} \quad (8)$$

and an easy computation leads to

$$b_c = \frac{\kappa(r, 0)}{\kappa(r, -1)} K \quad (9)$$

Therefore, the price is obtained.

**Proposition 1** The boundary of a perpetual American call is given by

$$b_c = \frac{\kappa(r, 0)}{\kappa(r, -1)} K$$

where  $\kappa$  is the ladder exponent of  $X$ .

**Comments** It can be remarked that Mordecki (1999) has proved that  $b_c = KE(e^{M_\theta})$  where  $M_t = \sup_{s \leq t} X_s$  and  $\theta$  is a random variable with exponential law of parameter 1, independent of  $X$ . The equality between  $E(e^{M_\theta})$  and  $\frac{\kappa(r, 0)}{\kappa(r, -1)}$  follows from Wiener–Hopf decomposition. (See Nguyen–Ngoc and Yor (2001)).

### 3.2. Particular case: $X$ without positive jumps

If the process  $X$  has no positive jumps, then, one obtains the well-known result derived by Zhang. Indeed, in that case

$$\kappa(\alpha, \beta) = \Psi^\sharp(\alpha) + \beta$$

where  $\Psi^\sharp(z)$  is the positive number  $y$  such that  $\Psi(y)=z$ . Hence

$$b_c = K \frac{\Psi^\sharp(r)}{\Psi^\sharp(r) - 1}$$

### 3.3. Particular case: $X$ without negative jumps

It is now assumed that the jumps of  $X$  are positive hence the dual process  $\widehat{X} = -X$  has no positive jumps. Then

$$\kappa(u, k) = \frac{u - \widehat{\Psi}(k)}{\widehat{\Psi}^\sharp(u) - k} \quad (10)$$

where  $\widehat{\Psi}$  is the Lévy exponent of  $-X$ , given by  $\widehat{\Psi}(k) = \Psi(-k)$ . Relying on Equations 9 and 10, the exercise boundary is given by

$$b_c = \frac{r - \widehat{\Psi}(0)}{\widehat{\Psi}^\sharp(r)} \frac{\widehat{\Psi}^\sharp(r) + 1}{r - \widehat{\Psi}(-1)} K$$

Using that  $\widehat{\Psi}(0) = 0$ ,  $\widehat{\Psi}(-1) = \Psi(1) = r - \delta$ , and  $\widehat{\Psi}^\sharp(r) = -\Psi^{\sharp, n}(r)$ , the following proposition is obtained:

**Proposition 2** The exercise boundary for a perpetual call is

$$b_c = \frac{r \Psi^{\sharp, n}(r) - 1}{\delta \Psi^{\sharp, n}(r)} K \quad (11)$$

where  $\Psi^{\sharp, n}(r)$  is the negative root of  $\Psi(k) = r$

It is straightforward to check that without jump ( $\phi = 0$ ) the formula coincides with the usual formula. Indeed, in this case,

$$\frac{r \Psi_0^{\sharp, n}(r) - 1}{\delta \Psi_0^{\sharp, n}(r)} = \frac{\Psi_0^\sharp(r)}{\Psi_0^\sharp(r) - 1} \quad (12)$$

where  $\Psi_0$  is the Lévy exponent in the case  $\phi = 0$ , so that  $\Psi_0^\sharp(r)$  is the positive root of  $bk + \frac{1}{2}\sigma^2 k(k-1) = r$  and  $\Psi_0^{\sharp, n}(r)$  is the negative root. Usual relations between the sum, the product of roots and the coefficients lead to the result.

## 4. The perpetual American currency put

The put case can be solved using the symmetrical relationship (see Fajardo and Mordecki (2003)) between the American call and put boundaries

$$b_p(K, r, \delta, \Psi) b_c(K, \delta, r, \widetilde{\Psi}) = K^2$$

where

$$\tilde{\Psi}(u) = \Psi(1-u) - \Psi(1) \quad (13)$$

and by relying on Equation 11, the exercise boundary  $b_p$  of the perpetual put in a jump diffusion setting with negative jumps can be obtained:

$$b_p = \frac{rK}{\delta} \frac{\tilde{\Psi}^{\#,n}(\delta)}{\tilde{\Psi}^{\#,n}(\delta) - 1} \quad (14)$$

In the case of jumps of constant size

$$b_p(K, r, \delta, \lambda, \varphi) b_c \left( K, \delta, r, \lambda(1+\varphi) - \frac{\varphi}{1+\varphi} \right) = K^2$$

The perpetual exercise boundary for the put can also be obtained by relying on the procedure used for the call. The proof is given as a check.

From definition

$$P_A(x) = \sup E \left( e^{-rT(L)} (K - S_{T(L)}) \right)$$

where

$$\begin{aligned} T(L) &= \inf\{t : S_t \leq L\} = \inf\{t : X_t \leq \ell\} \\ &= \inf\{t : \hat{X}_t \geq \hat{\ell}\} \end{aligned}$$

with  $\hat{\ell} = -\ell = \ln(S_0/L)$ . From the Pecherski–Rogozin (1969) formula

$$\int_0^\infty e^{-qx} E \left( e^{-\alpha \hat{T}(x) - \beta \hat{K}(x)} \right) dx = \frac{\hat{\kappa}(\alpha, q) - \hat{\kappa}(\alpha, \beta)}{(q - \beta) \hat{\kappa}(\alpha, q)}$$

where  $\hat{T}(x) = \inf\{t : \hat{X}_t \geq x\}$  and  $\hat{K}(x) = \hat{X}_{\hat{T}(x)} - x$ . Let  $\bar{h}(u, x) = KE \left( e^{-r\hat{T}(u)} \right) - xE \left( e^{-r\hat{T}(u) - \hat{K}(u)} \right)$ . Then  $\bar{h}(u, xe^{-u}) = f(x, xe^{-u})$  where

$$f(x, L) = E \left( e^{-rT(L)} (K - xe^{X_{T(L)}}) \right)$$

Then setting

$$H(q, x) = \int_0^\infty e^{-qu} \bar{h}(u, x) du$$

we get

$$H(q, x) = K \int_0^\infty e^{-qu} E \left( e^{-r\hat{T}(u)} \right) du - x \int_0^\infty e^{-qu} E \left( e^{-r\hat{T}(u) - \hat{K}(u)} \right) du$$

and a change of variables leads to

$$H(q, x) = \int_x^\infty e^{-q \ln(z/x)} f(z, x) \frac{1}{z} dz$$

Using the same framework as in the call case, one gets

$$\hat{\ell}^* = K \frac{\hat{\kappa}(1, 0)}{\hat{\kappa}(r, 1)}$$

It is now assumed that  $X$  has no positive jumps,

$$\hat{\kappa}(\alpha, \beta) = k\hat{k} \frac{\alpha - \Psi(\beta)}{\Psi^\#(\alpha) - \beta}$$

that leads to

$$\ell^* = K \frac{r \Psi^\#(r) - 1}{\delta \Psi^\#(r)}$$

**Proposition 3** Let  $X$  be a Lévy process without positive jumps:

$$b_p = \frac{r \Psi^\#(r) - 1}{\delta \Psi^\#(r)} K \quad (15)$$

This formula is the same as the one obtained by the symmetry principle: indeed both formulae are the same if

$$\frac{\tilde{\Psi}^{\#,n}(\delta)}{\tilde{\Psi}^{\#,n}(\delta) - 1} = \frac{\Psi^\#(r) - 1}{\Psi^\#(r)}$$

which reduces to

$$\tilde{\Psi}^{\#,n}(\delta) + \Psi^\#(r) = 1$$

this last equality is now obvious from

$$\tilde{\Psi}(1-z) = \Psi(z) - \Psi(1) = \Psi(z) - (r - \delta)$$

for  $z = \Psi^\#(r)$  one gets

$$\tilde{\Psi}(1-z) = \delta$$

hence  $1-z = \tilde{\Psi}^{\#,n}(\delta)$ .

Now that the perpetual exercise boundary for call and puts are known, option prices can be derived.

**Comments** It can be checked that, when there is no dividend and when the Lévy measure corresponds to a compound Poisson process, this formula is the same as in Gerber and Landry (1998). Indeed, if  $\delta=0$ , from (2),  $\Psi^\#(r)=1$  is obtained. Hence, we have to study the behaviour of the ratio  $(\Psi^\#(r)-1)/\delta$  when  $\delta$  goes to 0. By definition and using Equation 2,  $\Psi^\#(r)$  is the value of  $u$  such that

$$um + u^2 \frac{\sigma^2}{2} + \int (\mathrm{e}^{ux} - 1 - ux \mathbb{1}_{|x| \leq 1}) \nu(\mathrm{d}x) = m + \frac{\sigma^2}{2} + \int (\mathrm{e}^x - 1 - x \mathbb{1}_{|x| \leq 1}) \nu(\mathrm{d}x) + \delta$$

i.e. the value of  $u$  such that

$$G(u) = m(u-1) + (u^2-1) \frac{\sigma^2}{2} + \int (\mathrm{e}^{ux} - \mathrm{e}^x - (u-1)x \mathbb{1}_{|x| \leq 1}) \nu(\mathrm{d}x) = \delta$$

With this notation,

$$\frac{\Psi^\#(r) - 1}{\delta} = \frac{G^\#(\delta) - G^\#(0)}{\delta}$$

and this quantity converges to  $(G^\#)'(0)$  when  $\delta$  goes to 0. Now,

$$(G^\#)'(0) = \frac{1}{G'(G^\#)(0)} = \frac{1}{G'(1)} = \left( m + \sigma^2 + \int (xe^x - x \mathbb{1}_{|x| \leq 1}) \nu(dx) \right)^{-1}$$

and this is exactly the formula of Gerber and Landry (1998).

## 5. American and European

In a jump-diffusion setting, a decomposition of the American option price into the European price and the American premium can also be obtained. The optimal boundary is denoted by  $b_c(T-t)$ .

Let us assume that in the risk neutral economy the dynamics of the currency price are given by Equation 3 with constant coefficients. In the case where the jump part of  $X$  is a compound Poisson process, Gukhal (2001) establishes the following formula for an American call with maturity  $T$

$$\begin{aligned} C_A(x, 0) = & C_E(x, 0) + \int_0^T ds e^{-rs} E_Q(\delta S_s \mathbb{1}_{S_s \geq b_c(T-s)}) - rK \int_0^T ds e^{-rs} E_Q(\mathbb{1}_{S_s \geq b_c(T-s)}) \\ & - \lambda \int_0^T ds e^{-rs} E_Q((C(YS_s, T-s) - (YS_s - K)) \mathbb{1}_{S_s \geq b_c(T-s)} \mathbb{1}_{YS_s \leq b_c(T-s)}) \end{aligned}$$

In this formula,  $Y$  is a random variable with the same law as  $Y_k$ . In the case where  $Y > 1$ , the set  $\{S_s \geq b_c(T-s)\} \cap \{YS_s \leq b_c(T-s)\}$  is empty and the formula reduces to

$$\begin{aligned} C_A(x, 0) = & C_E(x, 0) + \int_0^T ds e^{-rs} E_Q(\delta S_s \mathbb{1}_{S_s \geq b_c(T-s)}) \\ & - rK \int_0^T ds e^{-rs} E_Q(\mathbb{1}_{S_s \geq b_c(T-s)}) \end{aligned}$$

In the case of a perpetual American call option, when the size of the jump is a strictly positive constant  $\phi$  (hence  $Y = \phi + 1 > 1$ ) one gets, in the case of constant jump size (see Appendix C)

$$\begin{aligned} C_A(x) = & \delta x \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^{+\infty} e^{-(\delta + \lambda(1+\phi))s} (\lambda s)^n \mathcal{N}(d_1(b_c, n; s)) (1+\phi)^n ds \\ & - rK \sum_{n=0}^{+\infty} \frac{1}{n!} \int_0^{+\infty} e^{-(r+\lambda)s} (\lambda s)^n \mathcal{N}(d_2(b_c, n; s)) ds \end{aligned} \tag{16}$$

with

$$d_1(z, n; s) = \frac{\ln(x/z) + (r - \delta - \lambda\phi + \sigma^2/2)s + n \ln(1+\phi)}{\sigma\sqrt{s}}$$

$$d_2(z, n; s) = d_1(z, n; s) - \sigma\sqrt{s}$$

Along the same lines a decomposition for the put can be obtained.

## 6. An approximation of the option value

Let us rely on the Barone-Adesi and Whaley (1987) approach, and on Bates' (1991) article. Let us assume that the jump size is constant and negative. If the American and European option values satisfied the same linear PDE (in the continuation region), their difference  $\Delta C$ , the American premium, must also satisfy this PDE in the same region. Let us write

$$\Delta C(S_0, T) = yh(S_0, y)$$

where

$$y = 1 - e^{-rT}$$

and where  $h$  is a two argument function that has to be determined. In the continuation region  $h$  satisfies the following p.d.e. obtained by a change of variables:

$$\frac{\sigma^2}{2} x^2 \frac{\partial^2 h}{\partial x^2} + (r - \delta)x \frac{\partial h}{\partial x} - \frac{rh}{y} - (1 - y)r \frac{\partial h}{\partial y} - \lambda \left( \frac{\partial h}{\partial x} \lambda x - h((1 + \phi)x, y) + h(x, y) \right) = 0$$

Like the authors, let it now be assumed that the term with the derivative of  $h$  with respect to  $y$  is negligible. Whether or not it is a good approximation is an empirical issue that will be considered in the following section. The following equation has to be solved:

$$\frac{\sigma^2}{2} x^2 \frac{\partial^2 h}{\partial x^2} + (r - \delta)x \frac{\partial h}{\partial x} - \frac{rh}{y} - \lambda \left( \frac{\partial h}{\partial x} \lambda x - h((1 + \phi)x, y) + h(x, y) \right) = 0$$

The perpetual option value satisfies almost the same differential equation. The only difference resides in the fact that  $y$  is equal to 1 in the perpetual case, and therefore we have  $\frac{x}{y}$  instead of  $r$  in the third term on the left-hand side. The form of the solution with a negative jump is known:

$$h = \eta x^\rho$$

where  $\eta$  is still unknown, and  $\rho = \Psi^\# \left( \frac{x}{y} \right)$ . But when  $S_0$  tends to the exercise boundary  $b_c(T)$ , by continuity of the option value, the following equation is satisfied:

$$b_c(T) - K = C_E(b_c(T), T) + y\eta b_c(T)^\rho \quad (17)$$

and by use of the smooth fit condition the following equation is obtained:

$$1 = \frac{\partial C_E}{\partial x}(b_c(T), T) + y\eta \rho b_c(T)^{\rho-1} \quad (18)$$

Within a jump-diffusion model, this condition was derived by Zhang (1994) in the context of variational inequalities and by Pham (1995) with a free boundary formulation. We thus have a system of two Equations 17 and 18 and two unknowns  $\eta$  and  $b_c(T)$  that can be solved.  $b_c(T)$  is the implicit solution of:

$$b_c(T) = K - C_E(b_c(T), T) + \left( 1 - \frac{\partial C_E}{\partial x}(b_c(T), T) \right) \frac{b_c(T)}{\rho}$$

and the approximation formula is the following:

$$C_A(S_0, T) = C_E(S_0, T) + A(S_0/b_c(T))^{\rho} \quad (19)$$

if  $S_0 < b_c$

$$C_A(S_0, T) = S_0 - K$$

otherwise, with

$$A = \left(1 - \frac{\partial C_E}{\partial x}(b_c(T), T)\right) \frac{b_c(T)}{\rho}$$

**Table 1.** Theoretical European and American call values.

Parameters	$S_0$	Call option prices							
		European			American				
		No jump	with jumps		no jump		with jumps		
	Garm. Kohlh.	Fin. diff. meth.	Merton	Fin. diff. meth.	Barone Adesi	Fin. diff. meth.	Bates	Approx	
(1)	80	0.05	0.08	0.08	0.05	0.05	0.08	0.08	0.08
$r=0.08$	90	0.85	1.07	1.07	0.85	0.85	1.07	1.07	1.07
$\sigma=0.20$	100	4.44	4.92	4.91	4.44	4.44	4.91	4.91	4.91
$T=0.25$	110	11.66	12.06	12.06	11.66	11.66	12.06	12.06	12.06
	120	20.90	21.08	21.07	20.90	20.90	21.09	21.07	21.07
(2)	80	0.05	0.08	0.08	0.05	0.05	0.08	0.08	0.08
$r=0.12$	90	0.84	1.06	1.06	0.84	0.84	1.06	1.06	1.06
$\sigma=0.20$	100	4.40	4.86	4.86	4.40	4.40	4.86	4.86	4.86
$T=0.25$	110	11.55	11.94	11.94	11.55	11.55	11.94	11.94	11.94
	120	20.69	20.88	20.86	20.69	20.69	20.88	20.87	20.87
(3)	80	1.29	1.40	1.40	1.29	1.29	1.40	1.40	1.40
$r=0.08$	90	3.82	4.02	4.01	3.82	3.82	4.02	4.02	4.01
$\sigma=0.40$	100	8.35	8.60	8.60	8.35	8.35	8.60	8.60	8.60
$T=0.25$	110	14.80	15.05	15.04	14.79	14.80	15.05	15.05	15.05
	120	22.71	22.93	22.92	22.71	22.72	22.93	22.92	22.92
(4)	80	0.41	0.59	0.60	0.41	0.41	0.59	0.59	0.59
$r=0.08$	90	2.18	2.67	2.65	2.18	2.18	2.67	2.65	2.65
$\sigma=0.20$	100	6.50	7.21	7.17	6.50	6.50	7.22	7.17	7.16
$T=0.50$	110	13.42	14.12	14.02	13.42	13.42	14.12	14.03	14.03
	120	22.06	22.45	22.45	22.06	22.06	22.45	22.45	22.45

Parameters:  $\lambda=1$ ,  $\phi=-0.1$ ,  $K=100$ ,  $r-\delta=0.04$ .

where (Merton, 1976)

$$C_E(S_0, T) = \sum_{n=0}^{+\infty} \frac{e^{-\lambda T} (\lambda T)^n}{n!} e^{-\left(\delta + \lambda\phi - \frac{n \ln(1+\phi)}{T}\right) T} C_{BS}\left(S_0, T, r - \delta - \lambda\phi + \frac{n \ln(1+\phi)}{T}, \sigma\right) \quad (20)$$

and

$$C_{BS}(S_0, T, \tau, \sigma) = S_0 N(d_1) - K e^{-\tau T} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + \left(\tau + \frac{\sigma^2}{2}\right) T}{\sigma T}$$

$$d_2 = d_1 - \sigma \sqrt{T}$$

**Table 2.** Theoretical European and American call values.

Parameters		Call option prices							
		European				American			
		No jump		with jumps		no jump		with jumps	
$S_0$	Garm. Kohlh.	Fin. diff. meth.	Merton	Fin. diff. meth.	Barone Adesi	Fin. diff. meth.	Bates	Approx	
(1)	80	0.03	0.05	0.05	0.03	0.03	0.05	0.05	0.05
$r=0.08$	90	0.57	0.74	0.74	0.58	0.59	0.75	0.76	0.76
$\sigma=0.20$	100	3.42	3.87	3.86	3.52	3.52	3.94	3.95	3.97
$T=0.25$	110	9.85	10.30	10.28	10.35	10.31	10.66	10.63	10.75
	120	18.62	18.87	18.84	20.00	20.00	20.00	20.00	20.00
(2)	80	0.03	0.05	0.05	0.03	0.03	0.05	0.05	0.05
$r=0.12$	90	0.56	0.74	0.74	0.58	0.59	0.74	0.76	0.75
$\sigma=0.20$	100	3.39	3.83	3.82	3.50	3.51	3.91	3.93	3.94
$T=0.25$	110	9.75	10.20	10.18	10.32	10.29	10.62	10.60	10.72
	120	18.43	18.68	18.65	20.00	20.00	20.00	20.00	20.00
(3)	80	1.05	1.14	1.14	1.06	1.07	1.15	1.16	1.16
$r=0.08$	90	3.23	3.41	3.41	3.27	3.28	3.45	3.47	3.47
$\sigma=0.40$	100	7.29	7.54	7.53	7.40	7.41	7.65	7.65	7.65
$T=0.25$	110	13.25	13.53	13.50	13.52	13.50	13.77	13.75	13.76
	120	20.73	20.97	20.94	21.29	21.23	21.48	21.42	21.45
(4)	80	0.21	0.32	0.32	0.21	0.23	0.32	0.34	0.34
$r=0.08$	90	1.31	1.69	1.68	1.36	1.39	1.73	1.75	1.75
$\sigma=0.20$	100	4.46	5.12	5.09	4.71	4.72	5.32	5.33	5.35
$T=0.50$	110	10.16	10.90	10.82	11.00	10.96	11.51	11.49	11.61
	120	17.85	18.47	18.34	20.00	20.00	20.05	20.03	20.00

Parameters:  $\lambda = 1$ ,  $\phi = -0.1$ ,  $K = 100$ ,  $r - \delta = -0.04$ .

The latter terms are given by Black and Scholes (1973). These results were obtained by Bates (1991) in a more general context in which  $1 + \phi$  is a log-normal random variable (for the put). However, as shown in Sections 2 and 3, the choice of the option pricing model should depend on the sign of the jumps. Furthermore, when the jump size is positive, as shown in Appendix B, the differential equation that the American option value satisfies, takes a specific form just below the exercise boundary, when the jumps are positive. Unfortunately there is no known solution to this equation. Positive jumps generate possible discontinuities in the process at the exercise boundary, and therefore the problem is more difficult to solve (see Sections 2 and 3). As shown in the following section, Bates' approximation gives better results when the size of the jump is negative or small.

**Table 3.** Theoretical European and American call values.

<i>Parameters</i>	<i>S<sub>0</sub></i>	<i>Call option prices</i>							
		<i>European</i>				<i>American</i>			
		<i>no jump</i>	<i>with jumps</i>			<i>no jump</i>	<i>with jumps</i>		
		<i>Garm. Kohlh.</i>	<i>Fin. diff. meth.</i>	<i>Merton</i>	<i>Fin. diff. meth.</i>	<i>Barone Adesi</i>	<i>Fin. diff. meth.</i>	<i>Bates</i>	<i>Approx</i>
(1)	80	0.05	0.14	0.14	0.05	0.05	0.14	0.14	0.14
<i>r</i> =0.08	90	0.85	1.19	1.19	0.85	0.85	1.19	1.19	1.19
<i>σ</i> =0.20	100	4.44	4.85	4.85	4.44	4.44	4.85	4.85	4.85
<i>T</i> =0.25	110	11.66	11.87	11.87	11.66	11.66	11.87	11.87	11.87
	120	20.90	20.95	20.95	20.90	20.90	20.95	20.95	20.95
(2)	80	0.05	0.14	0.14	0.05	0.05	0.14	0.14	0.14
<i>r</i> =0.12	90	0.84	1.17	1.18	0.84	0.84	1.17	1.18	1.18
<i>σ</i> =0.20	100	4.40	4.80	4.80	4.40	4.40	4.80	4.80	4.80
<i>T</i> =0.25	110	11.55	11.75	11.75	11.55	11.55	11.75	11.75	11.75
	120	20.69	20.74	20.74	20.69	20.69	20.74	20.74	20.75
(3)	80	1.29	1.41	1.43	1.29	1.29	1.41	1.43	1.43
<i>r</i> =0.08	90	3.82	4.00	4.03	3.82	3.82	4.00	4.03	4.03
<i>σ</i> =0.40	100	8.35	8.54	8.57	8.35	8.35	8.54	8.57	8.57
<i>T</i> =0.25	110	14.80	14.94	14.99	14.80	14.80	14.94	14.99	14.99
	120	22.71	22.80	22.85	22.71	22.72	22.80	22.85	22.85
(4)	80	0.41	0.67	0.69	0.41	0.41	0.67	0.69	0.69
<i>r</i> =0.08	90	2.18	2.66	2.72	2.18	2.18	2.66	2.72	2.72
<i>σ</i> =0.20	100	6.50	6.98	7.08	6.50	6.50	6.98	7.08	7.08
<i>T</i> =0.50	110	13.42	13.71	13.83	13.42	13.42	13.71	13.83	13.83
	120	22.06	22.12	22.26	22.06	22.06	22.12	22.26	22.26

Parameters:  $\lambda = 1$ ,  $\phi = 0.1$ ,  $K = 100$ ,  $r - \delta = 0.04$ .

## 7. Simulations

The simulations correspond to the case of a constant jump size. In the following tables with eight columns, the first column (Garman and and Kohlhagen model) corresponds to the European case without jumps. In the second third column (finite difference method and Merton model, equation 20) jumps are introduced. In the last columns, American options are considered: the fourth and fifth correspond respectively to the finite difference method and to the Barone-Adesi and Whalley model, without jumps (Equation 19) and the last three columns correspond to the American options values in the presence of jumps of constant size. The sixth column represents the benchmark: indeed results are generated by relying on a numerical method (explicit method). The seventh one corresponds to the Bates' model (i.e. Equation 19) used even if the size of the jumps is positive. Finally, the

**Table 4.** Theoretical European and American call values.

<i>Parameters</i>	$S_0$	<i>Call option prices</i>							
		<i>European</i>				<i>American</i>			
		<i>No jump</i>	<i>with jumps</i>			<i>no jump</i>		<i>with jumps</i>	
		<i>Garm. Kohlh.</i>	<i>Fin. diff. meth.</i>	<i>Merton</i>	<i>Fin. diff. meth.</i>	<i>Barone Adesi</i>	<i>Fin. diff. meth.</i>	<i>Bates</i>	<i>Approx</i>
(1)	80	0.03	0.10	0.10	0.03	0.03	0.10	0.10	0.10
$r=0.08$	90	0.57	0.86	0.86	0.58	0.59	0.88	0.87	0.88
$\sigma=0.20$	100	3.42	3.85	3.85	3.52	3.52	3.96	3.89	3.95
$T=0.25$	110	9.85	10.10	10.10	10.36	10.31	10.57	10.42	10.57
	120	18.62	18.69	18.69	20.00	20.00	20.00	20.00	20.00
(2)	80	0.03	0.09	0.09	0.03	0.03	0.10	0.10	0.10
$r=0.12$	90	0.56	0.85	0.85	0.57	0.59	0.87	0.86	0.88
$\sigma=0.20$	100	3.39	3.81	3.81	3.50	3.51	3.94	3.86	3.93
$T=0.25$	110	9.75	10.00	10.00	10.33	10.29	10.54	10.38	10.54
	120	18.43	18.51	18.51	20.00	20.00	20.00	20.00	20.00
(3)	80	1.05	1.16	1.17	1.05	1.07	1.17	1.18	1.19
$r=0.08$	90	3.23	3.41	3.43	3.27	3.28	3.45	3.46	3.48
$\sigma=0.40$	100	7.29	7.48	7.52	7.41	7.41	7.61	7.59	7.64
$T=0.25$	110	13.25	13.41	13.45	13.53	13.50	13.70	13.64	13.70
	120	20.73	20.83	20.88	21.29	21.23	21.40	21.29	21.38
(4)	80	0.21	0.39	0.40	0.21	0.23	0.40	0.41	0.42
$r=0.08$	90	1.31	1.73	1.77	1.36	1.39	1.80	1.79	1.85
$\sigma=0.20$	100	4.46	4.98	5.06	4.71	4.72	5.27	5.16	5.32
$T=0.50$	110	10.16	10.55	10.66	11.00	10.96	11.39	11.12	11.45
	120	17.85	18.01	18.14	20.00	20.00	20.00	20.00	20.00

Parameters:  $\lambda = 1$ ,  $\phi = 0.1$ ,  $K = 100$ ,  $r - \delta = -0.04$ .

pseudo-American call price is computed in the last column. It corresponds to Equation 19 where the jumps are taken into account only in the European part  $C_E$ .

In Table 1, it turns out that Formula 19 generates very accurate results. However, the positive sign of the interest rate differential induces an early exercise premium which is negligible. In this case, the European price already gives a very good approximation of the American one.

In Table 2, the interest rate differential is negative and the jump size is negative. Formula 19 seems to generate reasonably accurate results. Furthermore, these results are usually better than those generated by the pseudo American valuation. In Tables 3 and 4 the jump is positive, and its size is fairly high (10%). In Table 4, when the interest rate differential is negative, it is observed that Bates extension of the Barone-Adesi and Whaley formula generates relatively accurate results only when the option is out-of-the money. When the option is at and in-the-money the results are not good. Indeed, the probability for the underlying to reach the exercise boundary before maturity and to be

**Table 5.** Theoretical European and American call values.

Parameters	$S_0$	Call option prices							
		European			American				
		No jump	with jumps		no jump		with jumps		
			Garm. Kohlh.	Fin. diff. meth.	Merton	Fin. diff. meth.	Barone Adesi	Fin. diff. meth.	Bates
(1)	80	0.03	0.05	0.05	0.03	0.03	0.05	0.05	0.05
$r=0.08$	90	0.57	0.66	0.67	0.58	0.59	0.68	0.68	0.69
$\sigma=0.20$	100	3.42	3.57	3.59	3.52	3.52	3.68	3.65	3.69
$T=0.25$	110	9.85	9.92	9.95	10.36	10.31	10.43	10.34	10.42
	120	18.62	18.62	18.65	20.00	20.00	20.00	20.00	20.00
(2)	80	0.03	0.04	0.05	0.03	0.03	0.05	0.05	0.05
$r=0.12$	90	0.56	0.66	0.66	0.57	0.59	0.67	0.67	0.69
$\sigma=0.20$	100	3.39	3.53	3.55	3.50	3.51	3.65	3.63	3.67
$T=0.25$	110	9.75	9.82	9.85	10.33	10.29	10.40	10.30	10.39
	120	18.43	18.43	18.46	20.00	20.00	20.00	20.00	20.00
(3)	80	1.05	1.08	1.09	1.05	1.07	1.09	1.10	1.11
$r=0.08$	90	3.23	3.28	3.30	3.27	3.28	3.32	3.34	3.36
$\sigma=0.40$	100	7.29	7.33	7.37	7.41	7.41	7.45	7.47	7.49
$T=0.25$	110	13.25	13.26	13.32	13.53	13.50	13.56	13.54	13.58
	120	20.73	20.71	20.79	21.29	21.23	21.30	21.24	21.29
(4)	80	0.21	0.26	0.27	0.21	0.23	0.27	0.28	0.29
$r=0.08$	90	1.31	1.45	1.48	1.36	1.39	1.50	1.51	1.55
$\sigma=0.20$	100	4.46	4.63	4.69	4.71	4.72	4.90	4.84	4.95
$T=0.50$	110	10.16	10.27	10.36	11.00	10.95	11.12	10.95	11.15
	120	17.85	17.86	17.97	20.00	20.00	20.00	20.00	20.00

Parameters:  $\lambda=1$ ,  $\phi=0.06$ ,  $K=100$ ,  $r-\delta=-0.04$ .

discontinuous at this level is higher for in-the-money than for out-the-money calls. The fact that this model neglects possible overshoots at the exercise boundary is therefore a problem. For in-the-money calls, it is better to use the pseudo American call price. When the size of the jump is smaller (see Tables 5 and 6), the size of the bias also gets smaller, and the quality of the results improves.

## 8. Conclusions

In this paper, original results concerning the pricing of perpetual American currency options in a jump diffusion framework are obtained. It has been shown that the sign of the jump size is a relevant parameter. Without discontinuities at the exercise boundary, known results are obtained. With possible discontinuities, an overshoot is introduced and new results are derived. For finite life American currency calls, the formula given by Bates (1991) or Zhang (1995), is tested in the context of a negative

**Table 6.** Theoretical European and American call values.

<i>Parameters</i>	<i>S</i> <sub>0</sub>	<i>Call option prices</i>							
		<i>European</i>				<i>American</i>			
		<i>No jump</i>		<i>with jumps</i>		<i>no jump</i>		<i>with jumps</i>	
		<i>Garm. Kohlh.</i>	<i>Fin. diff. meth.</i>	<i>Merton</i>	<i>Fin. diff. meth.</i>	<i>Barone Adesi</i>	<i>Fin. diff. meth.</i>	<i>Bates</i>	<i>Approx</i>
(1)	80	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
<i>r</i> =0.08	90	0.57	0.59	0.58	0.58	0.59	0.60	0.60	0.60
$\sigma$ =0.20	100	3.42	3.45	3.44	3.52	3.52	3.55	3.53	3.54
<i>T</i> =0.25	110	9.85	9.86	9.86	10.36	10.37	10.37	10.30	10.33
	120	18.62	18.62	18.62	20.00	20.00	20.00	20.00	20.00
(2)	80	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
<i>r</i> =0.12	90	0.56	0.58	0.57	0.57	0.59	0.59	0.59	0.60
$\sigma$ =0.20	100	3.39	3.41	3.41	3.50	3.51	3.53	3.51	3.53
<i>T</i> =0.25	110	9.75	9.76	9.76	10.33	10.29	10.34	10.28	10.30
	120	18.43	18.44	18.44	20.00	20.00	20.00	20.00	20.00
(3)	80	1.05	1.05	1.05	1.05	1.07	1.06	1.07	1.07
<i>r</i> =0.08	90	3.23	3.24	3.24	3.27	3.28	3.28	3.29	3.29
$\sigma$ =0.40	100	7.29	7.30	7.30	7.41	7.41	7.42	7.41	7.42
<i>T</i> =0.25	110	13.25	13.26	13.26	13.53	13.50	13.54	13.50	13.51
	120	20.73	20.73	20.73	21.29	21.23	21.30	21.22	21.24
(4)	80	0.21	0.22	0.22	0.21	0.23	0.23	0.23	0.23
<i>r</i> =0.08	90	1.31	1.34	1.33	1.36	1.39	1.39	1.38	1.41
$\sigma$ =0.20	100	4.46	4.50	4.49	4.71	4.72	4.75	4.70	4.75
<i>T</i> =0.50	110	10.16	10.19	10.19	11.00	10.95	11.02	10.91	10.98
	120	17.85	17.86	17.86	20.00	20.00	20.00	20.00	20.00

Parameters:  $\lambda = 1$ ,  $\phi = 0.02$ ,  $K = 100$ ,  $r - \delta = -0.04$ .

and constant jump size. It is basically an extension of the Barone-Adesi and Whaley approach (1987). Results of the simulations are good only if the overshoot at the exercise boundary is nil with probability one; however, if the exchange rate can be discontinuous at the exercise boundary (positive jumps for the underlying value in the call case), the pricing problem is more difficult to tackle, and one should be very cautious in applying a Barone-Adesi extension.

## Acknowledgements

This research was supported by the national center of competence in research FINRISK, a research instrument of the Swiss National Science Foundation. We thank group HEC for their academic support.

This paper has benefited from the helpful comments of N. Bellamy, J. Bertoin, P. Collin Dufresne, R. Elliott, L. Gauthier, M. Yor and X. Zhang. We thank M. Suchaneki for pertinent comments on a first draft and L. Nguyen-Ngoc for a careful reading of the revised version. We thank also an anonymous referee for remarks which have helped us to improve this paper.

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## Appendix A

### *Pecherskii and Rogozin (1969) result*

Let us define  $T(\ell)$  the first passage time of the process  $X$  at  $\ell = \ln(L/S_0)$ . A result is recalled of Pecherskii and Rogozin which can be found in Bertoin: for every triple of positive numbers  $(\alpha, \beta, q)$ ,

$$\int_0^{\infty} e^{-q\ell} E\left(e^{-\alpha T(\ell) - \beta K(\ell)}\right) dx = \frac{\kappa(\alpha, q) - \kappa(\alpha, \beta)}{(q - \beta)\kappa(\alpha, q)} \quad (21)$$

where

$$\kappa(\alpha, \beta) = \exp\left(\int_0^{\infty} dt \int_0^{\infty} t^{-1} (e^{-t} - e^{-\alpha t - \beta x}) P(X_t \in dx)\right) \quad (22)$$

Let  $\hat{\Psi}$  the Lévy exponent of the dual process of  $X$ , i.e.  $\hat{X} = -X$ .

From the definition, the Laplace exponent of the dual process is  $\hat{\Psi}(u) = \Psi(-u)$ .

## Appendix B

When the jump size is constant, by applying generalized Ito's lemma to the discounted American call price, on the interval  $[t, T]$ , the following equation is obtained:

$$\begin{aligned} e^{-rt} C_A(S_t, T-t) &= C_A(S_0, T) + \int_t^T e^{-ru} \frac{\partial C_A}{\partial S}(S_u, T-u) du - \int_t^T r e^{-ru} C_A(S_u, T-u) du \\ &+ \int_t^T e^{-ru} \frac{\partial C_A}{\partial X}(S_u, T-u) S_u (r - \delta - \lambda\phi) du + \int_t^T e^{-ru} \frac{\partial C_A}{\partial X}(S_u, T-u) \sigma S_u dW_u \\ &+ \frac{\sigma^2}{2} \int_t^T e^{-ru} \frac{\partial^2 C_A}{\partial X^2}(S_u, T-u) S_u^2 du + \int_t^T e^{-ru} (C_A((1+\phi)S_{u-}, T-u) - C_A(S_{u-}, T-u)) dN_u \end{aligned}$$

In the risk-adjusted economy, the discounted American call price is a martingale in the continuation region.  $(\int_t^s \exp(-ru) \frac{\partial C_A}{\partial X}(S_{u-}, T-u) \sigma S_u dW_u, t \leq s \leq T)$  is also a martingale. Therefore, the drift term is

equal to zero. Hence, in the continuation region, the American call value satisfies the following differential equation:

$$\begin{aligned} \frac{\sigma^2}{2}x^2\frac{\partial^2 C_A}{\partial x^2}(x, T-u) + (r-\delta-\lambda\phi)x\frac{\partial C_A}{\partial x}(x, T-u) - rC_A(x, T-u) + \frac{\partial C_A}{\partial s}(x, T-u) \\ + \lambda(C_A((1+\phi)x, T-u) - C_A(x, T-u)) = 0 \end{aligned}$$

Now, if the jump is positive, and if  $S_t$  belongs to the interval:

$$\left[ \frac{b_c(T-t)}{1+\phi}, b_c(T-t) \right]$$

the value of the American call satisfies the following differential equation:

$$\begin{aligned} \frac{\sigma^2}{2}x^2\frac{\partial^2 C_A}{\partial x^2}(x, T-u) + (r-\delta-\lambda\phi)x\frac{\partial C_A}{\partial x}(x, T-u) - rC_A(x, T-u) + \frac{\partial C_A}{\partial s}(x, T-u) \\ + \lambda((1+\phi)x - K - C_A(x, T-u)) = 0 \end{aligned}$$

because in this case the value of the American option after the jump is equal to the intrinsic value.

## Appendix C

Let us compute  $E(1_{S_u > b_c(T-u)} | \mathcal{F}_t)$ , the computation shall be the same for  $E(S_u 1_{S_u > b_c(T-u)} | \mathcal{F}_t)$ . From

$$\begin{aligned} S_u &= S_t \exp\left((r-\delta-\lambda\phi-\sigma^2/2)(u-t) + \sigma(W_u - W_t) + (N_u - N_t)\ln(1+\phi)\right) \\ &= S_t \exp(v(u-t) + \sigma(W_u - W_t) + (N_u - N_t)\ln(1+\phi)) = S_t Z \end{aligned}$$

where  $Z$  is independent of  $S_t$ , we obtain  $E(1_{S_u > b_c(T-u)} | \mathcal{F}_t) = \Upsilon(S_t)$  where

$$\begin{aligned} \Upsilon(x) &= Q(x \exp(v(u-t) + \sigma(W_u - W_t) + (N_u - N_t)\ln(1+\phi)) > b_c(T-u)) \\ &= Q(v(u-t) + \sigma W_{u-t} + N_{u-t}\ln(1+\phi)) > \ln(b_c(T-u)) - \ln x \\ &= \sum_{n=0}^{\infty} Q(N_{u-t}=n) Q(v(u-t) + \sigma W_{u-t} + n \ln(1+\phi) > \ln(b_c(T-u)) - \ln x) \end{aligned}$$

Then, the result follows.