

Optimization of consumption with labor income

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Abstract. We present the solution of a portfolio optimization problem for an economic agent endowed with a stochastic insurable stream, under a liquidity constraint over the time interval $[0, T]$. Generally, the existence of labor income complicates the agent's decisions. Moreover, in the real world the economic agents are restricted in their ability to borrow against their future labor income. We deal with this kind of liquidity constraint following the lines of American option valuation which allows us to give a precise characterization of the optimal consumption as well as the terminal wealth. In a Markovian case, with infinite horizon and HARA utility, we obtain a closed form solution.

Key words: Portfolio optimization, labor income, American option, optimal stopping problem

JEL classification: E 21, G 11

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1 Introduction

This paper studies the individual's optimal consumption and portfolio strategies for an economic agent who receives or pays a stochastic wage income throughout his lifetime. The economic agents in the "real world" are restricted in their ability to borrow against their future labor income and the liquidity constraints and/or the presence of an income risk generally change the agent's tolerance to the risk in the financial market.

In a classical framework of consumption-investment problem, the terminal wealth is constrained to be non-negative. In many cases, when the investor is

not endowed with an income process, this assumption suffices to guarantee that the optimally invested wealth will never reach zero before terminal time, and the non-negativity constraint on current wealth is not binding as it was proved in Merton (1971), Cox and Huang (1989), Karatzas et al. (1986, 1991b).

When the investor receives an income, the non-negative terminal wealth constraint does not imply that the optimal non-constrained current wealth is non-negative: this optimal problem was studied in some special case in the seminal paper of Merton (1971) and in a general setting in Karatzas et al. (1991a) and Koo (1995). However, these authors ignore the effect of non-negative wealth constraint over the period of trading and allow the individual to capitalize the lifetime flow of wage income at some rate of interest which lead them to treat the capitalized value as an addition to the current stock of wealth. Bardhan (1993) studies the problem of stochastic income in an incomplete financial market by means of equivalent martingale measure and identifies a necessary and sufficient budget-feasibility condition that determines whether a consumption investment plan can be financed relative to the investor endowment; however, the income over the period allows the individual to overcome an initial negative capital and finance a consumption since the author does not impose non-negative wealth constraint. We will refer to the case without liquidity constraint as to the free case, the free wealth being its optimal solution.

We impose here the constraint on the investor's wealth to be non-negative over the lifetime interval. This prohibits the agent from selling his wage income in the financial market. As it is well known, this constraint implies that perfect hedging is not possible in general. The same problem is solved by Cuoco (1997) who proves the existence of an optimal strategy, using martingales techniques and working directly on the primal problem. However, he did not obtain closed-form solutions. This problem was also studied in He and Pagès (1993) via an application of duality theory. These authors transformed the problem into an unconstrained dual shadow price problem and established the existence of an optimal strategy for the primal problem. In their approach, when the security price processes follow a Markov-diffusion process, the shadow price problem can be solved by dynamic programming: the Bellman equation corresponding to the dual problem is a linear PDE subject to a free boundary condition. They solved this equation when the labor income is a function of the asset price. Duffie et al. (1997) solved the same kind of problem, when the income stream is assumed to be uninsurable, using the point of view of dynamic programming: in particular, they gave a "quasi-explicit" solution of the H.J.B. equation associated with an HARA utility function and deterministic coefficients. However, the assumption that the market is incomplete (the labor income cannot be duplicated by a portfolio) is essential. As a consequence of their study, they obtained that at zero wealth, a fixed fraction of wealth is consumed, the remainder being saved in the riskless asset.

In this paper, which is a new draft of an unpublished one, we define a dominance in terms of excess consumption over the period. Financing a consumption c and a terminal wealth ζ is achieved by a portfolio which allows a consumption

greater than c and a terminal wealth equal to ζ . The aim of this paper is to give, under the assumption that the financial market is complete and with some additional mild conditions, a characterization of the solution for the constrained problem, i.e., for the problem which takes into account the liquidity constraint. We first take into account the risk aversion of the agent by identifying a family of target consumption plans, which are the optimal consumption plans for the free problem before fitting an initial wealth. The first main result is a “separation” theorem: in the constrained case, the optimal solution consists to invest a part of the wealth in the unconstrained strategy and spend the remainder to finance an American Put written on the free wealth, in order to provide an insurance against the constraint. The second main result is that there is no needless consumption for the optimal constrained solution.

In a first part, we define the model and give a static characterization of the feasible consumption plans which are financeable from an initial wealth y less than or equal to x ; when the liquidity constraint $X_t \geq 0$ is required at any time $t \in [0, T]$ – more precisely at any stopping time $\tau \in [0, T]$ – an infinite number of budget constraints – associated with the different horizons τ – have to be satisfied (He and Pagès (1993)). The connections with optimal stopping times problem and American options are detailed. Let us remark that the liquidity constraint can be of the form $X \geq U$ where U is the benchmark portfolio process.

In a second step, we recall results in the case where the liquidity constraint does not hold (free-problem); this problem was completely solved in Cox and Huang (1989) and Karatzas et al. (1987) by mean of a family of consumption plan $(c^f(\nu), \zeta^f(\nu))$ which is optimal for the dual problem. The optimal solution of the primal problem is obtained by fitting the Kuhn-Tucker multiplier ν such that the budget constraint is saturated.

Then, we introduce the dual problem, in which the budget constraint is linearized by mean of the Kuhn-Tucker multipliers. We solve this dual problem, using some extended results on the related singular control problem. In particular, we show that the derivative of the dual value function is the value function of an optimal stopping times problem. The optimal solution of the dual problem is then constructed from the inverse of an optimal stopping times family. Necessary optimality conditions are used in a following step to show that the optimal solution for the dual problem leads to a solution for the primal problem, choosing a Kuhn-Tucker multiplier which saturates the constraint.

We introduce the stochastic boundary associated with the optimal stopping problem, and show that the optimal solution is obtained by forcing the shadow price of the free problem to stay below this boundary. In terms of the primal problem, when the shadow price reaches the boundary, the optimal wealth is equal to 0 and the constraint is active. However, the agent may consume a certain part of the income. Despite the fact that the wealth is equal to zero along this boundary, this boundary is not an absorbing one. These properties appear clearly in the particular case when all the parameters of the market are constant, the insurable income stream evolves according to a geometrical Brownian motion and the utility functions are HARA utility functions. Using some classical results

about American put options on dividend-paying assets in a Black and Scholes framework, we describe the solution in the last section. Moreover, when the horizon is infinite, we exhibit an explicit solution.

2 The optimization problem

We consider an investment-consumption problem for an agent who receives (or pays) a stochastic income stream and is submitted to liquidity constraints.

2.1 Statement of the problem

The model

We begin with the typical setup for continuous-time asset pricing. There are $n + 1$ financial assets which the agent can buy and/or sell without incurring any cost of trading. One of them is a non risky asset (the money market instrument) with price per unit $S_0(t)$ governed by the equation:

$$dS_0(t) = S_0(t) r_t dt, \quad S_0(0) = 1$$

where r_t is the short rate. In addition to the bond, n risky securities (the stocks) are continuously traded. The price process $(S_i(t), t \geq 0)$ for one share of the i^{th} stock is modeled by the linear stochastic differential equation:

$$dS_i(t) = S_i(t) \left[b_i(t) dt + \sum_{j=1}^n \sigma^{i,j}(t) dW_t^j \right]$$

where $W = (W^1, \dots, W^n)^*$ is a standard Brownian motion on \mathbb{R}^n , defined on a probability space (Ω, \mathcal{F}, P) . The information structure is given by the filtration $(\mathcal{F}_t; 0 \leq t \leq T)$ generated by the Brownian motion W and augmented. Throughout this paper, the usual following conditions are satisfied:

Hypotheses

- The short rate process $(r_t, t \geq 0)$, the column vector of the stock appreciation rates $(b_t = (b_1(t), \dots, b_n(t))^*, t \geq 0)$ and the volatility matrix $(\sigma_t = (\sigma^{i,j}(t)), t \geq 0)$ are predictable and bounded. For any time t , the matrix σ_t has a full rank.
- There exists a predictable and bounded vector $(\theta_t, t \geq 0)$, called the risk premium vector, such that

$$b_t - r_t \mathbf{1} = \sigma_t \theta_t \quad (dP \otimes dt) - a.s$$

where $\mathbf{1}$ is the vector in \mathbb{R}^n with all components equal to 1, therefore the financial market is arbitrage-free and complete.

Notation

We denote by T a fixed horizon, and introduce some integrability conditions :

- The set \mathcal{H} consists of the class of all (\mathcal{F}_t) -progressively measurable, \mathbb{R} -valued processes ϕ such that $\int_0^T |\phi(t)| dt < \infty, a.s..$ We define the subset $\mathcal{H}^+ \subset \mathcal{H}$ as the non-negative processes, $\mathcal{H}_2 \subset \mathcal{H}$ consists of processes such that $\int_0^T |\phi(t)|^2 dt < \infty,$ and \mathcal{H}_2^n corresponds to \mathbb{R}^n -valued processes.
- The set $H_2 \subset \mathcal{H}_2$ consists of the subclass of square integrable processes, i.e., such that $E\left(\int_0^T |\phi(t)|^2 dt\right) < \infty.$ With an obvious notation, H_2^+ is the subset of non-negative processes in $H_2,$ and H_2^n corresponds to \mathbb{R}^n -valued processes.
- The space of non-negative \mathcal{F}_T -measurable random variables ζ is L_0^+ and the subset of square integrable variables, i.e. such that $E(\zeta^2) < +\infty,$ is denoted by $L_2^+.$

The individual consumption problem

We consider an agent who receives an income at time t at the rate $e_t,$ and chooses at each time $t \in [0, T]$ his consumption rate c_t and the amount $\tilde{\pi}^i(t)$ to be invested in the i^{th} -stock ($i = 1, \dots, n$), using a self-financing strategy. The agent's financial wealth ($X_t, t \geq 0$) evolves according to the equation:

$$dX_t = r_t X_t dt - (c_t - e_t) dt + \tilde{\pi}_t^* \sigma_t (dW_t + \theta_t dt). \tag{2.1}$$

We shall refer to the wealth process solution of (2.1) with the initial endowment $X_0 = x$ as to $(X_t^{x,c,\tilde{\pi}}, t \geq 0).$

We assume that the income process $(e_t, t \geq 0)$ is spanned by the market assets and therefore is not a source of new uncertainty. In particular, e is (\mathcal{F}_t) -adapted. Similar assumptions are introduced in He and Pagès (1993) and more recently in Detemple and Serrat (1996), in opposite to the papers of Duffie et al. (1997) and Koo (1995) where the existence of a specific noise in the dynamics of the labor income is a key argument to solve the problem. In what follows, the income process $(e_t, t \geq 0)$ is assumed to be in $H_2.$

In this paper, we study the case where the *liquidity constraint* $X_t \geq 0$ is to be satisfied at any time t before $T.$ The existence of such a constraint implies that the agent cannot borrow against future labor income.

The set $\mathcal{A}^+(x)$ of *x-feasible* consumption profiles consists of consumption processes c in \mathcal{H}^+ associated with a portfolio $\tilde{\pi}$ in \mathcal{H}_2^n such that the liquidity constraint $(X_t^{x,c,\tilde{\pi}} \geq 0, t \in [0, T])$ is satisfied.

Remarks: a. When $(e_t \geq 0, t \geq 0),$ this stochastic flow may be interpreted as a labor wage. When $(e_t \leq 0, t \geq 0),$ the problem may be interpreted as a constraint on the consumption, by redefining the consumption in excess of the given income process $e: \tilde{c}_t = c_t - e_t \geq -e_t.$

b. If we restrict the consumption space to satisfy $(c_t \geq (e_t \vee 0), t \in [0, T])$, the problem is reduced to a portfolio management problem, and it is well-known that the assumption of a non-negative terminal wealth implies the liquidity constraint.

Utility functions

The agent's preferences over consumption and wealth profiles are given by time-additive utility functions u for consumption and g for the terminal wealth. We recall some classical definitions and properties of utility functions.

- A utility function $U :]0, \infty[\rightarrow \mathbb{R}$ is a strictly increasing, strictly concave, continuously differentiable function which satisfies $U'(\infty) \stackrel{\text{def}}{=} \lim_{x \rightarrow \infty} U'(x) = 0$, and possibly the Inada condition $U'(0) \stackrel{\text{def}}{=} \lim_{x \rightarrow 0} U'(x) = \infty$. If the Inada condition does not hold, we suppose that $U(0)$ is finite.

- We shall denote by \tilde{U} the convex conjugate of U , defined on $\mathbb{R}_+^* = \{x \in \mathbb{R}, x > 0\}$ as

$$\tilde{U}(y) = \sup_{x > 0} (U(x) - xy). \quad (2.2)$$

The function \tilde{U} is strictly convex and strictly decreasing on $]0, U'(0)[$. If $y > U'(0)$, the supremum being relative to non-negative values of x we obtain $\tilde{U}(y) = U(0)$. The function U' admits a strictly decreasing inverse, defined on $]0, U'(0)[$ by $\mathcal{F}(y) = \inf\{x \geq 0 \mid U'(x) \leq y\}$. We set $\mathcal{F}(y) = 0$ for $y \geq U'(0)$. It is well known that $\mathcal{F}(y) = -\tilde{U}'(y) \geq 0$ on $]0, U'(0)[$ and this equality extends to the whole interval $]0, +\infty[$ since $\mathcal{F}(y) = -\tilde{U}'(y) = 0$ for $y \geq U'(0)$. Furthermore, $\lim_{y \rightarrow \infty} -\tilde{U}'(y) = 0$ and $\lim_{y \rightarrow 0} -\tilde{U}'(y) = \infty$.

- The following relation holds for each $y \in]0, U'(0)[$

$$U(-\tilde{U}'(y)) + y\tilde{U}'(y) = \tilde{U}(y)$$

which implies that, for $y \in]0, U'(0)[$ the supremum in (2.2) is achieved for $x = \mathcal{F}(y)$. Otherwise, i.e., if $y \geq U'(0)$, the maximum is achieved for $x = 0 = -\tilde{U}'(y)$.

We shall consider throughout a map $u :]0, \infty[\times [0, T] \times \Omega \rightarrow \mathbb{R}$ such that for any given $(t, \omega) \in [0, T] \times \Omega$ the function $u(\cdot, t, \omega)$ is a utility function¹ and, for any $x \in \mathbb{R}_+^*$ the process $u(x, \cdot)$ is (\mathcal{F}_t) -adapted. We denote by u' the derivative of u with respect to the first argument, and by $\tilde{u}(\cdot, t)$ the convex conjugate of $u(\cdot, t)$. The function $g : \mathbb{R}_+^* \times \Omega \rightarrow \mathbb{R}$ is such that for any $x > 0$, the random variable $g(x, \cdot)$ is \mathcal{F}_T -measurable and $g(\cdot, \omega)$ is a utility function. The HARA functions $e^{-\beta t} \frac{c^{1-\gamma}}{1-\gamma}$ (with $-\tilde{u}(y, t) = (ye^{\beta t})^{-(1/\gamma)}$) and the logarithmic function are classical examples of deterministic utility functions.

Optimization criteria

Given an initial endowment $x \geq 0$ and an income stream $(e_t, t \leq T)$, an investor wishes to choose a consumption profile and an investment policy $\tilde{\pi}$ so as to max-

imize his total expected utility from consumption over the period and expected utility of investment at the end of the period $[0, T]$,

$$E \left(\int_0^T u(c_s, s) ds + g(X_T^{x,c,\tilde{\pi}}) \right)$$

using feasible policies.

2.2 Liquidity constraint

As it is now well known, the liquidity constraint $X_t^{x,c,\tilde{\pi}} \geq 0$ over the time interval $[0, T]$ can be formulated as a budget constraint which involves the value ζ of the terminal wealth.

Hedgeable consumption plan, free-value

When there is no income ($e \equiv 0$), it is well known that in a complete market, for a given consumption plan $(c, \zeta) \in H_2^+ \times L_2^+$, there exists a unique square integrable portfolio strategy $\tilde{\pi}$ and an initial endowment x such that the wealth process $X^{x,c,\tilde{\pi}}$ satisfies the liquidity constraint $X^{x,c,\tilde{\pi}} \geq 0$ and the terminal condition $X_T^{x,c,\tilde{\pi}} = \zeta$. In this case, the non-negative terminal wealth condition implies the liquidity constraint. If $(c, \zeta) \in \mathcal{H}^+ \times L_0^+$, Karatzas (1989) characterizes the minimal solution among the non-negative processes.

Let us now allow the presence of a stochastic income e . For notational simplicity, we denote by $\pi_t \stackrel{def}{=} \tilde{\pi}_t \sigma_t^*$ the stochastic part of a self-financing portfolio. Since the market is complete, all consumption plans $(c, \zeta) \in H_2^+ \times L_2^+$ are hedgeable with square-integrable processes, i.e., there exists a pair of processes $(X^{c,\zeta}, \pi) \in H_2 \times H_2^n$ such that,

$$\begin{aligned} dX_t^{c,\zeta} &= r_t X_t^{c,\zeta} dt - (c_t - e_t) dt + \pi_t^* (dW_t + \theta_t dt), \\ X_T^{c,\zeta} &= \zeta. \end{aligned} \tag{2.3}$$

Suppose now that $(c, \zeta) \in \mathcal{H}^+ \times L_0^+$. We consider the minimal equation of (2.3) which is

$$X_t^{c,\zeta} = E \left(\int_t^T H_s^t (c_s - e_s) ds + H_T^t \zeta \mid \mathcal{F}_t \right) \tag{2.4}$$

where $(H_s^t, s \geq t)$ is the shadow state-price process, also called the deflator process, defined by the following forward equation,

$$dH_s^t = -H_s^t (r_s ds + \theta_s dW_s), \quad s \geq t, \quad H_t^t = 1. \tag{2.5}$$

In this setting the liquidity constraint is on the form $(X_t^{c,\zeta} \geq -I_t, t \geq 0)$, where $I_t \stackrel{def}{=} E \left(\int_t^T H_s^t e_s ds \mid \mathcal{F}_t \right)$ is the unique solution of (2.3) when $c = e = 0$. From the linearity property, $X + I$ satisfies (2.3) when $e = 0$.

Remark: Liquidity constraints, of course, can take a different form, e.g. there are risky consumer loans available to the agent when the wealth of the agent is small or negative, but in general the loan amount is limited by $(M_t, t \geq 0)$ (supposed to be an Itô process). Such constraint may be easily modeled in our framework, by replacing the wealth by the cushion $\tilde{X}_t = X_t - M_t$. In the insurance portfolio problem the wealth is required to satisfy $X_t \geq U_t, \forall t$, where U is the benchmark portfolio. This problem can be studied by considering the process $X - U$. We will address this problem in a next paper.

The wealth $X_t^{c, \zeta}$, called hereafter the (c, ζ) -free-value at time t , is the t -time value of a contingent claim ζ paying a dividend rate $(e - c)$. We set $H_s = H_s^0$ and introduce the present value of the future labor income I_0 . The budget constraint is stated as a constraint on the consumption plans

$$E \left(\int_0^T H_s c_s ds + H_T \zeta \right) \leq x + I_0. \quad (2.6)$$

Let us remark that the investment-consumption free problem when an income is payed is in fact a classical optimization problem, where the initial wealth x is enlarged from the present value of the future income at time 0.

The set $\mathcal{A}(x)$ of x -hedgeable consumption plans, i.e., which are financeable from an initial wealth less or equal to x , consists of consumption plans (c, ζ) in $\mathcal{H}^+ \times L_0^+$ such that the budget constraint (2.6) holds.

Let us emphasize that a hedgeable consumption plan does not necessarily lead to a positive wealth; if the wealth process is non-negative over the time interval $[0, T]$, we call it the *free-price* process of the pair (c, ζ) and the pair (c, ζ) is said to be *feasible*.

Remarks: a. From the formula (2.4) follows the fact that, if the terminal wealth is non-negative and if the income process is non-positive, the free-value is non-negative and therefore is the free-price. However, the x -feasible consumption plans set may happen to be empty.

b. In an infinite horizon setting, there is only a consumption utility function, and we shall use a budget constraint on the form

$$E \left(\int_0^\infty H_s |c_s - e_s| ds \right) < \infty, \quad E \left(\int_0^\infty H_s (c_s - e_s) ds \right) \leq x. \quad (2.7)$$

We leave the details to the reader.

Superstrategies and fair value

Let us now study the liquidity constraint if the agent receives a marketed stochastic income $(e_t, t \leq T)$. In the case $e \geq 0$, if the individual is allowed to capitalize the lifetime flow of the wage income and treats the capitalized value as an addition to the current stock of wealth, the problem is easily solved (we recall the main results below, in the section devoted to the free case), but the current optimal wealth may be negative². At this stage, we can proceed in two ways:

- The first is to characterize the feasible consumption plans. This difficult way was studied in Saada (1994).
- The second is to introduce superstrategies which involve a cost process to cover the “uninsurable” liquidity risk induced by the liquidity constraint. We present now this method.

We give a less restricting definition for the value of a portfolio, which introduces an additional flow K which captures the possibility of free disposal of wealth: the agent is allowed not to reinvest some of his wealth if he chooses to do so. We shall refer to the triple (X, π, K) as a (c, ζ) -superstrategy if it satisfies

$$\begin{cases} dX_t^{K, \pi} = r_t X_t^{K, \pi} dt - dK_t - (c_t - e_t) dt + \pi_t (dW_t + \theta_t dt); & X_T^{K, \pi} = \zeta, \\ X_t^{K, \pi} \geq 0; \forall t \leq T, \end{cases} \quad (2.8)$$

where $(K_t, t \geq 0)$ belongs to \mathcal{K} , the set of adapted right-continuous non-decreasing processes.

Remarks: a. We can give the following equivalent formulations of this definition: there exists K such that the triple (X, π, K) is a (c, ζ) super-strategy

- if there exists a wealth process satisfying (2.3) such that $X \geq 0$ and $X_T \geq \zeta$: at the end of the period, there is excess wealth left over;
- or if there exists $\tilde{c}, \tilde{c} \geq c$ such that \tilde{c} is feasible.

b. By using the deflator process, we have $X_t = E(\int_t^T H_s^t (c_s - e_s) ds + \int_t^T H_s^t dK_s + H_T^t \zeta | \mathcal{F}_t)$ where $\int_0^T H_s dK_s \geq 0$. The cost process $(\int_0^t H_s dK_s, t \geq 0)$ finances the unhedgeable constraint of the consumption plan (c, ζ) . There is some needless consumption.

c. In Cuoco (1997), the optimisation problem is solved among the superstrategies.

For a given consumption plan there exist many superstrategies, and we are interested in a characterization of a minimal one. Such minimal superstrategies are extensively used in the arbitrage-free pricing of American options (Karatzas, 1997), where the American price for a pay-off square integrable process $(\xi_t, t \geq 0)$ is defined as the square integrable minimal superstrategy which dominates $(\xi_t, t \geq 0)$, when no dividends are payed ($c \equiv e$). The classical relation between American options and optimal stopping times theory leads us to give the following properties of the *fair-value* defined as the minimal investment required to replicate (c, ζ) with a superstrategy. We denote by $\mathcal{T}(t)$ (resp. \mathcal{T}) the set of stopping times τ such that $t \leq \tau \leq T$ (resp. $0 \leq \tau \leq T$).

Theorem 2.1 *Let $(c, \zeta) \in \mathcal{H}^+ \times L_0^+$ be a consumption plan.*

a) *There exists a triple $(Y^{c, \zeta}, K^{c, \zeta}, \pi^{c, \zeta})$ such that*

$$\begin{cases} dY_t^{c, \zeta} = r_t Y_t^{c, \zeta} dt - dK_t^{c, \zeta} - (c_t - e_t) dt + \pi_t^{c, \zeta} (dW_t + \theta_t dt) \\ Y_T^{c, \zeta} = \zeta \\ Y_t^{c, \zeta} \geq 0, \forall t \leq T \end{cases} \quad (2.9)$$

where $K^{c, \zeta} \in \mathcal{K}$ is continuous on $[0, T[$ and satisfies

$$\int_{[0, T[} Y_s^{c, \zeta} dK_s^{c, \zeta} = 0.$$

Furthermore, if $(c, \zeta) \in H_2^+ \times L_2^+$, there exists a unique solution in $H_2^+ \times \mathcal{H} \times H_2^n$.

b) The minimal solution, called the (c, ζ) -fair-value process, satisfies

$$Y_t^{c, \zeta} = \text{ess sup}_{\tau \in \mathcal{F}(t)} E \left(\int_t^\tau H_s^t (c_s - e_s) ds + H_T^t \zeta \mathbb{1}_{\tau=T} | \mathcal{F}_t \right), \quad (2.10)$$

its initial value

$$Y_0^{c, \zeta} = \text{ess sup}_{\tau \in \mathcal{F}} E \left(\int_0^\tau H_s (c_s - e_s) ds + H_T \zeta \mathbb{1}_{\tau=T} \right) \quad (2.11)$$

is called the (c, ζ) -fair-value.

The stopping time $\sigma^{c, \zeta} = T \wedge \inf\{s; 0 \leq s < T, Y_s^{c, \zeta} = 0\}$ achieves the supremum in (2.11) and is the smallest optimal stopping time.

The sketch of the proof is given in the Appendix. Let us remark that the (c, ζ) -fair value process $Y^{c, \zeta}$ is larger than the (c, ζ) -free value process $X^{c, \zeta}$. The positive process $Z^{c, \zeta} = Y^{c, \zeta} - X^{c, \zeta}$ satisfies by linearity the equation (2.9) without dividend ($c \equiv e$) and zero terminal wealth ($\zeta = 0$). Moreover, $Z_t^{c, \zeta} \geq \sup[-X_t^{c, \zeta}, 0] = (X_t^{c, \zeta})^-, a.s..$ It follows that $Z^{c, \zeta}$ is the price of an American put with exercise price equal to 0, written on the free-value process $X^{c, \zeta}$. Therefore the fair-value of a consumption plan can be split into two terms: the first one is the free-value process $X^{c, \zeta}$ of this consumption plan and the second one is an American put option $Z^{c, \zeta}$ with exercise price 0. A minimal superstrategy consists in investing a part of the initial wealth in a free portfolio and to use the remaining part for financing the American put option on this underlying asset.

Remark: In their 1989 paper (p.65), in a no-income situation, Cox and Huang, while studying a case where the non-negativity constraint is binding, have remarked that the agent invests some wealth in a European put option with a zero exercise price to obtain an insurance package.

A consumption plan (c, ζ) is said to be x -affordable if the (c, ζ) -fair-value is less than x , i.e. if the following budget constraint holds

$$Y_0^{c, \zeta} \leq x. \quad (2.12)$$

We characterize the x -affordable consumption plans by mean of \mathcal{D} , the class of adapted, right-continuous, non-increasing processes D such that $D_0 \leq 1$, $D_T = 0$, and we denote by D_T^- the left limit of D at T .

Proposition 2.2 *A consumption plan is x -affordable if and only if*

$$\sup_{D \in \mathcal{D}} E \left(\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T^- \zeta \right) \leq x. \quad (2.13)$$

Proof. The proof goes back to Pagès (1989) and can be found in the Appendix. \square

3 Resolution of the optimization problem

As we noticed, the solution of the constrained problem is constructed from the solution of the free problem. Therefore, we need some hypotheses on this free solution: in this paper, the following hypothesis³ prevails:

Hypothesis: We suppose that for any $\nu \in \mathbb{R}_+^*$, the process $(c_t^f(\nu) \stackrel{\text{def}}{=} -\tilde{u}'(\nu H_t, t), t \geq 0)$ belongs to H_2^+ and that $\zeta^f(\nu) \stackrel{\text{def}}{=} -\tilde{g}'(\nu H_T)$ is in L_2^+ .

The $(c^f(\nu), \zeta^f(\nu))$ -free-value process, denoted hereafter by $X^f(\nu)$ will play an important role in our study. This *free-value process*

$$X_t^f(\nu) \stackrel{\text{def}}{=} E \left(\int_t^T H_s^t (-\tilde{u}'(\nu H_s, s) - e_s) ds - H_T^t \tilde{g}'(\nu H_T) \mid \mathcal{F}_t \right), \quad (3.1)$$

has an initial value

$$X_0^f(\nu) = E \left(\int_0^T H_s (-\tilde{u}'(\nu H_s, s) - e_s) ds - H_T \tilde{g}'(\nu H_T) \right) \quad (3.2)$$

which is simply called the free value.

3.1 The free case

The *free problem* (where only the terminal wealth is assumed to be non-negative) has been studied by Karatzas et al. (1991a), Bardhan (1993), Koo (1995) and other authors. We review here those of their results which are of particular importance in the present paper.

The free problem can be restated as the equivalent problem of maximizing expected utility over all x -hedgeable consumption plans (c, ζ)

$$\sup_{(c, \zeta) \in \mathcal{L}(x)} E \left(\int_0^T u(c_s, s) ds + g(\zeta) \right).$$

It is well known that, under weak assumptions, there exists a complete solution to this optimization problem, constructed from a dual problem.

The free dual problem

The optimization problem may be linearized using Lagrangian method and is transformed in

$$\sup_{(c, \zeta) \in \mathcal{H}^+ \times L_0^+} E \left(\int_0^T u(c_s, s) ds + g(\zeta) + \nu \left(x - \int_0^T H_s (c_s - e_s) ds - H_T \zeta \right) \right)$$

where $\nu \geq 0$ is the Kuhn-Tucker multiplier. Some analytic considerations allow us to invert the supremum and the integral signs. Therefore, the free dual value function is

$$J(\nu) \stackrel{\text{def}}{=} E \left(\int_0^T \sup_{c \geq 0} (u(c_s, s) - \nu H_s(c_s - e_s)) ds + \sup_{\zeta \geq 0} (g(\zeta) - \nu H_T \zeta) \right)$$

and can be written in a form involving the convex conjugate functions

$$J(\nu) = E \left(\int_0^T (\tilde{u}(\nu H_s, s) + \nu H_s e_s) ds + \tilde{g}(\nu H_T) \right).$$

Theorem 3.1 *The consumption plan $(c^f(\nu), \zeta^f(\nu))$ is optimal for the dual problem. For a given endowment $x > -I_0$, there exists a unique $\nu = \nu_x \geq 0$ such that $X_0^f(\nu) = x$. The consumption plan $(c^f(\nu_x), \zeta^f(\nu_x))$ is optimal for the primal problem among the x -hedgeable consumption plans.*

Proof. The proof can be found in Karatzas et al. (1987) or Karatzas (1997). By integration with respect to ν of (3.2) and taking care to the integrability conditions, it follows that

Corollary 3.2 *If the Inada condition holds, the free value and the free dual value function are linked by*

$$\int_{\nu}^{\infty} (X_0^f(z) + I_0) dz = J(\nu) - \nu I_0. \quad (3.3)$$

If the Inada's condition does not hold, the same method leads to

$$\int_{\nu}^{\bar{\nu}} (X_0^f(z) + I_0) dz = J(\nu) - \nu I_0 - E \left(\int_0^T u(0, s) ds - g(0) \right)$$

where $\bar{\nu} = \inf\{\nu, X_0^f(\nu) = -I_0\}$. The needed presence of the expectation operator in the right member is due to the eventually stochastic character of the utility functions.

The corollary leads to the well known result $J'(\nu) = -X_0^f(\nu)$.

3.2 The general case: a singular problem

As in the free case, we apply the Kuhn-Tucker multiplier method in order to linearize the budget constraint and introduce a stochastic control problem which is the dual of the constrained problem. The budget constraint under the liquidity constraint stated as $\inf_{D \in \mathcal{D}} (x - E(\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T \zeta)) \leq 0$ involves a parameter D , therefore the dual problem is depending on this parameter. In this section, we solve this dual problem, establish the basic properties of the dual value function and prove the existence of an optimal control D^* .

Existence of an optimal control

The dual problem is a minimization problem, defined as

$$\Phi(\nu) = \inf_{D \in \mathcal{D}} J(\nu; D),$$

where

$$J(\nu; D) \stackrel{\text{def}}{=} \sup_{(c, \zeta) \in \tilde{\mathcal{H}}^+ \times L_0^+} E \left\{ \int_0^T u(c_t, t) dt + g(\zeta) - \nu \left(\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T^- \zeta \right) \right\}.$$

It follows from the properties of utility functions that

$$J(\nu; D) = E \left\{ \int_0^T [\tilde{u}(\nu H_t D_t, t) + \nu H_t D_t e_t] dt + \tilde{g}(\nu H_T D_T^-) \right\}. \quad (3.4)$$

The dual problem appears now as a singular control problem, with controlled dynamics $(H_t D_t; 0 \leq t \leq T)$ where $(D_t, t \geq 0)$ is a non-increasing process bounded by 1.

If $(r_t, t \geq 0)$ and $(\theta_t, t \geq 0)$ are deterministic, an analogous problem, known as fuel problem, has been solved in Taksar (1985), El Karoui and Karatzas (1991), and other authors using the fact that the first derivative of the value function is related with an optimal stopping problem. We use here similar arguments. Let us denote by $Y_0(\nu)$ the initial value of the fair-value process associated with the free optimal consumption plan $(c^f(\nu), \zeta^f(\nu))$, i.e.,

$$Y_0(\nu) \stackrel{\text{def}}{=} \sup_{\tau \in \mathcal{T}} E \left\{ \int_0^\tau H_s \{-\tilde{u}'(\nu H_s, s) - e_s\} ds - H_T \tilde{g}'(\nu H_T) \mathbb{1}_{\tau=T} \right\}. \quad (3.5)$$

In the sequel, $Y_0(\nu)$ is simply called the *fair-value*. We also introduce the process

$$Y_t(\nu) \stackrel{\text{def}}{=} \sup_{\tau \in \mathcal{T}_{(t)}} E \left\{ \int_t^\tau H_s^t \{-\tilde{u}'(\nu H_s^t, s) - e_s\} ds - H_T^t \tilde{g}'(\nu H_T^t) \mathbb{1}_{\tau=T} | \mathcal{F}_t \right\}. \quad (3.6)$$

Using this notation, the fair-value process associated with the free optimal consumption plan $(c^f(\nu), \zeta^f(\nu))$, defined as in (2.9) by

$$\sup_{\tau \in \mathcal{T}_{(t)}} E \left\{ \int_t^\tau H_s^t \{-\tilde{u}'(\nu H_s, s) - e_s\} ds - H_T^t \tilde{g}'(\nu H_T) \mathbb{1}_{\tau=T} | \mathcal{F}_t \right\} \quad (3.7)$$

is equal to $Y_t(\nu H_t)$ and is called the fair-value process.

Let us remark that the maps $\nu \rightarrow Y_0(\nu)$ and $\nu \rightarrow Y_t(\nu)$ are non-increasing, and that the optimal stopping time in (3.5) is defined as $\sigma(\nu) = \inf\{t, 0 \leq t < T \mid Y_t(\nu H_t) = 0\} \wedge T$. The main result of this section is the following

Theorem 3.3 *Let Y_0 be the fair-value defined in (3.5).*

a. *Let Φ be the value function of the dual problem, and J be the value function for the free dual problem. Then*

$$\Phi(\nu) = J(\nu) - \int_0^\nu (Y_0(z) - X_0^f(z)) dz . \tag{3.8}$$

b. *There exists an optimal control⁴ for the dual problem, that is a process $D^*(\nu) \in \mathcal{D}$ such that $\Phi(\nu) = J(\nu; D^*(\nu))$.*

Remarks: a. The function $z \rightarrow Y_0(z) - X_0^f(z)$ is integrable in a neighborhood of 0, this is no more the case for $X_0^f(z)$.

b. From the above theorem and Corollary 3.2, $-\Phi'(\nu) = Y_0(\nu) \geq 0$, an inequality which has a certain importance in He and Pagès (1993).

Proof. The proof of this main result follows from a change of variables formula. Instead of working directly with Φ , we study the difference between the free dual value function J and the dual value function Φ and we will express this quantity in terms of a primitive of

$$Y_0(z) - X_0^f(z) = \text{ess sup}_{\tau \in \mathcal{T}} E \left(\int_\tau^T H_s (\tilde{u}'(zH_s, s) + e_s) ds + \mathbb{1}_{\tau < T} H_T \tilde{g}'(zH_T) \right)$$

which, as we have seen in the comments following Theorem 2.1, gives the value of an American put on the negative part of the free wealth:

$$Y_0(z) - X_0^f(z) = \text{ess sup}_{\tau \in \mathcal{T}} E([X_\tau^f]^-).$$

From the definition of the free value function and of $J(\nu, D)$, we have:

$$J(\nu) - J(\nu; D) = E \left\{ \int_0^T (\tilde{u}(\nu H_s, s) + \nu H_s e_s) ds + \tilde{g}(\nu H_T) - \int_0^T (\tilde{u}(\nu H_s D_s, s) + \nu H_s D_s e_s) ds + \tilde{g}(\nu H_T D_{T-}) \right\}$$

Let us introduce the right-continuous, decreasing inverse $(\gamma_\theta, 0 \leq \theta \leq 1)$ of the decreasing process $(D_t, t \geq 0)$ to give an integral representation of quantities $\tilde{u}(\nu h, s) - \tilde{u}(\nu h D_s, s)$. The variable γ_θ is a stopping time and satisfies $\gamma_\theta = \inf\{s | D_s \leq \theta\}$. The set equality

$$\{D_s \leq \theta\} = \{\gamma_\theta \leq s\}$$

will be of constant use in what follows. We state without proof the obvious lemma which is the key for the main result.

Lemma 3.4 *We have*

$$\tilde{u}(\nu h, s) - \tilde{u}(\nu h D_s, s) = \int_{[0,1]} \nu h \tilde{u}'(\nu h y) \mathbb{1}_{\{\gamma_y \leq s\}} dy = \int_0^\nu h \tilde{u}'(hz) \mathbb{1}_{\{\gamma_{\frac{z}{\nu}} \leq s\}} dz .$$

Applying the lemma gives the inequality

$$\begin{aligned} & J(\nu) - J(\nu; D) \\ &= E \left\{ \int_0^\nu dz \left(\int_0^T H_s \mathbb{1}_{\{\gamma_{\frac{z}{\nu}} \leq s\}} (\tilde{u}'(zH_s, s) + e_s) ds \right) + H_T \tilde{g}'(zH_T) \mathbb{1}_{\{\gamma_{\frac{z}{\nu}} < T\}} \right\} \\ &\leq \int_0^\nu (Y_0(z) - X_0^f(z)) dz. \end{aligned}$$

We minimize $J(\nu, D)$ over D and obtain

$$\Phi(\nu) \geq J(\nu) - \int_0^\nu (Y_0(z) - X_0^f(z)) dz. \tag{3.9}$$

Conversely, let ν be given and introduce the family of optimal stopping times $(\sigma(z), z \leq \nu)$ such that

$$Y_0(z) - X_0^f(z) = E \left\{ \int_{\sigma(z)}^T H_s (\tilde{u}'(zH_s, s) + e_s) ds + H_T \tilde{g}'(zH_T) \mathbb{1}_{\sigma(z) < T} \right\}.$$

This identity implies that the equality holds in (3.9) after the identification of $\sigma(z)$ with a non-increasing process of the form $\gamma_{\frac{z}{\nu}}$, or in an equivalent way of $\sigma(\theta\nu)$ and γ_θ .

Since $(-\tilde{u}', -\tilde{g}')$ are non-increasing functions, the map $\nu \rightarrow \sigma(\nu)$ is non-increasing, and right continuous (see below for a proof and more complete properties). Let us denote by M the right-continuous inverse of σ ,

$$\{s < \sigma(\nu)\} = \{\nu < M(s)\}. \tag{3.10}$$

The right inverse of the map $\theta \in [0, 1] \rightarrow \sigma(\theta\nu)$ is the right continuous process $D^*(\nu)$

$$D_s^*(\nu) = \left[\frac{1}{\nu} M(s) \right] \wedge 1.$$

Applying the lemma, we have

$$\int_0^\nu [Y_0(z) - X_0^f(z)] dz = J(\nu) - J(\nu; D^*(\nu)).$$

Therefore, the decreasing process $D^*(\nu) \in \mathcal{S}$ is optimal. The map $\nu \rightarrow D_t^*(\nu)$ is non-increasing and continuous. Furthermore, $D_s^*(\nu) = 1$ for $s < \sigma(\nu)$ and $D_s^*(\nu) < 1$ if $s > \sigma(\nu)$. The process $D^*(\nu)$ decreases only on the set $Y_s(\nu H_s) = 0$, i.e., $\int_0^T Y_s(\nu H_s) dD_s^*(\nu) = 0$. \square

Remark: If $e \equiv 0$ and if the Inada condition prevails, $\sigma(\nu) = T, \forall \nu, M \equiv \infty$ and $D^* \equiv 1$. If $e \equiv 0$ and if the Inada condition does not hold, $\sigma(\nu) = T \wedge \inf\{s \mid \nu H_s \geq u'(0)\}$.

Furthermore, the optimal process $D^*(\nu)$ is the minimal one with the following meaning: let $\Delta(\nu)$ be an optimal process, i.e. such that $\Delta(\nu) \in \mathcal{S}$, $\Phi(\nu) = J(\nu, \Delta(\nu))$ and denote by γ its inverse. From right continuity of $\gamma(\cdot)$ and $\sigma(\cdot)$ and minimal character of σ , we deduce that $\sigma(\theta\nu) \leq \gamma(\theta)$ or equivalently $(D_t^*(\nu) \leq \Delta_t(\nu); t \geq 0)$. In particular, any optimal process $\Delta(\nu)$ is equal to 1 over $[0, \sigma(\nu)]$.

An expression for the fair value

We establish now properties of the process $(Y_t(\nu), t \geq 0)$ which will be important to prove that $(Y_t(\nu H_t D_t^*(\nu)), t \geq 0)$ is a fair price process, i.e. is a positive wealth which satisfies (2.3).

Proposition 3.5 *Let $D^*(\nu)$ be the optimal process for the dual problem defined in Theorem 3.3. The following relations hold, where $H_t^*(\nu) = \nu H_t D_t^*(\nu)$ and $H_{T-}^*(\nu) = \nu H_T D_{T-}^*(\nu)$:*

$$Y_0(\nu) = E \left(\int_0^T H_t D_t^*(\nu) \{-\tilde{u}'(H_t^*(\nu), t) - e_t\} dt - H_T D_{T-}^*(\nu) \tilde{g}'(H_{T-}^*(\nu)) \right), \quad (3.11)$$

$$Y_0(\nu) = E \left(\int_0^T H_t \{-\tilde{u}'(H_t^*(\nu), t) - e_t\} dt - H_T \tilde{g}'(H_{T-}^*(\nu)) \right). \quad (3.12)$$

In particular, the map $\nu \rightarrow Y_0(\nu)$ is non-increasing, continuous on \mathbb{R}_+^* and strictly decreasing up to its first hitting time to $-I_0$; furthermore, $\lim_{\nu \rightarrow 0} Y_0(\nu) = \infty$ and $\lim_{\nu \rightarrow \infty} Y_0(\nu) = -I_0$.

Proof. The stopping time $\sigma(z)$ is an optimal stopping time for the problem associated with $(-\tilde{u}'(zH_t, t), t \geq 0, -\tilde{g}'(zH_T))$. Therefore

$$\begin{aligned} & E \left(\int_0^{\sigma(\lambda z)} H_t (-\tilde{u}'(zH_t, t) - e_t) dt - H_T \tilde{g}'(zH_T) \mathbb{1}_{\{\sigma(\lambda z)=T\}} \right) \\ & \leq E \left(\int_0^{\sigma(z)} H_t (-\tilde{u}'(zH_t, t) - e_t) dt - H_T \tilde{g}'(zH_T) \mathbb{1}_{\{\sigma(z)=T\}} \right) = Y_0(z) < \infty. \end{aligned}$$

The non-increasing property of the map $-\tilde{u}'$ implies the non-increasing property of the map $z \rightarrow \sigma(z)$ which allows us to deduce that, for $\lambda > 1$,

$$E \left(\int_{\sigma(\lambda z)}^{\sigma(z)} H_t (-\tilde{u}'(zH_t, t) - e_t) dt - H_T \tilde{g}'(zH_T) \mathbb{1}_{\{\sigma(z)=T > \sigma(\lambda z)\}} \right) \geq 0.$$

We would like to integrate this inequality with respect to z on the interval $[0, \nu]$. However, the functions $-\tilde{u}'$ and $-\tilde{g}'$ are too large in a neighborhood of zero, and we need to integrate with respect to z only on the interval $[\epsilon, \nu]$ with $\epsilon > 0$ to obtain

$$E \left(\int_{\epsilon}^{\nu} dz \left[\int_{\sigma(\lambda z)}^{\sigma(z)} H_t (-\tilde{u}'(zH_t, t) - e_t) dt - H_T \tilde{g}'(zH_T) \mathbb{1}_{\{\sigma(z)=T > \sigma(\lambda z)\}} \right] \right) \geq 0,$$

which, from the non-increasing property of $-\tilde{u}'$ and $-\tilde{g}'$ and the link between σ and M expressed in (3.10) leads to

$$E \left(\int_{\epsilon}^{\nu} dz \left[\int_0^{\sigma(\epsilon)} H_t \mathbb{1}_{\{\frac{1}{\lambda} M_t \leq z < M_t\}} [-\tilde{u}'(\frac{1}{\lambda} M_t \vee \epsilon) H_t, t) - e_t] dt \right. \right.$$

$$-H_T \tilde{g}'\left(\left(\frac{1}{\lambda} M_T^- \vee \epsilon\right) H_T\right) \mathbb{1}_{\left\{\frac{1}{\lambda} M_T^- \leq z < M_T^-\right\}} \right] \geq 0.$$

Applying Fubini's theorem gives

$$E\left(\int_0^{\sigma(\epsilon)} H_t \left[-\tilde{u}'\left(\left(\frac{M_t}{\lambda} \vee \epsilon\right) H_t, t\right) - e_t\right] \left[\int_\epsilon^\nu \mathbb{1}_{\left\{\frac{1}{\lambda} M_t \leq z < M_t\right\}} dz\right] dt - H_T \tilde{g}'\left(\left(\frac{M_T^-}{\lambda} \vee \epsilon\right) H_T\right) \left[(\epsilon \vee [M_T^- \wedge \nu]) - \left(\left[\frac{M_T^-}{\lambda} \vee \epsilon\right] \wedge \nu\right)\right]\right) \geq 0. \tag{3.13}$$

On the set $\left\{\frac{M_t}{\lambda} < \nu\right\} \cap \{t < \sigma(\epsilon)\}$, the quantity $\int_\epsilon^\nu \mathbb{1}_{\left\{\frac{1}{\lambda} M_t \leq z < M_t\right\}} dz$ is equal to $(M_t \wedge \nu) - \left(\frac{M_t}{\lambda} \vee \epsilon\right)$, furthermore the inequalities

$$0 \leq \frac{\lambda}{\lambda - 1} \left(\int_\epsilon^\nu \mathbb{1}_{\left\{\frac{1}{\lambda} M_t \leq z < M_t\right\}} dz\right) \leq \nu \mathbb{1}_{\left\{\frac{1}{\lambda} M_t < \nu\right\}}$$

and $-\tilde{u}'\left(\left(\frac{1}{\lambda} M_t \vee \epsilon\right) H_t, t\right) \leq -\tilde{u}'(\epsilon H_t, t)$ hold. Let us multiply the left member of (3.13) by $\frac{\lambda}{\lambda - 1}$ and let λ go to 1; the right-continuity of $\sigma(\cdot)$, the boundness properties and the convergence towards $M_t \mathbb{1}_{[\sigma(\nu), T]}$ of $\frac{\lambda(M_t \wedge \nu) - M_t}{\lambda - 1} \mathbb{1}_{[\sigma(\lambda\nu), T]}$ when λ goes to 1 allow us to pass to the limit. We obtain

$$E\left(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_t M_t [-\tilde{u}'(M_t H_t, t) - e_t] dt - H_T (\epsilon \vee M_T^-) \tilde{g}'(H_T (M_T^- \vee \epsilon))\right) \geq 0.$$

We split the integral in two parts

$$E\left(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_t M_t (-\tilde{u}'(M_t H_t, t)) dt\right) \text{ and } E\left(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_t M_t e_t dt\right).$$

It remains to let ϵ go to zero and remark that $\sigma(\epsilon)$ converges towards T . The integral $E\left(\int_{\sigma(\nu)}^T H_t M_t e_t dt\right)$ converges, since on $\{\sigma(\nu) \leq t\}$, the process M is bounded by ν . Using the non-negativity of $-\tilde{u}'$, it follows

$$E\left(\int_{\sigma(\nu)}^T H_t M_t \{-\tilde{u}'(M_t H_t, t) - e_t\} dt - H_T M_T^- \tilde{g}'(H_T M_T^-)\right) \geq 0.$$

The inequality

$$E\left(\int_{\sigma(\nu)}^T H_t D_t^*(\nu) \{-\tilde{u}'(\nu H_t D_t^*(\nu), t) - e_t\} dt - H_T D_T^*(\nu) \tilde{g}'(\nu H_T D_T^*(\nu))\right) \geq 0$$

follows from the equality $\nu D_t^*(\nu) = M_t$ on the set $\{\sigma(\nu) < t\} = \{M_t < \nu\}$.

If $\lambda < 1$, the same arguments lead to an inequality similar to (3.13) where the lower bound of the integral does not depend on λ :

$$E\left(\int_{\sigma(\nu)}^T H_t \left(-\tilde{u}'\left(\nu \wedge \frac{M_t H_t}{\lambda}, t\right) - e_t\right) \left(\int_\epsilon^\nu \mathbb{1}_{M_t \leq z < \frac{1}{\lambda} M_t} dz\right) dt\right)$$

$$-H_T \left[\left(\frac{M_{T^-}}{\lambda} \wedge \nu \right) - M_{T^-} \right] \tilde{g}' \left(\frac{H_T M_{T^-}}{\lambda} \right) \leq 0.$$

By multiplying the left member by $\frac{\lambda}{1-\lambda}$ we obtain, letting λ go to 1 and ϵ to zero

$$E \left(\int_{\sigma(\nu)}^T H_t D_t^*(\nu) \{ -\tilde{u}'(\nu H_t D_t^*(\nu), t) - e_t \} dt - H_T D_{T^-}^*(\nu) \tilde{g}'(\nu H_T D_{T^-}^*(\nu)) \right) \leq 0.$$

The first equality in the theorem follows.

The second equality in the theorem is obtained in the same manner, using the family of stopping times $\sigma(z + \lambda)$ instead of $\sigma(\lambda z)$ which leads, for $\lambda > 0$ to the inequality

$$E \left[\int_{\sigma(\nu+\lambda)}^{\sigma(\epsilon)} H_t \left([(M_t \wedge \nu) \vee \epsilon] - [(M_t - \lambda) \vee \epsilon] \wedge \nu \right) \left(-\tilde{u}'(H_t [(M_t - \lambda) \vee \epsilon], t) - e_t \right) dt \right. \\ \left. - H_T \left([(M_{T^-} \wedge \nu) \vee \epsilon] - [(M_{T^-} - \lambda) \vee \epsilon] \wedge \nu \right) \tilde{g}'([(M_{T^-} - \lambda) \vee \epsilon]) \right] \geq 0,$$

and for $\lambda < 0$

$$E \left(\int_{\sigma(\nu)}^{\sigma(\epsilon)} H_t \left([(M_t + \lambda) \wedge \nu] - M_t \right) \left(-\tilde{u}'(H_t (M_t + \lambda), t) - e_t \right) dt \right. \\ \left. - H_T \left([(M_{T^-} + \lambda) \wedge \nu] - M_{T^-} \right) \tilde{g}'((M_{T^-} + \lambda)) \right) \leq 0.$$

Dividing by λ and letting λ go to zero gives the result.

We study the map $\nu \rightarrow Y_0(\nu)$. From (3.12), it can be proved that this map is continuous, since $\nu D_t^*(\nu) = M_t \wedge \nu$. When ν goes to zero, using (3.12) and the convergence of $\nu D_t^*(\nu)$ towards zero, $Y_0(\nu)$ goes to ∞ . When ν goes to ∞ , using (3.11) and the equality $\nu D_t^*(\nu) = M_t \wedge \nu$, we write (in the case $g = 0$ for simplicity)

$$Y_0(\nu) = E \left(\int_0^{\sigma(\nu)} H_t [-\tilde{u}'(\nu H_t, t)] dt \right) - E \left(\int_0^{\sigma(\nu)} H_t e_t dt \right).$$

The quantity $E \left(\int_0^{\sigma(\nu)} H_t [-\tilde{u}'(\nu H_t, t)] dt \right)$ is bounded by $E \left(\int_0^T H_t [-\tilde{u}'(\nu H_t, t)] dt \right)$ and converges to zero, whereas the quantity $-E \left(\int_0^{\sigma(\nu)} H_t e_t dt \right)$ converges to $-I_0$, due to the convergence of $\sigma(\nu)$ towards T .

3.3 Boundary and optimal stopping time

Let us recall that the optimal stopping time $\sigma(\nu)$ is such that

$$Y_0(\nu) = E \left\{ \int_0^{\sigma(\nu)} H_s \{-\tilde{u}'(\nu H_s, s) - e_s\} ds - H_T \tilde{g}'(\nu H_T) \mathbb{1}_{\sigma(\nu)=T} \right\}$$

is defined as the first time before T where the process $(Y_t(\nu H_t), t \geq 0)$ hits zero (if this time exists or T otherwise).

Proposition 3.6 *Define the stochastic boundary, for $t < T$*

$$b(t) = \inf\{\nu; Y_t(\nu) = 0\}.$$

Then $\sigma(\nu) = T \wedge \inf\{t < T; Y_t(\nu H_t) = 0\} = T \wedge \inf\{t < T; \nu H_t \geq b(t)\}$ and the inverse of the right-continuous map $\nu \rightarrow \sigma(\nu)$ satisfies $M_t = \inf_{s \leq t} \frac{b(s)}{H_s}$.

Proof. Let us study the properties of $\sigma(z)$. The function $z \rightarrow \sigma(z)$ is non-increasing: this follows from the non-increasing property of $z \rightarrow Y_0(z)$ and the characterisation of $\sigma(z)$ as $\sigma(z) = \inf\{t | Y_t(zH_t) = 0\}$. In order to establish the right-continuity of σ , let z_n be a sequence such that $z_n > z$ and $z_n \rightarrow z$. Then, denoting by σ^* the limit of $\sigma(z_n)$, the non-increasing property leads to $\sigma^* \leq \sigma(z)$. For any stopping time τ , the optimal character of $\sigma(z_n)$ and the non-increasing property of $-\tilde{u}'$ give the following inequalities

$$\begin{aligned} E \left(\int_0^\tau H_t (-\tilde{u}'(z_n H_t, t) - e_t) dt \right) &\leq E \left(\int_0^{\sigma(z_n)} H_t (-\tilde{u}'(z_n H_t, t) - e_t) dt \right) \\ &\leq E \left(\int_0^{\sigma(z_n)} H_t (-\tilde{u}'(z H_t, t) - e_t) dt \right) \end{aligned}$$

and, letting n go to infinity,

$$E \left(\int_0^\tau H_t (-\tilde{u}'(z H_t, t) - e_t) dt \right) \leq E \left(\int_0^{\sigma^*} H_t (-\tilde{u}'(z H_t, t) - e_t) dt \right).$$

The stopping time σ^* is therefore optimal, and from the minimal character of $\sigma(z)$ the inequality $\sigma^* \geq \sigma(z)$ follows. Since $-\tilde{u}'$ and $-\tilde{g}'$ are non-negative functions which are non-increasing with respect to the first argument, the process $Y(\nu)$ is non-negative and non-increasing with respect to ν , therefore $\{\nu; Y_t(\nu) = 0\}$ is an interval of the form $(b(t), +\infty]$ where $(b(t), t \geq 0)$ is an adapted process. Then, the equality $\inf\{t < T; Y_t(\nu H_t) = 0\} = \inf\{t, t < T | \nu H_t \geq b(t)\}$ follows. Let us remark that $b(t) > 0$, since for each t , $Y_t(\nu)$ converges to infinity when ν converges to zero.

It is obvious that, for any ν , the inequality $G_t \stackrel{\text{def}}{=} \inf_{s \leq t} \frac{b(s)}{H_s} \geq \nu$ holds on the set $\{\sigma(\nu) > t\}$, therefore $G_t \geq M_t$. On the other hand, consider that, on the set $G_t > \nu$, the inequality $\sigma(\nu) > t$ holds which implies that $M_t \geq G_t$, and the equality is now obvious for $t < T$. \square

Reflection to the boundary

The equality $M_t = \inf_{s \leq t} \frac{b(s)}{H_s}$ establishes that $\nu D_t^*(\nu) \leq \frac{b(t)}{H_t}$, therefore the process $H^*(\nu)$ remains below the boundary, i.e., $H_t^*(\nu) \leq b(t), \forall t \leq T$. The equality

$$\frac{d(H_t D_t^*)}{H_t D_t^*} = \frac{dD_t^*}{D_t^*} - (r_t dt + \theta_t dW_t)$$

can be interpreted as a reflection problem. The process $D^*(\nu)$ acts as a local time for the reflection along the stochastic boundary $b(t)$. The process $D^*(\nu)$ is the unique one to have this property.

This reflection is more obvious if we study the logarithm $\xi_t = \ln H_t D_t^*$ which is $\xi_t = \ln H_t + D_t^*$. If $\nu H_t \geq b(t)$, the optimal strategy is to throw the process at the boundary $b(t)$. \square

3.4 The primal problem

From now on, we introduce for simplicity the notation

$$c_t^*(\nu) = -\tilde{u}'(H_t^*(\nu), t), \quad \zeta^*(\nu) = -\tilde{g}'(H_{T-}^*(\nu))$$

and recall that

$$c_t^*(\nu) = \text{Arg max}\{u(c, t) - cH_t^*(\nu)\} \quad (3.14)$$

$$\zeta^*(\nu) = \text{Arg max}\{g(\zeta) - \zeta H_{T-}^*(\nu)\}. \quad (3.15)$$

We have proved that, for x large enough, we can fit ν such that $Y_0(\nu_x) = x$. Using the concise notation $D_t^{*x} = D_t^*(\nu_x)$ and similar notation for ζ^{*x} and c^{*x} , we will establish that (c^{*x}, ζ^{*x}) is optimal for the primal problem over all superstrategies and prove that this consumption plan is feasible, i.e. associated with a self-financing strategy and a non-negative current wealth. Using different approaches, He and Pagès (1993) and Cuoco (1997) obtained similar results. Let us emphasize that, from the two representations (3.11, 3.12), we are able to deduce the major fact that the optimality is achieved in the class of x -feasible strategies, which is not done in the above papers.

Optimal consumption plan

Theorem 3.7 *Let H^* be the optimal solution for the dual problem. For a given endowment $x > -I_0$, there exists a multiplier ν_x such that the consumption plan*

$$c^{*x} = -\tilde{u}'(H^{*x}, \cdot), \quad \zeta^{*x} = -\tilde{g}'(H_{T-}^{*x})$$

is an optimal feasible strategy: the associated wealth

$$X_t^* = E \left(\int_t^T H_s^t (c_s^* - e_s) ds + H_T^t \zeta^* | \mathcal{F}_t \right)$$

remains non-negative. The value function of the primal problem is $\Phi(\nu_x)$.

Proof. The definition of c^{*x} and ζ^{*x} implies that for each (c, ζ)

$$E \left(\int_0^T u(c_t, t) dt + g(\zeta) \right) \leq E \left(\int_0^T u(c_t^{*x}, t) dt + g(\zeta^{*x}) \right) - \nu_x \left\{ x - E \left(\int_0^T H_t D_t^{*x} (c_t - e_t) dt + H_T D_T^{*x} \zeta \right) \right\}.$$

Therefore, the consumption plan (c^{*x}, ζ^{*x}) is optimal among all the consumption plans (c, ζ) which satisfy

$$E \left(\int_0^T H_t D_t^{*x} (c_t - e_t) dt + H_T D_T^{*x} \zeta \right) \leq x. \quad (3.16)$$

From Proposition 2.2, it follows that any x -affordable consumption plan satisfies (3.16). At this stage, the consumption plan (c^{*x}, ζ^{*x}) is optimal in the class of superhedging strategies financed from an initial wealth equal to x . However, using similar arguments than in Proposition 3.5, it can be established that $Y_t(\nu)$ satisfies

$$\begin{aligned} Y_t(\nu) &= E \left(\int_t^T H_s^t D_s^{t*}(\nu) \{ -\tilde{u}'(H_s^{t*}(\nu), s) - e_s \} ds - H_T^t D_T^{t*}(\nu) \tilde{g}'(H_T^{t*}(\nu)) | \mathcal{F}_t \right) \\ &= E \left(\int_t^T H_s^t \{ -\tilde{u}'(H_s^{t*}(\nu), s) - e_s \} ds - H_T^t \tilde{g}'(H_T^{t*}(\nu)) | \mathcal{F}_t \right) \end{aligned} \quad (3.17)$$

where $\nu H_s^t D_s^{t*}(\nu) = H_s^{t*}(\nu)$, and $D_t^* D_s^{t*} = D_s^*$.

The equality (3.17) provides that the process $X_t^{*x} \stackrel{\text{def}}{=} Y_t(H_t^{*x})$ is not only the fair-value of the optimal consumption plan, but also the free price of this consumption plan – the minimal solution associated with (c^*, ζ^*) – since

$$X_t^{*x} = E \left(\int_t^T H_s^t (c_s^{*x} - e_s) ds + H_T^t \zeta^{*x} | \mathcal{F}_t \right)$$

and now obviously $X_t^* = Y_t(H_t^{*x}) \geq 0, \forall t$. In particular, there exists a portfolio π^* such that

$$\begin{cases} dX_t^{*x} &= r_t X_t^{*x} dt - (c_t^{*x} - e_t) dt + \pi_t^* (dW_t + \theta_t dt), \\ X_0^{*x} &= x, \quad X_T^{*x} = \zeta^{*x}. \end{cases}$$

□

Remark: The optimal pair does not satisfy the square integrability conditions. Nevertheless $E \left(\int_0^T H_t c_t^{*x} dt \right) < \infty$ and $E(H_T \zeta^{*x}) < \infty$.

Properties of the optimal consumption plan

Let us describe more precisely the links between the optimal wealth and the optimal consumption.

Theorem 3.8 *The optimal consumption plan*

$$c^{*x} = -\tilde{u}'(H^{*x}, \cdot), \quad \zeta^{*x} = -\tilde{g}'(H_T^{*x})$$

can be expressed in a feedback form as

$$c_t^{*x} = -\tilde{u}'(\mathcal{Y}(t, X_t^{*x}), t), \quad \zeta_t^{*x} = -\tilde{g}'(\mathcal{Y}(T, X_T^{*x}))$$

where $\mathcal{Y}(t, \cdot)$ is the inverse of $Y_t(\cdot)$. Furthermore, $0 \leq c_t^{*x} \leq e_t$ on the set $X_t^{*x} = 0$.

Proof. From the two equalities $X_t^{*x} = Y_t(H_t^{*x})$ and $c_t^{*x} = -\tilde{u}'(H_t^{*x}, t)$ we can deduce a feedback representation of the optimal consumption, as it is well known in the no-income case. We have studied the decreasing properties of $Y_0(\cdot)$ and established the decreasing property up to the first hitting time of zero, i.e., on the interval $]0, b(0)[$. From

$$Y_t(\nu) = E \left(\int_t^T H_s^t \{ -\tilde{u}'(H_s^{t*}(\nu), s) - e_s \} ds - H_t^t \tilde{g}'(H_T^{t*}(\nu)) \mid \mathcal{F}_t \right)$$

we deduce that $Y_t(\cdot)$ is strictly decreasing on $]0, b(t)[$ and, since $H_t^*(\nu)$ remains below the boundary, the map $\nu \rightarrow Y_t(\nu H_t^*(\nu))$ is strictly decreasing and admits an inverse $\mathcal{Y}(t, \cdot)$, which is adapted, such that $H_t^*(\nu_x) = \mathcal{Y}(t, X_t^*)$. The optimal consumption can be expressed in terms of the optimal wealth $c_t^* = -\tilde{u}'(\mathcal{Y}(t, X_t^*), t)$. As a check, remark that, for a given level of the wealth, this optimal consumption is smaller than the optimal consumption in the free problem. This representation is similar to the free-case, the only difference is the choice of the process X^* instead of the free wealth X^f . If the wealth is equal to zero (at the boundary), the agent can have a non zero consumption.

Let us be more precise: from Tanaka's formula, and using that X^* is non-negative

$$dX_t^* = \mathbb{1}_{X_t^* > 0} dX_t^* - \mathbb{1}_{X_t^* = 0} dX_t^* + \mathbb{1}_{X_t^* = 0} dL_t^*$$

where L^* is the increasing local time process. An identification with $\mathbb{1}_{X_t^* > 0} dX_t^* + \mathbb{1}_{X_t^* = 0} dX_t^*$ gives $\mathbb{1}_{X_t^* = 0} dL_t^* = 2\mathbb{1}_{X_t^* = 0} dX_t^*$. The equality of the bounded variation parts of the two decompositions of X^* leads to $-(c_t^* - e_t)\mathbb{1}_{X_t^* = 0} dt = \mathbb{1}_{X_t^* = 0} dL_t^*$, and $c_t^* \leq e_t$ on $X_t^* = 0$. \square

4 A classical Markovian situation

4.1 The general Markovian case

If the state price density H and the income process e are Markov processes with $de_t = \mu(t, e_t)dt + \sigma(t, e_t)dW_t$, the analogy with the results obtained in the

no-income case can be drawn further, as it was done in He and Pagès (1993). For notational simplicity, we restrict our attention to the case

$$dH_t = -H_t[r dt + \theta dW_t], \quad de_t = e_t[\mu dt + \sigma dW_t]$$

where r, θ, μ and σ are constant coefficients. The free dual value function J is defined in a dynamic form as

$$J(t, \nu, \varepsilon) \stackrel{\text{def}}{=} E\left(\int_t^T (\tilde{u}(\nu H_s^t, s) + \nu H_s^t e_s) ds + \tilde{g}(\nu H_T^t) | e_t = \varepsilon\right)$$

and satisfies

$$\tilde{u}(\nu, t) + \nu \varepsilon + \mathcal{L}(J) = 0 \tag{4.1}$$

where

$$\mathcal{L}(J) = J_t - \nu r J_\nu + \varepsilon \mu J_\varepsilon + \frac{1}{2} \nu^2 \theta^2 J_{\nu, \nu} + \frac{1}{2} \varepsilon^2 \sigma^2 J_{\varepsilon, \varepsilon} - \nu \theta \varepsilon \sigma J_{\nu, \varepsilon}$$

and the terminal condition $J(T, \nu, \varepsilon) = \tilde{g}(\nu)$.

We define in a dynamic way, the dual value function $\Phi(t, \nu, e_t)$ where

$$\Phi(t, \nu, \varepsilon) \stackrel{\text{def}}{=} \min_{D \in \mathcal{D}} E\left(\int_t^T (\tilde{u}(\nu H_s^t D_s, s) + \nu H_s^t D_s e_s) ds + \tilde{g}(\nu H_T^t D_T^-) | e_t = \varepsilon\right).$$

Restricting our attention to absolutely continuous parameters D , our problem can be characterized by mean of a variational inequality (cf. He and Pagès)

$$\begin{aligned} \Phi_\nu &\leq 0 \\ \tilde{u}(\nu, t) + \nu \varepsilon + \mathcal{L}(\Phi) &\leq 0 \\ (\Phi_\nu)(\tilde{u}(\nu, t) + \nu \varepsilon + \mathcal{L}(\Phi)) &= 0 \\ \Phi(T, \nu, \varepsilon) &= \tilde{g}(\nu) \end{aligned}$$

which we write of the form

$$\max(\Phi_\nu, \tilde{u}(\nu, t) + \nu \varepsilon + \mathcal{L}(\Phi)) = 0.$$

The previous boundary $b(t)$ is related to the manifold that separates $\{(t, \nu, \varepsilon); \Phi_\nu(t, \nu, \varepsilon) = 0\}$ and the continuation region $\{(t, \nu, \varepsilon); \Phi_\nu(t, \nu, \varepsilon) < 0\}$. (Recall that $\Phi_\nu(t, \nu H_t, e_t) = -Y_t(\nu H_t)$.)

From (4.1) we can deduce a variational inequality for $\Psi(t, \nu, \varepsilon) \stackrel{\text{def}}{=} \Phi_\nu(t, \nu, \varepsilon)$. However, let us show how we can obtain this inequality on a direct way. The variational equation associated with the American put can be deduced from

$$\Psi(t, \nu, \varepsilon) = \sup_{\tau \geq t} E\left(\int_t^\tau H_s^t (-\tilde{u}'(\nu H_s^t, s) - e_s) ds - H_\tau^t \mathbb{1}_{\tau=T} \tilde{g}'(\nu H_T^t) | e_t = \varepsilon\right).$$

Under the risk-neutral probability, the process $\hat{W}_t = W_t + \theta t$ is a Brownian motion

$$de_t = e_t[(\mu - \sigma\theta)dt + \sigma d\hat{W}_t], \quad dH_t = H_t[(-r + \theta^2)dt - \theta d\hat{W}_t]$$

and

$$\Psi(t, \nu, \varepsilon) = \sup_{\tau \geq t} E_Q \left(\int_t^\tau (-\tilde{u}'(\nu H_s^t, s) - e_s) ds - \mathbb{1}_{\tau=T} \tilde{g}'(\nu H_T^t) | e_t = \varepsilon \right)$$

is a solution of

$$\max(\Lambda \Psi(t, \nu), -\Psi(t, \nu)) = 0$$

where

$$\begin{aligned} \Lambda \Psi &= \Psi_t + (\theta^2 - r)\Phi_\nu + \varepsilon(\mu - \theta\sigma)\Psi_\varepsilon + \frac{1}{2}\nu^2\theta^2\Psi_{\nu,\nu} \\ &+ \frac{1}{2}\sigma^2(t, \varepsilon)\Psi_{\varepsilon,\varepsilon} - \varepsilon\nu\theta\sigma\Psi_{\nu,\varepsilon} - r\Psi - \tilde{u}' - \varepsilon. \end{aligned}$$

The optimal wealth is $X_t^* = \mathcal{X}(t, e_t, H_t^{*x})$ where

$$\mathcal{X}(t, \varepsilon, \nu) = E \left(\int_t^T H_s^t (-\tilde{u}'(H_s^{*x}(\nu), s) - e_s) ds - H_T^t \tilde{g}'(H_T^{*x}(\nu)) | e_t = \varepsilon \right).$$

Using Itô's formula, we obtain the optimal portfolio π^* by means of \mathcal{X}

$$\left[-H_t^{*x} \theta \frac{\partial \mathcal{X}}{\partial \nu} + \varepsilon \sigma \frac{\partial \mathcal{X}}{\partial \varepsilon} \right] (t, \varepsilon, H_t^{*x}) = \pi_t.$$

As in He and Pagès (1993) or Karatzas (1997), various formulae can be obtained. It suffices to change H into H^* .

4.2 An example

We give a closed formula for the optimal consumption plan in the infinite horizon case. It is easy to check that the properties we have established pass to the limit. We restrict our attention to the case where

- the dynamics of the state prices is given by $dH_t = -H_t(rdt + \theta dW_t)$.
- the income flow is a geometrical Brownian motion

$$de_t = -e_t(\mu_e dt + \sigma_e dW_t), \quad e_0 = \varepsilon.$$

- the coefficients r, θ, μ_e and σ_e are constant.
- the agent's preferences over consumption profiles are given by a HARA utility function $u(c, t)$ with convex conjugate function $\tilde{u}(\nu, t)$

$$\begin{cases} u(c, t) = e^{-\beta t} \frac{c^{1-\gamma}}{1-\gamma} & \text{if } \gamma \neq 1, \\ u(c, t) = e^{-\beta t} \text{Ln}(c) & \text{if } \gamma = 1, \\ -\tilde{u}'(\nu, t) = (\nu e^{\beta t})^{-\frac{1}{\gamma}}. \end{cases}$$

Let us introduce two conditions on the coefficients: under the first one, the income present value is finite, the second one implies that an optimal policy exists for the free case.

A1) The certainty equivalent present value I_0 of the lifetime labor income $I_0(\varepsilon) = E \int_0^{+\infty} H_t e_t dt$ is finite⁵, i.e., $r + \mu_e - \theta\sigma_e > 0$. In this case $I_0(\varepsilon) = B\varepsilon < +\infty$, where $B > 0$ is given by $B^{-1} \stackrel{\text{def}}{=} r + \mu_e - \theta\sigma_e$.

A2) The free dual problem is well-posed, that is $-I_0(\varepsilon) < X_0^f(\nu) < \infty$. This is equivalent to

$$0 < X_0^f(\nu) + I_0(\varepsilon) = E \left(\int_0^{+\infty} H_t (\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}} dt \right) < \infty$$

is non-negative and finite, or to $0 < A < \infty$ where

$$A^{-1} \stackrel{\text{def}}{=} \frac{\beta}{\gamma} + \left(\frac{\gamma - 1}{\gamma} \right) \left(r + \frac{\theta^2}{2\gamma} \right).$$

The free case

From Markovian properties, we deduce that at time t , the optimal free wealth is given by

$$X_t^f(\nu) = A(\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}} - B e_t. \tag{4.3}$$

To force the initial wealth to be equal to x , the multiplier ν must satisfy

$$A\nu^{-\frac{1}{\gamma}} = x + B\varepsilon.$$

In the free case, as it was shown by Svensson and Werner (1991), Merton (1991), He and Pagès (1993) and Koo (1995) the optimal consumption profile at time 0 given by

$$c_0(\nu) = \nu^{-\frac{1}{\gamma}} = \frac{x + B\varepsilon}{A}$$

is a fraction of the wealth plus the income present value. This relation also holds at time t on the form $c_t(\nu) = \frac{X_t^f(\nu) + B e_t}{A}$.

From the equation (4.3), we deduce the dynamics of the optimal wealth

$$\begin{aligned} dX_t^f(\nu) &= -Bde_t + Ad(\nu H_t e^{\beta t})^{-\frac{1}{\gamma}} \\ &= B e_t(\mu_e dt + \sigma_e dW_t) \\ &\quad + (X_t^f(\nu) + B e_t) \left[\left[\frac{1}{\gamma}(r - \beta) + \frac{1}{2\gamma} \left(\frac{\gamma + 1}{\gamma} \right) \theta^2 \right] dt + \frac{\theta}{\gamma} dW_t \right]. \end{aligned}$$

The optimal portfolio is $\pi_t = \frac{\theta}{\gamma} [X_t^f(\nu)] + \left(\sigma_e + \frac{\theta}{\gamma} \right) B e_t$.

The constrained case

We have seen that the fair value $Y_0(\nu)$ is easily deduced from the price of an American option written on the negative part of the free value, that is,

$$Y_0(\nu) = X_0^f(\nu) + V_0(\nu),$$

where

$$V_0(\nu) \stackrel{\text{def}}{=} \sup_{\tau \in \mathcal{T}} E \left(H_\tau \left[A(\nu e^{\beta\tau} H_\tau)^{-\frac{1}{\gamma}} - B e_\tau \right]^- \right).$$

The associated one-dimensional stopping time problem

The function V_0 is the value function of an optimal stopping problem associated with a two dimensional Markov diffusion process $(\nu e^{\beta t} H_t, e_t; t \geq 0)$. From the homogeneity property of the pay-off, it is easy to transform this problem into an optimal stopping problem for a one-dimensional Markov process

$$\frac{V_0(\nu)}{\varepsilon} = \sup_{\tau \in \mathcal{T}} E \left[\frac{H_\tau e_\tau}{\varepsilon} (AZ_\tau - B)^- \right], \quad \text{where} \quad Z_t = \frac{(\nu e^{\beta t} H_t)^{-\frac{1}{\gamma}}}{e_t}. \quad (4.4)$$

To make the transformation complete, let us introduce a change of probability measure and define a new probability measure Q^e , such that the Radon-Nikodym derivative of Q^e w.r.t. P is the exponential martingale generated by $-(\theta + \sigma_e)W$.

Proposition 4.1 *Let $\widehat{V}(z)$ be defined as*

$$\widehat{V}(z) \stackrel{\text{def}}{=} \frac{V_0(\nu)}{\varepsilon} = \sup_{\tau \in \mathcal{T}} E_{Q^e} [e^{-\frac{\tau}{B}} (AZ_\tau - B)^-]. \quad (4.5)$$

The function \widehat{V} is the value function of an optimal stopping time problem with respect to $(AZ - b)^-$ where the Markov process $(Z_t, 0 \leq t)$ is a geometrical Brownian motion under the probability measure Q^e with characteristics given by

$$\begin{cases} dZ_t = Z_t(\mu_Z dt + \sigma_Z dW_t^Z), & Z_0 = z = \nu^{-\frac{1}{\gamma}} \varepsilon^{-1}, \\ \mu_Z = B^{-1} - A^{-1}, & \sigma_Z^2 = \frac{\theta^2}{\gamma^2} + \sigma_e^2 + \frac{\theta \sigma_e}{\gamma}. \end{cases} \quad (4.6)$$

Proof. From (4.4)

$$dZ_t = Z_t \left[\left[\frac{r - \beta}{\gamma} + \mu_e + \frac{1 + \gamma}{2\gamma^2} \theta^2 + \sigma_e^2 + \frac{\theta \sigma_e}{\gamma} \right] dt + \left(\frac{\theta}{\gamma} + \sigma_e \right) dW_t \right].$$

Let us recall that $(H_t e_t, 0 \leq t)$ is a geometrical Brownian motion with discount rate $-B^{-1}$ and Gaussian part $-(\theta + \sigma_e)W$.

With respect to Q^e , W is a Brownian motion \widehat{W} with a drift given by $-(\theta + \sigma_e)$. Therefore,

$$\left(\frac{\theta}{\gamma} + \sigma_e\right)dW_t = \left(\frac{\theta}{\gamma} + \sigma_e\right)d\widehat{W}_t - \left(\frac{\theta^2}{\gamma} + \sigma_e^2 + \frac{\sigma_e\theta}{\gamma} + \theta\sigma_e\right)dt$$

and

$$\frac{dZ_t}{Z_t} = \left[\frac{r - \beta}{\gamma} + \mu_e + \frac{1 - \gamma}{2\gamma^2}\theta^2 - \theta\sigma_e\right]dt + \left(\frac{\theta}{\gamma} + \sigma_e\right)d\widehat{W}_t.$$

The proposition holds from the last decomposition and the remark that

$$\mu_Z = \frac{r - \beta}{\gamma} + \mu_e + \frac{1 - \gamma}{2\gamma^2}\theta^2 - \theta\sigma_e = B^{-1} - A^{-1}.$$

Taking the expectation with respect to this new probability measure, we easily prove that

$$\frac{V_0(\nu)}{\varepsilon} = \sup_{\tau \in \mathcal{T}} E\left[\frac{H_\tau e^{-\tau}}{\varepsilon}(AZ_\tau - B)^-\right] = \sup_{\tau \in \mathcal{T}} E_{Q^e}\left[e^{-\frac{\tau}{B}}(AZ_\tau - B)^-\right] \quad \square$$

Resolution of the optimal stopping problem

From the convexity and monotonicity of the value function \widehat{V} , the optimal stopping time is to be chosen in the class of entrance times $\tau(\alpha)$ such that

$$\tau(\alpha) = \inf\{t; Z_t \leq \alpha\}$$

Let us remark that if $\alpha \geq z$, then $\tau(\alpha) = 0$.

The reward associated with such a stopping time is given by

$$\Psi(z, \alpha) = E_{Q^e}(e^{-\tau(\alpha)/B}(AZ_{\tau(\alpha)} - B)^-) = (A(\alpha \wedge z) - B)^- E_{Q^e}(e^{-\tau(\alpha)/B}).$$

The computation of such an expression follows from the well known closed formula for the Laplace transform of the law of a Brownian entrance time: let $T(b, \mu) = \inf\{t | \mu t + \widehat{B}_t = b\}$, be the entrance time for the generalized Brownian motion $(\widehat{B}_t + \mu t, t \geq 0)$.

$$E(e^{-\lambda T(b, \mu)} \mathbb{1}_{T(b, \mu) < \infty}) = \exp\left(b\mu - |b| \sqrt{\mu^2 + 2\lambda}\right), \quad \lambda > 0. \quad (4.7)$$

The process $\ln(Z)$ is a Brownian motion with drift ∇ where

$$\nabla = \mu_Z - \frac{1}{2}(\sigma_Z)^2 = \frac{r - \beta - \theta^2/2}{\gamma} + \mu_e - \frac{\sigma_e^2}{2} - \theta\sigma_e - \frac{\theta\sigma_e}{2\gamma}.$$

Therefore, the stopping time $\tau(\alpha)$ may be seen as a Brownian entrance time, that is,

$$\tau(\alpha) = T\left(\frac{1}{\sigma_Z} \ln\left(\frac{\alpha}{z}\right), \frac{\nabla}{\sigma_Z}\right), \quad \text{for } \alpha < z.$$

The formula (4.7) with $\mu = \frac{\nabla}{\sigma_Z}$, $b = \frac{1}{\sigma_Z} \ln\left(\frac{\alpha}{z}\right)$ and $\lambda = \frac{1}{B}$ provides

$$E_{Q^\varepsilon}(\exp -\frac{\tau(\alpha)}{B}) = \left(\frac{\alpha \wedge z}{z}\right)^\Delta \quad (4.8)$$

with

$$\Delta = \frac{1}{(\sigma_z)^2} [\nabla + \sqrt{\nabla^2 + 2\sigma_z^2 B^{-1}}]. \quad (4.9)$$

The optimal stopping time is associated with the value of α which maximizes $\Psi(z, \alpha)$, that is, since $\Delta > 0$, with $\alpha^* \wedge z$, where

$$\alpha^* = \frac{\Delta B}{A(1 + \Delta)}$$

and we obtain

$$\widehat{V}(z) = \Psi(z, \alpha^* \wedge z) = (B - A(\alpha^* \wedge z))^+ \left(\frac{\alpha^* \wedge z}{z}\right)^\Delta.$$

It is easy to check that $\alpha^* \leq 1$.

Let us come back to the previous problem, and summarize the results concerning the dual constrained problem in the following theorem

Theorem 4.2 *The closed formula for the optimal wealth associated with the constrained dual problem is provided from the free boundary of this problem*

$$\begin{aligned} b(\varepsilon) &= \left(\frac{A(1 + \Delta)}{\Delta B \varepsilon}\right)^\gamma \\ Y_0(\nu) &= 0 \quad \text{if } \nu \geq b(\varepsilon) \end{aligned}$$

$$Y_0(\nu) = \frac{B\varepsilon}{1 + \Delta} \left(\frac{\Delta B \varepsilon \nu^{\frac{1}{\gamma}}}{A(1 + \Delta)}\right)^\Delta + A\nu^{-\frac{1}{\gamma}} - B\varepsilon \quad \text{otherwise.}$$

Proof. The form of $Y_0(\nu)$ follows from the equality $Y_0(\nu) = X_0^f(\nu) + V_0(\nu)$ (Check that $Y_0(\nu) \geq 0$).

Therefore, the boundary $b(t) = \inf\{\nu \mid Y_t(\nu) = 0\}$ is

$$b(t) = \left(\frac{A(1 + \Delta)}{\Delta B e_t}\right)^\gamma.$$

□

This allows us to obtain the solution of the primal problem

Theorem 4.3 *The optimal consumption function is given by $c^*(\nu_x) = (\nu_x \wedge b(\varepsilon))^{-1/\gamma} = \varepsilon(z \vee \alpha^*)$ where $z = \nu_x^{-1/\gamma} \varepsilon^{-1}$. The optimal consumption and optimal wealth are linked in a feedback formula*

$$x^* = \varepsilon \left(\frac{B}{1 + \Delta} \left(\frac{\Delta B}{A(1 + \Delta)} \frac{\varepsilon}{c^*} \right)^\Delta + A \frac{c^*}{\varepsilon} - B \right).$$

Associated with an initial wealth equal to 0, the maximal value for the consumption function is an α^* -fraction of the income flow, where $\alpha^* = \frac{\Delta B}{A(1 + \Delta)}$ is strictly smaller than the fraction $\frac{B}{A}$ associated with the free problem.

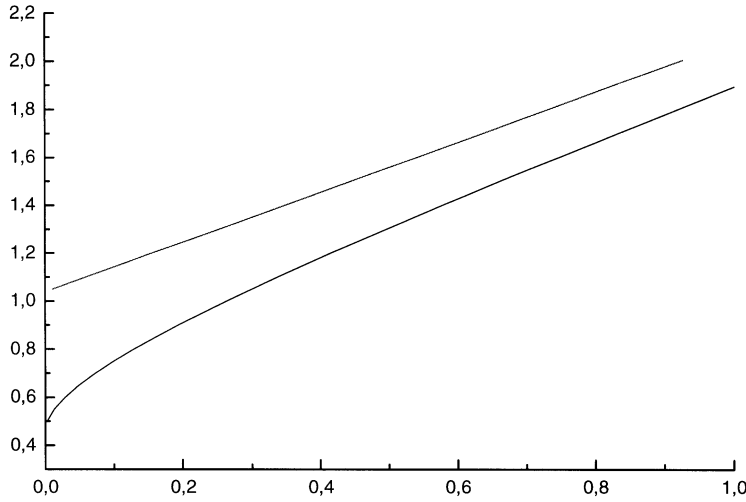


Fig. 1.

Proof. The value ν_x is such that $Y_0(\nu_x) = x$ and, if $\sigma = T \wedge \inf\{t : H_t \nu_x = b(t)\}$, we have

$$D_t^* = 1, t \leq \sigma, D_t^* = \inf_{\sigma \leq s \leq t} \frac{1}{H_s^\sigma} \left(\frac{e_\sigma}{e_s}\right)^{1-\gamma}.$$

The optimal consumption follows

$$\frac{dc_t^*}{c_t^*} = \frac{1}{\gamma - 1} \left(\frac{dD_t^*}{D_t^*} - \left[r + \frac{1}{2} \theta^2 \frac{\gamma - 2}{\gamma - 1} \right] dt - \theta dW_t \right).$$

□

We present, in Fig. 1, the curve c^*/ε as well as the straight line solution of the free problem in the plane $(x^*/\varepsilon, c^*/\varepsilon)$ for $r = 0.1, b = 0.2, \sigma = 0.1, \mu_e = 1, \sigma_e = 0.1, \gamma = 0.9, \beta = 1$.

5 Appendix

5.1 Sketch of the proof of Theorem 2.1

Uniqueness: It follows from Itô's lemma. Indeed, suppose that Y_1, K_1, π_1 and Y_2, K_2, π_2 are two solutions of (2.9) and denote by Y, K, π the differences $Y_1 - Y_2, \dots$. Then, Itô's formula applied to Y^2 provides

$$E \left(Y_t^2 + \int_t^T \|\pi_s\|^2 ds \right) = 2E \left(- \int_t^T r_s Y_s^2 ds + \int_t^T Y_s dK_s \right).$$

Since $\int_t^T Y_s dK_s = \int_t^T -Y_{2,s} dK_{1,s} - Y_{1,s} dK_{2,s}$, the right member of the preceding equality is negative and the uniqueness follows.

Existence: Let us consider $X^{c,\zeta}$ the free-value process of the consumption plan

(c, ζ) , and define the price $Z^{c, \zeta}$ of an American option with payoff $(X^{c, \zeta})^-$, that is,

$$Z_t^{c, \zeta} \stackrel{\text{def}}{=} \text{ess sup}_{\tau \in \mathcal{F}(t)} E \left[(X_\tau^{c, \zeta})^- | \mathcal{F}_t \right].$$

The theory of American options and the optimal stopping times provides that

$$\begin{cases} dZ_t^{c, \zeta} = r_t Z_t^{c, \zeta} dt - dK_t^{c, \zeta} + \pi_t^{c, \zeta} (dW_t + \theta_t dt), & Z_T^{c, \zeta} = 0, \\ Z_t^{c, \zeta} \geq (X_t^{c, \zeta})^-, \quad \forall t \leq T \end{cases} \quad (5.1)$$

where K is a continuous increasing process flat outside $\{t | Z_t^{c, \zeta} > (X_t^{c, \zeta})^-\}$.

The difficulty is to establish the square integrability of the processes $Z^{c, \zeta}$ and $\pi^{c, \zeta}$. This is done in El Karoui et al. (1997). Moreover, it is optimal to stop at the first time where $Z^{c, \zeta} = (X^{c, \zeta})^-$.

Let us define $Y_t^{c, \zeta} = Z_t^{c, \zeta} + X_t^{c, \zeta}$. Then

$$Y_t^{c, \zeta} \stackrel{\text{def}}{=} \text{ess sup}_{\tau \in \mathcal{F}(t)} E \left(\int_t^\tau H_s (c_s - e_s) ds + H_T^\tau \zeta \mathbb{1}_{\tau=T} | \mathcal{F}_t \right).$$

We deduce from the equation (2.9) that $(Y_t^{c, \zeta}, t \in [0, T])$ is a superstrategy against (c, ζ) , and from the properties of the increasing process K , that $(Y_t^{c, \zeta}, t \in [0, T])$ is the minimal superhedging strategy, i.e., by definition the fair value. The properties of the Snell envelope imply that the stopping time $\sigma = \inf\{s | Y_s^{c, \zeta} = \mathbb{1}_{s=T} \zeta\}$ is optimal, i.e.,

$$Y_0^{c, \zeta} = E \left(\int_0^\sigma H_s (c_s - e_s) ds + H_T \zeta \mathbb{1}_{\sigma=T} \right).$$

5.2 Proof of Proposition 2.2

In order to prove the characterisation of affordable plans by means of decreasing processes, it suffices to prove that

$$Y_0^{c, \zeta} = \sup_{D \in \mathcal{D}} E \left(\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T^- \zeta \right).$$

Let τ be a stopping time such that $\tau \leq T$, and define a sequence of decreasing processes by

$$D_s^n = \exp \left(- \int_0^s n \mathbb{1}_{\tau \leq u} du \right) \quad s < T, \quad D_T^n = 0.$$

Since $D_s^n \rightarrow \mathbb{1}_{s \leq \tau}$ and $D_{T-}^n \rightarrow \mathbb{1}_{\tau=T}$, the following inequality holds

$$\begin{aligned} \sup_{D \in \mathcal{D}} E \left(\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T^- \zeta \right) \\ \leq \sup_{\tau \in \mathcal{F}} E \left(\int_0^\tau H_s (c_s - e_s) ds + H_T \zeta \mathbb{1}_{\tau=T} \right). \end{aligned}$$

The reverse inequality is proved using the right-continuous inverse of D , $\Gamma_t = \inf\{s \mid D_s \leq t\}$. For each t , Γ_t is a stopping time. The set-equality $\{D_s > u\} = \{\Gamma_u > s\}$ implies that

$$\int_0^T H_s D_s (c_s - e_s) ds + H_T D_T^- \zeta = \int_0^1 du \left[\int_0^{\Gamma_u} H_s (c_s - e_s) ds + H_T \zeta \mathbb{1}_{\Gamma_u=T} \right]$$

and the result follows.

Endnotes

¹ We consider the case where $(u(x, t), t \geq 0)$ is an adapted process which allows us to deal with the case of insurance portfolio and cushion where the utility process is on the form $(u(e_t + c), t \geq 0)$.

² See He and Pagès (1993) for a counter-example.

³ We refer to He and Pagès' and Cuoco's papers, where the same kind of assumption is done, for an expression of this hypothesis in terms of the function u .

⁴ In the sequel, we give an explicit construction of this optimal control.

⁵ Let us recall that from Ito's formula

$$\begin{aligned} \frac{dH_t^a e_t^b e^{ct}}{H_t^a e_t^b e^{ct}} &= [c - ar - b\mu_e + \frac{1}{2}a(a-1)\theta^2 + \frac{1}{2}b(b-1)\sigma_e^2 + ab\theta\sigma_e]dt \\ &- (a\theta + b\sigma_e)dW_t. \end{aligned} \tag{4.2}$$

From this equality, it follows that

$$E(H_t^a e_t^b e^{ct}) = \varepsilon^b \exp([c - ar - b\mu_e + \frac{1}{2}a(a-1)\theta^2 + \frac{1}{2}b(b-1)\sigma_e^2 + ab\theta\sigma_e]t).$$

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