On the Starting and Stopping Problem: Application in reversible investments.

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Abstract

In this work we solve completely the *starting and stopping* problem when the dynamics of the system are a general adapted stochastic process. We use backward stochastic differential equations and Snell envelopes. Finally we give some numerical results.

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0. Introduction: First let us deal with an example in order to introduce the problem we consider in this paper.

Assume that a power station produces electricity whose selling price fluctuates and depends on many factors such as consumer demand, oil prices, weather and so on. It is well known that electricity cannot be stored and when produced it should be consumed. Now for obvious economic reasons we suppose that electricity is produced only when its profitability is satisfactory. Otherwise the power station is closed up to time when the profitability is coming back, *i.e.*, till the time when the market selling price of electricity reaches a level which makes the production profitable again.

So for the power station there are two modes: operating and closed. At the initial time, we assume it is in its operating mode. On the other hand, like every economic unit, there are expenditures when the station is in its operating mode as well as in the closed one. In addition, switching from a mode to another is not free and generates sunk costs.

The problem we are interested in is to find the sequence of stopping times where one should make decisions to stop the production and to resume it again successively in order to maximize the profitability of the station and then to determine the maximum profit.

More precisely suppose the electricity market selling price is given by a stochastic process $X = (X_t)_{t\leq T}$. As it is discussed previously, a management strategy of the power station is an increasing sequence of stopping times $\delta = (\tau_n)_{n\geq 1}$ where for $n \geq 1$, $\tau_n \leq \tau_{n+1}$ and τ_{2n-1} (resp. τ_{2n}) are the times where the station is switched from the operating to the closed mode (resp. conversely). Now let $J(\delta)$ be the profit of the power station provided by the implementation of the strategy δ. Naturally it depends on the given process X. Therefore we look for a strategy δ^* such that $J(\delta^*) \geq J(\delta)$ for any other δ.

The problem we consider in this paper is of *real options* type. It is usually called the *reversible investment problem*. In recent years, *real options* area has attracted considerable interest ([BO],[BS],[DP],[DZ],...). The motivations are mainly related to decision making in the economic sphere. For more details on this subject see *e.g.* the book by Dixit & Pindyck [DP] and the references therein.

In the literature, our problem is also called *starting and stopping* (or *switching*). In the previous example, we have considered electricity production. However there are many real cases where this problem intervenes. Among others, we can quote the management of oil tankers, oil fields,....

¿From the economic point of view, the problem of *starting and stopping* has been already considered by A.Dixit [D] in the case when T is infinite and X is a geometric Brownian motion. His approach is based on elliptic PDEs.

In this article we solve completely the *starting and stopping* problem when the dynamics of the system is a stochastic process $X = (X_t)_{t\leq T}$ adapted with respect to a Brownian filtration, whatever it may be and when T is finite. The main tools are the notions of reflected backward stochastic differential equation (BSDE in short) and Snell envelope. We show that our problem turns into the existence of a pair of adapted processes (Y^1, Y^2) which satisfies a system expressed by means of Snell envelopes. In a second step, we show that the existence of an optimal strategy and its expression is given. At the end we discuss a method to simulate the optimal strategy and we give some numerical results.

Another interest of our work is that we bring a new point of view to tackle the *starting and* *stopping* problem when the dynamic of the system is affected by the control. With respect to the above example, it means that the process X depends on the running implemented strategy δ. Such problems are met in the management of raw material mines like copper, gold, steel,... In [BO] and [DZ], the approach is based on PDEs, and solution are provided only under fairly stringent conditions. We think that our approach based on BSDEs could bring new results. Though this is a task with which we deal with in a forthcoming paper.

This paper is organized as follows: Section 1 is devoted to the setting of the *starting and stopping problem*. Further we show that our problem reduces to the existence of a pair of processes (Y^1, Y^2) solution of a system expressed by means of Snell envelopes. Then we construct the optimal strategy. Finally we prove the existence of (Y^1, Y^2). In Section 3, we study the case where the process X is a solution of a standard stochastic differential equation and we focus on some numerical aspects of the optimal strategy. Finally we consider some specific cases.

1 Setting of the problem. Preliminary results

Throughout this paper $(\Omega, \mathcal{F}, P)$ is a fixed probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{t\leq T}$ whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{t\leq T}$. Let $\mathbf{F} = (\mathcal{F}_t)_{t\leq T}$ be the completed filtration of $(\mathcal{F}_t^0)_{t\leq T}$ with the P-null sets of $\mathcal{F}$, hence $(\mathcal{F}_t)_{t\leq T}$ satisfies the usual conditions, *i.e.*, it is right continuous and complete. We now introduce the following notations: let

- $\mathcal{P}$ be the σ-algebra on $[0,T] \times \Omega$ of $\mathbf{F}$-progressively measurable sets

- $\mathcal{M}^{2,k}$ be the set of $\mathcal{P}$-measurable and $I\!\!R^k$-valued processes $w = (w_t)_{t\leq T}$ which belongs to $L^2(\Omega \times [0,T], dP \otimes dt)$

- $\mathcal{S}^2$ be the set of $\mathcal{P}$-measurable, continuous processes $w = (w_t)_{t\leq T}$ such that $E[\sup_{t\leq T} |w_t|^2] < \infty$

- $\mathcal{S}_i^2$ be the subset of $\mathcal{S}^2$ of processes $K := (K_t)_{t\leq T}$ which are non-decreasing and satisfy $K_0 = 0$. In particular, if $K \in \mathcal{S}_i^2$, then $E(K_T^2) < \infty$.

For any stopping time $\tau \in [0,T]$, $\mathcal{T}_\tau$ denotes the set of all stopping times θ such that $\tau \leq \theta \leq T$.

A management strategy is an increasing sequence of $\mathbf{F}$-stopping times $\delta := (\tau_n)_{n\geq 1}$ where for any $n \geq 1$, τ_{2n} (resp. τ_{2n-1}) are the moments where the production is frozen (resp. on).

A strategy $\delta := (\tau_n)_{n\geq 1}$ is called admissible if P-*a.s.*, $\lim_{n\to\infty} \tau_n = T$. The set of admissible strategies is denoted $\mathcal{D}_a$.

Now in a short period of time dt, when the production is open, it provides a profit which is equal to $\psi_1(t, X_t)dt$. The quantity $\psi_1(t, X_t)$ can be negative. Such a situation happens when the electricity price is low enough at point that management expenses are not recovered. On the other hand, when the production is frozen there are sunk costs which are equal to $\psi_2(t, X_t)dt$. At least because one should maintain the production equipment in a good state in order to operate in due time. Finally there are also costs linked to stop the production or to start it again. For example, one can think of the fees generated by laying of the workers or engaging them again.

So the outcome of those considerations is that when an admissible management strategy $\delta := (\tau_n)_{n\geq 1}$ is implemented, the average global profit is given by:

$$J(\delta) = E[\int_0^T \Phi(s, X_s, u_s)ds - \sum_{n\geq 1}\{D 1\!\!1_{[\tau_{2n-1}<T]} + a 1\!\!1_{[\tau_{2n}<T]}\}]$$

where :

[i] X_t is the electricity market price at t ; the process $(X_t)_{t\leq T}$ belongs to $\mathcal{M}^{2,1}$

[ii] for any $t \leq T$, $u_t = 1$ if the production is open and 0 otherwise. Actually the process $u = (u_t)_{t\leq T}$ is linked to the implemented strategy δ and for any $t \leq T$ we have $u_t = 1\!\!1_{[0,\tau_1]}(t) + \sum_{n\geq 1} 1\!\!1_{]\tau_{2n},\tau_{2n+1}]}(t)$

[iii] $\Phi(t, x, 0) = \psi_2(t, x)$ and $\Phi(t, x, 1) = \psi_1(t, x)$

[iv] D (resp. a) stands for the sunk cost when the production is stopped (resp. starts)

[v] the functions $\psi_j(t, x)$, $(t, x) \in [0, T] \times I\!\!R^d$ and $j = 1, 2$, are sub-linearly growing, *i.e*, there exists C such that $|\psi_j(t, x)| \leq C(1 + |x|)$.

Now the problem we are interested in is to find an optimal strategy for the manager, *i.e*, a strategy $\delta^* = (\tau_n^*)_{n\geq 1} \in \mathcal{D}_a$ such that $J(\delta^*) \geq J(\delta)$, for any admissible strategy δ. In a second stage we deal with the numerical results of the optimal profit and strategy.

Note that here, the function Φ may depend on time in a general way, which is not the case in Dixit [D].

1.1 The Snell envelope notion

Let $U = (U_t)_{t\leq T}$ be an $\mathbf{F}$-adapted $I\!\!R$-valued càdlàg process without negative jumps and which belongs to the class [D], *i.e.*, the set of random variables $\{U_\tau,\ \tau \in \mathcal{T}_0\}$ is uniformly integrable. Then there exists a unique $\mathbf{F}$-adapted $I\!\!R$-valued continuous process $Z := (Z_t)_{t\leq T}$ (see e.g. [CK], [EK], [H]), called the *Snell envelope* of U, such that :

Z is the smallest super-martingale which dominates U, *i.e*, if $(\bar{Z}_t)_{t\leq T}$ is another càdlàg super-martingale such that $\forall t \leq T$, $\bar{Z}_t \geq U_t$ then $\bar{Z}_t \geq Z_t$ for any $t \leq T$.

The following properties of the process Z hold true :

(i) Z can be expressed as : for any $\mathbf{F}$-stopping time γ,

$$Z_\gamma = \operatorname{esssup}_{\tau\in\mathcal{T}_\gamma} E[U_\tau|\mathcal{F}_\gamma] \ \text{ (and then } Z_T = U_T) \tag{1}$$

(ii) let γ be an $\mathbf{F}$-stopping time and $\tau_\gamma^* = \inf\{s \geq \gamma, Z_s = U_s\} \wedge T$ then τ_γ^* is optimal after γ, *i.e.*,

$$Z_\gamma = E[Z_{\tau_\gamma^*}|\mathcal{F}_\gamma] = E[U_{\tau_\gamma^*}|\mathcal{F}_\gamma] = \text{esssup}_{\tau \geq \gamma} E[U_\tau|\mathcal{F}_\gamma]. \tag{2}$$

(iii) if U_n, $n \geq 0$, and U are càdlàg and uniformly square integrable processes such that the sequence $(U_n)_{n\geq 0}$ converges increasingly and pointwisely to U then $(Z^{U_n})_{n\geq 0}$ converges increasingly and pointwisely to Z^U ; Z^{U_n} and Z^U are the Snell envelopes of respectively U_n and U.

The proof of (iii) is given in the appendix of [CK]. For more details on the Snell envelope notion, one can refer to [EK] $\diamond$

Let $\delta = (\tau_n)_{n\geq 1}$ be an admissible strategy. The strategy δ is called *finite* if during the time interval $[0, T]$ it allows to the manager to make only a finite number of decisions, *i.e*, $P(\omega, \tau_n(\omega) < T, \forall n \geq 1) = 0$. Hereafter the set of finite strategies will be denoted $\mathcal{D}$. Obviously optimal strategies should be necessarily finite, otherwise the sunk costs would be infinite. Therefore we have the following result whose proof is quite easy and then is omitted.

Proposition 1 *: The supremum over admissible strategies and finite strategies are the same:* $\sup_{\delta\in\mathcal{D}_a} J(\delta) = \sup_{\delta\in\mathcal{D}} J(\delta)$ $\diamond$

We now focus on the optimal profit and we have the following verification theorem.

Proposition 2 *: Assume there exist two* $I\!R$*-valued processes* $Y^1 = (Y_t^1)_{t\leq T}$ *and* $Y^2 = (Y_t^2)_{t\leq T}$ *of* $\mathcal{S}^2$ *such that* $\forall t \leq T$*,*

$$Y_t^1 = \text{esssup}_{\tau\in\mathcal{T}_t} E[\int_t^\tau \psi_1(s, X_s)ds + (-D + Y_\tau^2)1\!\!1_{[\tau<T]}|\mathcal{F}_t], \tag{3}$$

$$Y_t^2 = \text{esssup}_{\tau\in\mathcal{T}_t} E[\int_t^\tau \psi_2(s, X_s)ds + (-a + Y_\tau^1)1\!\!1_{[\tau<T]}|\mathcal{F}_t]. \tag{4}$$

Then $Y_0^1 = \sup_{\delta\in\mathcal{D}} J(\delta)$*. Moreover the strategy* $\delta^* = (\tau_n^*)_{n\geq 1}$ *defined as follows:*

$$\begin{aligned} \tau_0^* &= 0 \\ \forall n \geq 1,\ \tau_{2n-1}^* &= \inf\{s \geq \tau_{2n-2}^*, Y_s^1 = -D + Y_s^2\} \wedge T \\ \tau_{2n}^* &= \inf\{s \geq \tau_{2n-1}^*, Y_s^2 = -a + Y_s^1\} \wedge T \end{aligned}$$

is optimal.

Proof: First recall that Y^1 and Y^2 are continuous and verify $Y_T^1 = Y_T^2 = 0$. Therefore the jumps of $((-D + Y_t^2)1\!\!1_{[t<T]})_{t\leq T}$ and $((-a + Y_t^1)1\!\!1_{[t<T]})_{t\leq T}$ at T are non-negative. Now for any $t \leq T$, we have

$$Y_t^1 + \int_0^t \psi_1(s, X_s)ds = \text{esssup}_{\tau\geq t} E[\int_0^\tau \psi_1(s, X_s)ds + (-D + Y_\tau^2)1\!\!1_{[\tau<T]}|\mathcal{F}_t].$$

The random variable Y_0^1 is $\mathcal{F}_0$-measurable then it is $P-a.s.$ a constant and then $Y_0^1 = E[Y_0^1]$. On the other hand, according to (2),

$$Y_0^1 = E[\int_0^{\tau_1^*} \psi_1(s,X_s)ds + (-D + Y_{\tau_1^*}^2)1\!\!1_{[\tau_1^*<T]}] ,$$

where τ_1^* is given as in the proposition. From the properties of Snell's envelope

$$\begin{aligned} Y_{\tau_1^*}^2 &= \operatorname{esssup}_{\tau\in\mathcal{T}_{\tau_1^*}} E[\int_{\tau_1^*}^{\tau} \psi_2(s,X_s)ds + (-a+Y_\tau^1)1\!\!1_{[\tau<T]}|\mathcal{F}_{\tau_1^*}] \\ &= E[\int_{\tau_1^*}^{\tau_2^*} \psi_2(s,X_s)ds + (-a+Y_{\tau_2^*}^1)1\!\!1_{[\tau_2^*<T]}|\mathcal{F}_{\tau_1^*}]. \end{aligned}$$

It implies that

$$\begin{aligned} Y_0^1 &= E[\int_0^{\tau_1^*} \psi_1(s,X_s)ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s,X_s)ds - D1\!\!1_{[\tau_1^*<T]} + E[(-a+Y_{\tau_2^*}^1)1\!\!1_{[\tau_2^*<T]}|\mathcal{F}_{\tau_1^*}]\ 1\!\!1_{[\tau_1^*<T]}] \\ &= E[\int_0^{\tau_1^*} \psi_1(s,X_s)ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s,X_s)ds - D1\!\!1_{[\tau_1^*<T]} - a1\!\!1_{[\tau_2^*<T]} + Y_{\tau_2^*}^1 1\!\!1_{[\tau_2^*<T]}] \end{aligned}$$

since $[\tau_1^* < T] \in \mathcal{F}_{\tau_1^*}$ and $[\tau_2^* < T] \subset [\tau_1^* < T]$. Therefore

$$Y_0^1 = E[\int_0^{\tau_2^*} \Phi(s,X_s,u_s)ds - D1\!\!1_{[\tau_1^*<T]} - a1\!\!1_{[\tau_2^*<T]} + Y_{\tau_2^*}^1 1\!\!1_{[\tau_2^*<T]}]$$

since between 0 and τ_1^* (resp. τ_1^* and τ_2^*) the production is open (resp. suspended) and then $u_t = 1$ (resp. $u_t = 0$) which implies that

$$\int_0^{\tau_2^*} \Phi(s,X_s,u_s)ds = \int_0^{\tau_1^*} \psi_1(s,X_s)ds + \int_{\tau_1^*}^{\tau_2^*} \psi_2(s,X_s)ds.$$

Now following this reasoning as many times as necessary we obtain

$$Y_0^1 = E[\int_0^{\tau_{2n}^*} \Phi(s,X_s,u_s)ds - \sum_{1\le k\le n} (D1\!\!1_{[\tau_{2k-1}^*<T]} + a1\!\!1_{[\tau_{2k}^*<T]}) + Y_{\tau_{2n}^*}^1 1\!\!1_{[\tau_{2n}^*<T]}]. \tag{5}$$

But the strategy δ^* is finite. Indeed let $A = \{\omega, \tau_n^* < T, \forall n \ge 1\}$ and let us show that $P(A) = 0$. If $P(A) > 0$ then for any $n \ge 1$,

$$\begin{aligned} Y_0^1 \le\ & E[\int_0^T (|\psi_1(s,X_s)| \vee |\psi_2(s,X_s)|)ds - (\sum_{1\le k\le n} (D1\!\!1_{[\tau_{2k-1}^*<T]} + a1\!\!1_{[\tau_{2k}^*<T]}))1\!\!1_A \\ & -(\textstyle\sum_{1\le k\le n}(D1\!\!1_{[\tau_{2k-1}^*<T]} + a1\!\!1_{[\tau_{2k}^*<T]}))1\!\!1_{\bar{A}} + \sup_{s\le T}|Y_s^1|1\!\!1_{[\tau_{2n}^*<T]}]. \end{aligned}$$

The right-hand side converges to $-\infty$ as $n \to \infty$ since the process Y^1 belongs to $\mathcal{S}^2$ and $\psi_i(.,X_.) \in \mathcal{M}^{2,1}$, then $Y_0^1 = -\infty$. But this is contradictory because, once again, $Y^1 \in \mathcal{S}^2$. Henceforth the strategy δ^* is finite. Going back to (5) and taking the limit as $n \to \infty$ we obtain $Y_0^1 = J(\delta^*)$.

Now let us show that $Y_0^1 \ge J(\delta)$ for any $\delta \in \mathcal{D}$. Let $\delta = (\tau_n)_{n\ge1}$ be a finite strategy. According to (2), τ_1^* is optimal and then

$$Y_0^1 \ge E[\int_0^{\tau_1} \psi_1(s,X_s)ds + (-D + Y_{\tau_1}^2)1\!\!1_{[\tau_1<T]}].$$

On the other hand

$$Y^2_{\tau_1} \geq E[\int_{\tau_1}^{\tau_2} \psi_2(s,X_s)ds + (-a+Y^1_{\tau_2})1\!\!1_{[\tau_2<T]}|\mathcal{F}_{\tau_1}]$$

and then

$$\begin{aligned} Y^1_0 &\geq E[\int_0^{\tau_1} \psi_1(s,X_s)ds - D1\!\!1_{[\tau_1<T]} + E[(-a+Y^1_{\tau_2})1\!\!1_{[\tau_2<T]}|\mathcal{F}_{\tau_1}]1\!\!1_{[\tau_1<T]}] \\ &\geq E[\int_0^{\tau_1} \psi_1(s,X_s)ds + \int_{\tau_1}^{\tau_2} \psi_2(s,X_s)ds - D1\!\!1_{[\tau_1<T]} - a1\!\!1_{[\tau_2<T]} + Y^1_{\tau_2}1\!\!1_{[\tau_2<T]}] \end{aligned}$$

since $[\tau_1 < T] \in \mathcal{F}_{\tau_1}$ and $[\tau_2 < T] \subset [\tau_1 < T]$. Therefore we have,

$$Y^1_0 \geq E[\int_0^{\tau_2} \Phi(s,X_s,u_s)ds - D1\!\!1_{[\tau_1<T]} - a1\!\!1_{[\tau_2<T]} + Y^1_{\tau_2}1\!\!1_{[\tau_2<T]}].$$

Now making this reasoning as many times as necessary we obtain for any $n \geq 0$,

$$Y^1_0 \geq E[\int_0^{\tau_{2n}} \Phi(s,X_s,u_s)ds - \sum_{1\leq k\leq n}(D1\!\!1_{[\tau_{2k-1}<T]} + a1\!\!1_{[\tau_{2k}<T]}) + Y^1_{\tau_{2n}}1\!\!1_{[\tau_{2n}<T]}]. \qquad (6)$$

As the strategy δ is finite then the right-hand side of (6) converges to $J(\delta)$ as $n \to \infty$. Therefore we have $Y^1_0 = J(\delta^*) \geq J(\delta)$ which implies actually that the strategy δ^* is optimal.

Remark 1 *The random variable Y^1_t (resp. Y^2_t) stands for the optimal expected profit if at t the station is in its operating (resp. stopping) mode* $\diamond$

2 Existence of the pair (Y^1, Y^2).

We now focus on the existence of the pair (Y^1, Y^2). First let us recall the following result stated by El-Karoui et al. [EKal] and which is related to BSDEs with one reflecting barrier.

Let $\xi \in L^2(\Omega, \mathcal{F}_T, I\!\!R; dP)$, $(f_t)_{t\leq T}$ a process of $\mathcal{M}^{2,1}$ and finally let $S := (S_t)_{t\leq T}$ be an $I\!\!R$-valued process of $\mathcal{S}^2$ such that $S_T \leq \xi$. Then we have :

Theorem 1 (EKal) *: There exists a triple $(Y,Z,K) := (Y_t,Z_t,K_t)_{t\leq T}$ of $\mathcal{P}$-measurable processes, with values in $I\!\!R^1 \times I\!\!R^d \times I\!\!R^1$ such that:*

$$\begin{cases} Y \in \mathcal{S}^2,\ Z \in \mathcal{M}^{2,d} \text{ and } K \in \mathcal{S}^2_i\,(K_0=0) \\ Y_t = \xi + \int_t^T f_s ds + K_T - K_t - \int_t^T Z_s dB_s\, t \leq T \\ \forall t \leq T, Y_t \geq S_t \text{ and } \int_0^T (Y_t - S_t)dK_t = 0. \end{cases} \qquad (7)$$

In addition, Y can be interpreted as a Snell envelope in the following way: $\forall\ t \leq T$,

$$Y_t = esssup_{\tau\geq t} E[\int_t^\tau f_s ds + S_\tau 1\!\!1_{[\tau<T]} + \xi 1\!\!1_{[\tau=T]}|\mathcal{F}_t]\diamond \qquad (8)$$

We are going now to provide a solution for (3)-(4). It is based on BSDEs with two reflecting barriers studied by several authors (see e.g. [CK], [HL], [HLM], ...). Actually we have:

Theorem 2 *: There exists a pair of continuous processes* $(Y_t^1, Y_t^2)_{t\leq T}$ *which satisfies (3)-(4).*

Proof: Since $-D < a$ then there exists a unique quadruple of $\mathcal{P}$-measurable processes $(Y_t, Z_t, K_t^+, K_t^-)_{t\leq T}$ which satisfies :

$$\begin{cases} Y \in \mathcal{S}^2,\ Z \in \mathcal{M}^{2,d} \text{ and } K^{\pm} \in \mathcal{S}_i^2\,(K_0^{\pm} = 0) \\ Y_t = \int_t^T (\psi_1(s,X_s) - \psi_2(s,X_s))ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) - \int_t^T Z_s dB_s,\ \ t \leq T \\ \forall t \leq T, -D \leq Y_t \leq a \text{ and } \int_0^T (Y_t + D)dK_t^+ = \int_0^T (a - Y_t)dK_t^- = 0. \end{cases} \tag{9}$$

The existence of the quadruple $(Y, Z, K^{\pm})$ stems from a result by ([CK], pp.2034) or ([HLM], pp.165) since the barriers $-D$ and a are constants and then they are *regular* and satisfy also the so-called Mokobodzki's condition which means the location of a difference of non-negative supermartingales between $-D$ and a. Actually it is enough to choose those supermartingales null identically.

Now for $t \leq T$, let us set :

$$Y_t^1 = E[\int_t^T \psi_1(s,X_s)ds + K_T^+ - K_t^+ |\mathcal{F}_t] \text{ and } Y_t^2 = E[\int_t^T \psi_2(s,X_s)ds + K_T^- - K_t^- |\mathcal{F}_t].$$

Therefore for any $t \leq T$ we have $Y_t = Y_t^1 - Y_t^2$. Now let γ^1 and $(Z_t^1)_{t\leq T}$ be respectively the constant and the process of $\mathcal{M}^{2,d}$ such that :

$$\int_0^T \psi_1(s,X_s)ds + K_T^+ = \gamma^1 + \int_0^T Z_s^1 dB_s.$$

There is no existence problem for those items since $(\psi_1(t,X_t))_{t\leq T}$ belongs to $\mathcal{M}^{2,1}$ and K_T^+ is square integrable. Henceforth the triple (Y^1, Z^1, K^+) satisfies :

$$\begin{cases} -dY_t^1 = \psi_1(t,X_t)dt + dK_t^+ - Z_t^1 dB_t,\ Y_T^1 = 0 \\ Y_t^1 \geq -D + Y_t^2 \text{ and } (Y_t^1 - Y_t^2 + D)dK_t^+ = 0 \end{cases}$$

Now by (7)-(8) we have :

$$Y_t^1 = \text{esssup}_{\tau\in\mathcal{T}_t} E[\int_t^\tau \psi_1(s,X_s)ds + (-D + Y_\tau^2)1\!\!1_{[\tau<T]}|\mathcal{F}_t], t \leq T.$$

In the same way we can show that Y^2 verifies :

$$Y_t^2 = \text{esssup}_{\tau\in\mathcal{T}_t} E[\int_t^\tau \psi_2(s,X_s)ds + (-a + Y_\tau^1)1\!\!1_{[\tau<T]}|\mathcal{F}_t],\ t \leq T.$$

Henceforth the pair (Y^1, Y^2) satisfies actually the system (3)-(4) $\diamond$

Remark 2 *Let us consider the sequences* $(Y^{1,n})_{n\geq 0}$ *and* $(Y^{2,n})_{n\geq 1}$ *defined recursively as follows :* $\forall t \leq T$,

$$Y_t^{2,n} = esssup_{\tau\geq t} E[\int_t^\tau \psi_2(s,X_s)ds + (-a + Y_\tau^{1,n-1})1\!\!1_{[\tau<T]}|\mathcal{F}_t] \quad (10)$$

and

$$Y_t^{1,n} = esssup_{\tau\geq t} E[\int_t^\tau \psi_1(s,X_s)ds + (-D + Y_\tau^{2,n})1\!\!1_{[\tau<T]}|\mathcal{F}_t] \quad (11)$$

where $Y_t^{1,0} = E[\int_t^T \psi_1(s,X_s)ds|\mathcal{F}_t]$. *Then we have the following characterization of* $Y_t^{1,n}$ *:*

$$\forall t \leq T,\ Y_t^{1,n} = esssup_{\delta\in\mathcal{D}_t^n} E[\int_t^T \Phi(s,X_s,u_s)ds - \sum_{n\geq 1}(D1\!\!1_{[\tau_{2n-1}<T]} + a1\!\!1_{[\tau_{2n}<T]}|\mathcal{F}_t] \quad (12)$$

where $\mathcal{D}_t^n$ *is the set of admissible strategies* $\delta = (\tau_k)_{k\geq 1}$ *such that* $\tau_1 \geq t$ *and* $\tau_{2n+1} = T$*, P-a.s.. Therefore we can show that the sequence* $(Y^{1,n})_{n\geq 0}$ *(resp.* $(Y^{2,n})_{n\geq 1}$*) converges increasingly in* $\mathcal{S}^2$ *to* Y^1 *(resp.* Y^2*)* $\diamond$

3 Properties of the optimal strategy, numerical aspects and examples

Let once again $(Y_t)_{t\leq T}$ be the process of (9). Since $Y^1 - Y^2 = Y$ then it is easily seen that the stopping times τ_n^* which give the optimal strategy are the ones where the process Y reaches successively the barriers a and $-D$, *i.e*,

$$\forall n \geq 1,\ \tau_{2n-1}^* = \inf\{s \geq \tau_{2n-2}^*, Y_s = -D\} \wedge T\ (\tau_0^* = 0) \text{ and } \tau_{2n}^* = \inf\{s \geq \tau_{2n-1}^*, Y_s = a\} \wedge T.$$

Assume now that the process X is the unique solution of the following SDE :

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t,\ t \leq T;\ X_0 = x \in I\!R^k \quad (13)$$

where the functions b and σ, with appropriate dimensions, are jointly continuous and satisfy :

$$|b(t,x)| + |\sigma(t,x)| \leq k(1+|x|) \text{ and } |\sigma(t,x) - \sigma(t,x')| + |b(t,x) - b(t,x')| \leq k|x-x'|$$

for any $t \leq T$ and $x, x' \in I\!R^k$. Let us recall that under those assumptions on b, σ it is well known that for any $p \geq 1$ there exists a real constant C_p such that $E[(\sup_{t\leq T}|X_t|)^p] \leq C_p$.

In [HH], it has been shown the existence of a continuous function $u(t,x)$, $(t,x) \in [0,T] \times I\!R^k$, such that $Y_t = u(t,X_t)$, for any $t \leq T$. Moreover the function u is solution, in viscosity sense, of the following double obstacle partial differential inequality :

$$\begin{cases} \min\{u(t,x) + D, \max[-\frac{\partial u}{\partial t}(t,x) - L_t u(t,x) - \psi(t,x), u(t,x) - a]\} = 0, \\ u(T,x) = 0 \end{cases}$$

where $\psi = \psi_1 - \psi_2$ and the operator L is the generator associated with X, *i.e.*,

$$L_t = \frac{1}{2}\sum_{i,j=1}^{k}(\sigma\sigma^*(t,x))_{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{k} b_i(t,x)\frac{\partial}{\partial x_i}.$$

Therefore the optimal strategy can be expressed via the beginnings of some deterministic sets. Actually we have: $(\tau_0^* = 0)$,

$$\forall n \geq 1,\ \tau_{2n-1}^* = \inf\{s \geq \tau_{2n-2}^*, u(s,X_s) = -D\} \wedge T \ \text{ and } \tau_{2n}^* = \inf\{s \geq \tau_{2n-1}^*, u(s,X_s) = a\} \wedge T \ \diamond$$

Let us now focus on some numerical aspects of the optimal strategy of the *stopping and starting problem*. However keep in mind that our purpose in this article is not to provide an exhaustive treatment of this issue, which of course is a very interesting subject but is a bit far from the main objective of our work.

So for $n, k \geq 0$ let us consider the following standard or reflected BSDEs : for any $t \leq T$,

$$Y_t^{n,k} = \int_t^T \psi(s,X_s)ds - n\int_t^T (Y_s^{n,k} - a)^+ ds + k\int_t^T (Y_s^{n,k} + D)^- ds - \int_t^T Z_s^{n,k} dB_s,$$

$$\begin{cases} \tilde{Y}_t^k = \int_t^T \psi(s,X_s)ds - (\tilde{K}_T^{k,-} - \tilde{K}_t^{k,-}) + k\int_t^T (\tilde{Y}_s^k + D)^- ds - \int_t^T \tilde{Z}_s^k dB_s \\ \tilde{Y}_t^k \leq a \text{ and } (\tilde{Y}_t^k - a)dK_t^{k,-} = 0, \end{cases}$$

$$\begin{cases} Y_t^n = \int_t^T \psi(s,X_s)ds + (K_T^{n,+} - K_t^{n,+}) - n\int_t^T (Y_s^n - a)^+ ds - \int_t^T Z_s^n dB_s \\ Y_t^n \geq -D \text{ and } (Y_t^n + D)dK_t^{n,+} = 0. \end{cases}$$

It is well known that for any $n \geq 0$ the sequence $(Y^{n,k})_{k\geq 0}$ converges decreasingly in $\mathcal{S}^2$ to Y^n. On the other hand the sequence $(\tilde{Y}^k)_{k\geq 0}$ (resp. $((Y^n)_{n\geq 0})$ converges increasingly (resp. decreasingly) to Y in $\mathcal{S}^2$ (see e.g. [HLM]). So we are going to focus on the estimates of $Y^{n,m} - Y$ and $Y^n - Y$. To begin with let us recall the following properties related to $Y^{n,k}$, $\tilde{Y}^k$ and Y^n. The proofs can be found in [CK] or [HLM].

Proposition 3 *: The following properties hold true:*

(i) for any n, k we have $\tilde{Y}^k \leq Y^{n,k} \leq Y^n$

(ii) $\forall\ k \geq 0$ and $t \leq T$,

$$\tilde{Y}_t^k = essinf_{\nu \geq t} E[\int_t^\nu \{\psi(s,X_s) + k(\tilde{Y}_s^k + D)^-\}ds + a1_{[\nu < T]}|\mathcal{F}_t].$$

In addition the infimum is reached at the stopping time $\tilde{\nu}_t^k = \inf\{s \geq t, \tilde{Y}_s^k = a\} \wedge T$

(iii) let $\mathcal{U}$ be the set of $\mathcal{P}$-measurable processes $(v_t)_{t\leq T}$ with values in $[0,1]$. Then for any $n, k \geq 0$ and $t \leq T$ we have,

$$Y_t^{n,k} = essinf_{v \in \mathcal{U}} E[\int_t^T e^{-\int_t^s n v_s ds} \times \{\psi(s,X_s) + k(Y_s^{n,k} + D)^- + nav_s\}ds|\mathcal{F}_t].$$

In addition the infimum is reached at $(\tilde{v}_t = 1_{[a<Y_t^{n,k}]})_{t\leq T}$

(iv) there exists a constant C_ψ *which does depend only* $(\psi(t,X_t))_{t\leq T}$ *(and not on k, n) such that* $E[\sup_{s\leq T}((Y_s^{n,k}+D)^-)^2] \leq C_\psi k^{-2}$ *and* $E[\sup_{s\leq T}((Y_s^{n,k}-a)^+)^2] \leq C_\psi n^{-2}$ $\diamond$

Then we have :

Proposition 4 *For any* $n,k\geq 1$*, it holds true that*

$$E[\sup_{t\leq T}|Y_t^{n,k}-Y_t|^2] \leq C(n^{-2}+k^{-2}) \tag{14}$$

where C is a constant which depends only on ψ*.*

Proof: Let $t\leq T$ and let us focus on the difference $Y_t^{n,k}-\tilde{Y}_t^k$. Let θ_t be a stopping time such that $t\leq\theta_t\leq T$, P-$a.s.$. On the other hand for $s\leq T$ let us set $v_s = 1_{[s\geq\theta_t]}$. Then we have :

$$\begin{aligned}
&E[\int_t^T e^{-\int_t^s nv_r dr}\times\{\psi(s,X_s)+k(Y_s^{n,k}+D)^-+nav_s\}ds|\mathcal{F}_t]- \\
&\qquad\qquad E[\int_t^{\theta_t}\{\psi(s,X_s)+k(\tilde{Y}_s^k+D)^-\}ds+a1_{[\theta_t<T]}|\mathcal{F}_t] = \\
&E[\int_t^{\theta_t}\{k(Y_s^{n,k}+D)^- - k(\tilde{Y}_s^k+D)^-\}ds - a1_{[\theta_t<T]}|\mathcal{F}_t]+ \\
&\qquad\qquad E[\int_{\theta_t}^T e^{-n(s-\theta_t)}\{\psi(s,X_s)+k(Y_s^{n,k}+D)^-+na\}ds|\mathcal{F}_t] \leq \\
&E[-a1_{[\theta_t<T]}+\int_{\theta_t}^T e^{-n(s-\theta_t)}\{\psi(s,X_s)+k(Y_s^{n,k}+D)^-+na\}ds|\mathcal{F}_t]
\end{aligned}$$

since for any n,k, $\forall s\leq T$, $(Y_s^{n,k}+D)^- - (\tilde{Y}_s^k+D)^- \leq 0$, through Prop.3-$(i)$. Now let us deal with latter term.

$$\int_{\theta_t}^T e^{-n(s-\theta_t)}\{\psi(s,X_s)+k(Y_s^{n,k}+D)^-\}ds \leq \frac{1}{n}\sup_{s\leq T}\{|\psi(s,X_s)|+k(Y_s^{n,k}+D)^-\}$$

and

$$-a1_{[\theta_t<T]}+a\int_{\theta_t}^T ne^{-n(s-\theta_t)}ds = -a1_{[\theta_t<T]}+a(1-e^{-n(T-\theta_t)})1_{[\theta_t<T]}\leq 0$$

Therefore we have :

$$\begin{aligned}
&E[\int_t^T e^{-\int_t^s nv_r dr}\times\{\psi(s,X_s)+k(Y_s^{n,k}+D)^-+nav_s\}ds|\mathcal{F}_t]- \\
&\qquad E[\int_t^{\theta_t}\{\psi(s,X_s)+k(\tilde{Y}_s^k+D)^-\}ds+a1_{[\theta_t<T]}|\mathcal{F}_t] \leq \frac{1}{n}E[\sup_{s\leq T}\{|\psi(s,X_s)|+k(Y_s^{n,k}+D)^-\}|\mathcal{F}_t].
\end{aligned}$$

Taking $\theta_t = \tilde{\nu}_t^k$ (keep in mind that in the expression of $\tilde{Y}^k$ the infimum is reached at $\tilde{\nu}_t^k$) we deduce that:

$$Y_t^{n,k}\leq\tilde{Y}_t^k+\frac{1}{n}E[\sup_{s\leq T}\{|\psi(s,X_s)|+k(Y_s^{n,k}+D)^-\}|\mathcal{F}_t].$$

Now since $Y_t^{n,k}\geq\tilde{Y}_t^k$ and through the convergence of $(Y^{n,k})_{k\geq 0}$ (resp. $(\tilde{Y}^k)_{k\geq 0}$) to Y^n (resp. Y) in $\mathcal{S}^2$ and finally using Doob's inequality we deduce that

$$E[\sup_{t\leq T}|Y_t^n-Y_t|^2]\leq Cn^{-2}. \tag{15}$$

Now considering the BSDEs satisfied by $-Y^{n,k}$ and $-Y^n$ we can show exactly as previously that :

$$-Y_t^{n,k} \leq -Y_t^n + \frac{1}{k} E[\sup_{s \leq T} \{|\psi(s, X_s)| + n(-Y_s^{n,k} + a)^-\} | \mathcal{F}_t].$$

But $Y^n \geq Y^{n,k}$ and once again we have $E[\sup_{s \leq T} |n(-Y_s^{n,k} + a)^-|^2] \leq C_\psi$ since $-Y^{n,k}$ is an approximation scheme for a BSDE reflected in $-a$ and D respectively. Therefore we deduce that for any n, k we have

$$E[\sup_{t \leq T} |Y_t^{n,k} - Y_t^n|^2] \leq C k^{-2}.$$

This inequality combined with (15) complete the proof $\diamond$

Remark 3 *In the case when ψ is bounded, the estimate (14) is valid not only in expectation but also P-almost surely. On the other hand, we do not need to have X a diffusion but just to require that $\sup_{t \leq T} |\psi(t, X_t)|$ is a square integrable random variable* $\diamond$

Let us now focus on some numerical aspects of our problem, namely how to simulate the process Y and therefore the optimal strategy? Recall once again that here the process X is the diffusion of (13). Now for $n, k \geq 0$ let $Y^{n,k}$ be the process defined above. Under smoothness assumptions on b and σ, it is well known that there exists a deterministic function $u^{n,k}(t, x)$ such that for any $t \leq T$, $Y_t^{n,k} = u^{n,k}(t, X_t)$. In addition the function $u^{n,k}$ is a solution for the following PDE:

$$\begin{cases} \frac{\partial u^{n,k}}{\partial t}(t,x) + L_t u^{n,k}(t,x) + (\psi_1(t,x) - \psi_2(t,x)) - n(u^{n,k}(t,x) - a)^+ + k(u^{n,k}(t,x) + D)^- = 0 \\ u(T,x) = 0. \end{cases} \tag{16}$$

In the case when the dimension k of X is small (≤ 3), the solution $u^{n,k}$ of (16) can be simulated in using e.g. finite difference or element schemes. On the other hand, since X can be simulated, in using e.g. the Euler scheme, then we can simulate $Y^{n,k}$ which is a good approximation of Y through (14) when n, k are large.

Now if the dimension of X is greater than 4, it is well known that we have not a workable algorithm which allows the simulation of $u^{n,k}$. However, in recent years there have been many attempts in order to by-pass that obstacle and to provide approximation schemes for either Y^n or $Y^{n,k}$. Actually, among others, one can quote the papers by Bally-Pagès [BP] on the one hand and Bouchard-Touzi [BT] on the other hand. In [BP], Bally-Pagès use the quantization method while in [BT] the approach is linked to Malliavin calculus for the approximation of conditional expectation.

Now, as pointed out above, since we can simulate Y we obtain also simulations for the optimal strategy $(\tau_n^*)_{n \geq 1}$.

On the ground of those considerations, in Appendix are some simulations of Y obtained with X a geometric Brownian motion, *i.e.*, $dX_t = \alpha X_t dt + \sigma X_t dB_t,\ t \leq 1; X_0 = 1$ $\diamond$

Let us now deal with some particular cases. Assume we have $\psi_1 \geq \psi_2$. Therefore we have also $Y \geq 0$. Indeed let $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-)$ be the solution of the following BSDE (which exists and is unique see e.g. [HL] or [CK]):

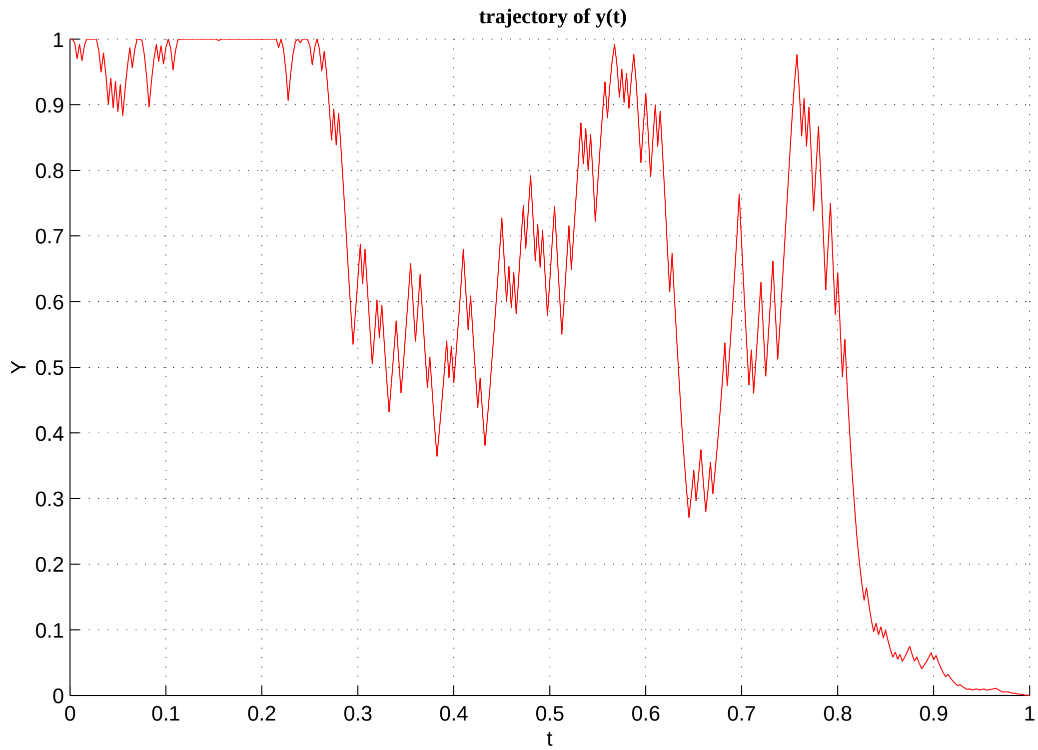
$$\begin{cases} -d\tilde{Y}_t = d\tilde{K}_t^+ - d\tilde{K}_t^- - \tilde{Z}_t dB_t, \ t \leq T \ ; \ \tilde{Y}_T = 0 \\ -D \leq \tilde{Y}_t \leq a \\ (\tilde{Y}_t + D)d\tilde{K}_t^+ = (a - \tilde{Y}_t)d\tilde{K}_t^- = 0. \end{cases}$$

Now the comparison theorem of solutions of BSDEs with two reflecting barriers (see e.g. [HH]) implies that $Y \geq \tilde{Y}$. But uniqueness implies also that $(\tilde{Y}, \tilde{Z}, \tilde{K}^+, \tilde{K}^-) \equiv (0, 0, 0, 0)$ therefore $Y \geq 0$. It implies that Y never reaches $-D$ and the production should be kept open all time up to T. In that case the optimal profit is $E[\int_0^T \psi_1(s, X_s)ds]$ $\diamond$

As a final remark let us point out that, in the resolution of the problem, there is no particular difficulty to replace D (resp. a) by a cost which depends on the state of the system, *i.e.*, $g(X_{\tau_{2n-1}})$ (resp. $\bar{g}(X_{\tau_{2n}})$) where g (resp. $\bar{g}$) is a positive function bonded by below by a positive constant γ. On the other hand we can deal with the situation where there are $k \geq 3$ modes for the power station. In that case, instead of $(Y^1$,$Y^2)$, we should construct Y^i, $i = 1, k$, processes which correspond to the optimal expected profit from t if at that time the station is in its $i - th$ mode $\diamond$

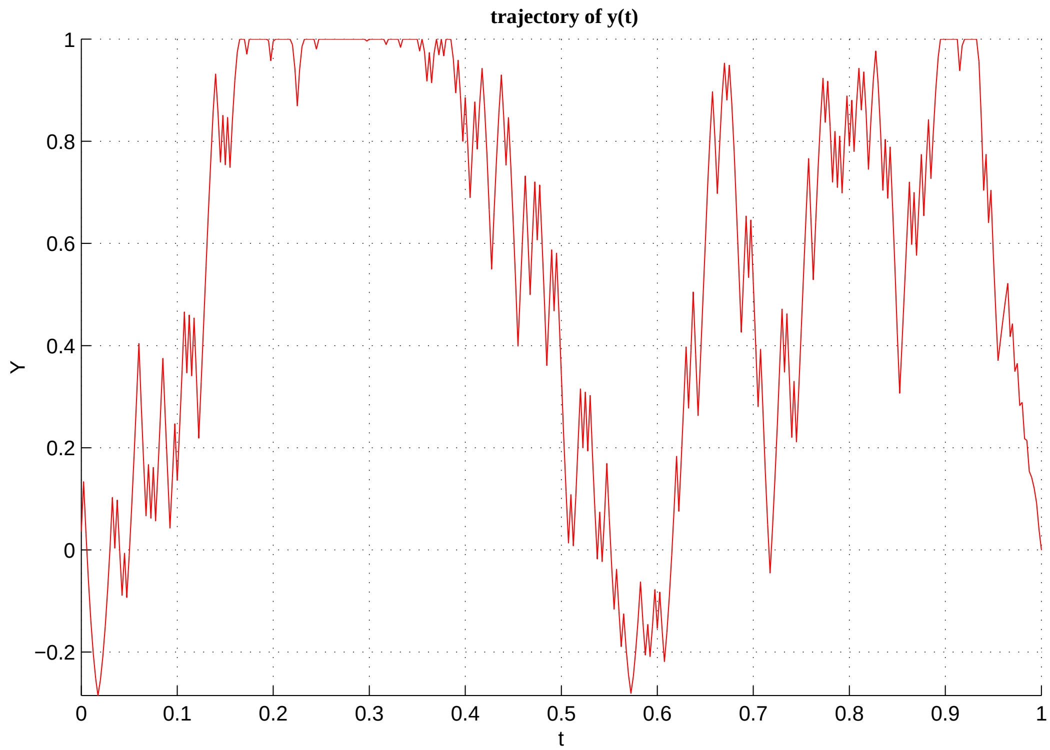
Appendix

As it is pointed out previously $Fig.1$ shows that when $\psi_1 \geq \psi_2$ then one should keep the production in its working mode. As for $Fig.2$, it shows that production must be stopped when Y reaches $-D$. Finally, $Fig.3$ and $Fig.4$ are related to the case where the sign of $\psi_1 - \psi_2$ is whatever.

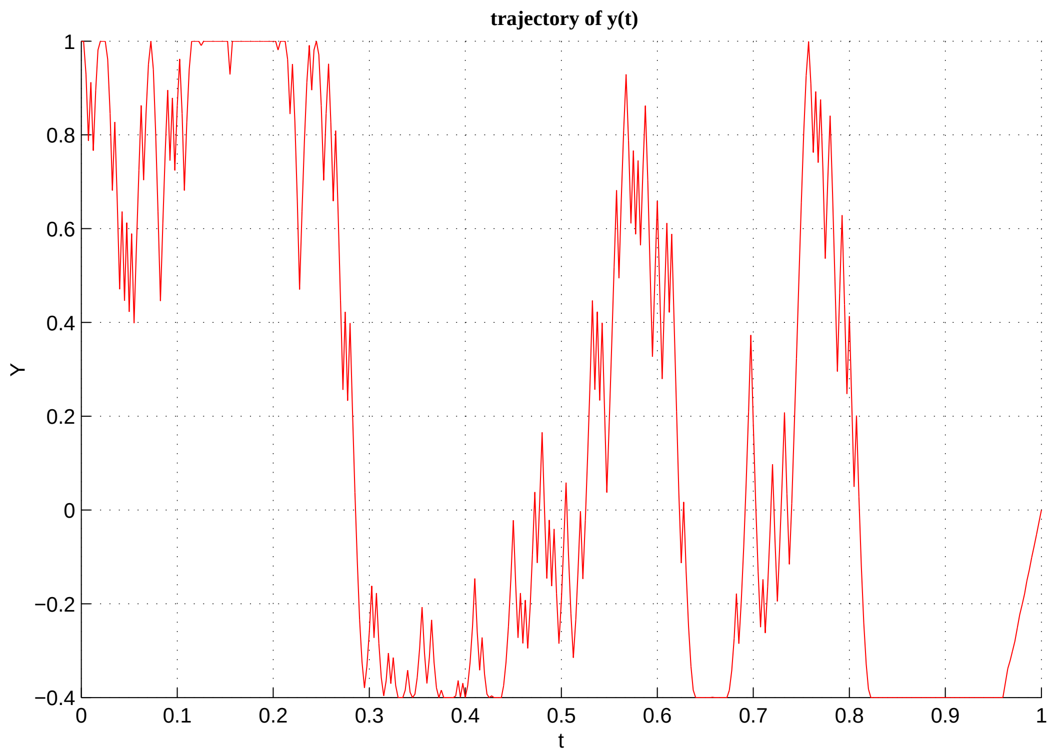


$Fig.1$: $X_0 = 1$,$\alpha = 1$, σ=-3, a=1, D=0.5, $(\psi_1 - \psi_2)$(x)=0.1x.

$Fig.2$: $X_0 = 1$,α=1, σ=1, a=1, D=2, $(\psi_1 - \psi_2)(x) = -0.01x - 0.5$.



$Fig.3$: $X_0 = 1$,α=1, σ=2, a=1, D=0.3, $(\psi_1 - \psi_2)(x) = 0.1x - 6$.



$Fig.4$: $X_0 = 1$,α=0.5, σ=-3, a=1, D=0.4, $(\psi_1 - \psi_2)(x) = 0.7x - 11$.

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