Progressive enlargement of filtrations with initial times*

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Abstract

The preservation of the semi-martingale property in progressive enlargement of filtrations has been studied by many authors. Most of them focus on progressive enlargement with a honest time, allowing for semi-martingale invariance and simple decomposition formulas. However, times allowing for semi-martingale invariance in initial enlargements preserve as well this property in progressive enlargements. This paper is devoted to the related canonical decomposition of the martingales in the reference filtration as semi-martingales in the enlarged filtration. Examples are given in credit risk modelling.

1 Introduction

The stability of the class of semi-martingales with respect to filtration shrinking or enlargement of filtration has been a field of research during the last decades. In the case $\mathbb{F} \subset \mathbb{G}$, it is known that any $\mathbb{G}$-semi-martingale which is $\mathbb{F}$-adapted is an $\mathbb{F}$-semi-martingale (Stricker's theorem [24]). This situation is what is called filtration shrinking. See also the recent work of P. Protter [22] for the specific case of local-martingales.
The situation of an enlargement of filtration is more complex, and the stability of the semi-martingale property does not always hold. In this framework, for $\mathbb{F} \subset \mathbb{G}$, it is usual to say that the hypothesis $(\mathcal{H}')$ holds between $\mathbb{F}$ and $\mathbb{G}$ if any $\mathbb{F}$-semi-martingale is a $\mathbb{G}$-semi-martingale.

In what follows, we denote by $\mathcal{M}(\mathbb{F})$ (resp. $\mathcal{M}(\mathbb{G})$) the set of $\mathbb{F}$-martingales (resp. $\mathbb{G}$-martingales). We start by recalling some well-known facts about the initial and progressive enlargements of filtrations.

• The initial enlargement of a reference filtration $\mathbb{F}$ by a random time τ (a non-negative random variable) is the filtration $\mathbb{G}^{(\tau)}$ defined by $\mathcal{G}_t^{(\tau)} = \cap_{\epsilon>0} \left(\mathcal{F}_{t+\epsilon} \vee \sigma(\tau)\right)$. In this framework, no general theorem guarantees that the hypothesis $(\mathcal{H}')$ holds. However, it is well known that if the conditional laws of the random time τ (with respect to the reference filtration) are absolutely continuous with respect to a probability measure η, then the hypothesis $(\mathcal{H}')$ holds (then, one can reduce attention to the case where η is the law of τ). This result is known as Jacod's theorem (see for example the paper of J. Jacod [8] or Chapter VI in the book of P. Protter [21]). Random times satisfying this property will be referred to as *initial times* in the sequel of this article (see Section 2).

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• The progressive enlargement of a reference filtration $\mathbb{F}$ by a random time τ is the smallest right-continuous filtration that contains $\mathbb{F}$ and makes τ a stopping time. This filtration $\mathbb{G}$ is defined by $\mathcal{G}_t = \cap_{\epsilon>0}\mathcal{G}^0_{t+\epsilon}$ where $\mathcal{G}^0_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$. Remark that, for fixed t, the σ-algebra $\mathcal{G}^0_t$ is generated by the set of random variables of the form $F_t h(t\wedge\tau)$, with h a Borel function, and F_t an $\mathcal{F}_t$-measurable random variable. It follows that the filtration $\mathbb{G}$ coincides with $\mathbb{F}$ before τ and with $\mathbb{G}^{(\tau)}$ after.

The study of the hypothesis $(\mathcal{H}')$ in progressive enlargements can be split in two different time intervals, before and after the occurrence of τ :

- On the set $\{t < \tau\}$, the hypothesis $(\mathcal{H}')$ always holds: precisely for any $\mathbb{F}$-martingale X, the stopped process X^τ is a $\mathbb{G}$-semi-martingale (a nice and short argument of M. Yor may be found in [26]).

  Moreover, the canonical decomposition of the $\mathbb{G}$-semi-martingale X^τ writes

  $$X^\tau_t = \mu_t + \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_s + dB_s}{G_{s-}}, \text{ with } \mu \in \mathcal{M}(\mathbb{G})$$

  where $G_t := \mathbb{P}(\tau > t|\mathcal{F}_t)$ is the conditional survival process[1] (also called the Azéma $\mathbb{F}$-super-martingale), and M denotes the martingale part of the Doob-Meyer decomposition[2] of the super-martingale G. The process B is the $\mathbb{F}$-predictable dual projection of the $\mathbb{G}$ -adapted process $(\varepsilon_u)_u$=$(H_u \Delta X_\tau)_u$, where $H_t = \mathbb{1}_{\tau\leq t}$. A proof of this decomposition can be found in the books of T. Jeulin [11] and [15] or in the papers of T. Jeulin and M. Yor [13], [12] or [14].

  If the random time τ avoids the $\mathbb{F}$-stopping times, that is if $\mathbb{P}(\tau = T) = 0$ for any $\mathbb{F}$-stopping time T (this assumption is often referred to as $(\mathcal{A})$), then $\Delta X_\tau = 0$ and $B = 0$. Under this condition, a proof of the above decomposition can be found in Chapter VI of the book of P. Protter [21].

- For the general case of non-stopped semi-martingales, semi-martingale invariance deeply depends on the properties of the random time. A natural extension of the proof leading to the last result - based on the structure of the $\mathbb{G}$-predictable σ-field $\mathcal{P}(\mathbb{G})$ and its links with $\mathcal{P}(\mathbb{F})$ - lies on the study of times allowing any $\mathbb{G}$-predictable process K to be written as $K = K^1 1_{[0,\tau]} + K^2 1_{]\tau,\infty[}$, where $\left(K^1, K^2\right)$ are $\mathbb{F}$-predictable processes (see M.T. Barlow [1], M. Yor [26], T. Jeulin [11] or C. Dellacherie and P-A. Meyer [4]). Such times are called honest times[3]: precisely the time τ is said "honest"if, for any $t > 0$, it is equal to an $\mathcal{F}_t$-measurable random variable on $\{\tau < t\}$.

  If the time is honest, the sequence of σ-algebras

  $$\mathcal{G}_t = \{A \in \mathcal{F}, \forall t, \exists A_t, B_t \in \mathcal{F}_t, A = (A_t \cap \{\tau > t\}) \cup (B_t \cap \{\tau \leq t\})\}$$

  is increasing (by honesty of the time) and forms a filtration. In that framework, the hypothesis $(\mathcal{H}')$ holds, and if $X \in \mathcal{M}(\mathbb{F})$:

  $$X_t = \mu_t + \int_0^{t\wedge\tau} \frac{d\langle X, M\rangle_s + dB_s}{G_{s-}} - 1_{\{\tau\leq t\}} \int_\tau^t \frac{d\langle X, M\rangle_s + dB_s}{1 - G_{s-}}, \text{ with } \mu \in \mathcal{M}(\mathbb{G}).$$

[1] It is well known (see Jeulin [11]) that G does not reach zero before τ. Indeed, if $T := \inf\{t > 0, G_t = 0$ or $G_{t-} = 0\}$, G_t is null after T (a non-negative super-martingale sticks at zero) and $\mathbb{P}(T < \tau) = \mathbb{E}(G_T 1_{\{T<\infty\}}) = 0$

[2] Remark that $M \in BMO$ (cf. discussions in the next section). The space BMO is defined as the subspace of $\mathcal{H}^2$ composed of the local martingales N such that $\mathbb{E}((N_\infty - N_{T-})^2|\mathcal{F}_T) \leq k$ for any $\mathbb{F}$-stopping time T. $\|N\|^2_{BMO}$ denotes the smallest k if it exists (i.e., $N \in BMO$) or ∞.

[3] Honest times coincide with the end of $\mathbb{F}$-predictable set in $[0,\infty]\times\Omega$, and finite honest times coincide with the end of $\mathbb{F}$-optional sets (non-finite honest times may be not the end of an $\mathbb{F}$-optional set, even if $\mathcal{F}_\infty$-measurable, see [11]).

The theory of progressive enlargement with an honest time presents nonetheless some major drawbacks in some application fields, such as credit derivatives modelling. Within the approach based on the enlargement of a reference filtration by the progressive knowledge of a credit event (see R. Elliott, M. Jeanblanc and M. Yor in [6] or M. Jeanblanc and Y. Le Cam in [10]), the hypothesis $(\mathcal{H}')$ is fundamental. Indeed, the absence of arbitrage in finance is closely linked to the property of semi-martingale satisfied by the assets (see F. Delbaen and W. Schachermayer in [3]), and it is necessary that the assets of the reference market (i.e., $\mathbb{F}$-semi-martingales) remain $\mathbb{G}$-semi-martingales.

The most important argument which makes impossible the application of the "honest" theory in credit modelling is the belonging of the honest time to $\mathcal{F}_\infty$: unfortunately the credit event (a change in the ranking of an obligation or an unpaid coupon for example) can neither be directly read on the market price of the asset of the reference filtration nor on its future and modelling it through an $\mathcal{F}_\infty$-measurable random variable is not consistent with reality. The widespread model of Cox construction of the credit event - in which $\tau = \inf\{t : \Lambda_t \geq \Theta\}$ with Λ an $\mathbb{F}$-adapted increasing process and Θ a random variable independent of $\mathcal{F}_\infty$ - strengthens this point (see Section 5).

The main goal in this paper is to present the progressive enlargement of a reference filtration $\mathbb{F}$ with an "initial" time τ, focusing on the canonical decomposition of the semi-martingale in the new filtration. The paper is organized as follows. The first section presents the definition of initial times and the features that make them a natural tool for the progressive enlargement of filtration in many applications. In the second section, we prove that if X is an $\mathbb{F}$-martingale, then X is a $\mathbb{G}$-semi-martingale with canonical decomposition

$$X_t = \mu_t + \int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_u + dB_u}{G_{u-}} - \int_{t\wedge\tau}^{t} \left.\frac{d\langle X, \alpha^\theta\rangle_u}{\alpha^\theta_{u-}}\right|_{\theta=\tau}, \text{ with } \mu \in \mathcal{M}(\mathbb{G}), \tag{1}$$

where α^θ is the density of the conditional laws of τ with respect to the law of τ, defined below by (2). We generalize our study to the case of enlargement with multiple times in Section 4. Section 5 gives examples of initial times that can be used in credit modelling, linked to Cox construction. Because of the important applications in this field, the random time τ will be called default time or credit event.

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F}$ satisfies the usual conditions ($\mathcal{F}_0$ contains the null sets of $\mathbb{P}$ and $\mathbb{F}$ is right continuous: $\forall t \geq 0, \mathcal{F}_t = \mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s$). We do not assume that $\mathcal{F} = \mathcal{F}_\infty$. Recall that under this condition,

(i) every $\mathbb{F}$-martingale admits a càdlàg version, and

(ii) the $\mathbb{F}$-predictable projection of any martingale $(M_t, t \geq 0)$ is $(M_{t-}, t \geq 0)$ [4].

We denote by $\mathcal{P}(\mathbb{F})$ the predictable σ-algebra on $\mathbb{R}^+ \times \Omega$.

2 Initial Times

As recalled in the introduction, the notion of initial times has been introduced by J. Jacod in [8] who proved that for initial enlargement with "initial" times, the hypothesis $(\mathcal{H}')$ holds. We chose the name *initial time* as a reference to this property.

[4] Recall that the $\mathbb{F}$-predictable projection of a bounded measurable process X (not necessarily $\mathbb{F}$-adapted) is the $\mathbb{F}$ -predictable process X^p that satisfies $X^p_T = \mathbb{E}(X_T|\mathcal{F}_{T-})$ on the set $\{T < \infty\}$, for any $\mathbb{F}$-predictable time T.

It will be useful to introduce the notation $\widehat{\Omega} = \mathbb{R}^+ \times \Omega$ and $\widehat{\mathcal{F}}_t$ for the right-continuous completion of $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$. The $\widehat{\mathbb{F}}$-optional σ-field $\mathcal{O}(\widehat{\mathbb{F}})$ (resp. the $\widehat{\mathbb{F}}$-predictable σ-field $\mathcal{P}(\widehat{\mathbb{F}})$) will therefore refer to the σ-field on $\mathbb{R}^+ \times \widehat{\Omega}$ generated by the càd (resp. càg) $\widehat{\mathbb{F}}$-adapted processes. Recall that[5]:

$$\mathcal{P}(\widehat{\mathbb{F}}) = \mathcal{B}\left(\mathbb{R}^+\right) \otimes \mathcal{P}\left(\mathbb{F}\right).$$

For any positive random time τ, and for any t, we write $Q_t(\omega, dT)$ the regular $\mathcal{F}_t$-conditional distribution of τ (that exists since the random variable τ is real-valued), and

$$G_t^T(\omega) := \mathbb{P}(\tau > T|\mathcal{F}_t)(\omega) = Q_t(\omega,]T, \infty[).$$

Definition 2.1 (Initial times) *A positive random time τ is called an initial time if there exists a measure η on $\mathcal{B}(\mathbb{R}^+)$ such that a.s. for each $t \geq 0$, $Q_t(\omega, dT) \ll \eta(dT)$.*

The density processes. By Doob's theorem of disintegration of measures, the definition of an initial time is equivalent to the existence of a family of positive $\mathbb{F}$-adapted processes $(\alpha_t^u, t \geq 0)$, such that:

- for any $t \geq 0$, one has $\alpha_t^u(\omega)\,\eta(du) = Q_t(\omega, du)$, i.e.,
$$G_t^T = \int_T^\infty \alpha_t^u\, \eta(du), \tag{2}$$
- for any $t \geq 0$ the mapping $(u, \omega) \mapsto \alpha_t^u(\omega)$ is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$ measurable.

Times satisfying (2) have been introduced to model the credit events by Y. Jiao [16]. In [5], the authors have studied these time from a "shrinkage" point of view: characterize the $\mathbb{G}$-martingales in terms of $\mathbb{F}$-martingales.

Existence of a "good version" of the density. The existence of a good version of the *processes* $(\alpha_t^u, t \geq 0)$ derives from the analysis developed in C. Striker and M. Yor in [25], and is carried out in [8]. These authors establish the existence of an $\widehat{\mathbb{F}}$-optional map $(u, \omega, t) \longmapsto \alpha_t^u(\omega)$ such that:

- it is càdlàg in t (i.e., for almost any (u, ω), $t \longmapsto \alpha_t^u(\omega)$ is c àdlàg),
- for any $u \geq 0$, α^u is an $\mathbb{F}$-martingale [6].

Remark 2.1 *We will consider this version of the density function from now on. From the martingale property for each $u \geq 0$, α^u "sticks to zero": if $T^u = \inf\{t \geq 0, \alpha_{t-}^u = 0$ or $\alpha_t^u = 0\}$, $\alpha^u > 0$ and $\alpha_-^u > 0$ on $[0, T^u[$ and $\alpha^u = 0$ on $[T^u, \infty[$.*

Applying the second point of the following Lemma 3.1 to the $\widehat{\mathbb{F}}$-predictable process $K_t^u = 1_{\{T^u < t\}}$,

$$\mathbb{P}\left(T^\tau < t\right) = \mathbb{E}\left(K_t^\tau\right) = \mathbb{E}\left(\int_0^\infty \alpha_{t-}^u 1_{\{T^u < t\}}\, \eta(du)\right) = 0.$$

It follows that the random variable T^τ is almost surely infinite.

[5] Remark that the optional σ-field generated by the càd augmentation of a filtration is in general strictly bigger than the optional σ-field generated by the filtration. In the opposite, the predictable σ-fields are the same. As $\widehat{\mathcal{F}}_t \supset \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$, it follows that $\mathcal{O}(\widehat{\mathbb{F}}) \supset \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{O}(\mathbb{F})$ and $\mathcal{P}(\widehat{\mathbb{F}}) = \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{P}(\mathbb{F})$.

[6] For any $t \in \mathbb{Q}^+$, there exists $\overline{\alpha}_t^u(\omega)$, $\mathcal{B}\ \mathbb{R}^+\ \otimes \mathcal{F}_t$-measurable, such that $\overline{\alpha}_t^u(\omega)\,\eta(du) = Q_t(\omega, du)$. For $s \leq t$, $s \in \mathbb{Q}^+$, there exists $\overline{\alpha}_s^u(\omega)$ with the same properties. By definition, $\{(u, \omega) : \overline{\alpha}_s^u \neq \mathbb{E}\left(\overline{\alpha}_t^u | F_s\right)\}$ has $\eta \otimes P$-measure 0. By Fubini, $\forall u \geq 0$, $\left(\overline{\alpha}_r^u\right)_{r \in \mathbb{Q}^+}$ is an $\mathbb{F}$-martingale. The construction of the càdlàg version α^u is classical and derives from the martingale property of $\overline{\alpha}^u$ and the right-continuity of the filtration $\mathbb{F}$.

Note that, if $f(u) := \mathbb{E}(\alpha_t^u) = \alpha_0^u$, then $\mathbb{P}(\tau \in du) = f(u)\eta(du)$. We shall consider, without loss of generality, the case where $f(u) = 1$.

Doob-Meyer decomposition of the survival process. In such a framework, we can write the survival process $G_t := G_t^t$ as

$$G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty \alpha_t^u \,\eta(du) = \int_0^\infty \alpha_{u\wedge t}^u \,\eta(du) - \int_0^t \alpha_u^u \,\eta(du) := M_t - \widetilde{A}_t.$$

We denote by A the $\mathbb{F}$-predictable increasing process: $A_t = \int_0^t \alpha_{u-}^u \eta(du)$. We have the

Proposition 2.1 *M is an $\mathbb{F}$-martingale and A is the compensator of G. If η has no atoms, $A_t = \widetilde{A}_t$.*

Proof. Let $0 \le t \le T$. From the positivity of the martingale densities and Fubini theorem:

$$\mathbb{E}\left(M_T|\mathcal{F}_t\right) = \mathbb{E}\left(\int_0^\infty \alpha_{u\wedge T}^u \eta(du) \middle| \mathcal{F}_t\right) = \int_0^\infty \mathbb{E}\left(\alpha_{u\wedge T}^u \middle| \mathcal{F}_t\right)\eta(du) = \int_0^\infty \alpha_{u\wedge t}^u \eta(du) = M_t,$$

where the third equality comes from the fact that for any $u \ge 0$, $\mathbb{E}\left(\alpha_{u\wedge T}^u \middle| \mathcal{F}_t\right) = \alpha_{u\wedge t}^u$ $\mathbb{P}$-a.s. It follows that M is an $\mathbb{F}$-martingale.

The process $\widetilde{A}$ is $\mathbb{F}$-adapted and increasing (from the positivity of the densities). If η does not have any atoms, $\widetilde{A}$ is continuous hence $\mathbb{F}$-predictable.

If η has atoms, $\widetilde{A}$ may be not predictable, for example if the process $u \longmapsto \alpha_u^u$ jumps at an atom t of η and if the size of the jump is not $\mathcal{F}_{t-}$-measurable[7]. In such a case, it is necessary to compensate the finite variation process $\widetilde{A}$. Since η is deterministic, for proving that A is the compensator of $\widetilde{A}$, it is enough to check that for any non-negative $\mathbb{F}$-predictable process K, one has $\mathbb{E}(K.\widetilde{A}) = \mathbb{E}(K.A)$. From the positivity of the processes, we have:

$$\mathbb{E}\left(K.\widetilde{A}\right) = \mathbb{E}\left(\int_0^\infty K_u \alpha_u^u \,\eta(du)\right) = \int_0^\infty \mathbb{E}\left(K_u \alpha_u^u\right)\,\eta(du).$$

For any $u \ge 0$, since $K \in \mathcal{P}(\mathbb{F})$, one has $K_u \in \mathcal{F}_{u-}$, hence

$$\mathbb{E}\left(K_u \alpha_u^u\right) = \mathbb{E}\left(K_u \mathbb{E}\left(\alpha_u^u|\mathcal{F}_{u-}\right)\right) = \mathbb{E}\left(K_u \alpha_{u-}^u\right),$$

(where we have used that, α^u being a martingale, $\mathbb{E}\left(\alpha_t^u|\mathcal{F}_{t-}\right) = \alpha_{t-}^u$ for any $t \ge 0$). ☐

Remark that A is also the $\mathbb{F}$-predictable dual projection of $H_t = 1_{\tau \le t}$.

Quadratic variations. Once the choice of the good version of the density processes has been made, it is possible to study the measurability of the quadratic covariation of α with some $\mathbb{F}$-martingales.

Let X be a local martingale. For any u, the covariance process $([\alpha^u, X]_t, t \ge 0)$ is càdlàg. Let $t \ge 0$ and $\mathcal{T}_n$ be a partition of $[0,t]$. For any $u \ge 0, n \ge 0$, the Riemann sum

$$S^n(u,\omega) = \sum_{\mathcal{T}_n} \left(\alpha_{t_{i+1}}^u(\omega) - \alpha_{t_i}^u(\omega)\right)\left(X_{t_{i+1}}(\omega) - X_{t_i}(\omega)\right)$$

[7] Recall that a process is $\mathbb{F}$-predictable if and only if it jumps at $\mathbb{F}$-predictable times and its jump at any $\mathbb{F}$-stopping time T belongs to $\mathcal{F}_{T-}$.

is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$ -measurable. As for any $u \geq 0$ $S^n(u,.)$ converges in probability when the mesh of the partition goes to zero (existence of the bracket), there exists a version $[\alpha^u, X]_t(\omega)$ which is $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}_t$-measurable (see the first proposition of [25] for a simple example of explicit construction). It follows that $(u,t,\omega) \mapsto [\alpha^u, X]_t(\omega)$ is $\widehat{\mathbb{F}}$-optional.

This $\widehat{\mathbb{F}}$-optional càdlàg process has paths of finite variation, hence its $\widehat{\mathbb{F}}$-predictable compensator $\langle \alpha^u, X\rangle$ exists as soon as its variation is locally integrable. In such a case, $(u,t,\omega) \mapsto \langle \alpha^u, X\rangle_t(\omega)$ is $\widehat{\mathbb{F}}$-predictable. Different cases may be considered:

- If the local martingale X is *locally in BMO*, for any $u \geq 0$ $\langle \alpha^u, X\rangle$ exists with no condition on α^u, from Fefferman's inequality. Indeed any local martingale α^u is locally in the space[8] $\mathcal{H}^1$, hence there exists a constant k such that

$$\mathbb{E}\int_0^\infty \left| d\left[X^{T_n}, (\alpha^u)^{T_n}\right]_s\right| \leq k \parallel (\alpha^u)^{T_n} \parallel_{\mathcal{H}^1} \|X^{T_n}\|_{BMO},$$

  and $[X, \alpha^u]$ is locally of integrable variation;

- If the local martingale X is *locally bounded*, the semi-martingale $X\alpha^u$ is special[9] for any $u \geq 0$, hence $\langle \alpha^u, X\rangle$ exists with no condition on α^u;

- If the local martingale X is *locally square integrable*, the angle bracket $\langle \alpha^u, X\rangle$ exists if α^u is *locally square integrable*;

- For a local martingale X with *no regularity condition*, the angle bracket $\langle X, \alpha^u\rangle$ exists as soon as for any $u \geq 0$, α^u is locally bounded.

Whereas these properties are quite general and do not depend on X, J. Jacod proved in [8] the following very interesting (and complex) result, central in the analysis:

Proposition 2.2 (Jacod, 1978) *If a local martingale X is given, there exists a subset of $\mathbb{R}^+ : \mathbb{R}^+_X$, satisfying $\eta\left(\mathbb{R}^+_X\right) = 1$ such that for any $u \in \mathbb{R}^+_X$, $\langle \alpha^u, X\rangle$ is defined on $\{\alpha^u_- > 0\}$.*

This fundamental result derives from the following property: On each increasing stochastic interval $[0, T^u_n]$ with $T^u_n = \inf\{t \geq 0, \alpha^u_{t-} \leq 1/n\}$, the stopped process $[\alpha^u, X]^{T^u_n}$ has locally integrable variation and its compensator $\langle \alpha^u, X\rangle^{T^u_n}$ is defined. As $\{t : \alpha^u_{t-} > 0\} = \cup_n [0, T^u_n]$, $\langle \alpha^u, X\rangle$ is defined by embedding. Beware that two very different cases may happen:

1. α^u jumps to zero: In this case the sequence T^u_n becomes constant, equal to T^u (and $\{t : \alpha^u_{t-} > 0\} = \cup_n [0, T^u_n] = [0, T^u]$).

2. α^u reaches zero continuously: In this case the sequence T^u_n increases strictly to T^u (and $\{t : \alpha^u_{t-} > 0\} = \cup_n [0, T^u_n] = [0, T^u[$).

[8] For any integer p, the space $\mathcal{H}^p$ is defined as the set of the local martingales N such that $\|N\|_{\mathcal{H}^p} < \infty$, with $\|N\|^p_{\mathcal{H}^p} = \mathbb{E}[N,N]^{p/2}_\infty$

[9] Indeed, if T_n is a sequence that bounds the local martingale X by a sequence x_n, and if $Y^n = X^{T_n}\alpha^u$,

$$\sup_{s\in[0,t]} |Y^n_s| \leq \sup_{s\in[0,t]} |X^{T_n}_s| \sup_{s\in[0,t]} |\alpha^u_s| \leq x_n \sup_{s\in[0,t]} |\alpha^u_s| \in L^1_{loc}$$

since α^u is a local martingale. It follows that Y^n, hence $X\alpha^u$ is special.

Let α^u be (the good version of) the family of density, with no added assumption on the regularity of the paths. The martingale part of the supermartingale G is in BMO, hence $\langle M, X\rangle$ exists for any X local martingale. As the process A is predictable with paths of finite variation, $[X, A]$ is a local martingale, hence $\langle A, X\rangle = 0$. It follows, from $G = M + A$, that $\langle X, G\rangle$ exists for any local martingale X and $\langle X, G\rangle = \langle X, M\rangle$. When the angle bracket between X and α exists (see discussion above), we have:

$$\begin{aligned}\langle X, G\rangle_t &= \langle X, M\rangle_t = \int_0^\infty \langle X, \alpha^u_{u\wedge .} - \Delta\alpha^u_u 1_{u\leq .}\rangle_t \,\eta(du) \\ &= \int_0^\infty \left(\langle X, \alpha^u\rangle_{u\wedge t} - \Delta \langle X, \alpha^u\rangle_u 1_{u\leq t}\right) \eta(du). \end{aligned} \tag{3}$$

Initial times have the very interesting feature (for credit modelling for example) to allow the existence of a probability under which the reference filtration and the random time are independent. Precisely, as proved in A. Grorud and M. Pontier [7] we have the:

Proposition 2.3 *If τ is an initial time with $\mathbb{E}_{\mathbb{P}}(1/\alpha^\tau_T) < \infty, \forall T$ there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which τ and $\mathcal{F}_\infty$ are independent.*

This result (obtained by choosing, for $t < T$, $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(1/\alpha^\tau_T|\mathcal{G}_t)/\mathbb{E}_{\mathbb{P}}(1/\alpha^\tau_T))$ leads to a straightforward proof of Jacod's theorem when the integrability assumption holds, since the hypothesis $(\mathcal{H}')$ is stable by a change of equivalent probability.

Under $\mathbb{Q}$, immersion property holds, i.e., any $(\mathbb{F}, \mathbb{Q})$-martingale remains a $(\mathbb{G}, \mathbb{Q})$-martingale. There exists - at our knowledge - no such result in a "honest"expansion. Characterizations of immersion itself are also very tractable in the framework of initial times, as it will be seen in the following Corollary 3.1.

3 Invariance of Semi-martingales

Let τ be an initial time. From Jacod's theorem, we know that the hypothesis $(\mathcal{H}')$ holds within the initial expansion $\mathbb{F} \subset \mathbb{G}^{(\tau)}$. If $\mathbb{G}$ denotes the progressive expansion of $\mathbb{F}$ by τ, $\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{(\tau)}$ and Stricker's theorem ensures that the hypothesis $(\mathcal{H}')$ holds between $\mathbb{F}$ and $\mathbb{G}$.

We now present the the $\mathbb{G}$-semi-martingale decomposition of an $\mathbb{F}$-martingale X.

It has been proved in the previous section that the compensator of G writes $A_t = \int_0^t \alpha^u_{u-}\,\eta(du)$, and that it is also the $\mathbb{F}$-predictable dual projection of $H_t = 1_{\tau\leq t}$. We start with a simple lemma that will be central in the following proofs.

Lemma 3.1 *Let $(K^u_t)_{t\geq 0}$ be a measurably indexed family of $\mathbb{F}$-predictable non-negative (or bounded) processes, i.e., such that the map $(\omega, t, u) \to K^u_t(\omega)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable (equivalently, with the notation on the product space, $K \in \mathcal{P}(\widehat{\mathbb{F}})$). Then:*

1. *The $\mathbb{F}$-optional projection of the process $t \mapsto K^\tau_t$ is the process $t \mapsto \int_0^\infty K^u_t \alpha^u_t\,\eta(du)$;*
2. *The $\mathbb{F}$-predictable projection of the process $t \mapsto K^\tau_t$ is the process $t \mapsto \int_0^\infty K^u_t \alpha^u_{t-}\,\eta(du)$.*

Proof. In the proof, we shall use, as a shortcut, $t \mapsto \alpha_t^u \in \mathcal{O}(\widehat{\mathbb{F}})$ for $(u,\omega,t) \mapsto \alpha_t^u(\omega)$ is $\mathcal{O}(\widehat{\mathbb{F}})$-measurable, and similar notation for $\mathcal{P}(\mathbb{F})$.

By the monotone class theorem, it is enough to prove properties 1. and 2. for $K_t^u = k(u)\,K_t$ where K is $\mathbb{F}$-predictable and non-negative and $k \geq 0$ is a non-negative Borel function. As $t \mapsto \alpha_t^u$ belongs to $\mathcal{O}(\widehat{\mathbb{F}})$ (resp. $t \mapsto \alpha_{t-}^u \in \mathcal{P}(\widehat{\mathbb{F}})$), Fubini's theorem implies that $t \mapsto \int_0^\infty k(u)\,\alpha_t^u\,\eta(du) \in \mathcal{O}(\mathbb{F})$ (resp. $t \mapsto \int_0^\infty k(u)\,\alpha_{t-}^u\,\eta(du) \in \mathcal{P}(\mathbb{F})$). Since K is predictable, it follows that

$$t \mapsto \int_0^\infty K_t^u \alpha_t^u\,\eta(du) \in \mathcal{O}(\mathbb{F}) \ \ (\text{resp. } t \mapsto \int_0^\infty K_t^u \alpha_{t-}^u\,\eta(du) \in \mathcal{P}(\mathbb{F})\,).$$

Moreover, $t \mapsto \int_0^\infty k(u)\,\alpha_t^u\,\eta(du)$ is the càd version of an $\mathbb{F}$-martingale

$$\mathbb{P}-a.s., \forall t \geq 0,\ \mathbb{E}\left(k(\tau)|\,\mathcal{F}_t\right) = \int_0^\infty k(u)\,\alpha_t^u\,\eta(du)\,. \tag{4}$$

It follows that:

- For any finite $\mathbb{F}$-stopping time T, we have from (4):

$$\mathbb{E}\left(K_T^\tau|\,\mathcal{F}_T\right) = K_T\mathbb{E}\left(k(\tau)|\,\mathcal{F}_T\right) = K_T \int_0^\infty k(u)\,\alpha_T^u\,\eta(du) = \int_0^\infty K_T^u\alpha_T^u\,\eta(du) \ \ \mathbb{P}-a.s.,$$

hence, the process $(\int_0^\infty K_t^u\alpha_t^u\,\eta(du)\,, t \geq 0)$ is the $\mathbb{F}$-optional projection of K^τ.

- For any finite $\mathbb{F}$-predictable time T, and increasing sequence of stopping times $T_n \uparrow T$, we have from (4):

$$\mathbb{E}\left(k(\tau)|\,\mathcal{F}_{T_n}\right) = \int_0^\infty k(u)\,\alpha_{T_n}^u\,\eta(du) \ \ \mathbb{P}-a.s.,$$

and letting n tend to ∞:

$$\mathbb{E}\left(k(\tau)|\,\mathcal{F}_{T-}\right) = \int_0^\infty k(u)\,\alpha_{T-}^u\,\eta(du) \ \ \mathbb{P}-a.s.$$

and from $K_T \in \mathcal{F}_{T-}$:

$$\begin{aligned}\mathbb{E}\left(K_T^\tau|\,\mathcal{F}_{T-}\right) &= K_T\mathbb{E}\left(k(\tau)|\,\mathcal{F}_{T-}\right) = K_T\int_0^\infty k(u)\,\alpha_{T-}^u\,\eta(du) \\ &= \int_0^\infty K_T^u\alpha_T^u\,\eta(du) \ \ \mathbb{P}-a.s.,\end{aligned}$$

hence, the process $(\int_0^\infty K_t^u\alpha_{t-}^u\,\eta(du)\,, t \geq 0)$ is the $\mathbb{F}$-predictable projection of K^τ. □

Remark 3.1 *The first point remains valid if K is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable. If it is $\mathcal{O}(\widehat{\mathbb{F}})$ measurable, the scheme of the proof does not hold anymore (see the first footnote of the last section).*

Remark 3.2 *Note that, if τ avoids the $\mathbb{F}$-stopping times and if immersion property holds, then, for any $\mathbb{F}$-predictable (bounded) process X*

$$\mathbb{E}\left(X_\tau \mathbb{1}_{\tau>t}|\,\mathcal{F}_t\right) = \int_t^\infty X_u\alpha_t^u\,\eta(du)\,.$$

Indeed, for any bounded $\mathcal{F}_t$-measurable random variable F_t, and any bounded $\mathbb{F}$-predictable process X :

$$\begin{aligned}\mathbb{E}\left(X_\tau \mathbb{1}_{\tau>t} F_t\right) &= \mathbb{E}\left(\int_t^\infty X_u F_t dH_u\right) = \mathbb{E}\left(\int_t^\infty X_u F_t dA_u\right) \\ &= \mathbb{E}\left(\int_t^\infty X_u \alpha_u^u \eta\left(du\right) F_t\right) = \mathbb{E}\left(\int_t^\infty X_u \alpha_t^u \eta\left(du\right) F_t\right),\end{aligned}$$

where the last equality comes from the characterization of immersion presented below in Corollary 3.1.

Let X be a $\mathbb{F}$-local martingale. We shall prove that there exist

(i) $J \in \mathcal{P}(\mathbb{F})$ with finite variation and

(ii) $K = (K_u(\theta), u \geq 0) \in \mathcal{P}(\widehat{\mathbb{F}})$ such that for any $\theta \geq 0$, the paths of the process $K(\theta)$ have finite variations, that satisfy:

$$Y_t = X_t - \int_0^{t\wedge\tau} dJ_u - \int_{t\wedge\tau}^t dK_u(\theta)\Big|_{\theta=\tau}$$

is a $\mathbb{G}$-martingale.

Remark that both integrals are Stieljes integrals. For any ω and $\theta \geq 0$, the process $t \longmapsto \int_0^t dK_u(\theta)$ is $\mathbb{F}$-predictable with finite variation. It follows that it is $\mathbb{G}$ -predictable and that

$$t \longmapsto \int_0^t dK_u(\theta)\Big|_{\theta=\tau\wedge t}$$

is $\mathbb{G}$-predictable with finite variation[10].

Remark also that if $K, H \in \mathcal{P}(\widehat{\mathbb{F}})$, we have for any $t \geq 0$,

$$\int_0^t H_u\left(\theta\right) dK_u(\theta)\Big|_{\theta=\tau} = \int_0^t H_u\left(\tau\right) dK_u(\tau), \mathbb{P} - a.s.$$

Such a result is clear for $K_u(\theta) = k\left(\theta\right) K_u,\ H_u(\theta) = h\left(\theta\right) H_u$ and derives from the monotone class theorem.

Before stating and proving the main result of this article, we start with two remarks about the implications of such a decomposition:

1. *Before default:*

If such a representation holds, it is necessary that

$$dJ_u = \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}}$$

(Recall that the $\mathbb{F}$-predictable process B refers to the dual $\mathbb{F}$-predictable projection of the $\mathbb{G}$-adapted process $\varepsilon_u = \Delta X_\tau H_u$).

Indeed from $Y \in \mathcal{M}(\mathbb{G})$ and since τ is a $\mathbb{G}$-stopping time, the stopped process Y^τ must be a $\mathbb{G}$ -martingale (by the optional sampling theorem), hence $X_t^\tau - \int_0^{t\wedge\tau} dJ_u \in \mathcal{M}(\mathbb{G})$. The result follows from Jeulin's formula and uniqueness of the canonical decomposition of a special semi-martingale.

[10]If $u \longmapsto k\left(u\right)$ is càg, and if for any ω, $t \longmapsto K_t$ is càg $\mathcal{F}_t$-adapted, it is $\mathcal{G}_t$-adapted and for any ω, $t \longmapsto k\left(\tau \wedge t\right) K_t$ is càg and $\mathcal{G}_t$-adapted.

2. *After default:*

Let s be fixed, F_s be a bounded non-negative $\mathcal{F}_s$ -measurable random variable and h be a bounded non-negative Borel function. Then the random variable $F_s h(\tau) 1_{\tau \leq s}$ is $\mathcal{G}_s$-measurable and if the above decomposition holds, the martingale property of Y implies that

$$\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(Y_t - Y_s\right)\right) = 0,$$

hence that:

$$\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(X_t - X_s\right)\right) = \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_u(\theta)\bigg|_{\theta=\tau}\right). \tag{5}$$

- We can write:

$$\begin{aligned}
\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(X_t - X_s\right)\right) &= \mathbb{E}\left(F_s\left(X_t - X_s\right) \int_0^s h(\theta) \alpha_t^\theta \eta(d\theta)\right) \\
&= \int_0^s h(\theta) \mathbb{E}\left(F_s\left(X_t \alpha_t^\theta - X_s \alpha_s^\theta\right)\right) \eta(d\theta)
\end{aligned}$$

where the first equality comes from a conditioning w.r.t. $\mathcal{F}_t$ and the second from the martingale property of α^θ for any $\theta \geq 0$. For any $\theta \geq 0$, the integration by parts formula implies

$$X_t \alpha_t^\theta - X_s \alpha_s^\theta = \int_s^t X_{u-} d\alpha_u^\theta + \int_s^t \alpha_{u-}^\theta dX_u + \int_s^t d\left[X, \alpha^\theta\right]_u \quad \mathbb{P} - a.s.$$

and, since the two first integrals are $\mathbb{F}$-martingales (X and α^θ are $\mathbb{F}$-martingales):

$$\mathbb{E}\left(F_s\left(X_t \alpha_t^\theta - X_s \alpha_s^\theta\right)\right) = \mathbb{E}\left(F_s \int_s^t d\left[X, \alpha^\theta\right]_u\right).$$

When the angle bracket always exists (see discussion above), we conclude

$$\mathbb{E}\left(F_s \int_s^t d\left[X, \alpha^\theta\right]_u\right) = \mathbb{E}\left(F_s \int_s^t d\left\langle X, \alpha^\theta\right\rangle_u\right),$$

but in the particular case where X is only a martingale (with no added condition), a special care must be brought. As $\alpha^\theta = 0$ on $[T^\theta, +\infty[$, $\left[X, \alpha^\theta\right]_u = \left[X, \alpha^\theta\right]_{u \wedge T^\theta}$, and it follows that:

$$\mathbb{E}\left(F_s \int_s^t d\left[X, \alpha^\theta\right]_u\right) = \mathbb{E}\left(F_s \int_s^t 1_{\{u \leq T^\theta\}} d\left[X, \alpha^\theta\right]_u\right).$$

Depending on the way α^θ reaches zero, the set $\{u \leq T^\theta\}$ may be decomposed in:

$$\{u \leq T^\theta\} = \{\alpha_{u-}^\theta > 0\} \cup \{u = T^\theta, \Delta\alpha_u^\theta = 0\},$$

hence

$$\begin{aligned}
\mathbb{E}\left(F_s \int_s^t 1_{\{u \leq T^\theta\}} d\left[X, \alpha^\theta\right]_u\right) &= \mathbb{E}\left(F_s \int_s^t 1_{\{\alpha_{u-}^\theta > 0\}} d\left[X, \alpha^\theta\right]_u\right) \\
&\quad + \mathbb{E}\left(F_s \int_s^t 1_{\{\Delta\alpha_u^\theta = 0, u = T^\theta\}} d\left[X, \alpha^\theta\right]_u\right).
\end{aligned}$$

From the definition of $\langle X, \alpha^\theta \rangle$ on each $[0, T_n^\theta]$, $\left[X, \alpha^\theta\right]_u^{T_n^\theta} - \left\langle X, \alpha^\theta\right\rangle_u^{T_n^\theta} \in \mathcal{M}\left(\mathbb{F}\right)$, and

$$\mathbb{E}\left(F_s \int_s^t 1_{\{u \leq T_n^\theta\}} d\left[X, \alpha^\theta\right]_u\right) = \mathbb{E}\left(F_s \int_s^t 1_{\{u \leq T_n^\theta\}} d\left\langle X, \alpha^\theta\right\rangle_u\right)$$

and since a.s. $1_{\{u \leq T_n^\theta\}} \uparrow 1_{\{u \leq T^\theta\}} \leq 1$, Lebesgue's theorem implies:

$$\mathbb{E}\left(F_s \int_s^t 1_{\{\alpha_{u-}^\theta > 0\}} d\left[X, \alpha^\theta\right]_u\right) = \mathbb{E}\left(F_s \int_s^t 1_{\{\alpha_{u-}^\theta > 0\}} d\left\langle X, \alpha^\theta\right\rangle_u\right).$$

Moreover,

$$\begin{aligned}\int_s^t 1_{\{\Delta\alpha_u^\theta = 0, u = T^\theta\}} d\left[X, \alpha^\theta\right]_u &= 1_{\{s \leq T^\theta \leq t, \Delta\alpha_{T^\theta}^\theta = 0\}} \Delta\left[X, \alpha^\theta\right]_{T^\theta} \\ &= 1_{\{s \leq T^\theta \leq t, \Delta\alpha_{T^\theta}^\theta = 0\}} \Delta X_{T^\theta} \Delta\alpha_{T^\theta}^\theta = 0,\end{aligned}$$

hence:

$$\mathbb{E}\left(F_s \int_s^t d\left[X, \alpha^\theta\right]_u\right) = \mathbb{E}\left(F_s \int_s^t 1_{\{\alpha_{u-}^\theta > 0\}} d\left\langle X, \alpha^\theta\right\rangle_u\right).$$

It follows[11] that (and the indicator function may be removed when the bracket always exists):

$$\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s}\left(X_t - X_s\right)\right) = \int_0^s h(\theta) \mathbb{E}\left(F_s \int_s^t 1_{\{\alpha_{v-}^\theta > 0\}} d\left\langle X, \alpha^\theta\right\rangle_v\right) \eta(d\theta). \tag{6}$$

- For the right hand member:

$$\begin{aligned}\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_v(\tau)\right) &= \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} K_t(\tau)\right) - \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} K_s(\tau)\right) \\ &= \mathbb{E}\left(F_s \int_0^s h(\theta)\left(K_t(\theta)\alpha_t^\theta - K_s(\theta)\alpha_s^\theta\right) \eta(d\theta)\right)\end{aligned}$$

by an application of Lemma 3.1 to the $\mathbb{F}$-predictable processes indexed by $u : t \mapsto h(u) 1_{u \leq s} K_t(u)$ $(s \leq t)$ and $t \mapsto h(u) 1_{u \leq t} K_t(u)$ (we use that F_s is $\mathcal{F}_t$ -measurable). For any $\theta \geq 0$, using integration by parts formula,

$$K_t(\theta)\alpha_t^\theta - K_s(\theta)\alpha_s^\theta - \int_s^t \alpha_{u-}^\theta dK_u^\theta \in \mathcal{M}\left(\mathbb{F}\right),$$

since α^θ is a martingale and $K(\theta)$ is $\mathbb{F}$-predictable. It follows, from Fubini's theorem, that:

$$\begin{aligned}\mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_v(\tau)\right) &= \int_0^s h(\theta) \mathbb{E}\left(F_s\left(K_t(\theta)\alpha_t^\theta - K_s(\theta)\alpha_s^\theta\right)\right) \eta(d\theta) \\ &= \int_0^s h(\theta) \mathbb{E}\left(F_s \int_s^t \alpha_{u-}^\theta dK_u^\theta\right) \eta(d\theta). \end{aligned} \tag{7}$$

[11] The measurability of the function $\theta \longmapsto \mathbb{E}(F_s \int_s^t d\langle X, \alpha^\theta\rangle_u)$ is insured by Fubini's theorem, since for any θ and almost any ω, $(\theta, \omega) \longmapsto F_s(\langle X, \alpha^\theta\rangle_t - \langle X, \alpha^\theta\rangle_s)$ is measurable. Existence of integrals like $\int_0^s \int_s^t d\langle X, \alpha^\theta\rangle_v \, \eta(d\theta)$ is insured by the existence of a measurable version for any ω of $(\theta, v) \mapsto \langle X, \alpha^\theta\rangle_v$. As pointed out by an Associate Editor of the journal, the question of the null sets associated to each θ is more tricky for Ito's integrals (the interested reader may refer to the work of A. Sznitman [23], where the question is addressed for non-finite variation integrals).

By equalization of (6) and (7), we obtain that it is necessary for (5) to hold, to have $\alpha_{u-}^{\theta} dK_u(\theta) = 1_{\{\alpha_{u-}^{\theta}>0\}} d\left\langle X, \alpha^{\theta}\right\rangle_u$:

$$K_t(\theta) = \int_0^t 1_{\{\alpha_{u-}^{\theta}>0\}} \frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}.$$

From Remark (2.1), $T^{\tau} = \infty$ a.s., hence $1_{\{\alpha_{u-}^{\tau}>0\}} = 1$ a.s. and we may remove the indicator in the statement of the theorem:

Theorem 3.1 *If X is an $\mathbb{F}$-local-martingale:*

$$Y_t = X_t - \int_0^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}} - \int_{t\wedge\tau}^t \left.\frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}\right|_{\theta=\tau} \in \mathcal{M}^{loc}(\mathbb{G}). \tag{8}$$

Proof. By localization, we prove the theorem for a martingale. Let s be fixed and consider a $\mathcal{G}_s$-measurable variable of the form $F_s h(\tau\wedge s)$ with F_s a bounded non-negative $\mathcal{F}_s$-measurable random variable and $h : \mathbb{R}^+ \to \mathbb{R}$ a non-negative bounded Borel function. Then, for $t > s$:

$$\mathbb{E}\left(F_s h(\tau\wedge s)\left(Y_t - Y_s\right)\right) = \underbrace{\mathbb{E}\left(F_s h(\tau) 1_{\tau\leq s}\left(Y_t - Y_s\right)\right)}_{a} + \underbrace{\mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(Y_t - Y_s\right)\right)}_{b}$$

and we compute the terms on the right-hand side separately:

- On $\{\tau \leq s\}$, $t\wedge\tau = s\wedge\tau = \tau$, hence

$$1_{\tau\leq s}\left(\int_0^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}} - \int_0^{s\wedge\tau} \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}}\right) = 0,$$

and it follows that

$$a = \mathbb{E}\left(F_s h(\tau) 1_{\tau\leq s}\left(X_t - X_s\right)\right) - \mathbb{E}\left(F_s h(\tau) 1_{\tau\leq s} \int_s^t \left.\frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}\right|_{\theta=\tau}\right)$$

which is equal to zero, as we have seen previously.

- We rewrite the quantity b as

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) \\ &\quad - \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_s^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}}\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_{t\wedge\theta}^t \left.\frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}\right|_{\theta=\tau}\right). \end{aligned}$$

Using Jeulin's formula before default, we have

$$\begin{aligned} \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_s\right)\right) &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_{t\wedge\tau} - X_{s\wedge\tau}\right)\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_s^{t\wedge\tau} \frac{d\left\langle X, G\right\rangle_u + dB_u}{G_{u-}}\right), \end{aligned}$$

and the expression of b follows:

$$\begin{aligned} b &= \mathbb{E}\left(F_s h(s) 1_{s<\tau}\left(X_t - X_{t\wedge\tau}\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau} \int_{t\wedge\theta}^t \left.\frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}\right|_{\theta=\tau}\right) \\ &= \mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t}\left(X_t - X_{\tau}\right)\right) - \mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t} \int_{\theta}^t \left.\frac{d\left\langle X, \alpha^{\theta}\right\rangle_u}{\alpha_{u-}^{\theta}}\right|_{\theta=\tau}\right). \end{aligned}$$

Moreover, we can write the decomposition:

$$\begin{aligned}
\mathbb{E}\left(F_s h(s) 1_{s<\tau\le t} X_\tau\right) &= \mathbb{E}\left(F_s h(s)\int_{v\in]s,t]} X_{v-} dH_v\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\Delta X_\tau\right)\\
&= \mathbb{E}\left(F_s h(s)\int_{v\in]s,t]} X_{v-} dA_v\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\Delta X_\tau\right)\\
&= \mathbb{E}\left(F_s h(s)\int_{v\in]s,t]} X_{v-}\alpha^v_{v-}\,\eta(dv)\right) + \mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\Delta X_\tau\right)
\end{aligned}$$

where the second equality comes from the definition of the predictable dual projection (recall that the process $(X_{v-}, v\ge 0)$ is $\mathbb{F}$ -predictable), and the third from the computation of the Doob-Meyer decomposition of G (see above). This computation is necessary, as emphasized in the remark following Lemma[12] 3.1. It follows that

$$\begin{aligned}
b &= \mathbb{E}\left(F_s h(s) X_t\int_{v\in]s,t]}\alpha^v_t\,\eta(dv)\right) - \mathbb{E}\left(F_s h(s)\int_{v\in]s,t]} X_{v-}\alpha^v_{v-}\,\eta(dv)\right)\\
&\quad -\mathbb{E}\left(F_s h(s) 1_{s<\tau\le t}\Delta X_\tau\right) - \mathbb{E}\left(F_s h(s)\int_{\theta\in]s,t]}\int_{u\in]\theta,t]} 1_{\{\alpha^\theta_{u-}>0\}}\frac{d\left\langle X,\alpha^\theta\right\rangle_u}{\alpha^\theta_{u-}}\,\alpha^\theta_t\,\eta(d\theta)\right)
\end{aligned}$$

where the last member comes from an application of Lemma 3.1 to the family (indexed by θ) of $\mathbb{F}$-predictable processes

$$J^\theta_t = 1_{s<\theta\le t}\int_0^t 1_{\{u>\theta\}} 1_{\{\alpha^\theta_{u-}>0\}}\frac{d\left\langle X,\alpha^\theta\right\rangle_u}{\alpha^\theta_{u-}}.$$

For any fixed θ, we have by Fubini's theorem,

$$\mathbb{E}\left(F_s h(s)\int_{\theta\in]s,t]}\int_{u\in]\theta,t]} 1_{\{\alpha^\theta_{u-}>0\}}\frac{d\left\langle X,\alpha^\theta\right\rangle_u}{\alpha^\theta_{u-}}\,\alpha^\theta_t\,\eta(d\theta)\right) = \int_{\theta\in]s,t]}\mathbb{E}\left(F_s h(s) J^\theta_t\alpha^\theta_t\right)\eta(d\theta)$$

and since as α^θ is a martingale and J^θ_t is predictable: $J^\theta_t\alpha^\theta_t - \int_0^t \alpha^\theta_{u-} dJ^\theta_u \in \mathcal{M}\left(\mathbb{F}\right)$, hence

$$\mathbb{E}\left(F_s h(s)\int_{u\in]\theta,t]} 1_{\{\alpha^\theta_{u-}>0\}}\frac{d\left\langle X,\alpha^\theta\right\rangle_u}{\alpha^\theta_{u-}}\alpha^\theta_t\right) = \mathbb{E}\left(F_s h(s)\int_{u\in]\theta,t]} 1_{\{\alpha^\theta_{u-}>0\}} d\left\langle X,\alpha^\theta\right\rangle_u\right).$$

It follows that

$$\begin{aligned}
b &= \mathbb{E}\left(F_s h(s)\int_{v\in]s,t]}\left(\left(X_t\alpha^v_t - X_v\alpha^v_v\right) - \int_{u\in]v,t]} 1_{\{\alpha^\theta_{v-}>0\}} d\left\langle X,\alpha^v\right\rangle_u\right)\eta(dv)\right)\\
&\quad +\mathbb{E}\left(F_s h(s)\left(\int_{v\in]s,t]}\Delta\left(X\alpha^v\right)_v\,\eta(dv) - 1_{s<\tau\le t}\Delta X_\tau\right)\right)\\
&= \mathbb{E}\left(F_s h(s)\left(\int_{v\in]s,t]}\Delta X_v\alpha^v_v\,\eta(dv) - 1_{s<\tau\le t}\Delta X_\tau\right)\right)
\end{aligned}$$

the first expectation being equal to zero, as seen in the introduction of the theorem.

[12]Indeed $\mathbb{E}\left(X_\tau|\,\mathcal{F}_t\right)\neq\int_0^\infty X_u\alpha^u_t\,\eta\left(du\right).$

•• If τ avoids the $\mathbb{F}$-stopping times (condition $(\mathcal{A})$), the proof is done: indeed the law η is therefore continuous and

$$\mathbb{E}\left(F_s h(s)\left(\int_{v\in]s,t]} \Delta X_v \alpha_v^v \,\eta(dv)\right)\right) = 0,$$

whereas $\mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t} \Delta X_\tau\right) = 0$ (the $\mathbb{F}$-martingale X jumps only at $\mathbb{F}$-stopping times, hence not at τ).

•• Let us consider from now on that condition $(\mathcal{A})$ does not hold. Assume the following result is true (this relation will be proved in the next lemma): for any $\mathbb{F}$-stopping time T,

$$\mathbb{E}\left(1_{\{\tau=T\}}\middle|\, \mathcal{F}_T\right) = \alpha_T^T \,\eta\left(T\right). \tag{9}$$

Let $(T_n)_{n\geq 0}$ be a sequence of $\mathbb{F}$-stopping times that exhausts the jumps of X. We have:

$$\begin{aligned}
\mathbb{E}\left(F_s h(s) 1_{s<\tau\leq t} \Delta X_\tau\right) &= \sum_{n\geq 0} \mathbb{E}\left(F_s h(s) 1_{s<T_n\leq t} \Delta X_{T_n} 1_{\{\tau=T_n\}}\right) \\
&= \sum_{n\geq 0} \mathbb{E}\left(F_s h(s) 1_{s<T_n\leq t} \Delta X_{T_n} \mathbb{E}\left(1_{\{\tau=T_n\}}\middle|\, \mathcal{F}_{T_n}\right)\right) \\
&= \sum_{n\geq 0} \mathbb{E}\left(F_s h(s) 1_{s<T_n\leq t} \Delta X_{T_n} \alpha_{T_n}^{T_n} \,\eta\left(T_n\right)\right) \\
&= \mathbb{E}\left(F_s h(s)\left(\int_{v\in]s,t]} \Delta X_v \alpha_v^v \,\eta(dv)\right)\right)
\end{aligned}$$

where the first and last equalities come from the definition of the times T_n, the second from the fact that $F_s h(s) 1_{s<T_n\leq t} \Delta X_{T_n} \in \mathcal{F}_{T_n}$ and the third from (9). This leads to the conclusion that $\mathbb{E}\left(F_s h(\tau\wedge s)\left(Y_t - Y_s\right)\right) = 0$. Using the monotone class theorem, $\mathbb{E}\left(G_s\left(Y_t - Y_s\right)\right) = 0$ for any $\mathcal{G}_s$ -measurable random variable G_s, hence Y is a $\mathbb{G}$-martingale. □

We have used in the proof the equality (9) that we now prove:

Lemma 3.2 *Let T be a finite $\mathbb{F}$-stopping time,*

$$\mathbb{E}\left(1_{\{\tau=T\}}\middle|\, \mathcal{F}_T\right) = \alpha_T^T \,\eta\left(\{T\}\right) \;\; a.s.$$

Proof. Let us first prove that for any $\varepsilon \geq 0$ and T finite $\mathbb{F}$-stopping time[13]:

$$\mathbb{E}\left(1_{\{\tau\leq T-\varepsilon\}}\middle|\, \mathcal{F}_T\right) = \int_0^{T-\varepsilon} \alpha_T^u \,\eta\left(du\right).$$

Recall that $\mathcal{F}_T = \{A \in \mathcal{F}, A\cap\{T\leq t\} \in \mathcal{F}_t\}$ also writes $\mathcal{F}_T = \{Z_T,\ Z$ bounded $\mathbb{F}$-optional process$\}$. Let Z be a right-continuous bounded $\mathbb{F}$-optional process. We now prove that

$$\mathbb{E}\left(Z_T 1_{\{\tau\leq T-\varepsilon\}}\right) = \mathbb{E}\left(Z_T \int_0^{T-\varepsilon} \alpha_T^u \,\eta\left(du\right)\right).$$

[13] We propose here a direct/constructive proof, but the result may be retrieved as and application of lemma (3.1).

For any $n \geq 0$, consider T_n defined by $T_n = (k+1)/2^n$ if $k/2^n < T \leq (k+1)/2^n$. This defines a sequence of $\mathbb{F}$-stopping times decreasing to T. We have

$$\begin{aligned}
\mathbb{E}\left(Z_{T_n} 1_{\{\tau \leq T_n - \varepsilon\}}\right) &= \sum_k \mathbb{E}\left(Z_{\frac{k+1}{2^n}} 1_{\left\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\right\}} 1_{\left\{\tau \leq \frac{k+1}{2^n} - \varepsilon\right\}}\right) \\
&= \sum_k \mathbb{E}\left(Z_{\frac{k+1}{2^n}} 1_{\left\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\right\}} \mathbb{E}\left(1_{\left\{\tau \leq \frac{k+1}{2^n} - \varepsilon\right\}} \middle| \mathcal{F}_{\frac{k+1}{2^n}}\right)\right) \\
&= \sum_k \mathbb{E}\left(Z_{\frac{k+1}{2^n}} 1_{\left\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\right\}} \int_0^{(k+1)/2^n - \varepsilon} \alpha^u_{\frac{k+1}{2^n}}\, \eta(du)\right) \\
&= \mathbb{E}\left(Z_{T_n} \int_0^{T_n - \varepsilon} \alpha^u_{T_n}\, \eta(du)\right),
\end{aligned}$$

where the third equality comes from the fact that τ is an initial time. Since all the processes involved are right-continuous and bounded, when n tends to infinity Lebesgue and Beppo-Levi theorems imply that

$$\begin{aligned}
\mathbb{E}\left(Z_T 1_{\{\tau \leq T - \varepsilon\}}\right) &= \mathbb{E}\left(Z_T \int_0^{T-\varepsilon} \alpha^u_T\, \eta(du)\right), \text{ hence} \\
\mathbb{E}\left(1_{\{\tau \leq T-\varepsilon\}} \middle| \mathcal{F}_T\right) &= \int_0^{T-\varepsilon} \alpha^u_T\, \eta(du).
\end{aligned}$$

By application to a null ε (the result has been proved for any $\varepsilon \geq 0$) and difference, it follows that

$$\mathbb{E}\left(1_{\{T-\varepsilon < \tau \leq T\}} \middle| \mathcal{F}_T\right) = \int_{T-\varepsilon}^{T} \alpha^u_T\, \eta(du)$$

for any ε, and if ε tends to 0, Lebesgue's theorem (left-hand side) and Beppo-Levi theorem (right-hand side) imply

$$\mathbb{E}\left(1_{\{\tau = T\}} \middle| \mathcal{F}_T\right) = \alpha^T_T\, \eta(T),$$

which concludes the proof of the lemma. □

In particular, note that, if η is non-atomic, then τ avoids $\mathbb{F}$-stopping times (see [5]. We close this section with a corollary, dealing with a simple characterization of immersion in the initial time set-up (that can be useful in credit modelling applications):

Corollary 3.1 *Under the assumption that the initial time τ avoids the $\mathbb{F}$-stopping times, there is equivalence between $\mathbb{F}$ is immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale α^u is constant after u.*

Proof.

• If for any $u \geq 0$, the martingale α^u is constant after u, then, the martingale part M of the Doob-Meyer decomposition of G is constant. Indeed,

$$M_t = \int_0^\infty \alpha^u_{u \wedge t}\, \eta(du) = \int_0^\infty \alpha^u_t\, \eta(du) = \mathbb{P}(\tau > 0 | \mathcal{F}_t) = 1,$$

hence for any $X \in \mathcal{M}(\mathbb{F})$, one has $\int_0^{t \wedge \tau} \frac{d\langle X, M \rangle_u}{G_{u-}} = 0$. Moreover, for any $F_t \in \mathcal{F}_t$ and bounded

Borel function h, setting $Z_t = F_t h(t \wedge \tau)$,

$$\begin{aligned}
\mathbb{E}\left(Z_t \int_{t\wedge\tau}^t \frac{d\left\langle X, \alpha^\theta \right\rangle_u}{\alpha^\theta_{u-}}\Bigg|_{\theta=\tau} \right) &= \mathbb{E}\left(F_t 1_{\tau \le t} h(\tau) \int_\tau^t \frac{d\left\langle X, \alpha^\theta \right\rangle_u}{\alpha^\theta_{u-}}\Bigg|_{\theta=\tau} \right) \\
&= \mathbb{E}\left(F_t \int_0^t h(s) \int_s^t \frac{d\left\langle X, \alpha^s \right\rangle_u}{\alpha^s_{u-}} \alpha^s_t \, \eta(ds) \right) = 0
\end{aligned}$$

since $d\left\langle X, \alpha^s \right\rangle_u = 0$ for $u \ge s$. It follows, from (1) and the fact that, under condition $(\mathcal{A})$, the B part in the decomposition disappears, that $X \in \mathcal{M}(\mathbb{G})$ and $\mathbb{F}$ is immersed in $\mathbb{G}$.

• Assume now that the hypothesis $(\mathcal{H})$ holds. For any $X \in \mathcal{M}(\mathbb{F})$, we have $X \in \mathcal{M}(\mathbb{G})$ hence with the notation of Theorem 3.1, the process $Y - X$ is a $\mathbb{G}$ martingale and is a predictable process with finite variation (see formula (1)), hence equal to zero. It follows that almost surely

$$\int_0^{t\wedge\tau} \frac{d\left\langle X, G \right\rangle_u}{G_{u-}} + \int_{t\wedge\tau}^t \frac{d\left\langle X, \alpha^\theta \right\rangle_u}{\alpha^\theta_{u-}}\Bigg|_{\theta=\tau} = 0,$$

hence each integral is equal to zero almost surely, and in particular the second. It follows that for any $\mathcal{G}_t$-measurable random variable Z_t of the form $Z_t = F_t h(t \wedge \tau)$,

$$\begin{aligned}
0 &= \mathbb{E}\left(Z_t \int_{t\wedge\tau}^t \frac{d\left\langle X, \alpha^\tau \right\rangle_u}{\alpha^\tau_u} \right) = \mathbb{E}\left(F_t 1_{\tau \le t} h(\tau) \int_\tau^t \frac{d\left\langle X, \alpha^\tau \right\rangle_u}{\alpha^\tau_u} \right) \\
&= \mathbb{E}\left(F_t \int_0^t h(s) \alpha^s_t \, \eta(ds) \int_s^t \frac{d\left\langle X, \alpha^s \right\rangle_u}{\alpha^s_u} \right)
\end{aligned}$$

hence $\alpha^s_t \int_s^t \left(d\left\langle X, \alpha^s \right\rangle_u / \alpha^s_u \right) = 0$, which implies $d\left\langle X, \alpha^s \right\rangle_u = 0$ for any $u \ge s$ and any $X \in \mathcal{M}(\mathbb{F})$. It follows that for any $u \ge 0$, the martingale α^u is constant after u. □

As seen in the proof, $M = cst$ if for any $u \ge 0$, the martingale α^u is constant after u. As we shall see in the examples presented in the third part, the assumption that the initial time avoids the $\mathbb{F}$ -stopping time is not too strong in applications such as credit modelling.

The same kind of characterization for immersion under $(\mathcal{A})$ condition is also easy to establish in a "honest"enlargement set up (there is equivalence between G is decreasing and predictable and $\mathbb{F}$ is immersed in $\mathbb{G}$) but the applications of such a result are very restrictive since as proved in [20], if G is a decreasing process, the time τ is a pseudo-stopping time, and it is known that under condition $(\mathcal{A})$ a honest time can not be a pseudo-stopping[14] (see for example [20]).

4 Successive Times

In aeras like credit derivatives modelling, a multi-dimensional version of this result is necessary for the modelling. The generalization of the theorem and its complexity depend deeply on the nature of the assumptions made on the random times (the credit events). First the quantities of interest in most of the situations are the ranked defaults - and not the default times themselves - and second, the expansion of the knowledge after the k^{th} default must be considered with respect to the filtration enlarged with the $k-1$ last credit events, and not to the reference filtration. We note $H^k_t = \mathbb{1}_{\tau^k \le t}$ and $\mathcal{H}^t = \sigma(H^k_s, s \le t)$.

The natural framework is therefore the following[15] . Starting from a vector of n random times $\theta_1, ..., \theta_n$, we define the vector of ranked times $\tau_1, ..., \tau_n$ (τ_1 is the smallest θ_i, etc.). We

[14] We would like to thank A. Nikeghbali for having pointed out this important remark.

[15] In [12], p. 86, Jeulin studies the same problem for honest times.

say such a vector of increasing times is recursively initial w.r.t. $\mathbb{F}$, if for any k the k^{th} time τ_k is an initial time w.r.t. $\mathbb{F}^{k-1} = \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^{k-1}$ (with $\mathbb{F}^0 = \mathbb{F}$), i.e., there exists a family of $\mathbb{F}^{k-1}$-martingales $\alpha^k(s)$ indexed by $\mathbb{R}^+$ (for any $s \geq 0$, $\left(\alpha_t^k(s)\right)_{t\geq 0} \in \mathcal{M}\left(\mathbb{F}^{k-1}\right)$) and a distribution η^k on $\mathbb{R}^+$, such that

$$\mathbb{P}\left(\tau_k > T | \mathcal{F}_t^{k-1}\right) = \int_T^\infty \alpha_t^k(s)\, \eta^k(ds).$$

The corollary follows (where $G_t^i = \mathbb{P}\left(\tau_i > t | \mathcal{F}_t^{i-1}\right)$).

Corollary 4.1 *Let $(\tau_1, ..., \tau_n)$ be a vector of increasing times recursively initial w.r.t $\mathbb{F}$, such that each martingale $\left(\alpha_t^i(u)\right)_{t\geq 0}$ is a square integrable $\mathbb{F}^{i-1}$-martingale for each $u \geq 0$ and $i \leq n$ (resp. in BMO w.r.t $\mathbb{F}^{i-1}$). For any square integrable $\mathbb{F}$ -martingale X (resp. $X \in \mathcal{M}^{loc}(\mathbb{F})$) and any $k \leq n$, the process Y^k defined as*

$$Y_t^k = X_t - \sum_{i=1}^k \int_0^{t\wedge\tau_i} \frac{d\left\langle X, G^i\right\rangle_u + dB_u^i}{G_{u-}^i} - \sum_{i=1}^k \int_{t\wedge\tau_i}^t \left.\frac{d\left\langle X, \alpha^i(\theta)\right\rangle_u}{\alpha_{u-}^i(\theta)}\right|_{\theta=\tau_i} \tag{10}$$

is an $\mathbb{F}^k$-martingale (resp. $Y^k \in \mathcal{M}^{loc}(\mathbb{F}^k)$).

Proof. This result is straightforward to prove by induction. Assume it holds for any vector of size $k-1$ (for $k=1$ it reduces to formula (1)).

If X is a square integrable martingale,

$$Y_t^{k-1} = X_t - \sum_{i=1}^{k-1} \int_0^{t\wedge\tau_i} \frac{d\left\langle X, G^i\right\rangle_u + dB_u^i}{G_{u-}^i} - \sum_{i=1}^{k-1} \int_{t\wedge\tau_i}^t \left.\frac{d\left\langle X, \alpha^i(\theta)\right\rangle_u}{\alpha_{u-}^i(\theta)}\right|_{\theta=\tau_i} \equiv X_t - A_t^k \in \mathcal{M}(\mathbb{F}^{k-1}),$$

with $\mathbb{F}^{k-1} = \mathbb{F} \vee \mathbb{H}^1 \vee ... \vee \mathbb{H}^{k-1}$ generated by the first $k-1$ times. The only thing to prove is the existence of $\left\langle Y^{k-1}, G^i\right\rangle = \left\langle Y^{k-1}, M^i\right\rangle$ and of $\left\langle Y^{k-1}, \alpha^i(\theta)\right\rangle$. The first one is obvious since M^i is BMO and Y_t^{k-1} is a martingale. For the second one, remark that $\left[Y^{k-1}, \alpha^i(\theta)\right] = \left[X, \alpha^i(\theta)\right] - \left[A^k, \alpha^i(\theta)\right]$. The process $\left[X, \alpha^i(\theta)\right]$ is integrable since the two semi-martingales are integrable, and $\left[A^k, \alpha^i(\theta)\right]$ is a local martingale (A^k is predictable with finite variation and $\alpha^i(\theta)$ is a martingale) hence locally integrable. It follows that $\left[Y^{k-1}, \alpha^i(\theta)\right]$ is locally integrable and the existence of the sharp bracket. The result follows by application of the proof of the theorem for the expansion of $\mathbb{F}^{k-1}$ by the $\mathbb{F}^{k-1}$-initial time τ^k.

If $X \in \mathcal{M}^{loc}(\mathbb{F})$, $Y_t^{k-1} \in \mathcal{M}^{loc}(\mathbb{F}^{k-1})$. The result follows from an application of the theorem to Y_t^{k-1} and τ^k which is initial with respect to filtration $\mathbb{F}^{k-1}$ and in $BMO\left(\mathbb{F}^{k-1}\right)$, since $\left\langle Y^{k-1}, G^i\right\rangle = \left\langle X, G^i\right\rangle$ and $\left\langle Y^{k-1}, \alpha^i(\theta)\right\rangle = \left\langle X, \alpha^i(\theta)\right\rangle$. □

5 Cox-like Examples

A random time constructed through the Cox method (see introduction or the following paragraph) is not $\mathcal{F}_\infty$-measurable, hence can not be honest. The question of the hypothesis $(\mathcal{H}')$ is therefore relevant. We shall prove that this property is in fact always satisfied, since these times are initial times, and that immersion also holds. We present after that two types of constructions derived from this methodology, still entering the initial times framework but without immersion.

First example: Cox construction. D. Lando was the first to propose in [17] the use of the Cox construction in credit modelling, in which a filtration $\mathbb{F}$ is given as well as a non-negative $\mathbb{F}$-adapted process λ and where the default time is defined as:

$$\tau = \inf\{t : \Lambda_t \geq \Theta\}$$

with $\Lambda_t = \int_0^t \lambda_s ds$, Θ is a given r.v. independent of $\mathcal{F}_\infty$ with unit exponential law.

For T and $t \geq 0$, under the hypothesis $\int_0^\infty \lambda_s ds = \infty$,

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \mathbb{P}(\Lambda_T < \Theta|\mathcal{F}_t) = \mathbb{E}(\exp -\Lambda_T|\mathcal{F}_t) = \mathbb{E}(\int_T^\infty \lambda_s \exp(-\Lambda_s)ds|\mathcal{F}_t).$$

Setting $\psi_s = \lambda_s \exp(-\Lambda_s)$ and $\gamma(s,t) = \mathbb{E}\left(\psi_s|\mathcal{F}_t\right)$,

$$G_t^T = \int_T^\infty \mathbb{E}\left(\psi_s|\mathcal{F}_t\right) ds = \int_T^\infty \gamma(s,t)ds,$$

Note that $\gamma(s,t) = \psi_s$ for $s \leq t$. Let $\eta(]0,t]) = \eta([0,t]) = \int_0^t \gamma(s,0)ds = \mathbb{P}(\tau \leq t)$ be the law of the random variable τ. In any cases, we can write

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty \alpha_t^s\, \eta(ds)$$

with $\alpha_t^s = \gamma(s,t)/\gamma(s,0)$ (remark that the non-negativity of λ implies that of ψ and that $\mathbb{E}\left(\psi_s\right) > 0$ for any s). This process is by construction for any s a positive martingale, hence the Cox time τ is an initial time. It follows that the hypothesis $(\mathcal{H}')$ holds, i.e., any $\mathbb{F}$-semi-martingale remains a $\mathbb{G}$-semi-martingale. Moreover, this time avoids the $\mathbb{F}$-stopping times and it is straightforward that for any $t \geq s$, $\alpha_t^s = \alpha_s^s$. It follows, from Proposition 3.1, that the hypothesis $(\mathcal{H})$ also holds. This feature of Cox times have already been pointed out in [16].

The main drawback in Cox construction in its application in credit modelling, is the corollary that the survival process $G = 1-F$ has no martingale part, which restrict the configurations that can be modelled in such a framework. Inspired by the property that in a progressive expansion of filtration, the hypothesis $(\mathcal{H})$ is equivalent to the conditionally independence of $\mathcal{F}_\infty$ and $\mathcal{H}_t$ given $\mathcal{F}_t$ (see P. Brémaud and M. Yor in [2]), M. Jeanblanc and Y. Le Cam proposed in [10] alternative constructions, in which the survival process of the credit event is not decreasing. The two main examples are recall in the sequel and we establish the property of initial times gathered by this type of construction.

Second example: "Cox-like"construction, a finite case.

Here again, we consider a reference filtration $\mathbb{F}$, a non-negative $\mathbb{F}$-adapted process λ and $\Lambda_t = \int_0^t \lambda_s ds$. The given r.v. Θ is independent of $\mathcal{F}_\infty$ with unit exponential law, and V is an $\mathcal{F}_\infty$-measurable non-negative random variable (such that λ_u/V is integrable for each u). We define:

$$\tau = \inf\{t : \Lambda_t \geq \Theta V\}.$$

The random variable $f(\Theta, V) = \Theta V$ is no longer independent from $\mathcal{F}_\infty$. For any T and t :

$$\begin{aligned} G_t^T &= \mathbb{E}\left(\mathbb{P}\left(\frac{\Lambda_T}{V} < \Theta \middle| \mathcal{F}_\infty\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\exp -\frac{\Lambda_T}{V}\middle| \mathcal{F}_t\right) \\ &= \int_T^\infty \mathbb{E}\left(\psi_s|\mathcal{F}_t\right) ds = \int_T^\infty \gamma(s,t)ds, \end{aligned}$$

with $\psi_s = (\lambda_s/V)\exp\left(-\int_0^s(\lambda_u/V)du\right)$, and $\gamma(s,t) = \mathbb{E}\left(\psi_s|\,\mathcal{F}_t\right)$ (remark that for any s, $(\gamma(s,t), t \geq 0)$ is an $\mathbb{F}$-martingale). If $\eta([0,t]) = \int_0^t \gamma(s,0)ds = \mathbb{P}(\tau \leq t)$ is the law of the random variable τ, we can write

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty \alpha_t^s\, \eta(ds)$$

with $\alpha_t^s = \gamma(s,t)/\gamma(s,0)$. This family of processes is by construction such that for any s the process α^s is a positive martingale, hence the Cox time τ is an initial time. It follows that the hypothesis $(\mathcal{H}')$ holds. Moreover, in this situation, if $t \geq s$,

$$\alpha_t^s = \frac{\gamma(s,t)}{\gamma(s,0)} = \frac{\mathbb{E}\left(\psi_s|\,\mathcal{F}_t\right)}{\mathbb{E}\left(\psi_s\right)} \neq \alpha_s^s,$$

since ψ_s is not $\mathcal{F}_s$ hence $\mathcal{F}_t$ -measurable. From Corollary 3.1, it follows that $(\mathcal{H})$ hypothesis does not hold in this framework (remark this time satisfies $(\mathcal{A})$ condition since $\mathbb{P}\left(\tau = T\right) = \mathbb{P}\left(\Lambda_T/V = \Theta\right) = 0$ for any $\mathbb{F}$ -stopping time T, since Θ is independent of $\mathcal{F}_\infty$).

Remark that this construction leads to very simple examples in which the initial times can have non square integrable α (cf. the first section). Consider for instance a non-negative $\mathbb{F}$-martingale M of mean $1/2$, not square integrable (i.e., there exist dates t such that $\mathbb{E}\left(M_t^2\right) = \infty$), a date $T > 0$, and an integrable $\mathcal{F}_\infty$-measurable random variable K, with mean $1/2$. We take for random time

$$\tau = \inf\{t : \Lambda_t \geq \Theta V\}.$$

with $\lambda_s = \lambda$ for any s, and $V = 1/\left(K + M_T\right) \in \mathcal{F}_\infty$ (which satisfies $1/V \in L^1$). From $\alpha_t^0 = \gamma(0,t)/\gamma(0,0)$,

$$\alpha_t^0 = \frac{\mathbb{E}\left(\psi_0|\,\mathcal{F}_t\right)}{\mathbb{E}\left(\psi_0\right)} = \frac{\mathbb{E}\left(\lambda/V|\,\mathcal{F}_t\right)}{\mathbb{E}\left(\lambda/V\right)} = \frac{\mathbb{E}\left(K + M_T|\,\mathcal{F}_t\right)}{\mathbb{E}\left(K\right) + \mathbb{E}\left(M_T\right)} = \mathbb{E}\left(K|\,\mathcal{F}_t\right) + M_t \geq M_t$$

hence is non square integrable since M is not so. It proves that such a property has to be imposed.

This type of construction may present two drawbacks, depending on the field of application: first it is difficult to derive the Doob-Meyer decomposition of the process F, and second the default time in this framework is always finite. A slight modification of the previous example may correct these two points.

Third example: "Cox-like"construction, a non-finite case. In this part, we force the random variable V to be bigger than 1 and denote by U a random variable uniformly distributed on $[0,1]$, independent of $\mathcal{F}_\infty$. We define

$$\tau = \inf\{t : 1 - \exp -\Lambda_t \geq UV\}.$$

The quantity $1 - \exp -\Lambda_t$ is increasing, starting from 0 and bounded by 1. In some cases, depending on the value of V, the barrier UV will be bigger than 1 and never crossed, so that the default time be non-finite. We have for any T and t :

$$\begin{aligned} G_t^T &= \mathbb{E}\left(\mathbb{P}\left(1 - \exp -\Lambda_T < UV|\,\mathcal{F}_\infty\right)|\mathcal{F}_t\right) = \mathbb{E}\left(\mathbb{P}\left(\frac{1 - \exp -\Lambda_T}{V} < U \middle|\, \mathcal{F}_\infty\right) \middle|\, \mathcal{F}_t\right) \\ &= 1 - \mathbb{E}\left(\frac{1 - \exp -\Lambda_T}{V} \middle|\, \mathcal{F}_t\right) = \mathbb{E}\left(\int_T^\infty \psi_s ds \middle|\, \mathcal{F}_t\right) = \int_T^\infty \gamma(s,t)ds, \end{aligned}$$

with $\psi_s = (\lambda_s/V)\exp - \int_0^s \lambda_u du$, and $\gamma(s,t) = \mathbb{E}\left(\psi_s|\,\mathcal{F}_t\right)$ (remark that for any s, $(\gamma(s,t), t \geq 0)$ is an $\mathbb{F}$-martingale). The law of τ writes: for any t (finite) $\eta\left([0,t]\right) = \mathbb{P}(\tau \leq t) = \int_0^t \gamma(s,0)ds$, hence $\eta\left([0,\infty[\right) = \int_0^\infty \gamma(s,0)ds = \mathbb{E}\left(1/V\right)$ and $\eta\left(\{\infty\}\right) = 1 - \mathbb{E}\left(1/V\right)$. We can write

$$G_t^T = \mathbb{P}(\tau > T|\mathcal{F}_t) = \int_T^\infty \alpha_t^s\, \eta(ds)$$

with $\alpha_t^s = \gamma(s,t)/\gamma(s,0)$. It follows that $(\mathcal{H}')$ hypothesis holds. In this situation again, if $t \geq s$,

$$\alpha_t^s = \frac{\gamma(s,t)}{\gamma(s,0)} = \frac{\mathbb{E}\left(\psi_s | \mathcal{F}_t\right)}{\mathbb{E}\left(\psi_s\right)} \neq \alpha_s^s$$

and $(\mathcal{H})$ hypothesis does not hold either in this framework (here again, condition $(\mathcal{A})$ holds).

We can compute the Doob-Meyer decomposition of $F = 1 - G$ in this framework, since

$$F_t = \int_0^t \mathbb{E}\left(\psi_s | \mathcal{F}_t\right) ds = \mathbb{E}\left(1/V | \mathcal{F}_t\right) \int_0^t \left(\lambda_s \exp - \int_0^s \lambda_u du\right) ds = N_t C_t$$

where $N_t = \mathbb{E}\left(1/V | \mathcal{F}_t\right) \in \mathcal{M}(\mathbb{F})$ and $C_t = \int_0^t \left(\lambda_s \exp - \int_0^s \lambda_u du\right) ds$ is an $\mathbb{F}$-predictable increasing process. It is therefore by uniqueness the multiplicative decomposition of F. If $F_t = M_t + A_t$ is the additive Doob-Meyer decomposition of F, we have

$$dA_t = N_t dC_t = F_t \frac{dC_t}{C_t} \text{ and } dM_t = C_t dN_t = F_t \frac{dN_t}{N_t}.$$

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