Brownian Excursions and Parisian Barrier Options

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July 95, Revised version : December 95
To appear in Adv. Appl. Proba. March 1997

Abstract: A new kind of option, called hereafter a Parisian barrier option, is studied in this paper. This option is the following variant of the so-called barrier option : a down-and-out barrier option becomes worthless as soon as a barrier is reached, whereas a down-and-out Parisian barrier option is lost by the owner if the underlying asset reaches a prespecified level and remains constantly below this level for a time interval longer than a fixed number, called the window. Properties of durations of Brownian excursions play an essential role. We also study another kind of option, called here cumulative Parisian option, which becomes worthless if the total time spent below a certain level is too long.

Keywords: Brownian Excursions, Brownian meander, Barrier Options.
AMS classification : 60 G 46, 60 H 60.

1 Introduction

The payoff of a standard European option only depends on the price of the underlying asset at the maturity date. In the barrier (or knock-out) option case, the payoff does not only depend on the final price of the underlying asset but also on whether or not the underlying asset has reached a barrier price during the life span of the option.
The more volatile the underlying value is, the less sense the concept of knock-out option makes. It is acceptable to lose the right to exercise a barrier option only if the probability that the standard option will be in the money at maturity is pretty low. This is not the case if the volatility is high when the option is lost (in this case the value of the option is pretty small). Actually, after a critical level of the volatility, the European knock-out option has a value which is a monotonic decreasing function of the volatility.

In this paper, we define a new kind of option which we shall call a Parisian barrier option. For conciseness, we shall often drop the term "barrier" in the sequel. A Parisian option is close to a barrier option, studied in [20], the difference lying in the fact that the owner doesn't lose the option if the value of the underlying value reaches the knock-out level, but only if it remains long enough under (or above) this level.
With a Parisian option, the joint probability to lose the right to exercise and to have at maturity an underlying value which is higher (resp. smaller) than the strike price for a call (resp. for a put) can be monitored.
The Parisian option combines the advantage of the knock-out option, i.e., to reduce the cost, with the advantage of standard options, i.e., to keep the right to exercise longer.
The referee also pointed out another possible advantage of Parisian options; we quote : "as far as barrier down-and-out options are concerned, an influential agent in the market who has written such options and sees the price approaching that limit could try to push the price below this limit, even momentarily. This will make the options worthless, so even if the agent loses some money while doing so, he might be compensated by the elimination of liabilities. However, in the case of Parisian barrier options, he will have to make sure that the price stays below that level for some time; this might prove more difficult or more expensive. This makes the market fair in that it protects the holder of such options from deliberate action taken by the writer."

The paper is organised as follows: in section II, we introduce the notation and define the assumptions. In section III, we obtain an upper and a lower bound for the value of the Parisian option. In section IV, we attempt to value these options, whereas in section V, we give precise formulae for the Laplace transforms of their values. In section VI, we prove the put-call parity. We study cumulative Parisian options in section VII. Technically, in pricing such options, we will rely upon ex-

cursion theory, the needed results from which are presented in the appendix.

The main computation made in this paper is an illustration of the following general "principle" : to estimate the usually complicated function

$$\phi(t,K) \stackrel{def}{=} E[\mathbb{1}_{\{t\geq T\}}(\Phi_t - K)^+]$$

for $(\Phi_t, t \geq 0)$ a Brownian functional, and T a stopping time, we compute the Laplace transform of $\phi(\cdot, K)$:

$$\int_0^\infty dt\, e^{-\lambda t}\phi(t,K) = E[e^{-\lambda T}\int_0^\infty du\, e^{-\lambda u}(\Phi_{T+u} - K)^+]\,,$$

and, in many instances, the right-hand side may be evaluated with the help of the strong Markov property. Two previous applications of this "principle" have been made for the computation on one hand of Asian options [10], on the other hand of double barrier options [11].

Acknowledgements: We are grateful to Glenn Kentwell and John Cornwall from Banker Trust Australia, for their comments and useful suggestions, to the anonymous referee for constructive remarks and to Laurent Gauthier for helpful discussions. We are of course responsible for any possibly remaining error.

2 Definitions

2.1 Notation

Let us briefly describe what is an excursion at (or away from) the level L for an Itô process S_t, i.e., a process of the form $dS_t = b(t,S_t)\,dt + a(t,S_t)\,dW_t$. For details, see the Appendix and Revuz-Yor [22], ch XII.
Let S_t be an Itô process and call L the level of the excursion we consider. We use the notation

$$g^S_{L,t} = \sup\{s \leq t\,|S_s = L\}, \quad d^S_{L,t} = \inf\{s \geq t|S_s = L\}$$

(we make the usual convention : $\sup(\emptyset) = 0$, and $\inf(\emptyset) = +\infty$).
The trajectory of S between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion of S at the level L, which straddles time t. The variables $g^S_{L,t}$ and $d^S_{L,t}$ are the left and the right ends of the excursion. If S remains below L during this excursion, i.e., if $S_t < L$, the excursion is said to be below L.
The length (or life duration) of the excursion which straddles t is $d^S_{L,t} - g^S_{L,t}$. We are interested here with $t - g^S_{L,t}$, which is the age of the excursion.

In the context of a Parisian option, S_t is the price of the underlying asset. We suppose that

$$dS_t = S_t(\mu\, dt + \sigma dB_t)\,, \quad S_0 = x \tag{1}$$

where σ is a non-negative constant and B a Brownian motion.
We will denote by r the interest rate and by δ the dividend rate if the underlying is a stock (otherwise, for a currency, δ will play the role of the foreign interest rate). We assume that both r and δ are constant.
The maturity of the option is denoted by T and the strike price is K.

2.2 Parisian out option

a. Down-and-out option The owner of the option loses the option if the underlying asset price S_t reaches the level L and remains constantly *below* this level for a time interval longer than a fixed number D, called the option window. If not, the owner will receive a payoff $\phi(S_T)$, with $\phi(x) = (x - K)^+$ for a call (resp. $(K - x)^+$ for a put) where K is a fixed value. For a down-and-out barrier option, the case of interest is when the initial price of the underlying asset is greater than the barrier; this is no more the case for a down-and-out Parisian option.

b. Up-and-out option The option is lost by its owner if there is an excursion *above* the level L which is older than D.

2.3 Parisian in option

a. Down-and-in option: The owner of a down-and-in option receives the payoff only if there is an excursion *below* the level L which is older than D. The down-and-in option (resp. down-and-out) refers to an option which appears (resp. disappears) when there is an excursion which lasts long enough below the level L.

b. Up-and-in option: The owner of the option receives the payoff only if there is an excursion *above* the level L which lasts longer than D.

We assume no rebate in our study; we could easily extend our computations to that case.

3 Upper and Lower Bounds of a Parisian down-out option

As presented in the introduction, the Parisian option combines different characteristics of the knock-out option, and of the standard option. More precisely, let us consider the two following limiting cases for down-and-out options:

- $S_t > L$ and $D \geq T - t$. In this case, the probability to have an excursion below L, between t and T , of length at least equal to D is zero.

The values of the Parisian call and put are the standard European options given by the Black and Scholes formula [3].

- $S_t > L$ and $D = 0$.
  In this case, as soon as the underlying value reaches the excursion level L, the option is lost. The option becomes a knock-out option and its value is well known [20].

Between those two limiting cases, when D decreases from $T-t$ to zero, the option value decreases, because the probability to lose the right to exercise increases. We thus have the following formulae for the call and put cases:

a. Call If $x > L$:

$$\left\{\begin{array}{rcl} C_o^d(x,T) & \leq & xe^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \\ \\ C_o^d(x,T) & \geq & xe^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) \\ -x\left(\frac{x}{L}\right)^{-2\epsilon}\mathcal{N}(y) & + & Ke^{-\delta T}e^{-rT}\left(\frac{x}{L}\right)^{2-2\epsilon}\mathcal{N}(y-\sigma\sqrt{T}) \end{array}\right. \tag{2}$$

b. Put If $x > L$:

$$\left\{\begin{array}{rcl} P_o^d(x,T) & \leq & Ke^{-rT}\mathcal{N}(-d_2) - xe^{-\delta T}\mathcal{N}(-d_1) \\ \\ P_o^d(x,T) & \geq & Ke^{-rT}\mathcal{N}(-d_2) - xe^{-\delta T}\mathcal{N}(-d_1) \\ & - & Ke^{-rT}e^{-\delta T}\left(\frac{x}{L}\right)^{2-2\epsilon}\mathcal{N}(-y+\sigma\sqrt{T}) + x\left(\frac{x}{L}\right)^{-2\epsilon}\mathcal{N}(-y) \end{array}\right. \tag{3}$$

with

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left(\ln(\frac{x}{K}) + (r-\delta)T + \frac{\sigma^2 T}{2}\right), \quad d_2 = d_1 - \sigma\sqrt{T}$$

and $\mathcal{N}$ the distribution function of the normalized gaussian variable,

$$\epsilon = \frac{1}{2} + \frac{r}{\sigma^2}\,; \quad y = \frac{1}{\sigma\sqrt{T}}\ln(\frac{L^2}{xK} + \epsilon\sigma\sqrt{T})$$

Notice that for each D belonging to $[0,T]$, a Parisian option can be defined. The two limit cases $D = 0$ and $D = T$ correspond respectively to the knock-out option and to the standard option.

4 The valuation of Parisian options

Let Q denote the risk neutral probability. Under Q, the dynamics of S is given by

$$dS_t = S_t\,((r-\delta)\,dt + \sigma\,dW_t)\,, \quad S_0 = x \tag{4}$$

where $(W_t, t \geq 0)$ is a Q-Brownian motion and $x > 0$. It follows that

$$S_t = x \exp\left[\left(r - \delta - \frac{\sigma^2}{2}\right) t + \sigma W_t\right] .$$

Let us introduce the following notation

$$m = \frac{1}{\sigma}\left(r - \delta - \frac{\sigma^2}{2}\right) \ , \ b = \frac{1}{\sigma} \ln\left(\frac{L}{x}\right)$$

where L is the excursion level.
The price of the asset is

$$S_t = x \exp(\sigma(mt + W_t)) .$$

4.1 Parisian out-option

4.1.1 Down-and-out option.

The owner of a down-and-out Parisian option loses it if S_t reaches L and remains constantly below the level L for a time interval longer than D.
This is the case if and only if $\tilde{H}^-_{L,D}(S) \leq T$, where $\tilde{H}^-_{L,D}(S)$ is the first time at which the age of an excursion below the level L for the process S_t is greater ("older") than or equal to D

$$\tilde{H}^-_{L,D}(S) = \inf\{t \geq 0 \,|\, 1\!\!1_{S_t < L}\,(t - g^S_{L,t}) \geq D\} \tag{5}$$

We can give a more convenient form to $\tilde{H}^-_{L,D}(S)$ by refering to the Brownian motion W_t. Actually, $\tilde{H}^-_{L,D}(S) = H^{m,-}_{b,D}$, where

$$H^{m,-}_{b,D} = \inf\{t \geq 0 \,|\, 1\!\!1_{W_t + mt < b}\,(t - g^m_{b,t}) \geq D\} \tag{6}$$

where $g^m_{b,t}$ is the left extremity of the excursion straddling time t and away from level b for the process $W_t + mt$:

$$g^m_{b,t} = \sup\{u < t |\, W_u + mu = b\} \tag{7}$$

Relying on Cameron-Martin-Girsanov theorem, we introduce a new probability P, which makes $(Z_t = W_t + mt, 0 \leq t \leq T)$ a P-Brownian motion. The price of a down-and-out option with payoff $\phi(S_T)$ is, in an arbitrage free model,

$$e^{-rT} E_Q\left(\phi(S_T) 1\!\!1_{H^{m,-}_{b,D} > T}\right) = \exp[-(r + \frac{m^2}{2})T]\, E_P\left(1\!\!1_{H^-_{b,D} > T}\, \phi(x e^{\sigma Z_T})\, e^{m Z_T}\right) \tag{8}$$

where

$$H^-_{b,D} = H^{0,-}_{b,D} = \inf\{t \geq 0 \,|\, 1\!\!1_{Z_t < b}\,(t - g_{b,t}) \geq D\}$$
$$g_{b,t} = \sup\{u \leq t |\, Z_u = b\} .$$

In many formulae involving a function Π of the maturity T, as in (8), the discounted factor $\exp[-(r+\frac{m^2}{2})T]$ appears. In order to give concise formulae, we introduce the following notation

$$^{\star}\Pi(T) = \exp[(r+\frac{m^2}{2})T]\,\Pi(T) \tag{9}$$

and we shall refer to this quantity as a (r,m)-discounted value.

a. Parisian down-and-out call : We now treat in details the Parisian down-and-out call. Let us denote by $C_o^d(x,T;K,L,D;r,\delta)$ the value of a Parisian down-and-out call.
This value depends on the parameters x (the value of the underlying asset), T (the maturity) and K,L,D,r,δ. When the parameters K,L,D,r,δ are fixed, we use the concise notation $C_o^d(x,T)$. From (8), we have

$$C_o^d(x,T;K,L,D;r,\delta) = \exp[-(r+\frac{m^2}{2})T]\,E_P\left(1\!\!1_{H_{b,D}^- > T}\,(x\exp(\sigma Z_T)\,-K)^+\,\exp(mZ_T)\right)$$

therefore, using notation (9), we obtain the following expression

$$^{\star}C_o^d(x,T;K,L,D;r,\delta) = E_P\left(1\!\!1_{H_{b,D}^- > T}\,(x\exp(\sigma Z_T)\,-K)^+\,\exp(mZ_T)\right)$$

b. Parisian down-and-out put : Using obvious notation, we get the following:

$$^{\star}P_o^d(x,T;K,L,D;r,\delta) = E_P\left(1\!\!1_{H_{b,D}^- > T}\,(K - x\exp(\sigma Z_T))^+\,\exp(mZ_T)\right)$$

4.1.2 Up-and-out options

The (r,m)-discounted value of an up-and-out option with payoff $\phi(S_T)$ is :

$$E_P\left(1\!\!1_{H_{b,D}^+ > T}\,\phi(xe^{\sigma Z_T})\,\exp(mZ_T)\right)$$

where

$$H_{b,D}^+ = \inf\{t \geq 0 \,|\, 1\!\!1_{Z_t > b}\,(t - g_{b,t}) \geq D\}$$
$$g_{b,t} = \sup\{u < t \,|\, Z_u = b\}$$

We denote by $C_o^u(x,T;K,L,D;r,\delta)$ (resp $P_o^u(x,T;K,L,D;r,\delta)$) the value of an up-and-out call (resp. put).

4.2 Parisian in options

The "in" case follows from the "out" case: for example, let us denote by

$$^{\star}C_i^d(x,T;K,L,D;r,\delta) = E_P\left(1\!\!1_{\{H_{b,D}^- \leq T\}}\,(xe^{\sigma Z_T} - K)^+\,\exp(mZ_T)\right)$$

the (r,m)-discounted value of a down-and-in call. Then,

$$^*C_o^d(x,T;K,L,D;r,\delta) = {}^*\mathcal{BS}(x,T) - {}^*C_i^d(x,T;K,L,D;r,\delta) \tag{10}$$

where $\mathcal{BS}(x,T)$ is the Black-Scholes price, i.e.,

$$\mathcal{BS}(x,T) = \exp[-(r+\frac{m^2}{2})T]\, E_P\left((xe^{\sigma Z_T}-K)^+ \exp(mZ_T)\right) .$$

In the same way, the values of an up-and-out call and of an up-and-in call

$$^*C_i^u(x,T;K,L,D;r,\delta) = E_P\left(1\!\!1_{\{H^+_{b,D}\le T\}}\,(xe^{\sigma Z_T}-K)^+ \exp(mZ_T)\right)$$

satisfy

$$^*C_o^u(x,T;K,L,D;r,\delta) = {}^*\mathcal{BS}(x,T) - {}^*C_i^u(x,T;K,L,D;r,\delta)$$

4.2.1 Some references

A number of papers [6, 13, 15, 16, 17, 21] and [24] have been devoted to the study of the longest duration of excursions of Brownian motion on the interval $[0,T]$, and more generally to the sequences

$$V_1(T) > V_2(T) > \ldots > V_n(T)\ldots$$

of ranked lengths of excursions during the interval $[0,T]$, including $T-g_T$. A more complete discussion of the existing literature is presented in [18], where this kind of study is furthered by showing that the law of the normalized sequence

$$\frac{1}{\tau}(V_1(\tau),V_2(\tau),\ldots,V_n(\tau),\ldots)$$

for certain very special random times τ, is identical to the law obtained for $\tau = T$, a constant time; this is in particular the case for

$$\tau = H_1(D) = \inf\{t\,:\,V_1(t) > D\}$$

and, more generally for $\tau = H_n(D) = \inf\{t : V_n(t) > D\}$. Such identities in law have originally motivated the writing of [18] and of the present paper. We remark that the scaling property implies that $\dfrac{T}{V_n(T)} \overset{(law)}{=} H_n(1)$, an identity which has been exploited by Wendel [24]. It would be possible to construct options based on $H_2(D)$, $H_3(D),\ldots$; however, we shall not develop this here.

4.3 Reduction to the case $x = L$

The case $x = L$ (or $b = 0$) is important, since it is possible to reduce the other cases to it. In this case, we use the following special notation : for excursions below the level L (with $x = L$ or $b = 0$)

$$C_o^-(x, T; K, D; r, \delta) = e^{-rT} E_Q\Big((S_T - K)^+ \mathbb{1}_{\hat{H}^-_{L,D}(S)>T}\Big)$$

or, using the notation (9):

$$^\star C_o^-(x, T; K, D; r, \delta) = E_P\left((xe^{\sigma Z_T} - K)^+ \exp(mZ_T)\, \mathbb{1}_{H^-_{0,D}>T}\right) \tag{11}$$

We refer to this case as a zero down-and-out call. For excursions above the level L (zero up-and-out call)

$$C_o^+(x, T; K, D; r, \delta) = e^{-rT}\, E_Q\left((S_T - K)^+ \mathbb{1}_{\tilde{H}^+_{L,D}(S)>T}\right) \tag{12}$$

4.3.1 Case $b > 0$

We show that the $b > 0$ (or $L > x$) case reduces to the case $b = 0$. We study the case of call options, the general case follows immediately. Let us denote by T_b the first hitting time of b, related to the Brownian motion $(B_t, t \geq 0)$

$$T_b = \inf\{t \geq 0 | B_t = b\}$$

and its well known law by $\mu_b(du)$:

$$\mu_b(du) = \frac{be^{-\frac{b^2}{2u}}}{\sqrt{2\pi u^3}}\, du$$

whose Laplace transform equals

$$E(\exp(-\frac{\lambda^2}{2}T_b)) = \exp -b\lambda\,, \quad \text{for } \lambda \geq 0\,.$$

a. Down call

In the case $b > 0$, if $H^-_{b,D} \geq T \geq D$, then $T_b \leq D$. Therefore, the (r, m)-discounted value of a down-and-out call in the case $L > x$ is

$$^\star C_o^d(x, T; K, L, D; r, \delta) =$$

$$E_P\Big(\mathbb{1}_{H^-_{b,D}\geq T}\mathbb{1}_{T_b\leq D}[x\exp(\sigma(Z_T - Z_{T_b} + b) - K]^+ \exp(m(Z_T - Z_{T_b} + b))\Big)$$

$$= \int_0^D \mu_b(du) E_P\Big(1\!\!1_{\tilde{H}^{b,-}_{b,D} \geq T-u} [x \exp(\sigma(W_{T-u}+b)) - K]^+ \exp(m(W_{T-u}+b))\Big)$$

where $\tilde{H}^{b,-}_{b,D}$ is the first time such that an excursion below b, for the Brownian motion $W_t + b$ issued from b, is older than D. It follows

$$^\star C^d_o(x,T;K,L,D;r,\delta) = \int_0^D \mu_b(du) E_P\Big(1\!\!1_{H^-_{0,D} \geq T-u} [x \exp(\sigma Z_{T-u}) - K]^+ \exp(m Z_{T-u})\Big)$$

therefore, we obtain :

Proposition 1 *The (r,m)-discounted value of a down-and-out call in the case $L > x$ can be expressed in terms of zero down-and-out calls*

$$^\star C^d_o(x,T;K,L,D;r,\delta) = \int_0^D \mu_b(du)\, ^\star C^-_o(x,T-u;K,D;r,\delta)$$

where

$$^\star C^-_o(x,T;K,D;r,\delta) = E_P(\, 1\!\!1_{H^-_{0,D} > T} (x e^{\sigma Z_T} - K)^+ \exp(m Z_T)\,)$$

and μ_b is the law of T_b.

b. Up call

If $b > 0$ and $H^+_{b,D} \leq T$, then $T_b \leq T$. Then, we prove the following

Proposition 2 *The (r,m)-discounted value of an up-and-in call in the case $L > x$ can be expressed in terms of zero up-and-in calls*

$$^\star C^u_i(x,T;K,L,D;r,\delta) = \int_0^T \mu_b(du)\, ^\star C^+_i(x,T-u;K,D;r,\delta)$$

where

$$^\star C^+_i(x,T) = E_P\left(1\!\!1_{\{H^+_{0,D} \leq T\}} (x e^{\sigma Z_T} - K)^+ \exp(m Z_T)\right)$$

corresponds to a zero up-and-in call.

4.3.2 Case $b < 0$.

The same method leads to the following formulae :

$$^\star C^d_i(x,T;K,L,D;r,\delta) = \int_0^T \mu_b(du)\, ^\star C^-_i(x,T-u;K,D;r,\delta)$$

where

$$^\star C^-_i(x,T) = E_P\left(1\!\!1_{\{H^-_{0,D} \leq T\}} (x e^{\sigma Z_T} - K)^+ \exp(m Z_T)\right) .$$

5 A formula for the valuation

We study, in this section, the valuation of a down-and-in option. The value of a down-and-out option was obtained in (10).

5.1 Parisian down-and-in call

Let us denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the completed filtration of the Brownian motion $(Z_t; t \geq 0)$. Remark that $H^-_{b,D}$ is an $\mathcal{F}_t$-stopping time. We have

$$^\star C^d_i(x,T) = E_P\left(1\!\!1_{H^-_{b,D} \leq T}\ E_P\left[e^{mZ_T}\,(xe^{\sigma Z_T} - K)^+ | \mathcal{F}_{H^-_{b,D}}\right]\right) \tag{13}$$

Thus by use of the strong Markov property

$$^\star C^d_i(x,T) = E_P(1\!\!1_{H^-_{b,D} \leq T}\ \mathcal{P}_{T-H^-_{b,D}}(f_x)(Z_{H^-_{b,D}})) \tag{14}$$

with

$$f_x(z) = e^{mz}(xe^{\sigma z} - K)^+ \tag{15}$$

and

$$\mathcal{P}_t f(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(u) \exp(-\frac{(u-z)^2}{2t})\, du$$

the Brownian semi-group acting on f. The same kind of formula applies for a down option with payoff $\phi(S_T)$.

As recalled in the Appendix, the random variables $Z_{H^-_{b,D}}$ and $H^-_{b,D}$ are independent. Let us denote the law of $Z_{H^-_{b,D}}$ by $\nu(dz)$.

Consequently, we obtain

$$\begin{aligned} ^\star C^d_i(x,T) &= \int_{-\infty}^{\infty} \nu(dz)\, E(1\!\!1_{H^-_{b,D} \leq T} \mathcal{P}_{T-H^-_{b,D}}(f_x)(z)) \\ &= \int_{-\infty}^{\infty} dy\, f_x(y)\, h_b(T,y) \end{aligned} \tag{16}$$

where

$$h_b(t,y) = \int_{-\infty}^{\infty} \nu(dz)\, E_P\left(1\!\!1_{H^-_{b,D} < t}\ \frac{\exp - \dfrac{(b-z-y)^2}{2(t-H^-_{b,D})}}{\sqrt{2\pi(t-H^-_{b,D})}}\right) \tag{17}$$

Let us remark that $h_b(t,y) = 0$ if $t < D$.

5.1.1 Case $b < 0$

In this case,

$$P(Z_{H^-_{b,D}} \in dx) = \frac{dx}{D}\,(b-x)\ \exp(-\frac{(x-b)^2}{2D}) 1\!\!1_{x \leq b}$$

We establish in the Appendix that, for $\lambda > 0$ the Laplace transform $\hat{h}_b(\lambda, y)$ of $h_b(\cdot, y)$ is

$$\int_0^\infty dt\, e^{-\lambda t} h_b(t, y) = \frac{e^{b\sqrt{2\lambda}}}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} \int_0^\infty dz\, z\, \exp(-\frac{z^2}{2D} - |b - z - y|\sqrt{2\lambda})$$

where

$$\Psi(z) = \int_0^\infty dx\; x\; \exp(-\frac{x^2}{2} + zx) = 1 + z\sqrt{2\pi}\exp(\frac{z^2}{2})\mathcal{N}(z)\,. \tag{18}$$

The integral

$$K_{\lambda,D}(y - b) \stackrel{def}{=} \int_0^\infty dz\, z\, \exp(-\frac{z^2}{2D} - |b - z - y|\sqrt{2\lambda})$$

can be easily evaluated.
If $a = y - b < 0$, we obtain, using change of variables

$$K_{\lambda,D}(a) = \exp(a\sqrt{2\lambda})\,[1 - 2e^{\lambda D}\sqrt{\pi\lambda D}\mathcal{N}(-\sqrt{2\lambda D})] = \exp(a\sqrt{2\lambda})\Psi(-\sqrt{2\lambda D})$$

If $a > 0$, a similar method leads to

$$K_{\lambda,D}(a) = e^{-a\sqrt{2\lambda}} + 2\sqrt{\pi\lambda D}$$

$$e^{\lambda D}\Big(e^{-a\sqrt{2\lambda}}\,[\mathcal{N}(\frac{a}{\sqrt{D}} - \sqrt{2\lambda D}) - \mathcal{N}(-\sqrt{2\lambda D})] - e^{a\sqrt{2\lambda}}\,[1 - \mathcal{N}(\frac{a}{\sqrt{D}} + \sqrt{2\lambda D})]\Big)$$

If $D = 0$, we obtain that $\hat{h}_b(\lambda, y) = \dfrac{e^{b\sqrt{2\lambda}}}{\sqrt{2\lambda}} e^{-|b-y|\sqrt{2\lambda}}$, therefore we have the formula for a barrier option (it is easy to invert $\hat{h}_b$ and thus to find the right-hand side of (2)).

Using the explicit expression (15) of f_x, we obtain

$$^\star C_i^d(x, T; K, L, D; r, \delta) = \int_{\beta(x)}^\infty dy\, e^{my}(xe^{\sigma y} - K)h_b(T, y)$$

where $\beta(x) = \dfrac{1}{\sigma}\ln\dfrac{K}{x}$.

Particular case: $K > L$ In this case, we need only the value of $h_b(t, y)$ for $y > b$ and we obtain

$$\hat{h}_b(\lambda, y) = \frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})}\,\frac{e^{(2b-y)\sqrt{2\lambda}}}{\sqrt{2\lambda}}$$

It can easily be proved, since that $e^{-\sqrt{2\lambda}}$ is a Laplace Transform, that $\Psi(-\sqrt{2\lambda D})$ is a Laplace transform. Therefore, in order to invert $\hat{h}_b$, it suffices to invert $\dfrac{1}{\Psi(\sqrt{2\lambda D})}$. This is not easy : see [5] for some computation.

Theorem 1 *In the case $x > L$ (i.e., $b < 0$), the (r,m)-discounted value of a Parisian down-and-in call with level L is*

$$^{\star}C_i^d(x,T;K,L,D;r,\delta) = \int_{\beta(x)}^{\infty} dy\, e^{my}(xe^{\sigma y} - K)h_b(T,y)$$

where $\beta(x) = \dfrac{1}{\sigma}\ln\dfrac{K}{x}$.
The function $h_b(t,y)$ is characterized by its Laplace transform

$$\hat{h}_b(\lambda,y) = \frac{e^{b\sqrt{2\lambda}}}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})}\int_0^{\infty} dz\, z\, \exp(-\frac{z^2}{2D} - |b-z-y|\sqrt{2\lambda})$$

where $\Psi(z)$ is defined in (18).
In the case $K > L$, the Laplace transform of $h_b(t,y)$ for $y > \beta(x)$ is

$$\hat{h}_b(\lambda,y) = \frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})}\,\frac{e^{(2b-y)\sqrt{2\lambda}}}{\sqrt{2\lambda}}\,.$$

Remark: In each case, it is possible to compute the Laplace transform of the "Delta" term, i.e., the derivative with respect to x of the value of the option. This Laplace transform has a somewhat complicated form. For example, in the case $K > L$, the Laplace transform of $\Delta(T) = \dfrac{\partial C_i^d}{\partial x}(x,T)$ is

$$\hat{\Delta}(\lambda) = \int_0^{\infty} e^{-\lambda t}\Delta(t)\,dt =$$

$$\frac{\Psi(-\sqrt{2\mu D})}{\Psi(\sqrt{2\mu D})}\,\frac{e^{2b\sqrt{2\mu}}}{\sqrt{2\mu}\;m+\sigma-\sqrt{2\mu}}\left(\frac{K}{x}\right)^{\dfrac{m+\sigma-\sqrt{2\mu}}{\sigma}}$$

where $\mu = \lambda + \frac{m^2}{2} + r$.

5.1.2 Case $b > 0$

In the case $b > 0$, the Laplace transform of $h_b(.,y)$ is complicated. We obtain

$$\int_0^{\infty} dt\, e^{-\lambda t}h_b(t,y) = E\Big(\int_{-\infty}^{\infty}\nu(dz)\,e^{-\lambda H_{b,D}^-}\,\frac{1}{\sqrt{2\lambda}}\,\exp(-|y-z|\sqrt{2\lambda})\Big)$$

Using the results of the Appendix,

$$\int_0^{\infty} dt\, e^{-\lambda t}h_b(t,y) = \tag{19}$$

$$\frac{1}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})}\int_0^{\infty} dz\, z\, \exp(-\frac{z^2}{2D} - |y-b+z|\sqrt{2\lambda})\int_0^{D}\mu_b(dx)\,e^{-\lambda x}$$
$$+\frac{e^{-\lambda D}}{2\sqrt{\lambda\pi D}}\int_{-\infty}^{b} dz\,\Big(\exp(-\frac{z^2}{2D}) - \exp -\frac{(z-2b)^2}{2D}\Big)\,e^{-|y-z|\sqrt{2\lambda}}$$

The second term of the right member of (19) is the Laplace transform of $g(\cdot, y)$ where

$$g(t,y) = \frac{1\!\!1_{t>D}}{2\pi\sqrt{t-D}} \int_{-\infty}^{b} \exp\Big(\frac{(y-z)^2}{2(t-D)}\Big) \Big(\exp(-\frac{z^2}{2D}) - \exp(-\frac{(z-2b)^2}{2D})\Big)\, dz\,. \tag{20}$$

Particular case If $y > b$, the first term on the right member of (19) is equal to

$$\frac{\Psi(-\sqrt{2\lambda D})}{\Psi(\sqrt{2\lambda D})} \frac{e^{-(y-b)\sqrt{2\lambda}}}{\sqrt{2\lambda}} \int_0^D \mu_b(dx) e^{-\lambda x} \tag{21}$$

This term is the product of four Laplace transforms, however the inverse of $\frac{1}{\Psi(\sqrt{2\lambda D})}$ is not identified.

Theorem 2 *In the case $x < L$ (i.e., $b > 0$), the (r,m)-discounted value of a Parisian down-and-in call with level L is*

$$^{\star}C_i^d(x,T;K,L,D;r,\delta) = \int_{\beta(x)}^{\infty} dy\, e^{my}(xe^{\sigma y} - K) h_b(T,y)$$

where $\beta(x) = \dfrac{1}{\sigma} \ln \dfrac{K}{x}$.

The function $h_b(t,y)$ is characterized by its Laplace transform $\hat{h}_b(\lambda, y) =$

$$\Big(\frac{1}{D\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} \int_0^{\infty} dz\, z\, \exp(-\frac{z^2}{2D} - |y-b+z|\sqrt{2\lambda}) \int_0^D \mu_b(dz)\, e^{-\lambda z}\Big) + \hat{g}(t,y)$$

where g is defined in (20).

6 Parisian put

We establish a put-call parity in the same manner as Grabbe [12]. The (r,m)-discounted value of a Parisian down-and-out put is

$$^{\star}P_o^d(x,T;K,L,D;r,\delta) = E_P\Big(1\!\!1_{H^-_{b,D}>T}\,(K - xe^{\sigma Z_T})^+ \exp(mZ_T)\Big)$$

The right side is equal to

$$xK\, E_P\Big(1\!\!1_{H^-_{b,D}>T}\,(\frac{e^{-\sigma Z_T}}{x} - \frac{1}{K})^+ \exp((\sigma+m)Z_T)\Big)$$

and this expression can be reduced, by introducing the Brownian motion $W_t = -Z_t$. Thus, the hitting time $H^-_{b,D}$ related to Z is the stopping time $H^+_{-b,D}$ related to W, and it follows that

$$^{\star}P_0^d(x,T) = xK\, E_P\Big(1\!\!1_{H^+_{-b,D}>T}\,(\frac{e^{\sigma W_T}}{x} - \frac{1}{K})^+ \exp(-(\sigma+m)W_T)\Big)$$

Writing the terms $(\sigma+m)$ and $\frac{m^2}{2}+r$ in an explicit form with (r,δ) as parameters, it is now easy to check that

$$P_o^d(x,T;K,L,D;r,\delta) = xKC_o^u(\frac{1}{x},T;\frac{1}{K},\frac{1}{L},D;\delta,r)$$

For other Parisian options, the same parity relation holds. For example, for a Parisian up-and-out put,

$$P_o^u(x,T;K,L,D;r,\delta) = xKC_o^u(\frac{1}{x},T;\frac{1}{K},\frac{1}{L},D;\delta,r)$$

7 Cumulative Parisian options

In this section, we define and study an option which is lost by its owner when the time spent by the underlying asset below the level L is greater than D.

More precisely, let $\Gamma_T^{-,L} = \int_0^T 1\!\!1_{S_t\le L}\,dt$ and $\Gamma_T^{+,L} = \int_0^T 1\!\!1_{S_t\ge L}\,dt$. The value of the option is

$$C_c^{\pm}(x,T;K,L,D;r,\delta) = e^{-rT}\,E_Q((S_T-K)^+ 1\!\!1_{\Gamma_T^{\pm,L}\ge D})\,.$$

The problem reduces to the computation of

$${}^{\star}C_c^{\pm}(x,T;K,L,D;r,\delta) = E_P(1\!\!1_{A_T^{b,\pm}\ge D}(xe^{\sigma Z_T}-K)^+\exp(mZ_T))$$

where $A_T^{b,-} = \int_0^T 1\!\!1_{Z_t\le b}\,dt$, and $A_T^{b,+} = \int_0^T 1\!\!1_{Z_t\ge b}\,dt$.

7.1 Case $b=0$

For $b=0$, the law of the pair (Z_T, A_T^+) may be obtained from the equality in law $(Z_T, g_T, A_{g_T}^+) = (\epsilon\sqrt{T-g_T}\,m_1, g_T, g_T U)$ where m_1 is the Brownian meander at time 1 (see Appendix), U is uniform on $[0,1]$, ϵ is a symmetric Bernoulli variable on $\{-1,+1\}$, the variables (m_1, g_T, U, ϵ) being independent.
We denote by $\psi_T(x)dx$ the sub-probability $P(Z_T\in dx, A_T^+ > D)$. It follows that, since $\{Z_T\in dx, A_T^+\} = \{Z_T\in dx, A_{g_T}^+\}$ for $x<0$,

$$\psi_T(x) = \frac{|x|}{2\pi}\int_D^T \frac{s-D}{\sqrt{s^3(T-s)^3}}\exp(-\frac{x^2}{2(T-s)})\,ds\,, \quad x<0\,.$$

For $x>0$, we use the decomposition

$$\{Z_T\in dx, A_T^+ > D\} = \{Z_T\in dx,\, T-g_T > D\}\cup\{Z_T\in dx,\, D > T-g_T > D-A_{g_T}^+\}$$

and it follows (see the Appendix for the law of (Z_T, g_T)) that, for $x > 0$:

$$\psi_T(x) = \frac{x}{2\pi}\int_0^{T-D} \frac{\exp(-\dfrac{x^2}{2(T-s)})}{\sqrt{s(T-s)^3}}\,ds + \frac{x(T-D)}{2\pi}\int_{T-D}^{T} \frac{\exp(-\frac{x^2}{2(T-s)})}{\sqrt{s^3(T-s)^3}}\,ds\,.$$

The (r,m)-discounted price of a cumulative Parisian zero down call ($b=0$) is

$$^\star C_c^+(x,T;K,x,D;r,\delta) = \int_{-\infty}^{\infty} dy(xe^{\sigma y}-K)^+\exp(my)\psi_T(y)\,.$$

Another proof of the pricing formula of cumulative options was proposed to us by the referee, involving some results on α-quantiles (see, e.g., [1, 7, 9]). Define the α-quantile of $\{Z_t,\, 0 \le t \le T\}$ as

$$M(\alpha,T) = \inf\{x : A_T^{x,-} > \alpha T\}$$

Then, in order to compute

$$E\Big(\mathbb{1}_{A_T^{b,-}>D}\Big(xe^{\sigma Z_T}-K\Big)^+\exp\Big(mZ_T\Big)\Big)\,,$$

we note that $\{A_T^{b,-} > D\} = \{M(\frac{D}{T},T) < b\}$. The decomposition in Dassios [7] and its extension in [9] state that

$$(M(\frac{D}{T},T)\,,Z_T) \overset{(law)}{=} (\sup_{0\le s\le D} Z_s + \inf_{0\le s\le T}\tilde{Z}_s\,, Z_D + \tilde{Z}_{T-D})$$

where $\tilde{Z}_t$ is an independent copy of Z_t. From this identity, a discrete time version of which goes back to Wendel [23] and Port [19], one can calculate the joint density of $M(\frac{D}{T},T)$ and Z_T and thus obtain an expression for the value of the option.

7.2 Down-and-out case, $b < 0$.

As in the case of the Parisian down-and-out call with $b < 0$, we establish that

$$^\star C_c^-(x,T;K,L,D;r,\delta) = \int_0^T \mu_b(du)\,{}^\star C_c^+(x,T-u;K,x,D;r,\delta)$$

7.3 Parity relations

Since $A_T^- + A_T^+ = T$, we have the parity relation :

$$C_c^-(x,T;K,L,D;r,\delta) + C_c^+(x,T;K,L,D;r,\delta) = \mathcal{BS}(x,K)$$

and

$$P_c^+(x,T;K,L,D;r,\delta) = xKC_c^-(\frac{1}{x},T;\frac{1}{K},L,D;\delta,r)$$

7.4 Some references

The law of the pair (Z_T, A_T^+) is studied in [1, 9, 25, 26]. The law of the triple (Z_T, L_T, A_T^+) where L_T is the local time at 0 is studied more completely in [14].

8 Appendix

8.1 Some definitions and notation

8.1.1 Excursions intervals

Let $(Z_t, t \geq 0)$ be a standard Brownian motion. For each $t > 0$, define

$$g_t = \sup\{s \mid s \leq t\,; Z_s = 0\} \tag{22}$$

$$d_t = \inf\{s \mid s \geq t\,; Z_s = 0\} \tag{23}$$

We have almost surely $g_t < t < d_t$.

The interval (g_t, d_t) is the interval of the excursion[1] which straddles time t. For u in this interval, $\operatorname{sgn} Z_u$ remains constant. The law of the pair (g_t, Z_t) is

$$1\!\!1_{s \leq t} \frac{|x|}{2\pi\sqrt{s(t-s)^3}} \exp -\frac{x^2}{2(t-s)}\, ds\, dx \tag{24}$$

More generally, we can define, for $b \in I\!R$,

$$g_{b,t} = \sup\{s \mid s \leq t\,; Z_s = b\} \tag{25}$$

$$d_{b,t} = \inf\{s \mid s \geq t\,; Z_s = b\} \tag{26}$$

Let $H_{b,D}^- = \inf\{t \,:\, 1\!\!1_{Z_t \leq b}\,(t - g_{b,t}) \geq D\}$ be the first time at which the age of an excursion below b is greater than or equal to D. In the same way, we define $H_{b,D}^+ = \inf\{t \,:\, 1\!\!1_{Z_t \geq b}\,(t - g_{b,t}) \geq D\}$ to be the first time at which the age of an excursion above b is greater than or equal to D. We are interested in the laws of the pairs $(Z_{H_{b,D}^-}, H_{b,D}^-)$ and $(Z_{H_{b,D}^+}, H_{b,D}^+)$ which are described below, in 8.3.

8.1.2 The slow Brownian filtration

We suppose that $b = 0$. Let us denote by $(\mathcal{F}_t, t \geq 0)$ the natural filtration of the Brownian motion Z.

If R is a random variable such that $R > 0$ a.s., we define the sigma-field $\mathcal{F}_R^-$ of the past up to R as the σ-algebra generated by the variables ζ_R, where ζ is a predictable process.

In particular, we consider the σ-algebra $\mathcal{F}_{g_t}^-$ which is included in $\mathcal{F}_t$ and is increasing with respect to t. Denote by $(\mathcal{F}_{g_t}^+, t \geq 0)$ the slow Brownian filtration[2] $\mathcal{F}_{g_t}^+ = \mathcal{F}_{g_t}^- \vee \sigma(\operatorname{sgn}(Z_t))$.

[1] For details, see Chung [4] and e.g., Revuz-Yor [22] p.107 Ex 3.23 and Ch. XII, par 3.

[2] See Dellacherie-Maisonneuve-Meyer [8] for details and comments.

8.2 The Brownian meander

We denote by $g = g_1 = \sup\{s \leq 1 \,:\, Z_s = 0\}$ the left extremity of the excursion which straddles time 1.
The Brownian meander[3] is defined as the process

$$m_u = \frac{1}{\sqrt{1-g}} |Z_{g+u(1-g)}| \,;\, u \leq 1 \tag{27}$$

The process m is a Brownian scaled part of the (normalized) Brownian excursion which straddles time 1.
The process m is independent of $\mathcal{F}_g^+$. The law of m_1 may be deduced from (24)

$$P(m_1 \in dx) = x \exp(-\frac{x^2}{2}) 1\!\!1_{x>0}\, dx \tag{28}$$

Using Brownian scaling again, we remark that for every t, the process

$$m_u^{(t)} = \frac{1}{\sqrt{t-g_t}} |Z_{g_t+u(t-g_t)}| \,;\, u \leq 1$$

is a Brownian meander independent of the σ-field $\mathcal{F}_{g_t}^+$; in particular, the law of $m^{(t)}$ does not depend on t.

8.2.1 The Azéma martingale

We now introduce the so-called Azéma martingale $\mu_t = (\mathrm{sgn} Z_t)\sqrt{t-g_t}$, which is a remarquable $\mathcal{F}_{g_t}^+$-martingale. Following [2] closely, we project the $\mathcal{F}_t$-martingale $\exp(\lambda Z_t - \frac{\lambda^2}{2}t)$ on the filtration $\mathcal{F}_{g_t}^+$:

$$E(\exp(\lambda Z_t - \frac{\lambda^2}{2}t)|\mathcal{F}_{g_t}^+) = E\left(\exp(\lambda\, m_1^{(t)}\, \mu_t - \frac{\lambda^2}{2}t)|\mathcal{F}_{g_t}^+\right)$$

and, from the independence property we just recalled, we get

$$E(\exp(\lambda Z_t - \frac{\lambda^2}{2}t)|\mathcal{F}_{g_t}^+) = \exp(-\frac{\lambda^2}{2}t)\Psi(\lambda\mu_t)$$

where $\Psi(z) = E(\exp(zm_1)) = \int_0^\infty x \exp(zx - \frac{x^2}{2})dx$.

[3] see Chung [4] or Revuz-Yor [22] Ex 3.8 ch. XII.

8.3 The law of $(H^-_{b,D}\,,\, Z_{H^-_{b,D}})$

8.3.1 Case $b = 0$

In this case, we denote $H^-_D = H^-_{0,D}$. The variable H^-_D is a $\mathcal{F}^+_{g_t}$, hence a $\mathcal{F}_t$, stopping time. The process

$$\left(\frac{1}{\sqrt{D}}\,|Z_{g_{H^-_D}} + uD|\,,\, u \le 1\right)$$

is a Brownian meander independent of $\mathcal{F}^+_{g_{H^-_D}}$. In particular $\frac{1}{\sqrt{D}} Z_{H^-_D}$ is distributed as $-m_1$:

$$P(Z_{H^-_D} \in dx) = \frac{-x}{D}\,\exp(-\frac{x^2}{2D})\,\mathbb{1}_{x<0}\,dx,$$

and the variables H^-_D and $Z_{H^-_D}$ are independent. For any $\lambda > 0$, the local martingale $(\Psi(-\lambda\mu_{t\wedge H^-_D})\,\exp(-\frac{\lambda^2}{2}(t\wedge H^-_D)\,,\, t \ge 0)$ is bounded. Hence, using the optional stopping theorem at H^-_D, we obtain

$$E\left(\Psi(-\lambda\mu_{H^-_D})\exp(-\frac{\lambda^2}{2}H^-_D)\right) = \Psi(0) = 1$$

and the left hand side is equal to $\Psi(\lambda\sqrt{D}\,)\,E(\exp(-\frac{\lambda^2}{2}H^-_D))$. The formula

$$E(\exp(-\frac{\lambda^2}{2}H^-_D)) = \frac{1}{\Psi(\lambda\sqrt{D}\,)}$$

follows. Such formulae go back to Wendel [24].

8.3.2 Case $b < 0$

This case study may be reduced to the previous one, with the help of the stopping time T_b. Since

$$H^-_{b,D} = T_b + \hat{H}^-_{0,D}(W)$$

where

$$\hat{H}^-_{0,D}(W) = \inf\{t \ge 0\,;\, \mathbb{1}_{W_t \le 0}(t - g^W_t) \ge D\} \overset{law}{=} H^-_{0,D}$$
$$W_t = Z_{T_b+t} - b\,;\; g^W_t = \sup\{u \le t\,;\, W_u = 0\}$$

it follows, from the independence of T_b and $\hat{H}^-_{0,D}(W)$, that

$$E(\exp -\frac{\lambda^2}{2}H^-_{b,D}) = E(\exp -\frac{\lambda^2}{2}T_b)\,E(\exp -\frac{\lambda^2}{2}\hat{H}^-_{0,D}(W))$$

and

$$P(Z_{H^-_{b,D}} \in dx) = P(Z_{H^-_{b,D}} - b \in dx - b) = \frac{dx}{D}\, 1\!\!1_{x<b}\,(b-x)\exp -\frac{(x-b)^2}{2D}\,.$$

We obtain finally

$$E(\exp -\frac{\lambda^2}{2} H^-_{b,D}) = \frac{\exp(b\lambda)}{\Psi(\lambda\sqrt{D})}\,.$$

Note in particular that $P(H^-_{b,D} < \infty) = 1$.

8.3.3 Case $b > 0$

In this case, the first excursion below b begins at $t = 0$. We now use the obvious equality

$$E(\exp -\lambda H^-_{b,D}) = E(1\!\!1_{T_b<D} \exp -\lambda H^-_{b,D}) + E(1\!\!1_{T_b>D} \exp -\lambda H^-_{b,D})$$

On the set $\{T_b > D\}$, we have $H^-_{b,D} = D$, therefore

$$\begin{aligned} E(1\!\!1_{T_b>D} \exp -\lambda H^-_{b,D}) &= \exp(-\lambda D) P(T_b > D) \\ &= \exp(-\lambda D)\, \mu_b(]D,\infty[)\,, \end{aligned}$$

whereas, on the set $\{T_b < D\}$, we write, as in the previous subsection 8.3.2 $H^-_{b,D} = T_b + \hat{H}^-_{0,D}$. Hence, on $(T_b < D)$, we have :

$$E(\exp -\lambda H^-_{b,D} \,|\, \mathcal{F}_{T_b}) = \exp(-\lambda T_b)\, E(\exp -\lambda \tilde{H}^{b,-}_{b,D})\,.$$

Therefore $E(1\!\!1_{T_b<D} \exp -\lambda H^-_{b,D}) = \dfrac{1}{\Psi(\sqrt{2\lambda D})} E(1\!\!1_{T_b<D} \exp(-\lambda T_b))$.

It follows that

$$E(1\!\!1_{T_b<D} \exp(-\lambda H^-_{b,D})\,) = \frac{1}{\Psi(\sqrt{2\lambda D})} \int_0^D \mu_b(dx) e^{-\lambda x}$$

hence

$$E(\exp -\lambda H^-_{b,D}) = e^{-\lambda D}\, \mu_b(]D,\infty[) + \frac{1}{\Psi(\sqrt{2\lambda D})} \int_0^D \mu_b(dx)\, e^{-\lambda x}$$

The law of $Z_{H^-_{b,D}}$ can easily be deduced from the three following equalities

$$\begin{aligned} Z_{H^-_{b,D}} &= (b + W_{\hat{H}^-_{b,D}}) 1\!\!1_{T_b<D} + Z_D 1\!\!1_{T_b>D} \\ P(b + W_{\hat{H}^-_{b,D}} \in dx, T_b < D) &= P(T_b < D) 1\!\!1_{x\le b}(b-x)\, \exp(-\frac{(x-b)^2}{2D})\frac{dx}{D} \\ P(Z_D \in dx, T_b > D) &= \frac{dx}{\sqrt{2\pi D}} \left(\exp(-\frac{x^2}{2D}) - \exp(-\frac{(x-2b)^2}{2D}) \right) \end{aligned}$$

8.4 The law of $(H^+_{b,D}\,,\,Z_{H^+_{b,D}})$

8.4.1 Case $b = 0$

In this case, we denote $H^+_D = H^+_{0,D}$.

Using the symmetry of the Brownian motion law, it follows that

$$E(\exp(-\frac{\lambda^2}{2}H^+_D)) = \frac{1}{\Psi(\lambda\sqrt{D}\,)}$$

8.4.2 Case $b > 0$

The computation of the law of $(H^+_{b,D}\,,\,Z_{H^+_{b,D}})$ is similar to that of $(H^-_{b,D}\,,\,Z_{H^-_{b,D}})$ in the case $b < 0$.

8.4.3 Case $b < 0$

The computation of the law of $(H^+_{b,D}\,,\,Z_{H^+_{b,D}})$ is similar to that of $(H^-_{b,D}\,,\,Z_{H^-_{b,D}})$ in the case $b > 0$. In particular, we find that

$$E(\exp -\lambda H^+_{b,D}) = e^{-\lambda D}\mu_{-b}(]D,\infty[) + \frac{1}{\Psi(\sqrt{2\lambda D})}\int_0^D \mu_{-b}(dx)\, e^{-\lambda x}$$

8.5 Some Laplace transforms

We have defined, in the case $b \leq 0$,

$$h(T,y) = \frac{1}{D}\int_0^\infty dz\, z\exp(-\frac{z^2}{2D})\,\gamma(T, b-z-y) \tag{29}$$

where

$$\gamma(t,x) = E\left(\frac{1\!\!1_{H^-_{b,D}<t}}{\sqrt{2\pi(t-H^-_{b,D})}}\exp\left(-\frac{x^2}{2(t-H^-_{b,D})}\right)\right)$$

We obtain

$$\begin{aligned}\int_0^\infty dt\, e^{-\lambda t}\gamma(t,x) &= E_P\left[\int_{H^-_{b,D}}^\infty dt\, \exp(-\frac{x^2}{2(t-H^-_{b,D})})\frac{e^{-\lambda t}}{\sqrt{2\pi(t-H^-_{b,D})}}\right] \\ &= E(e^{-\lambda H^-_{b,D}})\int_0^\infty du\, \exp(-\frac{x^2}{2u})\frac{e^{-\lambda u}}{\sqrt{2\pi u}}\,. \end{aligned} \tag{30}$$

The integral on the right of (30) is the resolvent kernel of Brownian motion, hence is equal to $\frac{1}{\sqrt{2\lambda}}\,e^{-|x|\sqrt{2\lambda}}$. By plugging this result in the equality (30), we obtain :

$$\int_0^\infty dt\, e^{-\lambda t}\gamma(t,x) = \frac{e^{-(|x|-b)\sqrt{2\lambda}}}{\sqrt{2\lambda}\,\Psi(\sqrt{2\lambda D})} \tag{31}$$

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