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Multipliers between Sobolev spaces and fractional differentiation

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Abstract

We characterize the pointwise multipliers which maps a Sobolev space $\dot{H}^r(\mathbb{R}^d)$ to a Sobolev space $\dot{H}^s(\mathbb{R}^d)$ in the case $|s| < r < d/2$.

Keywords: Sobolev spaces; Multipliers; Paraproducts; Singular integrals; Fractional differentiation

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Pointwise multipliers between Sobolev spaces have been examined by Maz'ya and his co-workers [9,10,12,13]. In this paper, we show how the use of paradifferential calculus allows one to characterize the multipliers from $\dot{H}^r$ to $\dot{H}^s$ ($|s| < r < d/2$) as fractional derivatives of paramultipliers of $\dot{H}^r$.

1. Homogeneous Sobolev spaces

We define the Fourier transform of a function f in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ as the function $\hat{f}$ (or $\mathcal{F}f$) defined as

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$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(x)e^{-ix\cdot\xi}\,dx. \tag{1}$$

The inverse Fourier transform then allows one to compute f from $\hat{f}$ by the formula

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = \frac{1}{(2\pi)^d}\int f(\xi)e^{ix\cdot\xi}\,d\xi. \tag{2}$$

For $|r| < d/2$, we define the homogeneous Sobolev space $\dot{H}^r(\mathbb{R}^d)$ as the closure of $\mathcal{S}(\mathbb{R}^d)$ for the norm

$$\|f\|_{\dot{H}^r} = \frac{1}{(2\pi)^{d/2}}\big\||\xi|^r\hat{f}(\xi)\big\|_2. \tag{3}$$

We then have the following dense embeddings:

$$\mathcal{S}(\mathbb{R}^d) \subset \dot{H}^r(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d). \tag{4}$$

Moreover, the scalar product in L^2 allows one to identify $\dot{H}^{-r}(\mathbb{R}^d)$ to the dual space of $\dot{H}^r(\mathbb{R}^d)$: using the Plancherel formula

$$\int f(x)\bar{g}(x)\,dx = \frac{1}{(2\pi)^d}\int \hat{f}(\xi)\bar{\hat{g}}(\xi)\,d\xi, \tag{5}$$

we get that

$$\dot{H}^{-r}(\mathbb{R}^d) = \big\{T \in \mathcal{S}'(\mathbb{R}^d)\ \big|\ \exists C \geqslant 0\ \forall\varphi \in \mathcal{S}(\mathbb{R}^d)\ \big|\langle T|\varphi\rangle\big| \leqslant C\|\varphi\|_{\dot{H}^r}\big\} \tag{6}$$

and

$$\|T\|_{\dot{H}^{-r}} = \sup_{\varphi\in\mathcal{S}}\frac{|\langle T|\varphi\rangle|}{\|\varphi\|_{\dot{H}^r}}. \tag{7}$$

Finally, we quote the following well-known Sobolev inequalities:

$$\text{for } r \in [0, d/2) \text{ and } 1/p = 1/2 - r/d,\ \exists C_r \geqslant 0\ \forall\varphi \in \mathcal{S}\quad \|\varphi\|_p \leqslant C_r\|\varphi\|_{\dot{H}^r}. \tag{8}$$

We now check that the product between a distribution in $\dot{H}^r$ and a distribution in $\dot{H}^s$ is well defined as a distribution in $\mathcal{S}'(\mathbb{R}^d)$ provided that $r + s \geqslant 0$:

Lemma 1. *Let $r, s, t \in (-d/2, d/2)$ such that $r + s \geqslant 0$.*

(i) *If $r + s > 0$, let $t = d/2 - r - s$. Then there exists a constant $C_{r,s} \geqslant 0$ such that, for all φ, ψ and ω in $\mathcal{S}(\mathbb{R}^d)$, we have*

$$\left|\int \varphi\psi\omega\,dx\right| \leqslant C_{r,s}\|\varphi\|_{\dot{H}^r}\|\psi\|_{\dot{H}^s}\|\omega\|_{\dot{H}^t}. \tag{9}$$

(ii) *If $r + s = 0$, then there exists a constant $C_r \geqslant 0$ such that, for all φ, ψ and ω in $\mathcal{S}(\mathbb{R}^d)$, we have*

$$\left|\int \varphi\psi\omega\,dx\right| \leqslant C_r\|\varphi\|_{\dot{H}^r}\|\psi\|_{\dot{H}^{-r}}\big(\big\||\xi|^{d/2}\hat{\omega}\big\|_2 + \|\hat{\omega}\|_1\big). \tag{10}$$

Proof. (i) Since we have $r+s+t=d/2$ with $r,s,t\in(-d/2,d/2)$, we have not only $r+s>0$ but $s+t>0$ and $t+r>0$, so that at least two of the numbers r,s,t are positive. We may assume that $r>0$ and $s>0$ and we consider two cases:

(α) $t\geqslant 0$. We use the Sobolev embeddings $\dot H^r\subset L^{p_1}$, $\dot H^s\subset L^{p_2}$ and $\dot H^t\subset L^{p_3}$ with $1/p_1+1/p_2+1/p_3=1/2-r/d+1/2-s/d+1/2-t/d=1$.

(β) $t<0$. We write

$$\begin{aligned}\|\varphi\psi\|_{\dot H^{-t}}&=\frac{1}{(2\pi)^{3d/2}}\big\||\xi|^{-t}(\hat\varphi*\hat\psi)\big\|_2\\&\leqslant\frac{2^{-t}}{(2\pi)^{3d/2}}\big(\big\|\big(|\xi|^{-t}|\hat\varphi|\big)*|\hat\psi|\big\|_2+\big\||\hat\varphi|*\big(|\xi|^{-t}|\hat\psi|\big)\big\|_2\big).\end{aligned}$$

We then use Lorentz spaces: for $\alpha\in(0,d)$, $|\xi|^{-\alpha}\in L^{d/\alpha,\infty}$; for p_1, p_2 in $(1,\infty)$ and q_1, q_2 in $[1,\infty]$ the pointwise product $(f,g)\mapsto fg$ maps $L^{p_1,q_1}\times L^{p_2,q_2}$ to $L^{p,q}$ with $1/p=1/p_1+1/p_2$, $1/q=1/q_1+1/q_2$ (provided that $1/p<1$ and $1/q\leqslant 1$) and the convolution product $(f,g)\mapsto f*g$ maps $L^{p_1,q_1}\times L^{p_2,q_2}$ to $L^{p,q}$ with $1/p'=1/p_1+1/p_2-1$, $1/q=1/q_1+1/q_2$ (provided that $1/p'>0$ and $1/q\leqslant 1$). Since $L^{2,2}=L^2$ and $L^{p,q}\subset L^{p,q'}$ for $q\leqslant q'$, we get, defining $1/p_1=1/2+(r+t)/d$ and $1/p_2=1/2+s/d$ (so that $1/p_1+1/p_2=1/2$),

$$\begin{aligned}\big\|\big(|\xi|^{-t}|\hat\varphi|\big)*|\hat\psi|\big\|_2&\leqslant C\big\||\xi|^{-t}\hat\varphi\big\|_{L^{p_1,2}}\|\hat\psi\|_{L^{p_2,2}}\leqslant C'\big\||\xi|^r\hat\varphi\big\|_2\big\||\xi|^s\hat\psi\big\|_2\\&\leqslant C''\|\varphi\|_{\dot H^r}\|\psi\|_{\dot H^s}.\end{aligned}$$

A similar estimates holds for $\||\hat\varphi|*(|\xi|^{-t}|\hat\psi|)\|_2$. Thus, we get (9) by duality between $\dot H^t$ and $\dot H^{-t}$.

(ii) The proof is similar to the case (β): if $r=0$ the result is obvious (since $\|\omega\|_\infty\leqslant(2\pi)^{-d}\|\hat\omega\|_1$); if $r>0$, we write

$$\begin{aligned}\|\varphi\omega\|_{\dot H^r}&=\frac{1}{(2\pi)^{3d/2}}\big\||\xi|^r(\hat\varphi*\hat\omega)\big\|_2\\&\leqslant\frac{2^r}{(2\pi)^{3d/2}}\big(\big\|\big(|\xi|^r|\hat\varphi|\big)*|\hat\omega|\big\|_2+\big\||\hat\varphi|*\big(|\xi|^r|\hat\omega|\big)\big\|_2\big)\end{aligned}$$

and we use the embeddings $L^2*L^1\subset L^2$ and $L^{p_1,2}*L^{p_2,2}\subset L^2$ with $1/p_1=1/2+r/d$ and $1/p_2=1/2+(d/2-r)/d$. □

Corollary 1. *Let $r,s\in(-d/2,d/2)$ such that $r+s\geqslant 0$. Then the pointwise product $(f,g)\mapsto fg$ can be extended as a bounded bilinear map from $\dot H^r\times\dot H^s$ to $\mathcal S'(\mathbb R^d)$.*

Estimate (10) is far from being optimal: indeed, it is quite easy to see that we may replace the norm $\|\hat\omega\|_1$ by the weaker norm $\|\omega\|_\infty$. In order to prove this (classical) estimate, we shall use the paraproduct operators. First, we introduce the well-known Littlewood–Paley decomposition of distributions into dyadic blocks of frequencies.

Definition 1. Let $\varphi_0\in\mathcal D(\mathbb R^d)$ be a non-negative radial function such that

$$|\xi|\leqslant\frac12\quad\Rightarrow\quad\varphi_0(\xi)=1\quad\text{and}\quad|\xi|\geqslant 1\quad\Rightarrow\quad\varphi_0(\xi)=0.$$

Let ψ_0 be defined as $\psi_0(\xi) = \varphi_0(\xi/2) - \varphi_0(\xi)$. Let S_j and Δ_j be defined as the Fourier multipliers $\mathcal{F}(S_j f) = \varphi_0(\xi/2^j)\mathcal{F}f$ and $\mathcal{F}(\Delta_j f) = \psi_0(\xi/2^j)\mathcal{F}f$. The distribution $\Delta_j f$ is called the jth dyadic block of the Littlewood–Paley decomposition of f.

Lemma 2. *For all $N \in \mathbb{Z}$ and all $f \in \mathcal{S}'(\mathbb{R}^d)$ we have*

$$f = S_N f + \sum_{j \geqslant N} \Delta_j f \quad \text{in } \mathcal{S}'(\mathbb{R}^d).$$

This equality is called the Littlewood–Paley decomposition of the distribution f. If, moreover, $\lim_{N\to-\infty} S_N f = 0$ in $\mathcal{S}'$, then the equality $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ is called the homogeneous Littlewood–Paley decomposition of f.

Proof. Clearly, we have:

$$\left\langle S_N f + \sum_{N \leqslant j < N+K} \Delta_j f \middle| g \right\rangle_{\mathcal{S}',\mathcal{S}} = \langle S_{N+K} f | g\rangle_{\mathcal{S}',\mathcal{S}} = \langle f | S_{N+K} g\rangle_{\mathcal{S}',\mathcal{S}}.$$

Thus, taking the Fourier transform $h = \hat{g}$ of g, it is enough to check that, for any $h \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\lim_{N\to\infty} \varphi_0\left(\frac{\xi}{2^n}\right) h(\xi) = h(\xi) \quad \text{strongly in } \mathcal{S}. \qquad \square$$

We have the following classical characterization of homogeneous Sobolev spaces.

Lemma 3. *Let $r \in (-d/2, d/2)$. Then:*

(i) *the following Littlewood–Paley decomposition of $\dot{H}^r$ holds:*

$$f \in \dot{H}^r \quad \Leftrightarrow \quad f = \sum_{j\in\mathbb{Z}} \Delta_j f \quad \text{in } \mathcal{S}' \quad \text{and} \quad \sum_{j\in\mathbb{Z}} 4^{jr} \|\Delta_j f\|_2^2 < \infty \tag{11}$$

and there exists two positive constants A_r, B_r such that

$$\forall f \in \dot{H}^r \quad A_r \|f\|_{\dot{H}^r} \leqslant \sqrt{\sum_{j\in\mathbb{Z}} 4^{jr} \|\Delta_j f\|_2^2} \leqslant B_r \|f\|_{\dot{H}^r}. \tag{12}$$

(ii) *More generally, if A and B are two positive constants and if $f = \sum_{j\in\mathbb{Z}} f_j$ in $\mathcal{S}'$ where*

$$\hat{f}_j \text{ is supported in } \left\{\xi \in \mathbb{R}^d \mid A2^j \leqslant |\xi| \leqslant B2^j\right\}$$

and

$$\sum_{j\in\mathbb{Z}} 4^{jr} \|\Delta_j f\|_2^2 < \infty$$

then $f \in \dot{H}^r$ and

$$\|f\|_{\dot{H}^r} \leqslant C \sqrt{\sum_{j\in\mathbb{Z}} 4^{jr} \|\Delta_j f\|_2^2}$$

for some constant C (which depends only on A, B, r).

(iii) *Similarly, if $r > 0$ and if R is some positive constant and if $f = \sum_{j\in\mathbb{Z}} f_j$ in $\mathcal{S}'$ where*

$$\hat{f}_j \text{ is supported in } \left\{\xi \in \mathbb{R}^d \;\middle|\; |\xi| \leqslant R2^j\right\}$$

and

$$\sum_{j\in\mathbb{Z}} 4^{jr}\|\Delta_j f\|_2^2 < \infty$$

then $f \in \dot{H}^r$ and

$$\|f\|_{\dot{H}^r} \leqslant C\sqrt{\sum_{j\in\mathbb{Z}} 4^{jr}\|\Delta_j f\|_2^2}$$

for some constant C (which depends only on R and r).

Proof. By Plancherel, we just need to check that

$$\forall\xi \neq 0 \quad A_r^2|\xi|^{2r} \leqslant \sum_{j\in\mathbb{Z}} 4^{jr}\psi_0(\xi/2^j)^2 \leqslant B_r^2|\xi|^{2r}$$

and, by Plancherel and Cauchy–Schwarz, that, writing $\chi_{A,B}(\xi) = 1_{A\leqslant|\xi|\leqslant B}$

$$\forall\xi \neq 0 \quad |\xi|^{2r}\sum_{j\in\mathbb{Z}} 4^{-jr}\chi_{A,B}(\xi/2^j) \leqslant C^2$$

and, for $r > 0$, writing $\chi_R(\xi) = 1_{|\xi|\leqslant R}$, that

$$\forall\xi \neq 0 \quad |\xi|^{2r}\sum_{j\in\mathbb{Z}} 4^{-jr}\chi_R(\xi/2^j) \leqslant C^2. \qquad \square$$

The Littlewood–Paley decomposition of a product is an useful tool in non-linear analysis.

Lemma 4. *Let f and g in $\mathcal{S}(\mathbb{R}^d)$. Then, for $j \in \mathbb{Z}$, we have*

$$\begin{aligned}\Delta_j(fg) = {}&\sum_{l=-3}^{3}\Delta_j(S_{j-2}f\,\Delta_{j+l}g) + \sum_{l=-3}^{3}\Delta_j(\Delta_{j+l}f\,S_{j-2}g) + \sum_{k=j-2}^{\infty}\Delta_j(\Delta_k f\,\Delta_k g)\\ &+ \sum_{k=j-2}^{\infty}\sum_{l=1}^{5}\Delta_j(\Delta_k f\,\Delta_{k+l}g) + \sum_{k=j-2}^{\infty}\sum_{l=1}^{5}\Delta_j(\Delta_{k+l}f\,\Delta_k g).\end{aligned} \tag{13}$$

Proof. It is enough to write

$$\Delta_j(fg) = \Delta_j\left(\left(S_{j-2}f + \sum_{k=j-2}^{\infty}\Delta_k f\right)\left(S_{j-2}g + \sum_{l=j-2}^{\infty}\Delta_l g\right)\right)$$

and to study the support of the Fourier transforms. $\square$

We now give the improvement of (10) with use of the L^∞ norm.

Lemma 5. *Let $r \in (0, d/2)$. There exists a constant $C_r \geqslant 0$ such that, for all φ, ψ and ω in $\mathcal{S}(\mathbb{R}^d)$, we have*

$$\left|\int \varphi\psi\omega\,dx\right| \leqslant C_r \|\varphi\|_{\dot H^r} \|\psi\|_{\dot H^{-r}} \big(\|\omega\|_{\dot H^{d/2}} + \|\omega\|_\infty\big). \tag{14}$$

Proof. We use (13) to estimate $\|\Delta_j(\varphi\omega)\|_2$. We have

$$2^{j(r-d/2)}\|\Delta_j\varphi\|_\infty \leqslant C2^{jr}\|\Delta_j\varphi\|_2$$

and thus from the inequality

$$\begin{aligned}2^{jr}\|\Delta_j(\varphi\omega)\|_2 \leqslant C2^{jr}\Bigg(&\sum_{l=-3}^{3}\|S_{j-2}\varphi\|_\infty\|\Delta_{j+l}\omega\|_2 + \sum_{l=-3}^{3}\|\Delta_{j+l}\varphi\|_2\|S_{j-2}\omega\|_\infty\\ &+\sum_{k=j-2}^{\infty}\|\Delta_k\varphi\|_2\|\Delta_k\omega\|_\infty + \sum_{k=j-2}^{\infty}\sum_{l=1}^{5}\|\Delta_k\varphi\|_2\|\Delta_{k+l}\omega\|_\infty\\ &+\sum_{k=j-2}^{\infty}\sum_{l=1}^{5}\|\Delta_{k+l}\varphi\|_2\|\Delta_k\omega\|_\infty\Bigg)\end{aligned}$$

we get that

$$\begin{aligned}&2^{jr}\|\Delta_j(\varphi\omega)\|_2\\ &\quad\leqslant C\left(\sum_{k=-\infty}^{j-3}2^{kr}\|\Delta_k\varphi\|_2 2^{(j-k)(d/2-r)}\|\omega\|_{\dot H^{d/2}} + \sum_{k=j-3}^{\infty}2^{kr}\|\Delta_k\varphi\|_2 2^{(j-k)r}\|\omega\|_\infty\right)\end{aligned}$$

and finally

$$\|\varphi\omega\|_{\dot H^r} \leqslant C\|\varphi\|_{\dot H^r}\big(\|\omega\|_{\dot H^{d/2}} + \|\omega\|_\infty\big). \qquad \square$$

We shall need two other properties of Sobolev spaces. By using the Fourier transform, those properties are based on the properties of weighted Lebesgue's spaces, namely the spaces $L^2(|\xi|^{2r}\,d\xi)$, where the weight $|\xi|^{2r}$ belongs to the Muckenhoupt class $\mathcal{A}_2$. First, the theory of complex interpolation of weighted Lebesgue spaces [1] gives us the following interpolation property:

$$\text{for } r_0 \text{ and } r_1 \in (-d/2, d/2) \text{ and } \theta \in (0,1), \quad [\dot H^{r_0}, \dot H^{r_1}]_\theta = \dot H^{(1-\theta)r_0+\theta r_1}. \tag{15}$$

Second, the Littlewood–Paley decomposition theory for weighted Lebesgue spaces with Muckenhoupt weights gives us that for the function ψ_0 in Definition 1, we have the following equivalence of norms: there exists two positive constants A_r and B_r such that, for every $f \in \dot H^r$,

$$A_r\|f\|_{\dot H^r}^2 \leqslant \sum_{j\in\mathbb{Z}}\big\|\psi_0(2^{-j}x)f(x)\big\|_{\dot H^r}^2 \leqslant B_r\|f\|_{\dot H^r}^2. \tag{16}$$

2. The space of pointwise multipliers $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$

We now introduce the spaces of pointwise singular multipliers between the Sobolev spaces $\dot{H}^r$ and $\dot{H}^s$.

Definition 2. Let $r, s \in (-d/2, d/2)$ such that $s \leqslant r$. Then we define $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$ as the space of the distributions T such that there exists a constant C such that for all $\varphi \in \mathcal{S}$ we have $\varphi T \in \dot{H}^s$ and $\|\varphi T\|_{\dot{H}^s} \leqslant C\|\varphi\|_{\dot{H}^r}$. We define the norm $\|\ \|_{\mathcal{M}(\dot{H}^r \to \dot{H}^s)}$ as

$$\|T\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^s)} = \sup_{\varphi \in \mathcal{S}} \frac{\|\varphi T\|_{\dot{H}^s}}{\|\varphi\|_{\dot{H}^r}}. \tag{17}$$

Since we have

$$\|T\|_{\mathcal{M}(\dot{H}^r \to \dot{H}^s)} = \sup_{\varphi, \psi \in \mathcal{S}} \frac{|\langle \varphi T | \psi \rangle|}{\|\varphi\|_{\dot{H}^r} \|\psi\|_{\dot{H}^{-s}}} \tag{18}$$

we find that

$$\mathcal{M}(\dot{H}^r \to \dot{H}^s) = \mathcal{M}(\dot{H}^{-s} \to \dot{H}^{-r}) \tag{19}$$

so that we shall always assume that $r + s \geqslant 0$.

Those spaces of multipliers are useful to give minimal regularity requirements for nonlinear estimates in PDEs. For instance, the space $\mathcal{M}(\dot{H}^1 \mapsto \dot{H}^{-1})$ has been considered as the space of potentials V such that the Schrödinger operator $-\Delta + V$ is bounded from $\dot{H}^1$ to $\dot{H}^{-1}$ [12]. Another example is the celebrated uniqueness criterion of Serrin for the Leray solutions of the 3D Navier–Stokes equations. The solutions $\vec{u}$, $\vec{v}$ are assumed to belong to $L^\infty L^2 \cap L^2 \dot{H}^1$, hence to $L^t \dot{H}^r$ for $0 \leqslant r \leqslant 1$ and $1/t = r/2$, then Serrin's criterion grants uniqueness provided that at least one solution $\vec{u}$ belongs to $L^p L^q$ with $2/p = 1 - 3/q$. The proof relies on the fact that the quantity

$$\int_0^T \int_{\mathbb{R}^3} \vec{u} \cdot \big((\vec{u} - \vec{v}) \cdot \vec{\nabla}\big)(\vec{u} - \vec{v})\, dx\, dt$$

is well defined for every finite positive T, as it may be seen by writing $\dot{H}^r \subset L^\sigma$ for $1/\sigma = 1/2 - r/3$, so that $\int f g h\, dx$ is well defined for $f \in L^q$, $g \in \dot{H}^r$ and $h \in L^2$ with $1/q = r/3$, and $\iint f g h\, dx\, dt$ is well defined for $f \in L^p L^q$, $g \in \dot{L}^t H^r$ and $h \in L^2 L^2$ with $2/p = 1 - 3/q$, $1/q = r/3$ and $1/t = 1/2 - 1/p = r/2$. This criterion extends in a straightforward manner by replacing the requirement $\vec{u} \in L^p L^q$ with $2/p = 1 - 3/q$ by the requirement $\vec{u} \in L^p \mathcal{M}(\dot{H}^r \mapsto L^2)$ with $2/p = 1 - r$ [8]. Further applications to PDEs are described in [3,4,11–13].

We now recall some classical examples of multipliers.

Lemma 6.

(i) *Let r and s in $(-d/2, d/2)$ with $s < r$. Then the function $f(x) = \|x\|^{s-r}$ belongs to $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$.*

(ii) *Let* $r \in (-d/2, d/2)$. *Then, for* $\gamma \in \mathbb{R}$, *the function* $f(x) = \|x\|^{i\gamma}$ *belongs to* $\mathcal{M}(\dot{H}^r \to \dot{H}^r)$.

Proof. (i) We use inequalities (16) and write for $g \in \dot{H}^s$

$$A_s \|fg\|^2_{\dot{H}^s} \leqslant \sum_{j\in\mathbb{Z}} \|\psi_0(2^{-j}x)f(x)g(x)\|^2_{\dot{H}^s} = \sum_{j\in\mathbb{Z}} 2^{j(d/2-s)} \|\psi_0(x)f(2^jx)g(2^jx)\|^2_{\dot{H}^s}$$

then we use the embedding

$$\{h \in \dot{H}^r \mid h(x) = 0 \text{ for } \|x\| > 4\} \subset \dot{H}^r$$

and the property that $f(2^jx) = 2^{j(s-r)}f(x)$ to get

$$\|fg\|^2_{\dot{H}^s} \leqslant C(r,s) \sum_{j\in\mathbb{Z}} 2^{j(d/2-r)} \|\psi_0(x)f(x)g(2^jx)\|^2_{\dot{H}^r};$$

moreover, in the neighbourhood of the support of ψ_0, f is a smooth bounded function so that we may use (10) or (14) to get that there exists a constant C_r such that for every $h \in \dot{H}^r$ we have $\|\psi_0 fh\|_{\dot{H}^r} \leqslant C_R \|\psi_0 h\|_{\dot{H}^r}$ and finally we get

$$\begin{aligned}\|fg\|^2_{\dot{H}^s} &\leqslant C(r,s)C_r \sum_{j\in\mathbb{Z}} 2^{j(d/2-r)} \|\psi_0(x)g(2^jx)\|^2_{\dot{H}^r} \\ &\leqslant C(r,s)C_r \sum_{j\in\mathbb{Z}} \|\psi_0(2^{-j}x)g(x)\|^2_{\dot{H}^r}\end{aligned}$$

and thus

$$\|fg\|^2_{\dot{H}^s} \leqslant C(r,s)C_r B_r \|g\|^2_{\dot{H}^r}.$$

The proof of (ii) follows the same lines, since $\|2^jx\|^{i\gamma} = 2^{ij\gamma}\|x\|^{i\gamma}$. □

One important tool in studying pointwise products is the paraproduct operator, based on the Littlewood–Paley decomposition and more precisely on the spectral analysis of $\Delta_j(fg)$ given in Lemma 4 and formula (13). We shall use formula (13) when dealing with distributions in Besov spaces.

Definition 3. For $s < 0$ we define the Besov space $\dot{B}^{s,\infty}_\infty$ as

$$\dot{B}^{s,\infty}_\infty = \Big\{ f \in \mathcal{S}'(\mathbb{R}^d) \ \Big|\ \sup_{j\in\mathbb{Z}} 2^{js}\|S_jf\|_\infty < \infty \Big\} \tag{20}$$

normed with

$$\|f\|_{\dot{B}^{s,\infty}_\infty} = \sup_{j\in\mathbb{Z}} 2^{js}\|S_jf\|_\infty. \tag{21}$$

Definition 4. For $f \in \dot{B}^{s,\infty}_\infty$, $s < 0$, we define the paraproduct operator $\pi(f,\cdot)$ and the remainder operator $\rho(f,\cdot)$ as

$$\pi(f,g)=\sum_{j\in\mathbb{Z}} S_{j-2}g\Delta_j f=\sum_{j\in\mathbb{Z}}\sum_{l=-3}^{3}\Delta_{j+l}(S_{j-2}g\Delta_j f) \tag{22}$$

and

$$\rho(f,g)=\sum_{j\in\mathbb{Z}}\frac{1}{2}\Delta_j(S_{j+3}f\Delta_j g)+\sum_{j\in\mathbb{Z}}\sum_{l=1}^{5}\Delta_{j+l}(S_{j+3}f\Delta_j g). \tag{23}$$

We may rewrite Lemma 4 as follows.

Lemma 7. *Let f and g in $\mathcal{S}(\mathbb{R}^d)$ and let $h\in\dot{B}_\infty^{s,\infty}$, $s<0$. Then, we have*

$$\int fgh\,dx=\int\pi(h,f)g\,dx+\int\rho(h,f)g\,dx+\int\pi(h,g)f\,dx+\int\rho(h,g)f\,dx \tag{24}$$

The role of Besov spaces and paraproduct operators is explained in the following proposition.

Proposition 1. *Let $r\in(0,d/2)$ and $s\in(-r,r)$. Then the following assertions are equivalent*:

(A) $h\in\mathcal{M}(\dot{H}^r\to\dot{H}^s)$;
(B) $h\in\dot{B}_\infty^{s-r,\infty}$ *and* $\pi(h,\cdot)$ *maps boundedly* $\dot{H}^r$ *to* $\dot{H}^s$.

Proof. (A) $\Rightarrow$ (B). We may write $\varphi_0=\mathcal{F}(\theta\omega)$ with $\omega\in\mathcal{S}$ and $\theta(x)=\frac{1}{(1+x^2)^d}$. Thus, we have

$$\begin{aligned}\left|S_j h(x)\right|&=\left|\int h(y)2^{jd}\theta\big(2^j(x-y)\big)\omega\big(2^j(x-y)\big)\,dy\right|\\&\leqslant\|h\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}\|\theta\|_{\dot{H}^r}\|\omega\|_{\dot{H}^{-s}}2^{j(r-s)}.\end{aligned}$$

Hence, $h\in\dot{B}_\infty^{s-r,\infty}$.

Moreover, if $h\in\dot{B}_\infty^{s-r,\infty}$, then it is obvious that $\rho(h,\cdot)$ maps $\dot{H}^r$ to $\dot{H}^{r+s-r}=\dot{H}^s$ and $\dot{H}^{-s}$ to $\dot{H}^{-s+s-r}=\dot{H}^{-r}$. If $h\in\mathcal{M}(\dot{H}^r\to\dot{H}^s)$, then $\pi(h,\cdot)$ maps boundedly $\dot{H}^{-s}$ to $\dot{H}^{-r}$. Indeed, we have

$$\begin{aligned}\left\|\pi(h,g)\right\|_{\dot{H}^{-r}}^2&\leqslant C\sum_{j\in\mathbb{Z}}\sum_{l=-3}^{3}4^{-jr}\left\|\Delta_{j+l}(S_{j-2}g\Delta_j h)\right\|_2^2\\&\leqslant C'\sum_{j\in\mathbb{Z}}4^{-j(r+s)}\|S_{j-2}g\Delta_j h\|_{\dot{H}^s}^2\end{aligned}$$

so that, using the fact that

$$\|\Delta_j h\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}\leqslant C\|h\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}$$

(due to the easily checked fact that the convolution is a bounded bilinear operator from $L^1 \times \mathcal{M}(\dot{H}^r \to \dot{H}^s)$ to $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$, we get

$$\|\pi(h,g)\|^2_{\dot{H}^{-r}} \leqslant C\|h\|^2_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)} \sum_{j\in\mathbb{Z}} 4^{-j(r+s)}\|S_{j-2}g\|^2_{\dot{H}^r}$$

and we conclude easily since

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 4^{-j(r+s)}\|S_{j-2}g\|^2_{\dot{H}^r} &\leqslant C\sum_{j\in\mathbb{Z}} 4^{-j(r+s)}\Bigl(\sum_{k\leqslant j-3} 4^{kr}\|\Delta_k g\|_2^2\Bigr)\\ &= C\sum_{k\in\mathbb{Z}} 4^{-ks}\|\Delta_k g\|_2^2\Bigl(\sum_{j\geqslant k+3} 4^{(k-j)(r+s)}\Bigr)\end{aligned}$$

with $r+s>0$. Using (24), we get that $\pi(h,\cdot) = M_h - {}^t\pi(h,\cdot) - \rho(h,\cdot) - {}^t\rho(h,\cdot)$ (where M_h is the pointwise product operator with h: $M_h f = hf$), hence $\pi(h,\cdot)$ maps $\dot{H}^r$ to $\dot{H}^s$.

(B) ⇒ (A). If $h \in \dot{B}^{s-r,\infty}_\infty$, then we know that $\rho(h,\cdot)$ maps $\dot{H}^r$ to $\dot{H}^s$ and $\dot{H}^{-s}$ to $\dot{H}^{-r}$. Moreover, $\pi(h,\cdot)$ maps $\dot{H}^\sigma$ to $\dot{H}^{\sigma+s-r}$ for every $\sigma<0$:

$$\|\pi(h,g)\|^2_{\dot{H}^{\sigma+s-r}} \leqslant C\sum_{j\in\mathbb{Z}} 4^{j(\sigma+s-r)}\|S_{j-2}g\Delta_j h\|_2^2 \leqslant C'\|h\|^2_{\dot{B}^{s-r,\infty}_\infty}\sum_{j\in\mathbb{Z}} 4^{j\sigma}\|S_{j-2}g\|_2^2$$

and we conclude easily since

$$\begin{aligned}\sum_{j\in\mathbb{Z}} 4^{j\sigma}\|S_{j-2}g\|_2^2 &\leqslant C\sum_{j\in\mathbb{Z}} 4^{j\sigma}\Bigl(\sum_{k\leqslant j-3}\|\Delta_k g\|_2^2\Bigr)\\ &= C\sum_{k\in\mathbb{Z}} 4^{k\sigma}\|\Delta_k g\|_2^2\Bigl(\sum_{j\geqslant k+3} 4^{(j-k)\sigma}\Bigr)\end{aligned}$$

with $\sigma<0$. If we assume, moreover, that $\pi(h,\cdot)$ maps $\dot{H}^r$ to $\dot{H}^s$, then by interpolation between r and $\sigma<0$ if $s\leqslant 0$ or directly if $s>0$ we find that $\pi(h,\cdot)$ maps $\dot{H}^{-s}$ to $\dot{H}^{-r}$. Thus, using (24), we find that M_h maps $\dot{H}^r$ to $\dot{H}^s$. □

3. Fractional differentiation and fractional integration

We first introduce the predual of $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$. Due to Lemma 1, we may introduce the following Banach space.

Definition 5. Let $r,s\in(-d/2,d/2)$ such that $r+s\geqslant 0$. Then we define $\mathcal{N}_{r,s}$ as the subspace of $\mathcal{S}'(\mathbb{R}^d)$ of the distributions T that can be written as a series

$$T=\sum_{n\in\mathbb{N}} f_n g_n \quad \text{with} \sum_{n\in\mathbb{N}}\|f_n\|_{\dot{H}^r}\|g_n\|_{\dot{H}^s}<\infty$$

and the norm $\|\cdot\|_{\mathcal{N}_{r,s}}$ as

$$\|T\|_{\mathcal{N}_{r,s}} = \min_{T=\sum_{n\in\mathbb{N}} f_n g_n}\sum_{n\in\mathbb{N}}\|f_n\|_{\dot{H}^r}\|g_n\|_{\dot{H}^s}. \tag{25}$$

Proposition 2. *Let* $r \in [0, d/2)$ *and* $s \in [-r, r]$. *Then* $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$ *is a Banach space. More precisely, this is the dual space of* $\mathcal{N}_{r,-s}$.

Proof. We have

$$(\mathcal{N}_{r,-s})^* = \bigl\{T \in \mathcal{S}' \mid \exists C \geqslant 0\ \forall \varphi \in \mathcal{S}\ \bigl|\langle T|\varphi\rangle\bigr| \leqslant C\|\varphi\|_{\mathcal{N}_{r,-s}}\bigr\}.$$

Since $\|\varphi\psi\|_{\mathcal{N}_{r,-s}} \leqslant \|\varphi\|_{\dot{H}^r}\|\psi\|_{\dot{H}^{-s}}$, we have obviously $(\mathcal{N}_{r,-s})^* \subset \mathcal{M}(\dot{H}^r \to \dot{H}^s)$.

Conversely, it is easy to see that each $\varphi \in \mathcal{S}$ belongs to $\mathcal{N}_{r,-s}$ and that one may write (in $\mathcal{S}'$)

$$\varphi = \sum_{n\in\mathbb{N}} \varphi_n\psi_n \quad \text{with } \varphi_n, \psi_n \text{ in } \mathcal{S} \quad \text{where}$$

$$\sum_{n\in\mathbb{N}} \|\varphi_n\|_{\dot{H}^r}\|\psi_n\|_{\dot{H}^{-s}} \leqslant 2\|\varphi\|_{\mathcal{N}_{r,-s}}.$$

Thus, we have to prove that

$$\langle T|\varphi\rangle = \sum_{n\in\mathbb{N}} \langle T|\varphi_n\psi_n\rangle.$$

Let $\omega \in \mathcal{S}$ with $\omega(0) = 1$ and $\int \omega\,dx = 1$. For $R > 0$, we define $\omega_R(x) = \omega(x/R)$ and $\omega^{\{R\}}(x) = R^d\omega(Rx)$. Let $T_R = \omega^{\{R\}} * (\omega_R T)$. Then $T_R \in \mathcal{S}$, $T_R \to T$ in $\mathcal{S}'$ as $R \to \infty$ and

$$\|T_R\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)} \leqslant C_r\bigl(\|\omega\|_1 + \bigl\||\xi|^{d/2}\hat{\omega}\bigr\|_1 + \|\hat{\omega}\|_1\bigr)\|T\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}.$$

Thus, we find

$$\langle T_R|\varphi\rangle = \sum_{n\in\mathbb{N}} \langle T_R|\varphi_n\psi_n\rangle, \qquad \lim_{R\to\infty} \langle T_R|\varphi\rangle = \langle T|\varphi\rangle,$$

$$\lim_{R\to\infty} \langle T_R|\varphi_n\psi_n\rangle = \langle T|\varphi_n\psi_n\rangle \quad \text{and} \quad \sum_{n\in\mathbb{N}} \sup_{R>0}\bigl|\langle T_R|\varphi_n\psi_n\rangle\bigr| < \infty.$$

We then conclude by dominated convergence. □

We now introduce the fractional differentiation and fractional integration operators. Formally, the fractional differentiation operator D^ρ is the operator associated with the Fourier multiplier $|\xi|^\rho$ and, similarly, the fractional integration operator I^ρ is the operator associated with the Fourier multiplier $|\xi|^{-\rho}$. However, we shall deal with distributions and, since our Fourier multipliers are not smooth, we shall give a definition which can be applied to distributions in Besov spaces.

Definition 6. Let $\omega_0 \in \mathcal{D}$ such that $\omega_0\psi_0 = \psi_0$ and $0 \notin \operatorname{supp}\omega_0$. Then

(i) On $\dot{B}^{s,\infty}_\infty$, $s < 0$, we define the fractional differentiation operator D^ρ $(0 < \rho)$ as

$$D^\rho f = \sum_{j\in\mathbb{Z}} \mathcal{F}^{-1}\bigl(|\xi|^\rho\omega_0(\xi/2^j)\bigr) * \Delta_j f. \tag{26}$$

The operator D^ρ maps $\dot{B}^{s,\infty}_\infty$ to $\dot{B}^{s-\rho,\infty}_\infty$.

(ii) On $\dot{B}^{s,\infty}_{\infty}$, $s < 0$, we define the fractional integration operator I^{ρ} $(0 < \rho < -s)$ as

$$I^{\rho} f = \sum_{j \in \mathbb{Z}} \mathcal{F}^{-1}\big(|\xi|^{-\rho} \omega_0(\xi/2^j)\big) * \Delta_j f. \tag{27}$$

The operator I^{ρ} maps $\dot{B}^{s,\infty}_{\infty}$ to $\dot{B}^{s+\rho,\infty}_{\infty}$.

It is very easy to check that, for $s < 0$ and $0 < \rho < -s$, we have $I^{\rho} D^{\rho} = D^{\rho} I^{\rho} = id$ on $\dot{B}^{s,\infty}_{\infty}$. Moreover, we have

$$D^{\rho_1} D^{\rho_2} = D^{\rho_1 + \rho_2} \quad \text{and} \quad I^{\rho_1} I^{\rho_2} = I^{\rho_1 + \rho_2}.$$

Moreover, I^{ρ} and D^{ρ} are self-adjoint: if $T \in \dot{B}^{s,\infty}_{\infty}$ and $f \in \mathcal{S}$ then

$$\int D^{\rho} T\, f\, dx = \sum_{j \in \mathbb{Z}} \int D^{\rho} \Delta_j T\, f\, dx = \sum_{j \in \mathbb{Z}} \int T\, D^{\rho} \Delta_j f\, dx$$

and

$$\int I^{\rho} T\, f\, dx = \sum_{j \in \mathbb{Z}} \int I^{\rho} \Delta_j T\, f\, dx = \sum_{j \in \mathbb{Z}} \int T\, I^{\rho} \Delta_j f\, dx.$$

We may now state the main result in this paper.

Theorem 1. *Let* $0 < s < r < d/2$. *Then*

(i) $f \in \mathcal{M}(\dot{H}^r \to \dot{H}^s)$ *if and only if* $D^s f \in \mathcal{M}(\dot{H}^r \to L^2)$;
(ii) $f \in \mathcal{M}(\dot{H}^r \to \dot{H}^{-s})$ *if and only if* $I^s f \in \mathcal{M}(\dot{H}^r \to L^2)$.

We shall prove the result by considering the predual $\mathcal{N}_{r,-s}$ of $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$.

Proposition 3. *Let* $0 < r < d/2$ *and* $s \in (-r, r)$. *Then*

(i) *if* $\rho \in (0, s + r)$, D^{ρ} *maps* $\mathcal{N}_{r,s}$ *to* $\mathcal{N}_{r,s-\rho}$.
(ii) *if* $\rho \in (0, r - s)$, I^{ρ} *maps* $\mathcal{N}_{r,s}$ *to* $\mathcal{N}_{r,s+\rho}$.

Moreover, if $\rho \in (0, 2r)$, D^{ρ} *maps* $\mathcal{N}_{r,r}$ *to* $\mathcal{N}_{r,r-\rho}$.

Proof. We may assume that $0 < \rho < 1$ (since $D^{\rho} = (D^{\rho/N})^N$ and $I^{\rho} = (I^{\rho/N})^N$). We then have a formula for computing D^{ρ}, when $f \in \mathcal{S}$,

$$D^{\rho} f(x) = C_{\rho} \int \frac{f(x) - f(y)}{|x - y|^{d+\rho}}\, dy, \tag{28}$$

where C_{ρ} is a positive constant. This will allow us to compute $D^{\rho}(fg)$:

$$\begin{aligned} D^{\rho}(fg)(x) = {} & f(x) D^{\rho} g(x) + g(x) D^{\rho} f(x) \\ & - C_{\rho} \int \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{d+\rho}}\, dy. \end{aligned} \tag{29}$$

We then use the paraproduct and write (if $s < \rho$)

$$fg = \pi(g, f) + R(f, g)$$

with

$$\pi(g, f) = \sum_{j\in\mathbb{Z}} S_{j-2} f \Delta_j g = \sum_{j\in\mathbb{Z}} \sum_{l=-3}^{3} \Delta_{j+l}(S_{j-2} f \Delta_j g) \tag{30}$$

and

$$R(f, g) = \sum_{j\in\mathbb{Z}} S_{j+3} g \Delta_j f = \sum_{j\in\mathbb{Z}} S_{j+4}(S_{j-2} f \Delta_j g). \tag{31}$$

This gives

$$D^\rho(fg) = f D^\rho g + \pi(g, D^\rho f) + R(D^\rho f, g) - C_\rho(A + B) \tag{32}$$

with

$$A(x) = \sum_{j\in\mathbb{Z}} \int \frac{(S_{j-2} f(x) - S_{j-2} f(x-y))(\Delta_j g(x) - \Delta_j g(x-y))}{|y|^{d+\rho}}\, dy \tag{33}$$

and

$$B(x) = \sum_{j\in\mathbb{Z}} \int \frac{(S_{j+3} g(x) - S_{j+3} g(x-y))(\Delta_j f(x) - \Delta_j f(x-y))}{|y|^{d+\rho}}\, dy. \tag{34}$$

We then write

$$\|f D^\rho g\|_{\mathcal{N}_{r,s-\rho}} \leqslant \|f\|_{\dot H^r} \|D^\rho g\|_{\dot H^{s-\rho}} \leqslant C\|f\|_{\dot H^r}\|g\|_{\dot H^s}, \tag{35}$$

$$\begin{aligned}\big\|\pi(g, D^\rho f)\big\|_{\mathcal{N}_{r,s-\rho}} &\leqslant \sum_{j\in\mathbb{Z}} \big\|S_{j-2} D^\rho f\big\|_{\dot H^r} \|\Delta_j g\|_{\dot H^{s-\rho}} \\ &\leqslant C\bigg(\sum_{j\in\mathbb{Z}} 4^{-j\rho} \big\|S_{j-2} D^\rho f\big\|_{\dot H^r}^2\bigg)^{1/2} \|g\|_{\dot H^s}\end{aligned}$$

and

$$\sum_{j\in\mathbb{Z}} 4^{-j\rho} \big\|S_{j-2} D^\rho f\big\|_{\dot H^r}^2 \leqslant C \sum_{j\in\mathbb{Z}} 4^{-j\rho} \sum_{k\leqslant j-3} 4^{k(r+\rho)} \|\Delta_k f\|_2^2 = C' \sum_{k\in\mathbb{Z}} 4^{kr} \|\Delta_k f\|_2^2$$

so that

$$\big\|\pi(g, D^\rho f)\big\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\|f\|_{\dot H^r}\|g\|_{\dot H^s}. \tag{36}$$

Similarly, we write

$$\begin{aligned}\big\|R(D^\rho f, g)\big\|_{\mathcal{N}_{r,s-\rho}} &\leqslant \sum_{j\in\mathbb{Z}} \|S_{j+3} g\|_{\dot H^r} \big\|\Delta_j D^\rho f\big\|_{\dot H^{s-\rho}} \\ &\leqslant C\bigg(\sum_{j\in\mathbb{Z}} 4^{j(s-r)} \|S_{j+3} g\|_{\dot H^r}^2\bigg)^{1/2} \|f\|_{\dot H^r}\end{aligned}$$

and

$$\sum_{j\in\mathbb{Z}} 4^{j(s-r)}\|S_{j+3}g\|_{\dot H^r}^2 \leqslant C\sum_{j\in\mathbb{Z}} 4^{j(s-r)}\sum_{k\leqslant j+2} 4^{kr}\|\Delta_k g\|_2^2 = C'\sum_{k\in\mathbb{Z}} 4^{ks}\|\Delta_k g\|_2^2,$$

so that

$$\big\|R(D^\rho f, g)\big\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\|f\|_{\dot H^r}\|g\|_{\dot H^s}. \tag{37}$$

In order to estimate A, we write

$$\|A\|_{\mathcal{N}_{r,s-\rho}} \leqslant \sum_{j\in\mathbb{Z}} \int \frac{\|S_{j-2}f(x) - S_{j-2}f(x-y)\|_{\dot H^r}\|\Delta_j g(x) - \Delta_j g(x-y)\|_{\dot H^{s-\rho}}}{|y|^{d+\rho}}\,dy,$$

then for a function $h \in \mathcal{S}$ and $t \in (-d/2, d/2)$

$$\int \frac{\|h(x) - h(x-y)\|_{\dot H^t}^2}{|y|^{d+\rho}}\,dy = \frac{1}{(2\pi)^d}\iint \big|\hat h(\xi)\big|^2|\xi|^{2t}\frac{|1-e^{i\xi\cdot y}|^2}{|y|^{d+\rho}}\,dy\,d\xi = C\int \big|\hat h(\xi)\big|^2|\xi|^{2t+\rho}\,d\xi,$$

where the constant C depends on t and ρ. Thus, we find

$$\|A\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\sum_{j\in\mathbb{Z}} \|S_{j-2}f\|_{\dot H^{r+\rho/2}}\|\Delta_j g\|_{\dot H^{s-\rho/2}} \leqslant C'\Big(\sum_{j\in\mathbb{Z}} 4^{-j\rho/2}\|S_{j-2}f\|_{\dot H^{r+\rho/2}}^2\Big)^{1/2}\|g\|_{\dot H^s}$$

which gives

$$\|A\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\|f\|_{\dot H^r}\|g\|_{\dot H^s}. \tag{38}$$

Similarly, we write

$$\|B\|_{\mathcal{N}_{r,s-\rho}} \leqslant \sum_{j\in\mathbb{Z}} \int \frac{\|S_{j+3}g(x) - S_{j+3}g(x-y)\|_{\dot H^r}\|\Delta_j f(x) - \Delta_j f(x-y)\|_{\dot H^{s-\rho}}}{|y|^{d+\rho}}\,dy,$$

so that

$$\|B\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\sum_{j\in\mathbb{Z}} \|S_{j+3}g\|_{\dot H^{r+\rho/2}}\|\Delta_j f\|_{\dot H^{s-\rho/2}} \leqslant C'\Big(\sum_{j\in\mathbb{Z}} 4^{j(s-r-\rho)/2}\|S_{j+3}g\|_{\dot H^{r+\rho/2}}^2\Big)^{1/2}\|f\|_{\dot H^r}$$

and

$$\|B\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\|f\|_{\dot H^r}\|g\|_{\dot H^s}. \tag{39}$$

Summing up estimates (35)–(39), we find that

$$\left\| D^\rho(fg)\right\|_{\mathcal{N}_{r,s-\rho}} \leqslant C\|f\|_{\dot{H}^r}\|g\|_{\dot{H}^s}. \tag{40}$$

The same proof works for $s=r$, by writing

$$D^r(fg) = fD^r g + gD^r f - C_r(A+B) \tag{41}$$

instead of (32).

We now consider $I^\rho(fg)$. We write

$$fg = \sum_{j\in\mathbb{Z}} S_{j-2}g\Delta_j f + \sum_{j\in\mathbb{Z}}\sum_{l=-2}^{2}\Delta_j f\Delta_{j+l}g + \sum_{j\in\mathbb{Z}} S_{j-2}f\Delta_j g = \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{42}$$

We control easily $I^\rho(\mathrm{I})$: we write

$$\Psi_\rho = \sum_{l=-2}^{2}\mathcal{F}^{-1}\big(|\xi|^{-\rho}\psi_0(\xi/2^l)\big)$$

and

$$I^\rho(\mathrm{I})(x) = \sum_{j\in\mathbb{Z}}\int 2^{j(d-\rho)}\Psi_\rho(2^j y)S_{j-2}g(x-y)\Delta_j f(x-y)\,dy$$

hence

$$\begin{aligned}\left\|I^\rho(\mathrm{I})\right\|_{\mathcal{N}_{r,s+\rho}} &\leqslant \|\Psi_\rho\|_1\sum_{j\in\mathbb{Z}}2^{-j\rho}\|S_{j-2}g\|_{\dot{H}^r}\|\Delta_j f\|_{\dot{H}^{s+\rho}}\\ &\leqslant C\|f\|_{\dot{H}^r}\sqrt{\sum_{j\in\mathbb{Z}}4^{j(s-r)}\|S_{j-2}g\|^2_{\dot{H}^r}}\end{aligned}$$

and thus

$$\left\|I^\rho(\mathrm{I})\right\|_{\mathcal{N}_{r,s+\rho}} \leqslant C\|f\|_{\dot{H}^r}\|g\|_{\dot{H}^s}. \tag{43}$$

The control of $I^\rho(\mathrm{II})$ is easy as well. Indeed, the embedding

$$\mathcal{M}(\dot{H}^r \to H^{-s-\rho}) \subset \dot{B}_\infty^{-s-r-\rho,\infty}$$

and the density of $\mathcal{S}$ in the predual $\dot{B}_1^{s+r+\rho,1}$ of $B_\infty^{-s-r-\rho,\infty}$ shows that

$$\left\|I^\rho(\mathrm{II})\right\|_{\mathcal{N}_{r,s+\rho}} \leqslant C\left\|I^\rho(\mathrm{II})\right\|_{\dot{B}_1^{s+r+\rho,1}} \leqslant C\sum_{j\in\mathbb{Z}}\sum_{l=-2}^{2}\left\|I^\rho\Delta_j f\Delta_{j+l}g\right\|_{\dot{B}_1^{s+r+\rho,1}}.$$

We start from the inequality

$$\forall F\in L^1,\quad \left\|I^\rho S_{j+5}F\right\|_{\dot{B}_1^{s+r+\rho,1}} \leqslant C2^{j(s+r)}\|F\|_1$$

to get

$$\left\| I^\rho(\mathrm{II}) \right\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \sum_{j \in \mathbb{Z}} \sum_{l=-2}^{2} 2^{j(r+s)} \|\Delta_j f\|_2 \|\Delta_{j+l} g\|_2$$

hence

$$\left\| I^\rho(\mathrm{II}) \right\|_{\mathcal{N}_{r,s+\rho}} \leqslant 5C4^{(r+s)} \sqrt{\sum_{j \in \mathbb{Z}} 4^{jr} \|\Delta_j f\|_2^2} \sqrt{\sum_{j \in \mathbb{Z}} 4^{js} \|\Delta_j g\|_2^2}.$$

Thus, we have

$$\left\| I^\rho(\mathrm{II}) \right\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|f\|_{\dot{H}^r} \|g\|_{\dot{H}^s}. \tag{44}$$

In order to deal with $I^\rho(\mathrm{III})$, we write $G = I^\rho g$ (and $g = D^\rho G$). We then write

$$\begin{aligned} &D^\rho(S_{j-2} f \Delta_j G)(x) \\ &\quad = S_{j-2} f(x) \Delta_j g(x) + S_{j-2} D^\rho f(x) \Delta_j G(x) \\ &\qquad - C_\rho \int \frac{(S_{j-2} f(x) - S_{j-2} f(x-y))(\Delta_j G(x) - \Delta_j G(x-y))}{|y|^{d+\rho}}\, dy. \end{aligned}$$

Thus, we find

$$I^\rho(\mathrm{III}) = \mathrm{IV} - \mathrm{V} + C_\rho \mathrm{VI}$$

with

$$\mathrm{IV} = \sum_{j \in \mathbb{Z}} S_{j-2} f \Delta_j G = fG - \sum_{j \in \mathbb{Z}} S_{j+3} G \Delta_j f,$$

$$\mathrm{V} = \sum_{j \in \mathbb{Z}} \int 2^{j(d-\rho)} \Psi_\rho(2^j y) S_{j-2} D^\rho f(x-y) \Delta_j G(x-y)\, dy$$

and

$$\begin{aligned} \mathrm{VI} = \sum_{j \in \mathbb{Z}} \iint & 2^{j(d-\rho)} \Psi_\rho(2^j z) \big(S_{j-2} f(x-z) - S_{j-2} f(x-z-y) \big) \\ & \times \frac{(\Delta_j G(x-z) - \Delta_j G(x-z-y))}{|y|^{d+\rho}}\, dy\, dz. \end{aligned}$$

We then write

$$\|\mathrm{IV}\|_{\mathcal{N}_{r,s+\rho}} \leqslant \|f\|_{\dot{H}^r} \|G\|_{\dot{H}^{s+\rho}} + \sum_{j \in \mathbb{Z}} \|S_{j+3} G\|_{\dot{H}^r} \|\Delta_j f\|_{\dot{H}^{s+\rho}},$$

hence

$$\|\mathrm{IV}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|f\|_{\dot{H}^r} \left(\|G\|_{\dot{H}^{s+\rho}} + \sqrt{\sum_{j \in \mathbb{Z}} 4^{j(s+\rho-r)} \|S_{j+3} G\|_{\dot{H}^r}^2} \right)$$

and finally, since $s + \rho < r$,

$$\|\mathrm{IV}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|f\|_{\dot{H}^r} \|G\|_{\dot{H}^{s+\rho}} \leqslant C' \|f\|_{\dot{H}^r} \|g\|_{\dot{H}^s}. \tag{45}$$

Next, we have

$$\|\mathrm{V}\|_{\mathcal{N}_{r,s+\rho}} \leqslant \|\Psi_\rho\|_1 \sum_{j\in\mathbb{Z}} 2^{-j\rho} \big\| S_{j-2} D^\rho f \big\|_{\dot H^r} \|\Delta_j G\|_{\dot H^{s+\rho}},$$

hence

$$\|\mathrm{V}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|G\|_{\dot H^{s+\rho}} \sqrt{\sum_{j\in\mathbb{Z}} 4^{-j\rho} \big\| S_{j-2} D^\rho f \big\|^2_{\dot H^r}}$$

and finally

$$\|\mathrm{V}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|D^\rho f\|_{\dot H^{r-\rho}} \|G\|_{\dot H^{s+\rho}} \leqslant C' \|f\|_{\dot H^r} \|g\|_{\dot H^s}. \tag{46}$$

In order to control VI, we use the shift-invariance of the Sobolev norms to write that, for all $z \in \mathbb{R}^d$, all $t \in (-d/2, d/2)$ and all $F \in \dot H^d$,

$$\big\| F(x-z) \big\|_{\dot H^t} = \|F\|_{\dot H^t},$$

so that

$$\|\mathrm{VI}\|_{\mathcal{N}_{r,s+\rho}} \leqslant \|\Psi_\rho\|_1 \sum_{j\in\mathbb{Z}} \int 2^{-j\rho} \big\| S_{j-2} f(x) - S_{j-2} f(x-y) \big\|_{\dot H^r} \times \frac{\|\Delta_j G(x) - \Delta_j G(x-y)\|_{\dot H^{s+\rho}}}{|y|^{d+\rho}}\, dy.$$

Next, we write

$$\int \frac{\|S_{j-2} f(x) - S_{j-2} f(x-y)\|^2_{\dot H^r}}{|y|^{d+\rho}}\, dy \leqslant C \|S_{j-2} f\|^2_{\dot H^{r+\rho/2}}$$

and

$$\int \frac{\|\Delta_j G(x) - \Delta_j G(x-y)\|^2_{\dot H^{s+\rho}}}{|y|^{d+\rho}}\, dy \leqslant C \|\Delta_j G\|^2_{\dot H^{r+3\rho/2}}$$

and we get

$$\|\mathrm{VI}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \sum_{j\in\mathbb{Z}} 2^{-j\rho} \|S_{j-2} f\|_{\dot H^{r+\rho/2}} \|\Delta_j G\|_{\dot H^{r+3\rho/2}},$$

so that

$$\|\mathrm{VI}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \Big(\sum_{j\in\mathbb{Z}} 2^{-j\rho} \|S_{j-2} f\|^2_{\dot H^{r+\rho/2}} \Big)^{1/2} \Big(\sum_{j\in\mathbb{Z}} 2^{-j\rho} \|\Delta_j G\|^2_{\dot H^{r+3\rho/2}} \Big)^{1/2}$$

and finally

$$\|\mathrm{VI}\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|f\|_{\dot H^r} \|G\|_{\dot H^{s+\rho}} \leqslant C' \|f\|_{\dot H^r} \|g\|_{\dot H^s}. \tag{47}$$

Summing up estimates (43)–(47), we find that

$$\big\| I^\rho(fg) \big\|_{\mathcal{N}_{r,s+\rho}} \leqslant C \|f\|_{\dot H^r} \|g\|_{\dot H^s}. \tag{48}$$

Thus, Proposition 3 is proved. □

Proof of Theorem 1. Theorem 1 is a direct consequence of Proposition 3. Let $-r < \sigma < s < r$. Let $f \in \mathcal{M}(\dot{H}^r \to \dot{H}^\sigma)$ and let $g \in \mathcal{M}(\dot{H}^r \to \dot{H}^s)$. Finally, let $\rho = s - \sigma$.

Let φ and $\psi \in \mathcal{S}$. Then, we have

$$\langle I^\rho f | \varphi\psi\rangle = \bigl\langle f | I^\rho(\varphi\psi)\bigr\rangle,$$

hence Proposition 3 gives

$$\begin{aligned}\bigl|\langle I^\rho f|\varphi\psi\rangle\bigr| &\leqslant \|f\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^\sigma)}\bigl\|I^\rho(\varphi\psi)\bigr\|_{\mathcal{N}_{r,-\sigma}}\\ &\leqslant C\|f\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^\sigma)}\|\varphi\|_{\dot{H}^r}\|\psi\|_{\dot{H}^{-s}}.\end{aligned}\tag{49}$$

Thus, I^ρ maps $\mathcal{M}(\dot{H}^r \to \dot{H}^\sigma)$ to $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$.

Similarly, we have

$$\langle D^\rho g | \varphi\psi\rangle = \bigl\langle g | D^\rho(\varphi\psi)\bigr\rangle,$$

hence Proposition 3 gives

$$\begin{aligned}\bigl|\langle D^\rho g|\varphi\psi\rangle\bigr| &\leqslant \|g\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}\bigl\|D^\rho(\varphi\psi)\bigr\|_{\mathcal{N}_{r,-s}}\\ &\leqslant C\|g\|_{\mathcal{M}(\dot{H}^r\to\dot{H}^s)}\|\varphi\|_{\dot{H}^r}\|\psi\|_{\dot{H}^{-\sigma}}.\end{aligned}\tag{50}$$

Thus, D^ρ maps $\mathcal{M}(\dot{H}^r \to \dot{H}^s)$ to $\mathcal{M}(\dot{H}^r \to \dot{H}^\sigma)$. □

4. Singular integrals and Sobolev spaces

The boundedness of singular integral operators on Sobolev spaces has been studied by Lemarié [6,7] and Meyer [14].

Definition 7. A singular integral operator is a continuous linear operator T from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ such that there exists a continuous function K defined on $\mathbb{R}^d \times \mathbb{R}^d - \Delta$ (where Δ is the diagonal set $x = y$) which satisfies:

(i) $\exists C_0 > 0\ \forall x\ \forall y\ |K(x,y)| \leqslant C_0 \frac{1}{|x-y|^d}$;
(ii) $\exists C_1 > 0\ \forall x\ \forall y\ |\vec{\nabla}_x K(x,y)| \leqslant C_1 \frac{1}{|x-y|^{d+1}}$;
(iii) $\forall f, g \in \mathcal{D}(\mathbb{R}^d)$, $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset \Rightarrow \langle Tf|g\rangle = \iint K(x,y) f(y)\bar{g}(x)\,dx\,dy$.

We shall now introduce two useful notions associated to singular integral operators: the *weak boundedness property* and the distribution $T(1)$.

Definition 8. A continuous linear operator T from $\mathcal{D}(\mathbb{R}^d)$ to $\mathcal{D}'(\mathbb{R}^d)$ satisfies the weak boundedness property (what we shall write $T \in \mathit{WBP}$) if there exist a constant C_2 and a number N such that for all $\phi, \psi \in \mathcal{D}(\mathbb{R}^d)$ with support in $B(0,1)$, all $x_0 \in \mathbb{R}^d$ and all $R > 0$, writing $\phi_{x_0,R}(x) = \phi(\frac{x-x_0}{R})$ and $\psi_{x_0,R}(x) = \psi(\frac{x-x_0}{R})$, we have

$$\bigl|\bigl\langle T(\phi_{x_0,R})|\psi_{x_0,R}\bigr\rangle\bigr| \leqslant C_2\biggl(\sum_{|\alpha|\leqslant N} R^{|\alpha|}\bigl\|\partial^\alpha(\phi_{x_0,R})\bigr\|_2\biggr)\biggl(\sum_{|\alpha|\leqslant N} R^{|\alpha|}\bigl\|\partial^\alpha(\psi_{x_0,R})\bigr\|_2\biggr).$$

Definition 9. If T is a singular integral operator, we may define $T(1) \in \mathcal{D}'(\mathbb{R}^d)/\mathbb{C}$ by choosing $\varphi \in \mathcal{D}(\mathbb{R}^d)$ equal to 1 in a neighbourhood of 0 and computing for $\psi \in \mathcal{D}(\mathbb{R}^d)$ with $\int \psi\, dx = 0$ $\langle T(1)|\psi\rangle$ as $\langle T(1)|\psi\rangle = \lim_{R\to\infty}\langle T(\varphi(\frac{x}{R}))|\psi\rangle$. We may see easily that, if $\omega \in \mathcal{D}(\mathbb{R}^d)$ is equal to 1 in a neighbourhood of $\operatorname{supp}\psi$ and $x_0 \in \operatorname{supp}\psi$, then

$$\bigl\langle T(1)|\psi\bigr\rangle = \bigl\langle T(\omega)|\psi\bigr\rangle + \iint \bigl(K(x,y) - K(x_0,y)\bigr)\bigl(1-\omega(y)\bigr)\psi(x)\,dx\,dy.$$

We recall now the theorem of Lemarié [6,7] on the boundedness of singular integral operators on Sobolev spaces.

Proposition 4. *Let T be a singular integral operator. If $T \in WBP$ and $T(1) = 0$, then T is bounded from $\dot H^r$ to $\dot H^r$ for every $0 < r < \min(d/2, 1)$.*

The Besov space $\dot B^{0,\infty}_\infty$ allows one to give a better description of the distribution $T(1)$.

Definition 10. We define the Besov space $\dot B^{0,\infty}_\infty$ as

$$\dot B^{0,\infty}_\infty = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) \mid \sup_{j\in\mathbb{Z}} \|\Delta_j f\|_\infty < \infty \quad \text{and} \quad \lim_{x\to\infty} \frac{S_0(x)}{|x|} = 0 \right\} \tag{51}$$

and the semi-norm $\|\cdot\|_{\dot B^{0,\infty}_\infty}$ as

$$\|f\|_{\dot B^{0,\infty}_\infty} = \sup_{j\in\mathbb{Z}} \|\Delta_j f\|_\infty. \tag{52}$$

It is easy to see that, when $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\sup_{j\in\mathbb{Z}} \|\Delta_j f\|_\infty < \infty$, then the series

$$\sum_{j<0} \Delta_j f(x) - \Delta_j f(0) + \sum_{j\geqslant 0} \Delta_j f(x)$$

converges in $\mathcal{S}'$ to a distribution $\tilde f \in \dot B^{0,\infty}_\infty$ such that, for all $j \in \mathbb{Z}$, $\Delta_j f = \Delta_j \tilde f$. If, moreover, $f \in \dot B^{0,\infty}_\infty$, the $f - \tilde f$ is a constant distribution: $f - \tilde f \in \mathbb{C}$. Thus, $\dot B^{0,\infty}_\infty$ with the norm $\|\cdot\|_{\dot B^{0,\infty}_\infty}$ is a Banach space of distributions modulo the constants:

$$\dot B^{0,\infty}_\infty \subset \mathcal{D}'/\mathbb{C}.$$

Another way to characterize $\dot B^{0,\infty}_\infty$ is that

$$f \in \dot B^{0,\infty}_\infty \quad \Leftrightarrow \quad \vec\nabla f \in \bigl(\dot B^{-1,\infty}_\infty\bigr)^d. \tag{53}$$

Lemma 8.

(i) *Let T be a singular integral operator. If $T \in WBP$, then $T(1) \in \dot B^{0,\infty}_\infty$.*
(ii) *Conversely, let $h \in \dot B^{0,\infty}_\infty$. Then the paraproduct operator $\pi(h,\cdot)$ is a singular integral operator, $T \in WBP$ and $T(1) = h$.*

Proof. (i) We must check first that $\sup_{j\in\mathbb{Z}}\|\Delta_j(T(1))\|_\infty<\infty$. It is enough to check that there exists a constant C_3 which depends only on the constants C_0, C_1 and C_2 in Definitions 7 and 8 such that

$$\left|\Delta_0\bigl(T(1)\bigr)(0)\right|\leqslant C_3.$$

Then we shall use the invariance of the constants C_0, C_1 and C_2 through dilations and translations to get a similar control on $\Delta_j(T(1))(x_0)$ (changing the kernel $K(x,y)$ into $2^{-jn}K(x_0+2^{-j}x,x_0+2^{-j}y)$). In order to estimate $\Delta_0(T(1))(0)$, we call Ψ the inverse Fourier transform of ψ_0. Since $\Psi\in\mathcal{S}$ and $\int\Psi\,dx=0$, we may write $\Psi=\sum_{k\in\mathbb{N}}\Psi_k$ with:

- $\Psi_k\in\mathcal{D}$ and $\operatorname{supp}\Psi_K\subset B(0,2^k)$;
- $\int\Psi_k\,dx=0$;
- $\sum_{k\in\mathbb{N}}4^{kN}\|\Psi_k\|_{H^N}<\infty$;
- $\sum_{k\in\mathbb{N}}\|\Psi_k\|_1<\infty$.

Then, choosing $\omega\in\mathcal{D}$ such that $\omega=1$ on $B(0,2)$, we write

$$\Delta_0\bigl(T(1)\bigr)(0)=\int\Psi(-y)T(1)(y)\,dy$$

and thus

$$\Delta_0\bigl(T(1)\bigr)(0)=\sum_{k\in\mathbb{N}}\bigl\langle T\bigl(\omega(2^{-k}y)\bigr)\big|\Psi_k(-y)\bigr\rangle + \iint\bigl(K(x,y)-K(0,y)\bigr)\bigl(1-\omega(2^{-k}y)\bigr)\Psi_k(x)\,dx\,dy.$$

This is enough to get the required control.

A similar proof gives that $\sup_{j\in\mathbb{Z}}2^{-j}\|\vec\nabla S_jf\|_\infty<\infty$, so that $\vec\nabla S_jf\in(\dot B_\infty^{-1,\infty})^d$ and finally $f\in\dot B_\infty^{0,\infty}$.

(ii) We check easily that $\pi(h,f)$ is well defined when $f\in\mathcal{S}$ and $h\in\dot B_\infty^{0,\infty}$. Indeed, $\pi(h,\cdot)\in\mathcal{L}(\dot B_\infty^{\sigma,\infty},\dot B_\infty^{\sigma,\infty})$ for every $\sigma<0$ and $\mathcal{S}\subset\dot B_\infty^{\sigma,\infty}$ for every $\sigma<0$. Moreover, $\pi(h,\cdot)\in$ *WBP*: it is enough to check that, for all ϕ and ψ in $\mathcal{D}$ with support in $B(0,1)$ we have

$$\left|\bigl\langle\pi(h,\phi)\big|\psi\bigr\rangle\right|\leqslant C\|h\|_{\dot B_\infty^{0,\infty}}\|\phi\|_{H^N}\|\psi\|_{H^N}. \tag{54}$$

We shall then conclude by using the invariance of the norm in $\dot B_\infty^{0,\infty}$ through dilations and translations. (54) is easy to prove: if

$$H^N(B)=\bigl\{f\in H^n\mid\operatorname{supp}f\subset B(0,1)\bigr\},$$

then we have $H^N(B)\subset L^1\subset\dot B_\infty^{-d/2,\infty}$ and, when $N>d/2$,

$$H^N(B)\subset W^{N,1}\subset\dot B_1^{d/2,1}$$

and (54) is thus a direct consequence of

$$\left|\bigl\langle\pi(h,\phi)\big|\psi\bigr\rangle\right|\leqslant C\bigl\|\pi(h,\phi)\bigr\|_{\dot B_\infty^{-d/2,\infty}}\|\psi\|_{\dot B_1^{d/2,1}}\leqslant C\|h\|_{\dot B_\infty^{0,\infty}}\|\phi\|_{\dot B_\infty^{-d/2,\infty}}\|\psi\|_{\dot B_1^{d/2,1}}.$$

The kernel of $\pi(h,\cdot)$ is easily computed: if $\Phi = \mathcal{F}^{-1}\varphi_0$, then

$$K(x,y) = \sum_{j\in\mathbb{Z}} \Delta_{j+2}h(x)2^{jd}\Phi\big(2^j(x-y)\big)$$

with

- $|\Delta_{j+2}h(x)| \leqslant C\|h\|_{\dot{B}^{0,\infty}_\infty}$;
- $|\vec{\nabla}\Delta_{j+2}h(x)| \leqslant C2^j\|h\|_{\dot{B}^{0,\infty}_\infty}$;
- $|\Phi(x)| \leqslant C(1+|x|)^{-d-2}$;
- $|\vec{\nabla}\Phi(x)| \leqslant C(1+|x|)^{-d-2}$.

Those estimates allows us to control the size of K and of $\vec{\nabla}_x K$. We finish by checking that, in $\mathcal{D}'/\mathbb{C}$, we have $\pi(h,1) = \sum_{j\in\mathbb{Z}}\Delta_j h = h$. $\square$

A direct consequence of Proposition 4 and of Lemma 8 is the following result of Meyer [14].

Corollary 2. *Let T be a singular integral operator. Then, for $0 < r < \min(1, d/2)$, the following assertions are equivalent*:

(A) *T is bounded from $\dot{H}^r$ to $\dot{H}^r$*;
(B) *$T \in WBP$ and $\pi(T(1),\cdot)$ is bounded from $\dot{H}^r$ to $\dot{H}^r$.*

A *Calderón–Zygmund operator* is an operator T such that both T and T^* are singular integral operators and are bounded on $L^2(\mathbb{R}^d)$ ($T \in \mathcal{L}(L^2, L^2)$). We define the Calderón–Zygmund norm of T as the sum

$$\|T\|_{CZO} = \|T\|_{\mathcal{L}(L^2,L^2)} + \big\||x-y|^d K(x,y)\big\|_\infty + \big\||x-y|^{d+1}\vec{\nabla}_x K(x,y)\big\|_\infty + \big\||x-y|^{d+1}\vec{\nabla}_y K(x,y)\big\|_\infty. \tag{55}$$

A classical property of Calderón–Zygmund operators is that they are bounded on $L^2(\omega\,dx)$ for every weight ω in the Muckenhoupt class $\mathcal{A}_2$ [2,5,15]. Thus, Verbitsky's theorem [11] can be applied and we have the following.

Proposition 5. *Let T be a Calderón–Zygmund operator. Then T is bounded on the space $\mathcal{M}(\dot{H}^r \to L^2)$ for all $r \in (0, d/2)$. Moreover, there exists a constant C_r such that for all Calderón–Zygmund operator T, we have*

$$\|T\|_{\mathcal{L}(\mathcal{M}(\dot{H}^r\to L^2),\mathcal{M}(\dot{H}^r\to L^2))} \leqslant C_r\|T\|_{CZO}. \tag{56}$$

5. The limit case $r = s$

The link between multipliers of $\dot{H}^r$ and paramultiplication has been studied by Meyer [14] and Youssfi [16]. Proposition 1 can be extended to the case $s = r$ provided that we change the Besov space $\dot{B}^{0,\infty}_\infty$ into L^∞:

Proposition 6. *Let* $r \in (0, d/2)$. *Then the following assertions are equivalent*:

(A) $h \in \mathcal{M}(\dot H^r \to \dot H^r)$;
(B) $h \in L^\infty$ *and* $\pi(h, \cdot)$ *maps boundedly* $\dot H^r$ *to* $\dot H^r$.

Proof. (A) ⇒ (B). We write $\phi_0 = \mathcal{F}(\theta\omega)$ with $\omega \in \mathcal{S}$ and $\theta(x) = \frac{1}{(1+x^2)^d}$. Thus, we have

$$\begin{aligned}\bigl|S_j h(x)\bigr| &= \left|\int h(y) 2^{jd}\theta\bigl(2^j(x-y)\bigr)\omega\bigl(2^j(x-y)\bigr)\,dy\right| \\ &\leqslant \|h\|_{\mathcal{M}(\dot H^r \to \dot H^r)}\|\theta\|_{\dot H^r}\|\omega\|_{\dot H^{-r}}.\end{aligned}$$

Hence, letting j go to $+\infty$, we find that $h \in L^\infty$.

Using (24), we write that $\pi(h,\cdot) = M_h - {}^t\pi(h,\cdot) - \rho(h,\cdot) - {}^t\rho(h,\cdot)$ (where M_h is the pointwise product operator with h: $M_h f = hf$). If $h \in L^\infty$, it is obvious that $\rho(h,\cdot)$ maps $\dot H^r$ to $\dot H^r$ and $\dot H^{-r}$ to $\dot H^{-r}$. If $h \in \mathcal{M}(\dot H^r \to \dot H^r)$, then $\pi(h,\cdot)$ maps boundedly $\dot H^{-r}$ to $\dot H^{-r}$:

$$\begin{aligned}\bigl\|\pi(h,g)\bigr\|_{\dot H^{-r}}^2 &\leqslant C\sum_{j\in\mathbb{Z}}\sum_{l=-3}^{3} 4^{-jr}\bigl\|\Delta_{j+l}(S_{j-2}g\Delta_j h)\bigr\|_2^2 \\ &\leqslant C'\sum_{j\in\mathbb{Z}} 4^{-2jr}\|S_{j-2}g\Delta_j h\|_{\dot H^r}^2,\end{aligned}$$

so that

$$\bigl\|\pi(h,g)\bigr\|_{\dot H^{-r}}^2 \leqslant C\|h\|_{\mathcal{M}(\dot H^r\to\dot H^r)}^2\sum_{j\in\mathbb{Z}}4^{-2jr}\|S_{j-2}g\|_{\dot H^r}^2 \leqslant C'\|h\|_{\mathcal{M}(\dot H^r\to\dot H^r)}^2\|g\|_{\dot H^{-r}}^2.$$

Thus, $\pi(h,\cdot)$ maps boundedly $\dot H^r$ to $\dot H^r$.

(B) ⇒ (A). If $h \in L^\infty$, then we see easily that $\rho(h,\cdot)$ maps $\dot H^r$ to $\dot H^r$ and $\dot H^{-r}$ to $\dot H^{-r}$. Moreover, $\pi(h,\cdot)$ maps $\dot H^{-r}$ to $\dot H^{-r}$:

$$\begin{aligned}\bigl\|\pi(h,g)\bigr\|_{\dot H^{-r}}^2 &\leqslant C\sum_{j\in\mathbb{Z}}4^{-jr}\|S_{j-2}g\Delta_j h\|_2^2 \leqslant C'\|h\|_\infty^2\sum_{j\in\mathbb{Z}}4^{-jr}\|S_{j-2}g\|_2^2 \\ &\leqslant C''\|h\|_\infty^2\|g\|_{\dot H^{-r}}^2.\end{aligned}$$

If we assume, moreover, that $\pi(h,\cdot)$ maps $\dot H^r$ to $\dot H^r$, then we find that M_h maps $\dot H^r$ to $\dot H^r$. □

Theorem 1 can be extended as well to the case $s = r$ provided that we replace multipliers by paramultipliers.

Theorem 2. *Let* $0 < r < d/2$. *Then* $f \in \mathcal{M}(\dot H^r \to L^2)$ *if and only if there exists* $F \in \dot B_\infty^{0,\infty}$ *such that* $f = D^r F$ *and* $\pi(F,\cdot)$ *is bounded on* $\dot H^r$.

Proof. If f belongs to $\dot B_\infty^{-r,\infty}$, then $\sum_{j\in\mathbb{Z}} I^r\Delta_j f$ converges in $\mathcal{D}'/\mathbb{C}$ to a distribution $I^r f \in \dot B_\infty^{0,\infty}$ and I^r is an isomorphism between $\dot B_\infty^{-r,\infty}$ and $\dot B_\infty^{0,\infty}$. We shall now distinguish between the cases $r < 1$ and $r \geqslant 1$.

(i) $0 < r < 1$. For $f \in \dot{B}_\infty^{-r,\infty}$, we know (Proposition 1) that $f \in \mathcal{M}(\dot{H}^r \to L^2)$ if and only if $\pi(f,\cdot) \in \mathcal{L}(\dot{H}^r, L^2)$, or equivalently if and only if $I^r\pi(f,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^r)$. But it is easy to see that $I^r\pi(f,\cdot)$ is a singular integral operator with kernel

$$K(x,y) = \sum_{j\in\mathbb{Z}} \int 2^{j(d-r)}\Psi_r\big(2^j(x-z)\big)\Delta_j f(z) 2^{jd}\Phi\big(2^j(y-z)\big)\,dz$$

and we may apply Corollary 2 to get that $T = I^r\pi(f,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^r)$ if and only if $\pi(T(1),\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^r)$. But we have

$$T(1) = \sum_{j\in\mathbb{Z}} I^r(\Delta_j f S_{j-2}1) = \sum_{j\in\mathbb{Z}} I^r\Delta_j f = F.$$

This proves Theorem 2 in the case $r < 1$.

(ii) $1 \leqslant r < d/2$. Using the Riesz transforms, we see from Proposition 5 (Verbitsky's theorem) that $f \in \mathcal{M}(\dot{H}^r \to L^2)$ if and only if the Riesz transforms $I^1\partial_j f$ belong to $\mathcal{M}(\dot{H}^r \to L^2)$ for $j = 1,\dots,d$. Using Theorem 1, we find that $I^1\partial_j f$ belongs to $\mathcal{M}(\dot{H}^r \to L^2)$ if and only if $I^r\partial_j f$ belongs to $\mathcal{M}(\dot{H}^r \to \dot{H}^{r-1})$. Thus, we have

$$f \in \mathcal{M}(\dot{H}^r \to L^2) \quad \Leftrightarrow \quad \text{for } j \in \{1,\dots,d\} \quad \pi(\partial_j F,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^{r-1}).$$

Since we have, for all $g \in \dot{H}^r$,

$$\partial_j\pi(F,g) = \pi(\partial_j F, g) + \pi(F, \partial_j g)$$

we can easily conclude:

- if $\pi(F,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^r)$, then we see that $\pi(F,\cdot) \in \mathcal{L}(\dot{H}^{r-1}, \dot{H}^{r-1})$ (by interpolation between $\dot{H}^r$ and $\dot{H}^\sigma$ with $\sigma < 0$) and thus we get that $\pi(\partial_j F,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^{r-1})$;
- conversely, if $\pi(\partial_j F,\cdot) \in \mathcal{L}(\dot{H}^r, \dot{H}^{r-1})$ for $j = 1,\dots,d$, then $f \in \mathcal{M}(\dot{H}^r \to L^2)$, hence $\pi(F,\cdot) \in \mathcal{L}(\dot{H}^{r-1}, \dot{H}^{r-1})$:

$$\begin{aligned}\big\|\pi(F,g)\big\|_{\dot{H}^{r-1}}^2 &\leqslant C\sum_{j\in\mathbb{Z}} 4^{j(r-1)}\|\Delta_j F S_{j-2}g\|_2^2 \\ &\leqslant C\sum_{j\in\mathbb{Z}} 4^{j(r-1)}\|\Delta_j F\|_{\mathcal{M}(\dot{H}^r\to L^2)}^2 \|S_{j-2}g\|_{\dot{H}^r}^2.\end{aligned}$$

Using the stability of $\mathcal{M}(\dot{H}^r \to L^2)$ under convolution with L^1, we have

$$\big\|\Delta_j I^r f\big\|_{\mathcal{M}(\dot{H}^r\to L^2)} \leqslant C2^{-jr}\|f\|_{\mathcal{M}(\dot{H}^r\to L^2)}$$

and thus

$$\big\|\pi(F,g)\big\|_{\dot{H}^{r-1}}^2 \leqslant C\|f\|_{\mathcal{M}(\dot{H}^r\to L^2)}^2 \sum_{j\in\mathbb{Z}} 4^{-j}\|S_{j-2}g\|_{\dot{H}^r}^2 \leqslant C'\|f\|_{\mathcal{M}(\dot{H}^r\to L^2)}^2\|g\|_{\dot{H}^{r-1}}^2.$$

Finally, we get that $\partial_j\pi(F,\cdot)$ maps boundedly $\dot{H}^r$ to $\dot{H}^{r-1}$ for $j = 1,\dots,d$, and thus $\pi(F,\cdot)$ maps boundedly $\dot{H}^r$ to $\dot{H}^r$. $\square$

The class of paramultipliers of $\dot{H}^r$ is slightly bigger than the class $\mathcal{M}(\dot{H}^r \to \dot{H}^r)$; for instance, the function f defined as $f(x) = \ln|x|$ belongs to the first class and not to the second one.

6. The limit case $r = -s$

As a direct consequence of Proposition 3, we have

Theorem 3. *Let* $0 < r < d/2$. *Then, if* $f \in \mathcal{M}(\dot{H}^r \to L^2)$, *we have* $D^r f \in \mathcal{M}(\dot{H}^r \to \dot{H}^{-r})$.

Proof. We just use the fact that D^r maps $\mathcal{N}_{r,r}$ to $\mathcal{N}_{r,0}$. □

The converse is most probably true. It has been proved for $r = 1$ by Maz'ya and Verbitsky [12], and it is easy to deduce from the case $r = 1$ that it is true for the case $r = 1/2$ by using trace theorems [3,13].

Theorem 4. *Let* $r = 1$ *or* $r = 1/2$, *and* $r < d/2$. *Then* D^r *is an isomorphism from* $\mathcal{M}(\dot{H}^r \to L^2)$ *onto* $\mathcal{M}(\dot{H}^r \to \dot{H}^{-r})$.

The proof for $r = 1$ relies on potential theory and uses some fine properties of equilibrium measures. As far as we know, the case of $r \notin \{1, 1/2\}$ is still open.

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