Propagation of Sobolev regularity for the critical dissipative quasi-geostrophic equation

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1 Introduction

The 2D quasi-geostrophic equations (QG)$_{\alpha}$, $0 < \alpha \leq 1$, for a function $\theta(t,x)$ defined on $[0, +\infty) \times \mathbb{R}^2$ are

$$
\begin{cases}
\theta_t + u \cdot \nabla \theta + k \Lambda^{2\alpha} \theta = 0 \\
u = -R^2 \theta = -(-R_2 \theta, R_1 \theta) \\
\theta(0, \cdot) = \theta_0
\end{cases}
$$

where $k > 0$ and $\Lambda$ is the operator $(-\Delta)^{\frac{1}{2}}$, defined at the Fourier level by

$$
\hat{\Lambda} f(\xi) = |\xi| \hat{f}(\xi).
$$

Thus,

$$
\hat{\Lambda}^{2\alpha} f(\xi) = |\xi|^{2\alpha} \hat{f}(\xi)
$$

where the Fourier transform $\hat{f} = \mathcal{F}(f)$ is defined by

$$
\hat{f}(\xi) = \int_{\mathbb{R}^2} f(x)e^{-ix \cdot \xi} dx.
$$

The Riesz transforms $R_1$ and $R_2$ are defined by

$$
\hat{R_k} f(\xi) = -\frac{i\xi_k}{|\xi|} \hat{f}(\xi).
$$

This equation comes from more general quasi-geostrophic models of atmospheric and ocean fluid flow; the scalar $\theta$ represents the temperature and $u$ the divergence free velocity field.

The mathematical study of the non-dissipative case has first been proposed by Constantin, Majda and Tabak in [5] where it is shown to be an analogue to the 3D Euler equations. The dissipative case has then been studied by Constantin and Wu in [6] when $\alpha > 1/2$ and global existence in Sobolev spaces is studied by Constantin, Cordoba and Wu in [4] when $\alpha = 1/2.$

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The case $1/2 < \alpha \leq 1$, is called sub-critical since smooth solutions are known to exists globally in time whereas the uniqueness of $L^2$ weak solutions is an unsolved problem.

According to [6], $(QG)_{1/2}$ describes the evolution of the temperature on the 2D boundary of a rapidly rotating half-space with small Rossby and Ekman numbers. This case is called critical since it is not known whether smooth solutions exists globally in time, therefore is a good model for the 3D Navier-Stokes equations.

The case $0 < \alpha < 1/2$ is called super-critical and is harder to deal with compared to the other cases.

The aim in studying the fractional power $0 < \alpha \leq 1$ is to provide a better understanding of the analysis of the 3D Navier-Stokes equations.

In the following, the symbol $\preceq$ will stand for, $a \preceq b$ if there exists a constant $C > 0$ (independent of all relevant parameters) such that $a \leq C b$.

The norm in $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, will be denoted by $\| \cdot \|_p$.

In this paper we will use the two notions of weak and mild solutions, we now give a precise meaning of these notions.

Let $T$ belongs to $(0, +\infty]$; because of the Riesz transforms in the non-linear term and the fractional derivative, we cannot consider, as in the Navier-Stokes equations, the case of a distribution $\theta(t,x) \in L^2_{loc}((0,T) \times \mathbb{R}^2)$ in order to define weak solutions to $(QG)_\alpha$. To avoid the definition of the Riesz transform on bad spaces, we define the space $F_2$ by the space of all functions $f \in L^2_{aaloc}(\mathbb{R}^2)$ ($f \in L^2_{loc}(\mathbb{R}^2)$ and $\| f \|_{L^2_{aaloc}} = \sup_{x \in \mathbb{R}^2} (\int_{|x-y|<1} |f|^2 dy)^{1/2} < \infty$) such that the low frequencies blocks (see section 2) $S_j f$ goes to 0 in $\mathcal{S}'$ when $j \to -\infty$ and such that $R_1(Id - S_j)f$ and $R_2(Id - S_j)f$ admit limits in $\mathcal{S}'$ (when $j \to -\infty$) which belongs to $L^2_{aaloc}(\mathbb{R}^2)$; for such a function we can write $R_1 f \in L^2_{aaloc}(\mathbb{R}^2)$ and $R_2 f \in L^2_{aaloc}(\mathbb{R}^2)$. The norm on $F_2$ is then defined by $\| f \|_{F_2} = \| f \|_{L^2_{aaloc}} + \| R_1 f \|_{L^2_{aaloc}} + \| R_2 f \|_{L^2_{aaloc}}$.

**Definition 1.1.** Let $0 < \alpha \leq 1$,

a. A **weak solution** for the quasi-geostrophic equation $(QG)_\alpha$ on $(0,T) \times \mathbb{R}^2$ is a distribution $\theta(t,x) \in L^2_{aaloc}((0,T), F_2)$ which satisfies in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$:

$$\partial_t \theta + \nabla \cdot (\theta u) + \Lambda^{2\alpha} \theta = 0$$

with $u = \mathcal{R}^\perp \theta$ and $0 < \alpha \leq 1$.

b. A **mild solution** for the quasi-geostrophic equation $(QG)_\alpha$ on $(0,T) \times \mathbb{R}^2$ with initial data $\theta_0 \in \mathcal{S}'(\mathbb{R}^2)$ is a distribution $\theta(t,x) \in L^2((0,T), F_2)$ which satisfies in $\mathcal{D}'((0,T) \times \mathbb{R}^2)$:

$$\theta = e^{-t \Lambda^{2\alpha}} \theta_0 - \int_0^t e^{-(t-s) \Lambda^{2\alpha}} \nabla \cdot (\theta u) ds.$$  

Under the assumption $\theta \in L^2((0,T), F_2)$, using the same arguments than Furioli, Lemarié-Rieusset and Terraneo in [8] it is not hard to prove that these two definitions are equivalent.
Remark 1.1. We point out that with the assumption $\theta \in L^2_{loc}((0, T), F_2)$ we can give sense to $\Lambda^{2\alpha} \theta$. Indeed the embedding $L^2_{loc} \hookrightarrow B^{-1,\infty}_\infty$ (see section 4.2 for the definition of Besov spaces) implies that $\Lambda^{2\alpha} \theta \in L^2_{loc}((0, T), B^{1-2\alpha,\infty}_\infty)$.

The quasi-geostrophic equations have a remarkable property of maximum principle which can be seen as the consequence of the following lemma proved in [13]:

**Lemma 1.1.** Let $p \in [2, +\infty)$ and $\theta$ be a $H^\infty = \cap_{k \geq 0} H^k$ function on $\mathbb{R}^n$, we have:

$$2 \int |\Lambda^\alpha(|\theta|^{p/2})|^2 \, dx \leq p \int |\theta|^{p-2} \theta \Lambda^{2\alpha} \theta \, dx$$

**Corollary 1.2.** (maximum principle) Let $\theta$ be a smooth function decaying sufficiently rapidly at infinity and satisfying $\partial_t \theta + \mathcal{R}^\perp \theta \cdot \nabla \theta + k \Lambda^{2\alpha} \theta = 0$; we have:

$$\|\theta(t)\|_p + 2k \int_0^t \int |\Lambda^{\alpha/2}(|\theta|^{p/2})|^2 \, dx \, ds \leq \|\theta_0\|_p$$

$$\|\theta(t)\|_{\infty} \leq \|\theta_0\|_{\infty}$$

for all $p \in [2, +\infty)$ and for all $t \geq 0$.

This property is the starting point of several works made on these equations especially when studying existence in Sobolev spaces [4], [7] or weak solutions [13].

The existence of weak $L^2$ solutions for $(QG)_{1/2}$ is very similar to the existence of Leray solutions to the Navier-Stokes equations. However, in [13] we construct global weak solutions to the quasi-geostrophic equations (dissipative or not) in other spaces than $L^2$, i.e. $\dot{H}^{-1/2}$ and $L^p$, $1 < p < \infty$. For our purpose, we only give a statement when the initial data is in $L^2$ or $\dot{H}^{-1/2}$.

**Theorem 1.3.** (Weak solutions)

Let $T \in (0, \infty]$.

(i) if $\theta_0 \in L^2(\mathbb{R}^2)$, there exists a weak solution $\theta$ on $(0, T) \times \mathbb{R}^2$ to $(QG)_{1/2}$ such that $\theta \in L^\infty((0, T), L^2) \cap L^2((0, T), \dot{H}^{1/2})$ and satisfies the global inequality:

$$\text{for all } t \in (0, T), \quad \|\theta(t, \cdot)\|_2^2 + 2k \int_0^t \int_{\mathbb{R}^2} |\Lambda^{1/2} \theta|^2 \, dx \, ds \leq \|\theta_0\|_2^2. \quad (4)$$

Such a solution will be called a $L^2$-solution.

(ii) if $\theta_0 \in \dot{H}^{-1/2}(\mathbb{R}^2)$, there exists a weak solution $\theta$ on $(0, T) \times \mathbb{R}^2$ to $(QG)_{1/2}$ such that $\theta \in L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2)$ and satisfies the global inequality:

$$\text{for all } t \in (0, T), \quad \|\theta(t, \cdot)\|_{\dot{H}^{-1/2}}^2 + 2k \int_0^t \int_{\mathbb{R}^2} |\theta|^2 \, dx \, ds \leq \|\theta_0\|_{\dot{H}^{-1/2}}^2. \quad (5)$$

Such a solution will be called a $\dot{H}^{-1/2}$-solution.
We point out that the existence of $\dot{H}^{-1/2}$-solutions is a new result; examples of such solutions are provided when we take, in particular, the initial data in $L^{4/3}$.

In this paper we first generalize some previous results given by Constantin, Cordoba, Wu in [4] and by Cordoba in [7] for the study of global existence in the Sobolev spaces $H^s$. These results show that under a condition of smallness of the initial data in $L^\infty$, if, moreover, the initial data is in $H^s$, $s \in \{1, 3/2, 2\}$, there exists a weak $L^2$-solution which remains in $H^s$. More precisely:

**Theorem 1.4.** Let $s \in \{1, 3/2, 2\}$, there exists a constant $C_\infty > 0$, such that for all initial data $\theta_0$ in $H^s$ with $\|\theta_0\|_\infty < C_\infty k$ there exists a $L^2$-solution $\theta$ such that $\theta \in C^0_0((0, \infty), H^s) \cap L^2((0, \infty), \dot{H}^{s+1/2})$ and

$$\|\theta(t, \cdot)\|_{H^s} \leq \|\theta_0\|_{H^s}$$

for all $t \in (0, \infty)$.

Here $C^0_0$ stands for continuous and bounded. These results are improvements (at least when the regularity is large enough i.e. $s > n/2$) of the known results for the Navier-Stokes equations where global solutions in Sobolev spaces are proved to exists with a small $H^s$ norm of the initial data.

As observed by Ju in [10], these solutions are unique since they belong to $L^\infty((0, \infty), H^s)$ when $s > 1$ or $L^2((0, T), \dot{H}^{3/2})$ when $s = 1$.

**Theorem 1.5.** Let $\theta_0 \in L^2$ and $T > 0$, assume that there exists a weak solution to $(QG)_{1/2}$ on $(0, T) \times \mathbb{R}^2$ with initial value $\theta_0$ such that:

$$\theta \in L^\infty((0, T), L^2) \cap L^2((0, T), \dot{H}^{1/2})$$

and

$$\theta \in L^\infty((0, T), \dot{H}^s) \text{ if } s > 1$$

or

$$\theta \in L^2((0, T), \dot{H}^{3/2}) \text{ if } s = 1.$$  

Then $\theta$ satisfies the energy inequality (4) and is the unique $L^2$-solution on $(0, T)$ with initial data $\theta_0$.

The proof of this result relies on estimating the $L^2$ norm of the difference of two solutions to $(QG)_{1/2}$, and therefore proves that the solutions are stable in the $L^2$ norm.

The main point in the proof of theorem 1.4 is the non-linear estimate

$$\left| \int \Lambda^{2s} \theta \mathcal{R} \cdot \nabla \theta dx \right| \leq \|\theta\|_\infty \|\Lambda^{s+1/2} \theta\|_2^2. \quad (6)$$
In [4], \( s \in \{1, 2\} \) and the proof relies on the Leibniz rule and integration by parts. In the case where \( s = 3/2 \), [7], the proof relies on the Leibniz rule, integration by parts and a clever computation using the duality between \( \text{BMO} \) and the Hardy space \( \mathcal{H}^1 \).

In section 3, we generalize this kind of non-linear estimates and prove more precisely the following inequality:

\[
\left| \int L^{2s} \theta \cdot \nabla \theta dx \right| \leq \| \theta_0 \|_\infty \| L^{s+1/2} \theta \|_2^2
\]  

(7)

for all \( s > -1/2 \) and all function \( \theta \) smooth enough (see proposition 3.1).

The proof of this estimate is quite technical, the main difficulties are the non-existence of Leibniz rule for fractional derivatives and the fact that the Riesz transforms are not bounded on \( L^\infty \), but from \( L^\infty \) to \( \text{BMO} \). To deal with these problems, we will use a substitute to the Leibniz rule for fractional derivatives (corollary 2.2).

Estimate (7) leads us to our main result which is the global existence of \( H^s \) weak solutions, \( s > -1/2 \), when the initial data has a small \( L^\infty \) norm:

**Theorem 1.6.** Let \( s > -1/2 \).

(i) if \(-1/2 < s < 0\), there exists a constant \( C_\infty > 0 \) such that for all initial data \( \theta_0 \) in \( \dot{H}^{1/2} \cap \dot{H}^s \cap L^\infty \) with \( \| \theta_0 \|_\infty < C_\infty k \), there exists a \( \dot{H}^{1/2} \)-solution \( \theta \), to \( (QG)_{1/2} \), with initial value \( \theta_0 \) such that \( \theta \in C_0^0((0, \infty), \dot{H}^s) \cap L^2((0, \infty), \dot{H}^{s+1/2}) \) and

\[
\| \theta(t, \cdot) \|_{\dot{H}^s} \leq \| \theta_0 \|_{H^s}
\]

for all \( t \in (0, \infty) \).

(ii) if \( s \geq 0 \), there exists a constant \( C_\infty > 0 \) such that for all initial data \( \theta_0 \) in \( H^s \cap L^\infty \) with \( \| \theta_0 \|_\infty < C_\infty k \), there exists a \( L^2 \)-solution \( \theta \), to \( (QG)_{1/2} \), with initial value \( \theta_0 \) such that \( \theta \in C_0^0((0, \infty), H^s) \cap L^2((0, \infty), H^{s+1/2}) \) and

\[
\| \theta(t, \cdot) \|_{H^s} \leq \| \theta_0 \|_{H^s}
\]

for all \( t \in (0, \infty) \).

The propagation of Sobolev regularity with a regularity below the critical regularity (i.e. \(-1/2 < s < 1\)) has no analogue with the 3D Navier-Stokes equations; we prove this theorem in section 3. We also point out that in the case \( s \geq 0 \), Wu obtained in [17] the same conclusion under the smallness assumption \( \| \theta(t) \|_\infty + \| R\theta(t) \|_\infty \leq C_0 k, C_0 > 0 \) (small), for all \( t \geq 0 \) which is a much more stronger assumption than the one given in our theorem.

Theorem 1.5 does not provide uniqueness of these solutions when \(-1/2 < s < 1\); however, we have proved in [14] that we still have a uniqueness result if we consider the more restrictive class of solutions with small \( L^\infty \) norm:

**Theorem 1.7.** Let \( \theta_0 \in L^\infty \cap \dot{H}^{-1/2} \), assume there exists a solution \( \theta \) to the quasi-geostrophic equation, \( (QG)_{1/2} \), on \((0, T)\) with initial value \( \theta_0 \) such that:

\[
\theta \in L^\infty((0, T), \dot{H}^{-1/2}) \cap L^2((0, T), L^2) \cap L^\infty((0, T), L^\infty).
\]
Then $\theta$ satisfies the equality in (5); moreover, there exists a positive constant $D_\infty$ such that, if the norm $\|\theta\|_{L^\infty(L^\infty)} < D_\infty k$, $\theta$ is the unique $H^{-1/2}$-solution to $(QG)_{1/2}$ on $(0, T)$ with initial value $\theta_0$. In particular for a solution which satisfies the maximum principle, we have $\|\theta(t)\|_{\infty} \leq \|\theta_0\|_{\infty}$ for all $t \in (0, T)$ and the smallness assumption on the $L^\infty(L^\infty)$ norm is implies by a smallness assumption on the $L^\infty$ norm of the initial data.

Solutions of theorem 1.6 are thus unique for every $s > -1/2$ (at least if $\theta_0$ is small enough).

Remark 1.2. All the above results are still valid when we work on the 2D torus; in that case, due to the embedding $L^2(\mathbb{T}^2) \hookrightarrow H^{-1/2}(\mathbb{T}^2)$ (where the dot stands for mean value equal to zero on $\mathbb{T}^2$) the results have a simpler statement.

In the last section, we leave energy inequalities and turn to the existence of global mild solutions. Mild solutions are provided by fixed points theorems which allow us to deal with more general spaces than energy inequality methods (homogeneous spaces for example). In the case of the study of Sobolev spaces, mild solutions give weaker results than energy method since we do not know how to take into account the maximum principle in the fixed point theorem and we can’t deal with regularity less than the critical regularity $s = 1$. Using the Picard contraction principle on the fluctuation $\theta - e^{-t\Lambda}\theta_0$ we can prove:

Theorem 1.8. Let $0 < \eta < 1$ we define $E_\eta$ as the Banach space of locally integrable functions, $f(t, x)$, on $]0, +\infty[ \times \mathbb{R}^2$ which satisfies $f \in L^\infty((0, \infty), \dot{B}^{1,1}_2) \cap L^2((0, \infty), H^{3/2})$ and $t^n f(t, \cdot) \in L^\infty((0, \infty), \dot{H}^{1+\eta}) \cap L^2((0, \infty), \dot{H}^{3/2+\eta})$.

For all $\theta_0 \in L^\infty \cap \dot{H}^1$ such that $R_1\theta_0, R_2\theta_0 \in L^\infty$; there exists $\epsilon > 0$ and $C > 0$ such that, if $\|\theta_0\|_{\infty} + \|R_1\theta_0\|_{\infty} + \|R_2\theta_0\|_{\infty} + \|\theta_0\|_{\dot{H}^1} < \epsilon$, the equation $\theta = e^{-t\Lambda}\theta_0 - B(\theta, \theta)$ has a unique solution which satisfies $\theta - e^{-t\Lambda}\theta_0 \in E_\eta$ and $\|\theta - e^{-t\Lambda}\theta_0\|_{E_\eta} \leq C(\|\theta_0\|_{\infty} + \|R_1\theta_0\|_{\infty} + \|R_2\theta_0\|_{\infty} + \|\theta_0\|_{\dot{H}^1})$.

Remark 1.3.
- If we take an initial data $\theta_0$ in $\dot{B}^{1,1}_2$ or in $H^s$, $s > 1$, with a small norm we can apply the theorem; however, in the last case it gives a weaker statement than theorem 1.6.
- This theorem is still valid if we switch the homogeneous spaces by their inhomogeneous counterparts; the result is then close to the one given in [3] where the authors consider an initial data with small norm in the Besov space $\dot{B}^{1,1}_2$.
- The proof relies on the fundamental fact that the fluctuation $\theta - e^{-t\Lambda}\theta_0$ is more regular than the tendency $e^{-t\Lambda}\theta_0$; this fact was first underlined, for the Navier-Stokes equations, by Cannone in [2] and used in [8] by Lemarié-Rieusset, Furioli and Terraneo to prove the uniqueness of mild solutions in $C([0, T], (L^n)^n)$.
- As far as we know, with the result given in [12], this is the first result of mild solutions for the equation $(QG)_{1/2}$. In [12], we give a result of global existence in a space close to $L^\infty$; in particular its gives the existence of non-radial self-similar solutions to $(QG)_{1/2}$.
- In contrast with the Navier-Stokes equations, it is not easy to get mild solutions results for the critical quasi-geostrophic equation and the results obtain in this paper and in [12] do not provide local existence with large initial data norm. Results of local existence with large initial norm has been obtain in [7] and [10] by energy inequalities but, until now, in contrast to the Navier-Stokes equation, there is no local existence results with large initial data norm in $H^1$ or $\dot{H}^1$. 6
Remark 1.4. For the simplicity of the presentation all the proofs will be made with the assumption $k = 1$.

2 Some technical preliminaries

We start with a very useful formula for fractional derivatives that we can find in [15] or more recently in [7].

Proposition 2.1. Let $0 < \alpha < 2$, there exists a constant $c_\alpha > 0$ such that for all function $f$ in $H^\infty(\mathbb{R}^n) = \bigcap_{k \geq 0} H^k(\mathbb{R}^n)$ we have the formula

$$\Lambda^\alpha f(x) = c_\alpha \ p.v. \int_{\mathbb{R}^n} \frac{\delta_y f(x)}{|y|^{n+\alpha}} dy$$

for all $x \in \mathbb{R}^n$ with $\delta_y f(x) = f(x - y) - f(x)$.

This formula allows us to compute $\Lambda^\alpha(\theta \varphi)$ and get a substitute for the Leibniz formula :

Corollary 2.2. Let $0 < \alpha < 2$ and $f, g$ be two functions in $H^\infty(\mathbb{R}^n)$ ; we have

$$\Lambda^\alpha (fg) = f \Lambda^\alpha g + g \Lambda^\alpha f - C_\alpha(f, g) \quad (8)$$

with

$$C_\alpha(f, g) = c_\alpha \ p.v. \int_{\mathbb{R}^n} \frac{(\delta_y f)(\delta_y g)}{|y|^{n+\alpha}} dy.$$

We will make use of the Littlewood-Paley theory which relies on dyadic frequencies cut-off, we will see it is a powerful tool in order to deal with non-linear estimates.

Let $C$ be the ring of center 0, of small radius $3/4$ and great radius $8/3$. There exists two radial functions $\varphi$ and $\psi$ with values in $[0, 1]$, belonging respectively to $\mathcal{D}(B(0, 4/3))$ and to $\mathcal{D}(C)$ such that

$$\varphi(\xi) + \sum_{j \geq 0} \psi(2^{-j}) = 1 \ \text{ for all } \xi \in \mathbb{R}^2,$$

$$\sum_{j \in \mathbb{Z}} \psi(2^{-j}) = 1 \ \text{ for all } \xi \in \mathbb{R}^2 \setminus \{0\}.$$

For $j \in \mathbb{Z}$, we define the $j$-th dyadic block of the Littlewood-Paley decomposition of a temperate distribution $f$ by

$$\Delta_j f = \mathcal{F}^{-1} \left( \psi(\xi/2^j) \mathcal{F}(f) \right).$$

We also define the low-frequencies cutoff operators by

$$S_j f = \mathcal{F}^{-1} \left( \varphi(\xi/2^k) \mathcal{F}(f) \right) = \sum_{k \leq j-1} \Delta_k f.$$

With these operators we have the following decomposition [11]:
Proposition 2.3. (Littlewood-Paley decomposition) For all $N \in \mathbb{N}$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$, we have the Littlewood-Paley decomposition of $f : f = S_N f + \sum_{j \geq N} \Delta_j f$ in $\mathcal{S}'(\mathbb{R}^n)$.

If, moreover, $\lim_{N \to -\infty} S_N f = 0$ in $\mathcal{S}'(\mathbb{R}^n)$, then we have the homogeneous Littlewood-Paley decomposition of $f : f = \sum_{j \in \mathbb{Z}} \Delta_j f$.

The interest of such a decomposition relies on the well-known following Bernstein inequalities

Lemma 2.4. Let $\sigma \in \mathbb{R}$, $j \in \mathbb{Z}$, $1 \leq p \leq q \leq \infty$ and all $f \in \mathcal{S}'(\mathbb{R}^n)$; we have,

$$2^{\sigma j} \| \Delta_j f \|_p \leq \| \Lambda^\sigma \Delta_j f \|_p \leq 2^{\sigma j} \| \Delta_j f \|_p,$$

$$\| \Lambda^\sigma S_j f \|_p \leq 2^{\sigma j} \| S_j f \|_p.$$

Using these inequalities it is not hard to get a characterization of the Sobolev spaces in terms of the Littlewood-Paley decomposition :

Proposition 2.5. (characterization of inhomogeneous Sobolev spaces)

For every real number $s$, a function $f$ belongs to the inhomogeneous Sobolev space $H^s$ if and only if $S_0 f \in L^2$ and $\sum_{j \geq 0} 2^{js} \| \Delta_j f \|_2^2 < \infty$.

Moreover, we have the equivalence of norms

$$\| f \|_{H^s} \sim \| S_0 f \|_2 + \left( \sum_{j \geq 0} 2^{js} \| \Delta_j f \|_2^2 \right)^{1/2},$$

where $\| \cdot \|_{H^s}$ stands for the classical norm of the Sobolev spaces : $\| f \|_{H^s} = \left( \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi \right)^{1/2}$.

Definition 2.1. (inhomogeneous Besov spaces)

For $s \in \mathbb{R}$, $(p, q) \in [1, \infty]$ and $N \in \mathbb{Z}$, the distribution $f \in \mathcal{S}'$ belongs to the the Besov spaces $B^{s,q}_p$ if and only if $S_N f \in L^p$, for all $j \geq N \Delta_j f \in L^p$ and $(2^{js} \| \Delta_j f \|_p)_{j \geq N} \in l^q$. The norm on this space is defined by

$$\| f \|_{B^{s,q}_p} = \| S_N f \|_p + \left( \sum_{j \geq N} 2^{jq} \| \Delta_j f \|_p^q \right)^{1/q}$$

and with this norm $B^{s,q}_p$ is a Banach space.

We will make use of homogeneous spaces, let first define the homogeneous Sobolev semi-norm :

Definition 2.2. Let $f$ be a tempered distribution and $s$ a real number. Then we set

$$\| f \|_{\dot{H}^s} = \left( \sum_{j \in \mathbb{Z}} 2^{2js} \| \Delta_j f \|_2^2 \right)^{1/2}.$$

Moreover, when $\hat{f} \in L^1_{\text{loc}}$, we have

$$\| f \|_{\dot{H}^s} \sim \left( \int |\xi|^{2s} |\hat{f}|^2 dx \right)^{1/2}.$$
It is important to point out that $\| \cdot \|_{\dot{H}^s}$ is a semi-norm in the sense that, if $f$ is a polynomial, the support of its Fourier transform is exactly the origin. Thus, for all $j \in \mathbb{Z}$, we have $\Delta_j f = 0$, and so $\| f \|_{\dot{H}^s} = 0$.

We have the following immediate properties:

**Lemma 2.6.** For every function $f \in \mathcal{S}(\mathbb{R})$ we have:

\[
\| f(\lambda \cdot) \|_{\dot{H}^s} = \lambda^{s-\frac{n}{2}} \| f \|_{\dot{H}^s} \quad \forall \lambda > 0, \forall s \in \mathbb{R};
\]

\[
\| f g \|_{\dot{H}^s} \leq C(\| f \|_{\infty} \| g \|_{\dot{H}^s} + \| g \|_{\infty} \| f \|_{\dot{H}^s}) \quad \forall s \geq 0;
\]

\[
\| \Lambda^s f \|_{\dot{H}^s} \sim \| f \|_{\dot{H}^{s-\sigma}} \quad \forall \sigma \in \mathbb{R}, \forall s \in \mathbb{R};
\]

\[
\| f \|_{\dot{H}^s} \leq C \| f \|_{\dot{H}^s} \quad \forall s \geq 0.
\]

We will also make use of the homogeneous Besov semi-norm:

**Definition 2.3.** Let $f$ be a tempered distribution, $s$ a real number and $(p,q) \in [1, \infty]^2$. Then, we set

\[
\| f \|_{\dot{B}^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{js} \| \Delta_j f \|_p^q \right)^{1/q}.
\]

Moreover, if $0 < s < 1$ and $\hat{f} \in L^1_{loc}$ we have the equivalence

\[
\| f \|_{\dot{B}^s_{p,q}} \sim \left( \int \frac{\| \delta_y f \|_p^q}{|y|^{n+sq}} dy \right)^{1/q}.
\]

We avoid the general definition of homogeneous Besov spaces and refer the reader to the book of Triebel [16] for details. For our purpose, we only define the homogeneous Besov space $\dot{B}^{1,1}_2(\mathbb{R}^2)$:

**Definition 2.4.** We define the Besov space $\dot{B}^{1,1}_2(\mathbb{R}^2)$ as the space of all temperate distribution $f$ such that $f = \sum_{j \in \mathbb{Z}} \Delta_j f$ in $\mathcal{S}'$ and $\| f \|_{\dot{B}^{1,1}_2} = \sum_{j \in \mathbb{Z}} 2^j \| \Delta_j f \|_2 < \infty$.

This normed space will be convenient for the analysis of $(QG)_{1/2}$ in the critical space $\dot{H}^1$ since the Riesz transforms are obviously bounded on this space and this space is continuously embedded in $L^\infty \cap \dot{H}^1$.

Moreover, we will use the easy interpolation inequality:

\[
\| f \|_{\dot{B}^{1,1}_2} \leq \| f \|_{\dot{H}^{\alpha_1}}^{\frac{\eta}{\alpha_2}} \| f \|_{\dot{H}^{\alpha_2}}^{1-\frac{\eta}{\alpha_2}} \tag{9}
\]

with $\alpha_1 < 1 < \alpha_2$ and $\eta \alpha_1 + (1-\eta) \alpha_2 = 1$.

The Littlewood-Paley analysis turns out to be very useful in the study of non-linearities; more precisely, if we consider two temperate distributions $u$ and $v$, we can formally write:
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad v = \sum_{j \in \mathbb{Z}} \Delta_j v \quad \text{and} \quad uv = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}} \Delta_j u \Delta_k v. \]

If we split the last sum in the following three sums we get the so-called Bony’s decomposition [1]:

\[ uv = \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v + \sum_{j \in \mathbb{Z}} S_{j-1} v \Delta_j u + \sum_{j \in \mathbb{Z}} \Delta_j u (\Delta_{j-1} v + \Delta_j v + \Delta_{j+1} v) \]

The interest of such a decomposition relies on the fact that it is easier to study each of these sums separately than the whole product.

For further details on Littlewood-Paley analysis and homogeneous Besov spaces, the interested reader may consult [9], [11] or [16].

We will make use of the maximal \( L^p(L^q) \) regularity theorem for the Poisson kernel which is the analog to the maximal \( L^p(L^q) \) regularity theorem for the heat kernel [11]:

**Theorem 2.7.** (maximal \( L^p(L^q) \) regularity for the Poisson Kernel)

The operator \( f(t, x) \to \int_0^t e^{-(t-s)\Lambda} f(s, \cdot) ds \) is bounded from \( L^p((0, T), L^q(\mathbb{R}^n)) \) to \( L^p((0, T), L^q(\mathbb{R}^n)) \) for every \( T \in (0, \infty), \quad 1 < p < \infty \) and \( 1 < q < \infty \).

The proof is essentially the same as for the heat kernel and is left to the reader.

We recall the definition of the Hardy space \( \mathcal{H}^1 \) and the space \( \text{BMO} \):

\[ \mathcal{H}^1(\mathbb{R}^n) = \{ f \in L^1(\mathbb{R}^n), f^* \in L^1(\mathbb{R}^n) \} \]

with \( f^*(x) = \sup_{0<r<\infty} \frac{1}{r^n} \int f(y) \eta(\frac{x-y}{r}) dy \) and \( \eta \in \mathcal{D}(\mathbb{R}^n), \quad \int \eta dx = 1 \).

We define \( \text{BMO}(\mathbb{R}^n) \) as the dual space of \( \mathcal{H}^1(\mathbb{R}^n) \). More details on these spaces can be found in [9] or in [11].

We end this section with one of the key ingredients in the proof of the estimate (7); this is the commutator between a singular integral and a function in \( \text{BMO} \):

**Theorem 2.8.** Let \( T \) be a singular integral operator given by a convolution kernel and \( b \) be a function in the space \( \text{BMO} \). Then, for all \( 1 < p < \infty \), the commutator between \( T \) and \( b \) is bounded on \( L^p \):

\[ \| T(bf) - bT(f) \|_p \leq \| b \|_{\text{BMO}} \| f \|_p \]

for all \( f \in L^p \).

For a proof of this result one may consult [9].

**Remark 2.1.** In particular, when \( T \) has a odd kernel, \( fT(f) \) belongs to the Hardy space \( \mathcal{H}^1 \) for every \( f \in L^2 \). We will use this remark when \( T \) is a Riesz transform which has a odd kernel.
3 Global existence in Sobolev spaces

In this section we prove theorem 1.6. The proof of this result relies on the following non-linear estimate :

**Proposition 3.1.** Let \( s \geq -\frac{1}{2} \), for all function \( \theta \) in \( H^\infty \) if \( s \geq 0 \) or \( H^\infty \cap \dot{H}^{-1/2} \) if \( -1/2 < s < 0 \), we have

\[
\left| \int \Lambda^{2s} \theta \mathcal{R}^\perp \theta \cdot \nabla \theta dx \right| \leq \| \theta \|_\infty \| \Lambda^{s+1/2} \theta \|_2^2.
\]

**proof.** In order to prove this inequality we need to split the proof in two main cases, \( s \in \frac{1}{2} \mathbb{N} \) and \( s \in ] -\frac{1}{2}, \frac{1}{2}[ \cup ]1/2, 3/2[ + \mathbb{N} \).

We start with this last case and especially with the case \( s \in ] -\frac{1}{2}, \frac{1}{2}[ \cup ]1/2, 3/2[ ; for this purpose we use the Bony’s decomposition introduced in section 2.

We can write,

\[
\int \Lambda^{2s} \theta \nabla \omega \cdot \mathcal{R}^\perp \theta = \sum_{j \in \mathbb{Z}} \int S_{j-1} \Lambda^{2s} \theta \Delta_j \nabla \theta \cdot \tilde{\Delta}_j \mathcal{R}^\perp \theta + \sum_{j \in \mathbb{Z}} \int \Delta_j \Lambda^{2s} \theta \cdot \tilde{\Delta}_j \mathcal{R}^\perp \theta + \sum_{j \in \mathbb{Z}} \int \Delta_j \Lambda^{2s} \theta \cdot \mathcal{R}^\perp \theta
\]

\[
- \sum_{j \in \mathbb{Z}} \int \Delta_j \Lambda^{2s} \theta \cdot \mathcal{R}^\perp \theta + \sum_{j \in \mathbb{Z}} \int \Delta_j \Lambda^{2s} \theta \cdot \mathcal{R}^\perp \theta
\]

\[
= (1) + (2) - (3) + (4)
\]

with \( \tilde{\Delta}_j = \Delta_j + \Delta_{j+1} \), \( \mathcal{F}(\tilde{\Delta}_j \mathcal{R}^\perp \theta)(\xi) = \tilde{\psi}(\xi/2^j) \frac{\xi^j}{|\xi|^2} \mathcal{F}(\theta) \) and \( \tilde{\psi} \in \mathcal{D} \) is supported in a ring and \( \tilde{\psi} \) is equal to 1 on \( \supp(\phi(2\cdot)) + \supp(\psi) \).

With the use of Bernstein inequalities, the control of the first two sums is easy :

\[
|(1)| \leq \sum_{j \in \mathbb{Z}} \| S_{j-1} \Lambda^{2s} \theta \|_\infty \| \Delta_j \nabla \theta \|_2 \| \tilde{\Delta}_j \mathcal{R}^\perp \theta \|_2
\]

\[
\leq \| \theta \|_\infty \sum_{j \in \mathbb{Z}} 2^{(2s+1)j} \| \Delta_j \theta \|_2 \| \tilde{\Delta}_j \theta \|_2
\]

\[
\leq \| \theta \|_\infty \| \Lambda^{s+1/2} \theta \|_2^2
\]

and, in a similar way we have,

\[
|(2)| \leq \| \theta \|_\infty \| \Lambda^{s+1/2} \theta \|_2^2.
\]
Since we do not have \( \|S_j \mathcal{R}^{-1} \theta\|_\infty \leq \|\theta\|_\infty \) (Riesz transforms are not bounded on \( L^\infty \) but from \( L^\infty \) into \( BMO \)), we can’t deal with the last terms as for the others. The idea is to “put some (fractional) derivative” on \( \mathcal{R}^{-1} \theta \) by the means of formula (8) and use the duality between \( \mathcal{H}^1 \) and \( BMO \) to deal with the bad term. We start with the case \(-1/2 < s < 3/2\), the terms (3) and (4) can be both treated in exactly the same way, we thus only deal with term (3).

Let \( j \in \mathbb{Z} \), since \( \nabla \cdot (S_{j+10} \mathcal{R}^{-1} \theta) = 0 \), an integration by parts gives:

\[
I_j = \int \Delta_j \Lambda^{2s} \Delta_j \nabla \theta \cdot S_{j+10} \mathcal{R}^{-1} \theta = - \int \mathcal{R} \Delta_j \Lambda^{2s+1} \theta \Delta_j \theta \cdot S_{j+10} \mathcal{R}^{-1} \theta.
\]

Using the formula (8) we get,

\[
I_j = - \int \Lambda^{s+1/2} \Delta_j \theta \mathcal{R} \Lambda^{s+1/2} \Delta_j \theta \cdot S_{j+10} \mathcal{R}^{-1} \theta - \int \Delta_j \theta \mathcal{R} \Lambda^{s+1/2} \Delta_j \theta \cdot \Lambda^{s+1/2} S_{j+10} \mathcal{R}^{-1} \theta - \int \mathcal{R} \Delta_j \Lambda^{s+1/2} \theta \cdot C_{s+1/2} (\Delta_j \theta, S_{j+10} \mathcal{R}^{-1} \theta).
\]

We are now going to estimate each of these terms; for the first term, the bad one, we use remark 2.1 to get

\[
\left| \int \Lambda^{s+1/2} \Delta_j \theta \mathcal{R} \Lambda^{s+1/2} \Delta_j \theta \cdot S_{j+10} \mathcal{R}^{-1} \theta \right| \leq \|\Lambda^{s+1/2} \Delta_j \theta \mathcal{R} \Lambda^{s+1/2} \Delta_j \theta\|_{\mathcal{H}^1} \|S_{j+10} \mathcal{R}^{-1} \theta\|_{BMO} \leq 2^{(2s+1)j} \|\theta\|_\infty \|\Delta_j \theta\|^2_2.
\]

The estimate of the second term is easy,

\[
\left| \int \Delta_j \theta \mathcal{R} \Lambda^{s+1/2} \Delta_j \theta \cdot \Lambda^{s+1/2} S_{j+10} \mathcal{R}^{-1} \theta \right| \leq \|S_{j+10} \Lambda^{s+1/2} \mathcal{R}^{-1} \theta\|_\infty \|\Delta_j \theta\|_2 \|\mathcal{R} \Lambda^{s+1/2} \Delta_j \theta\|_2 \leq (\sum_{k \leq j+10} \|\Delta_k \Lambda^{s+1/2} \mathcal{R}^{-1} \theta\|_\infty) \|\Delta_j \theta\|^2_2 \leq 2^{(2s+1)j} \|\theta\|_\infty \|\Delta_j \theta\|^2_2.
\]

We now deal with the last term and write,

\[
\left| \int \mathcal{R} \Delta_j \Lambda^{s+1/2} \cdot C_{s+1/2} (\Delta_j \theta, S_{j+10} \mathcal{R}^{-1} \theta) \right| \leq \|\mathcal{R} \Delta_j \Lambda^{s+1/2} \theta\|_2 \|C_{s+1/2} (\Delta_j \theta, S_{j+10} \mathcal{R}^{-1} \theta)\|_2.
\]

The difficulty relies on the estimate of the \( L^2 \) norm of \( C_{s+1/2} \); we have two cases to consider, \(-1/2 < s < 1/2 \) and \( 1/2 < s < 3/2 \).

When \(-1/2 < s < 1/2\), we write

\[
\|C_{s+1/2} (\Delta_j \theta, S_{j+10} \mathcal{R}^{-1} \theta)\|_2 \leq 2 \|\Delta_j \theta\|_2 \int \frac{\|\delta_y S_{j+10} \mathcal{R}^{-1} \theta\|_\infty}{|y|^{s+1/2}} \, dy \leq 2 \|\Delta_j \theta\|_2 \|S_{j+10} \mathcal{R}^{-1} \theta\|_{B^{s+1/2,1}_n}.
\]
where we used the equivalent representation of Besov semi-norm by the mean of differences (definition 2.3).

We can estimate the Besov norm:

$$\|S_{j+10} R^{\perp} \theta\|_{B_{\infty}^{s+1/2,1}} = \sum_{k \leq j + 3} 2^{(s+1/2)k} \|\Delta_k S_{j+10} R^{\perp} \theta\|_\infty \leq \|\theta\|_\infty \sum_{k \leq j + 3} 2^{(s+1/2)k} \leq 2^{(s+1/2)j} \|\theta\|_\infty.$$ 

We thus get,

$$\|C_{s+1/2}(\Delta_j \theta, S_{j+10} R^{\perp} \theta)\|_2 \leq 2^{(s+1/2)j} \|\theta\|_\infty \|\Delta_j \theta\|_2.$$ 

Now, for the case $1/2 < s < 3/2$, we use that $\|\delta y \Delta_j \theta\|_2 \leq 2^j y \|\Delta_j \theta\|_2$ to get,

$$\|C_{s+1/2}(\Delta_j \theta, S_{j+10} R^{\perp} \theta)\|_2 \leq 2^j \|\Delta_j \theta\|_2 \int \frac{\|\delta y S_{j+10} R^{\perp} \theta\|_\infty}{|y|^{2(s-1/2)}} dy \leq 2^j \|\Delta_j \theta\|_2 \|S_{j+10} R^{\perp} \theta\|_{B_{\infty}^{s-1/2,1}}.$$ 

We estimate the Besov norm in the same way as before,

$$\|S_{j+10} R^{\perp} \theta\|_{B_{\infty}^{s-1/2,1}} \leq 2^{(s-1/2)j} \|\theta\|_\infty,$$

and get

$$\|C_{s+1/2}(\Delta_j \theta, S_{j+10} R^{\perp} \theta)\|_2 \leq 2^{(s+1/2)j} \|\theta\|_\infty \|\Delta_j \theta\|_2.$$ 

Finally, collecting all the previous estimates, we get the desired estimate for the term (3):

$$|\langle 3 \rangle| \leq \|\theta\|_\infty \|\Lambda^{s+1/2} \theta\|_2^2.$$ 

Now, if $s = k + \epsilon$ for a $k \in \mathbb{N}$ and $\epsilon \in [-1/2, 1/2[ \cup ]1/2, 3/2]$, we write that $\Lambda^{2k+1} = (-\Delta)^{k} \Lambda^{2k+1}$ then, after integration by parts and the use of the Leibniz formula we get that the quantity

$$\int R \Delta_j (-\Delta)^k \Lambda^{2k+1} \theta \cdot \Delta_j \theta \cdot S_{j+10} R^{\perp} \theta$$

is equal to a sum of integrals where there is some derivatives on $S_{j+10} R^{\perp} \theta$, these terms are thus easy to estimate, and a last integral with no derivatives on $S_{j+10} R^{\perp} \theta$:

$$\int R \Delta_j (-\Delta)^{\frac{k}{2}} \Lambda^{2k+1} \theta (-\Delta)^{\frac{k}{2}} \Delta_j \theta \cdot S_{j+10} R^{\perp} \theta$$

if $k$ is even; or,

$$\int R \Delta_j \nabla (-\Delta)^{\frac{k-1}{2}} \Lambda^{2k+1} \theta \nabla (-\Delta)^{\frac{k-1}{2}} \Delta_j \theta \cdot S_{j+10} R^{\perp} \theta$$

if $k$ is odd.

The end of the proof is essentially the same as before, we use formula (8) to compute

$$\Lambda^{s+1/2}((-\Delta)^{\frac{k}{2}} \Delta_j \theta \cdot S_{j+10} R^{\perp} \theta)$$

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in the even case (odd case is similar) and we estimate the $L^2$ norm of

$$C_{\epsilon+1/2}((-\Delta)^{k_2} \Delta_j \theta, S_{j+16} \mathcal{R}^{\perp} \theta)$$

in exactly the same way as before.

We now turn to the case where $\epsilon = 1/2 + k$ for a $k \in \mathbb{N}$ and write,

$$\int \Lambda^{2\epsilon} \mathcal{R}^{\perp} \theta \cdot \nabla \theta dx = \int (-\Delta)^k \Lambda \theta \cdot \mathcal{R}^{\perp} \theta dx$$

where we used that $\nabla = \mathcal{R} \Lambda$.

Then, with the use of the Leibniz rule, the previous quantity can be written as a sum of several integrals where there is some derivatives on $\mathcal{R}^{\perp} \theta$ so that it is easy to deal with (using, for example, Bony’s decomposition as before) and another integral :

$$\int (-\Delta)^{k_2} \Lambda \theta \mathcal{R}((-\Delta)^{k_2} \Lambda \theta \cdot \mathcal{R}^{\perp} \theta dx$$

if $k$ is even; or

$$\int \nabla(-\Delta)^{k_2} \Lambda \theta \mathcal{R}\nabla((-\Delta)^{k_2} \Lambda \theta \cdot \mathcal{R}^{\perp} \theta dx$$

if $k$ is odd.

We then use remark 2.1 to conclude :

$$\left| \int (-\Delta)^{k_2} \Lambda \theta \mathcal{R}((-\Delta)^{k_2} \Lambda \theta \cdot \mathcal{R}^{\perp} \theta dx \right| \leq \|\Lambda^{s+1/2} \mathcal{R} \Lambda^{s+1/2} \theta \|_{H^1} \|\mathcal{R}^{\perp} \theta \|_{BMO}$$

(13)

if $k$ is even; the case where $k$ is odd is similar if we write $\nabla = \mathcal{R} \Lambda$ and use the continuity of Riesz transforms on $L^2$.

We can now prove theorem 1.6 :

**proof.** Let $\theta_0$ be a function as in theorem 1.6, we construct the solution as the weak limit of a sequence of solutions to the following problems :

$$\begin{cases} 
\partial_t \theta^\epsilon - \mathcal{R}^{\perp} \theta^\epsilon \cdot \nabla \theta^\epsilon + \Lambda \theta^\epsilon - \epsilon \Delta \theta^\epsilon = 0 \\
\theta^\epsilon(0, \cdot) = \omega(x) * \theta_0
\end{cases}$$

(14)

with $0 < \epsilon < 1$, $\omega(x) = e^{-4\omega(x)}$, $\omega \in \mathcal{D}(\mathbb{R}^2)$ and $\int \omega dx = 1$.

Using energy inequalities and lemma 3.1, it is easy to prove that there exists a unique solution of (14) which belongs to $C^\infty((0, \infty) \times \mathbb{R}^2)$ and satisfies $\theta^\epsilon(t) \in H^\infty(\mathbb{R}^2)$ for every $t > 0$.  

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According to [13], we can extract a sequence \((\theta^n)_{n \geq 1}\) which is \(*\)-weakly convergent to a solution \(\theta\) to \((QG)_{1/2}\) in the space \(L^\infty((0, \infty), \dot{H}^{-1/2})\) if \(-1/2 < s < 0\), or in the space \(L^\infty((0, \infty), L^2)\) if \(s \geq 0\).

Now, with the notation \(\theta_n = \theta^n, n \geq 1\), multiplying the evolution equation of \(\theta_n\) by \(\Lambda^s \theta_n\) and integrating by parts, we obtain the equality
\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^s \theta_n\|^2 + \|\Lambda^{s+1/2} \theta_n\|^2 + \epsilon_n \|\Lambda^{s+1} \theta_n\|^2 = \int \Lambda^s \theta_n \cdot \nabla \theta_n dx.
\]
Using lemma 3.1 and integrating over time, we get that
\[
\|\Lambda^s \theta_n\|^2 + 2 \int_0^t \|\Lambda^{s+1/2} \theta_n(s)\|^2 ds \leq C_\infty \|\theta_0\|_\infty \int_0^t \|\Lambda^{s+1} \theta_n(s)\|^2 ds
\]
and letting, \(n \to +\infty\), in this inequality we get (see [11] pp. 136 for details)
\[
\|\Lambda^s \theta\|^2 + 2 \int_0^t \|\Lambda^{s+1/2} \theta(s)\|^2 ds \leq C_\infty \|\theta_0\|_\infty \int_0^t \|\Lambda^{s+1} \theta(s)\|^2 ds.
\]
If \(\|\theta_0\|_\infty < C_\infty\) we get that \(\theta \in L^\infty([0, \infty), \dot{H}^s) \cap L^2((0, \infty), \dot{H}^{s+1/2}).\)

Finally, we prove the continuity in time of the \(\dot{H}^s\) or \(H^s\) norm. Since \(\theta\) is a weak solution to \((QG)_{1/2}\) on \((0, \infty) \times \mathbb{R}^2\), it is also a mild solution to \((QG)_{1/2}\) on \((0, \infty) \times \mathbb{R}^2\) with initial data \(\theta_0:\)
\[
\theta = e^{-t\Lambda} \theta_0 - \int_0^t e^{-(t-s)\Lambda} \nabla \cdot (\theta u) ds
\]
for all \(t \in (0, \infty)\). Then, using the fact that \(H^{s+1/2} \cap L^\infty\) is an algebra for pointwise multiplication (when \(s \geq -1/2\)), the following lemma gives the conclusion :

**Lemma 3.2.** Let \(T \in (0, +\infty)\) and \(r \in \mathbb{R}\), for all \(f \in L^2((0, T), H^{r+1/2})\) we have
\[
(x, t) \mapsto \int_0^t e^{-(t-s)\Lambda} f(x, s) ds \in C^0([0, T], H^r)
\]

**Remark 3.1.** The same conclusion holds if one switches the Sobolev spaces \(H^{r+1/2}\) and \(H^r\) by their homogeneous counterparts \(\dot{H}^{r+1/2}\) and \(\dot{H}^r\).

We prove the lemma :
Let \(f \in L^2((0, T), H^{r+1/2})\), \(g(t) = \int_0^t e^{-(t-s)\Lambda} f ds\) and \(h \in H^{-r}\). We first write
\[
\langle g(t) \rangle_{H^{r+1/2}} = \int_0^t \langle f(s) \rangle_{H^{-1/2}} ds.
\]
Then, the Cauchy-Schwarz inequality gives
\[
\|\langle g(t) \rangle_{H^{r+1/2}}\| \leq \left( \int_0^t \| (Id - \Delta)^{r+1/2} f \|_{L^2}^2 ds \right)^{1/2} \left( \int_0^t \| (Id - \Delta)^{-r-1/2} e^{-(t-s)\Lambda} h \|_{L^2}^2 ds \right)^{1/2} \leq \|f\|_{L^2([0, T], H^{r+1/2})} \left( \int_0^{+\infty} \| (Id - \Delta)^{-r-1/2} e^{-(t-s)\Lambda} h \|_{L^2}^2 ds \right)^{1/2}.
\]
Finally, with the Plancherel equality we have
\[
\int_0^{+\infty} \| (Id - \Delta)^{-r-1/2} e^{-(t-s)\Lambda} h \|^2 ds = \frac{1}{(2\pi)^2} \int_0^{+\infty} \int (1 + |\xi|^2)^{-r-1/2} e^{-2|\xi|/|\xi|^2} |\hat{h}(\xi)|^2 d\xi dt
= \int \frac{|\xi|}{(1 + |\xi|^2)^{1/2}} (1 + |\xi|^2)^{-r} |\hat{h}(\xi)|^2 d\xi
\leq \| h \|_{H^{-r}}.
\]

The mapping \( f \to g \) is thus bounded from \( L^2((0, T), H^{r+1/2}) \) to \( L^\infty([0, T), H^r) \). When \( f \) belongs to \( D(((0, T) \times \mathbb{R}^2) \) we easily check that \( g \in C^0([0, T], H^r) \) and by the density of \( D((0, T) \times \mathbb{R}^2) \) in \( L^2((0, T), H^{r+1/2}) \) we get that the mapping \( f \to g \) is bounded from \( L^2((0, T), H^{r+1/2}) \) to \( C^0([0, T], H^r) \).

\( \diamond \)

4 Mild solutions

In this section we are going to prove theorem 1.8. Let \( B \) be the bilinear operator defined on \( L^2((0, T), F_2) \times L^2((0, T), F_2) \) (see [13]) by
\[
B(f, g)(t) = \int_0^t e^{-(t-s)\Lambda} \nabla \cdot (f R^1 g) dr.
\]

We will use the fixed point result proved in [2]:

**Proposition 4.1.** Let \( E \) be a Banach space, \( B \) a bilinear map from \( E \times E \) in \( E \) and \( L \) a linear map from \( E \) in \( E \) such that
\[
\forall (e, f) \in E^2, \quad \| B(e, f) \|_E \leq C_B \| e \|_E \| f \|_E,
\]
\[
\forall e \in E, \quad \| L(e) \|_E \leq C_L \| e \|_E.
\]

Under the conditions \( 0 \leq C_L < 1 \) and \( 4C_B \| e_0 \| < (1 - C_L)^2 \), if \( e_0 \in E \) and
\[
R = \frac{(1 - C_L) - \sqrt{(1 - C_L)^2 - 4C_B e_0}}{2C_B},
\]
the equation \( e = e_0 + L(e) + B(e, e) \) has a solution \( \| e \|_E \leq R \) and this solution is unique in the closed ball \( \overline{B}(0, R) \).

In order to prove theorem 1.8, we shall define some function spaces:
- \( X \) is the space of locally integrable functions on \( \mathbb{R}^2 \) which belongs to \( \dot{H}^1 \cap L^\infty \) and whose Riesz transforms are bounded on \( L^\infty \).
- for \( 0 < \eta < 1 \), \( X_\eta \) is the Banach space of locally integrable functions, \( f(t, x) \), on \( ]0, +\infty[ \times \mathbb{R}^2 \) which satisfies \( f \in L^\infty((0, \infty), X) \cap L^2((0, \infty), \dot{H}^{3/2}) \) and \( t^\eta f(t, \cdot) \in L^\infty((0, \infty), \dot{H}^{1+\eta}) \cap L^2((0, \infty), \dot{H}^{3/2+\eta}) \).
for \(0 < \eta < 1\), \(E_\eta\) is the Banach space of locally integrable functions, \(f(t, x)\), on \([0, +\infty] \times \mathbb{R}^2\) which satisfies \(f \in L^\infty((0, \infty), \dot{B}^{1+1}_2)\) and \(t^n f(t, \cdot) \in L^\infty((0, \infty), \dot{B}^{3/2+\eta})\) and \(t^n f(t, \cdot) \in L^\infty((0, \infty), \dot{B}^{3/2+\eta})\).

**Remark 4.1.** Since \(\dot{B}^{1+1}_2\) is included in \(X\), \(E_\eta\) is obviously included in \(X_\eta\).

We have the following lemma,

**Lemma 4.2.** The bilinear operator \(B\) is bounded from \(X_\eta \times X_\eta\) into \(E_\eta\).

**Proof.** Let \(f, g \in X_\eta\), we note \(h = B(f, g)\). In order to prove the lemma, we will prove the two following estimates,

\[
\sup_{t > 0} t^{-1} \|h\|_{L^2} + t^n \|h\|_{\dot{H}^{1+\eta}} \leq \|f\|_{X_\eta} \|g\|_{X_\eta}
\]

and

\[
\int_0^\infty r^{-2} \|h(r)\|_{\dot{H}^{1/2}}^2 + \|r^n h(r)\|_{\dot{H}^{3/2+\eta}}^2 dr \leq \|f\|_{X_\eta}^2 \|g\|_{X_\eta}^2.
\]

Then, using the interpolation argument (9), we immediately get \(h \in E_\eta\).

We start with the estimate on \(\sup_{t > 0} t^{-1} \|h\|_{L^2}\):

\[
\|h(t)\|_2 \leq \int_0^t \|\nabla \cdot (f R^\perp g)(r)\|_2 dr \leq t \sup_{t > 0} \|\nabla \cdot (f R^\perp g)(t)\|_{\dot{H}^{1/2}} \leq t \sup_{t > 0} \|f(t)\|_X \sup_{t > 0} \|g(t)\|_X.
\]

We now deal with the integral \(\int_0^\infty r^{-2} \|h(r)\|_{\dot{H}^{1/2}}^2 dr\) and write:

\[
t^{-1} \|h(t)\|_{\dot{H}^{1/2}} \leq t^{-1} \int_0^t \|\nabla \cdot (f R^\perp g)(r)\|_{\dot{H}^{1/2}} dr = t^{-1} \int_0^t \|\nabla \cdot (f R^\perp g)(r)\|_{\dot{H}^{3/2}} dr.
\]

Then, with the observation that

\[
\|\nabla \cdot (f R^\perp g)(r)\|_{\dot{H}^{3/2}} \leq \|f\|_X \|g\|_{\dot{H}^{3/2}} + \|g\|_X \|f\|_{\dot{H}^{3/2}},
\]

the continuity of the Hardy-Littlewood maximal function on \(L^2\) gives:

\[
\int_0^\infty r^{-2} \|h(r)\|_{\dot{H}^{1/2}}^2 dr \leq \|\nabla \cdot (f R^\perp g)(r)\|_{\dot{H}^{3/2}}^2 \int_0^\infty dr \leq \|f\|_{X_\eta}^2 \|g\|_{X_\eta}^2.
\]

Let now deal with \(\sup_{t > 0} t^n \|h\|_{\dot{H}^{1+\eta}}\); we split \(h\) in two terms:

\[
h(t) = \int_0^{t/2} e^{-(t-r)\Lambda} \nabla \cdot (f R^\perp g) dr + \int_{t/2}^t e^{-(t-r)\Lambda} \nabla \cdot (f R^\perp g) dr = h_1(t) + h_2(t).
\]
For the first term we write,
\[
\|h_1(t)\|_{H^{1+\eta}} \leq \int_0^{t/2} \frac{1}{t-r} \|(f\mathcal{R}^\perp g)(r)\|_{H^{1+\eta}} dr \\
\leq t^{-1} \int_0^{t/2} \frac{dr}{r^\eta} \|f\|_{X^{\eta}} \|g\|_{X^{\eta}} \\
\leq t^{-\eta} \|f\|_{X^{\eta}} \|g\|_{X^{\eta}}.
\]
To deal with \(h_2\) we proceed to the change of variable \(\tau = r - t/2\),
\[
h_2(t) = \int_0^{t/2} e^{-(t/2-\tau)\Lambda} \nabla \cdot (f\mathcal{R}^\perp g)(\tau + t/2) d\tau.
\]
Then, using lemma 3.2 (homogeneous version) we get,
\[
\|h_1(t)\|_{H^{1+\eta}} \leq \left( \int_0^{t/2} \|(f\mathcal{R}^\perp g)(\tau + t/2)\|_{H^{\eta+3/2}}^2 d\tau \right)^{1/2}
\]
we then write
\[
\|t^n(f\mathcal{R}^\perp g)\|_{L^2(H^{\eta+3/2})} \leq \|f\|_{L^\infty(X)} \|t^n g\|_{L^2(H^{\eta+3/2})} + \|g\|_{L^\infty(X)} \|t^n f\|_{L^2(H^{\eta+3/2})}
\]
and we conclude with the following bound:
\[
\int_0^{t/2} \|(f\mathcal{R}^\perp g)(\tau + t/2)\|_{H^{\eta+3/2}}^2 d\tau = \int_0^{t/2} \frac{1}{(\tau + t/2)^{2n}} (\tau + t/2)^2 \|(f\mathcal{R}^\perp g)(\tau + t/2)\|_{H^{\eta+3/2}}^2 d\tau \\
\leq t^{-2n} \|t^n(f\mathcal{R}^\perp g)\|_{L^2(H^{\eta+3/2})}^2.
\]
Finally, the last estimate, \(\int_0^{\infty} \|r^n h(r)\|_{H^{\eta+3/2}}^2 dr \leq \|f\|_{X^{\eta}} \|g\|_{X^{\eta}},\) is obvious for \(h_2\) by the mean of the maximal \(L^p(L^q)\) regularity theorem for the Poisson kernel (theorem 2.7). For the term \(h_1\), we write
\[
h_1(t) = \int_0^{t/2} \Lambda^{\eta+1} e^{-(t-r)\Lambda} \Lambda^{-\eta-1} \nabla \cdot (f\mathcal{R}^\perp g)(r) dr
\]
then
\[
t^n \|h_1(t)\|_{H^{\eta/2+\eta}} \leq t^n \int_0^{t/2} \frac{1}{(t-r)^{1+\eta}} \|(f\mathcal{R}^\perp g)(r)\|_{H^{\eta/2}} dr \\
\leq t^{-1} \int_0^{t} \|(f\mathcal{R}^\perp g)(r)\|_{H^{\eta/2}} dr
\]
and, as before, we use the continuity of the Hardy-Littlewood maximal function on \(L^2\) to conclude the proof of the lemma. \(\diamond\)
Theorem 1.8 is then an immediate consequence of proposition 4.1 and lemma 4.2.
Références


