Optimal position targeting with stochastic linear-quadratic costs

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We consider the dynamic control problem of attaining a target position at a finite time T, while minimizing a linear-quadratic cost functional depending on the position and speed. We assume that the coefficients of the linear-quadratic cost functional are *stochastic* processes adapted to a Brownian filtration. We provide a probabilistic solution in terms of two coupled backward stochastic differential equations possessing a singularity at the terminal time T. We verify optimality of the candidate control by using a penalization argument. Special cases for which the problem has explicit solutions are discussed. Finally we illustrate our results in financial applications, where we derive optimal trading strategies for closing financial asset positions in markets with stochastic price impact and non-zero returns.

1 Introduction

We consider the control problem of attaining a target position at some finite time horizon T. We allow only for position paths X_t, $t \in [0,T]$, that are absolutely continuous, and we interpret the derivative $\dot{X}_t$ as the *position speed*. We suppose that a position path entails costs that depend linear-quadratically on the position X and the position speed $\dot{X}$. The linear-quadratic cost functional is assumed to have random coefficients, modeled as stochastic processes γ, π and η that are progressively measurable with respect to a Brownian filtration on some probability space. The coefficient processes satisfy some nice

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integrability conditions specified in Section 2 below. Moreover, η and γ are nonnegative. The cost functional to be minimized is given by

$$J(X) = E\left[\int_0^T \left(\gamma_t X_t^2 + \pi_t \dot{X}_t + \eta_t \dot{X}_t^2\right) dt\right]. \tag{1}$$

Our goal is to find the position process X that minimizes the costs (1) among all progressively measurable processes attaining a given target at time T. For simplicity we first assume that $X_T = 0$ a.s. We refer to Section 3.2 for how to incorporate stochastic terminal state constraints.

We choose a probabilistic approach, based on a maximum principle, in order to solve the control problem. We show that one can characterize the associated value function and the optimal control in terms of two backward stochastic differential equations (BSDEs). One of the BSDEs satisfies the dynamics

$$dY_t = \left(\frac{Y_t^2}{\eta_t} - \gamma_t\right) dt + Z_t dW_t, \tag{2}$$

where W is a Brownian motion, and possesses the singular terminal condition

$$\lim_{t \nearrow T} Y_t = \infty. \tag{3}$$

BSDEs with singular terminal conditions have been first studied by Popier in [12] and [13]. In [2] singular BSDEs have been employed for solving the control problem of minimizing the costs (1) without the linear term $\pi\dot{X}$. More precisely, the objective functional imposed in [2] is given by

$$\bar{J}(X) = E\left[\int_0^T \left(\gamma_t |X_t|^p + \eta_t |\dot{X}_t|^p\right) dt\right], \tag{4}$$

for some power $p > 1$. It is shown that the value function and the optimal control can be characterized in terms of the solution process of a BSDE with a singular terminal condition. By introducing a linear component in the objective functional, the optimal control and the value function can no longer be characterized in terms of a *single BSDE*. For the case $p = 2$, there exists a probabilistic characterization in terms of a system of two coupled BSDEs. The aim of the present paper is to show this.

A variant of the stochastic control problem of minimizing the cost functional $\bar{J}$ has been studied by Graewe, Horst and Séré in [7]. Position paths are allowed to be discontinuous at jump times of a Poisson process. The coefficients η and γ are Markov processes. Under a non-vanishing volatility assumption the authors establish existence and uniqueness of a classical solution to the associated HJB equation. In [6] these results are extended beyond the Markovian framework by using backward stochastic partial differential equations with singular terminal conditions.

The paper is organized as follows. In Section 2 we describe the precise model setup and present the main result, stated in Theorem 2.4. In Section 3 we illustrate the main result in two economic applications. The first application is concerned with the optimal closure of a financial asset position in a market with a stochastic price impact. We can derive optimal trading rates by directly appealing to Theorem 2.4. In the second application we consider an agent aiming at building up an asset position, e.g. for risk management purposes, but not knowing the precise target at the beginning. We assume that the agent learns the target position *gradually* up to time T. We show that one can reduce the control problem with an uncertain target to a problem with position target 0.

Section 4 is devoted to a penalized version of our control problem: position controls are no longer required to attain the target, but any deviation entails extra costs weighted with a factor L. Letting the weight factor L tend to infinity, allows us, in Section 5, to perform the verification for the control problem with the target constraint and to prove the main result.

2 The main result: the optimal control in terms of coupled BSDEs

Fix a deterministic, finite time horizon $0 < T < \infty$. Let $d \in \mathbb{N}$ and $(W_t)_{t\in[0,T]}$ a d-dimensional Brownian motion on a probability space $(\Omega, P, \mathcal{F})$. Let $(\mathcal{F}_t)_{t\in[0,T]}$ denote the filtration generated by $(W_t)_{t\in[0,T]}$ and completed by the P-null sets. We denote by $\mathcal{A}(x)$ the set of all progressively measurable processes $X : \Omega \times [0,T] \to \mathbb{R}$ with absolutely continuous paths that start in $X_0 = x \in \mathbb{R}$ and satisfy $E[\sup_{0\le t\le T} |X_t|^4] < \infty$. The set of controls, denoted by $\mathcal{A}_0(x)$, consists of all processes in $\mathcal{A}(x)$ that additionally meet the terminal state constraint $X_T = 0$ a.s.

We aim at finding the control that minimizes the objective functional (1). We need to specify the conditions imposed on the coefficient processes γ, π and η. We assume that γ, π and η are progressively measurable with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Moreover, to specify some integrability conditions, we introduce the following spaces of processes. For $i = 1, 2$ and $t \le T$

$$\mathcal{M}^i(0,t) = L^i(\Omega \times [0,t], \mathcal{P}, P \otimes \text{Leb})$$

where Leb is the Lebesgue measure and $\mathcal{P}$ denotes the σ-algebra of $(\mathcal{F}_t)$-progressively measurable subsets of $\Omega \times [0,T]$. Throughout we make the following assumption.

Assumption 2.1. *a) The processes η and γ are nonnegative (this guarantees that the cost functional J is convex).*

b) $\eta, \gamma \in \mathcal{M}^2(0,T)$ and $E\left[\left(\int_0^T \frac{1}{\eta_s^2} ds\right)^4\right] < \infty$.

c) π is a semimartingale with semimartingale decomposition

$$\pi_t = \pi_0 + b_t + \int_0^t \psi_t dW_t,$$

such that π_0 *is a real,* $E\left[\left(\int_0^T \psi_t^2 dt\right)^4\right] < \infty$ *and* $E\left[\left(\int_0^T |db_t|\right)^8\right] < \infty$[1]*.*

Assumption 2.1 guarantees that $J(X)$ is defined for any $X \in \mathcal{A}_0(x)$. Indeed, the expectation $E\left[\int_0^T \left(\gamma_t X_t^2 + \eta_t \dot{X}_t^2\right) dt\right]$ is defined (possibly taking the value $+\infty$). Moreover, the integral $\int_0^T \pi_t \dot{X}_t dt$ exists and has finite expectation. To show this, note that the paths of $X \in \mathcal{A}_0(x)$ are absolutely continuous, and hence have finite variation on $[0, T]$. Since every path of π is bounded, we further obtain that the integral $\int_0^T |\pi_t \dot{X}_t| dt$ is finite, a.s. Integration by parts yields $\int_0^T \pi_t \dot{X}_t dt = \pi_T X_T - \pi_0 X_0 - \int_0^T X_t d\pi_t$. Consequently,

$$\begin{aligned} E\left[\left|\int_0^T \pi_t \dot{X}_t dt\right|\right] &\leq |\pi_0 X_0| + E\left[\left|\int_0^T X_t d\pi_t\right|\right] \\ &\leq |\pi_0 X_0| + E\left[\sup_{t\in[0,T]} |X_t|^2\right]^{\frac{1}{2}} E\left[(\pi_T - \pi_0)^2\right]^{\frac{1}{2}} < \infty. \end{aligned}$$

We now define the value function

$$v(x) = \inf_{X \in \mathcal{A}_0(x)} J(X). \tag{5}$$

We give a purely probabilistic solution of the control problem (5) in terms of two processes (Y, Z) satisfying

$$dY_t = \left(\frac{Y_t^2}{\eta_t} - \gamma_t\right) dt + Z_t dW_t \tag{6}$$

with singular terminal condition

$$\lim_{t \nearrow T} Y_t = \infty. \tag{7}$$

We first recall the notion of a solution of a BSDE with singular terminal condition in the style of [12].

Definition 2.2. We say that a pair of progressively measurable processes (Y, Z) with values in $\mathbb{R} \times \mathbb{R}^d$ solves the BSDE (6) with singular terminal condition ξ if it satisfies

(i) for all $0 \leq s \leq t < T$: $Y_s = Y_t - \int_s^t \left(\frac{Y_r^2}{\eta_r} - \gamma_r\right) dr - \int_s^t Z_r dW_r$;

(ii) for all $0 \leq t < T$: $E\left[\sup_{0\leq s\leq t} |Y_s|^2 + \int_0^t |Z_r|^2 dr\right] < \infty$;

(iii) $\lim_{t\nearrow T} Y_t = \infty$, a.s.

In [2] the following existence result is established.

[1]Here $|db_t|$ denotes the integral with respect to the absolute variation of the path b.

Proposition 2.3. *There exists a minimal*[2] *solution* (Y, Z) *of the BSDE* (6) *with* $\lim_{t \nearrow T} Y_t = \infty$.

Proof. See [2, Theorem 1.2]. □

Let (Y, Z) be the minimal solution of (6) and (7). Moreover, let $H_t = \exp\left(-\int_0^t \frac{Y_s}{\eta_s} ds\right)$. From [2] we know that $H \in \mathcal{A}_0(1)$ is optimal in (5) if $\pi = 0$. Define

$$U_t = -\frac{1}{2} E\left[\int_t^T \frac{H_s}{H_t} d\pi_s \middle| \mathcal{F}_t\right], \tag{8}$$

for $t \in [0, T)$.

Observe that U can be understood as a solution to a BSDE. Indeed, U satisfies $\lim_{t \nearrow T} U_t = 0$ and evolves according to the dynamics

$$dU_t = \frac{1}{2} d\pi_t + \frac{Y_t U_t}{\eta_t} dt - \frac{\Phi_t}{2H_t} dW_t \tag{9}$$

on $[0, T)$, where the process $\Phi \in \mathcal{M}^2(0, T)$ is the integrand of the martingale representation of $E\left[\int_0^T H_s d\pi_s | \mathcal{F}_t\right]$.

We next define a control process that will turn out to be the optimal one. Let $\widehat{X}$ be the pathwise solution of the ODE

$$\dot{\widehat{X}}_t = -\frac{1}{\eta_t}\left(U_t + Y_t \widehat{X}_t\right), \tag{10}$$

with initial condition $\widehat{X}_0 = x$. Notice that $\widehat{X}_t = H_t\left(x - \int_0^t \frac{U_s}{H_s \eta_s} ds\right)$.

The next theorem summarizes the main findings of the paper. It shows that the value function and the optimal control are determined by the solution of the two coupled differential equations (6) and (9).

Theorem 2.4. *The strategy* $\widehat{X}$ *belongs to* $\mathcal{A}_0(x)$ *and is optimal in* (5). *Moreover, we have*

$$v(x) = Y_0 x^2 + (2U_0 - \pi_0)x - E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right].$$

We prove Theorem 2.4 in Section 5. In the following section we give an interpretation of Problem (5) and Theorem 2.4 in the context of optimal trade execution.

3 Application to optimal trade execution

Control problems of the type (5) arise when economic agents have to close or build up, during a fixed time span $[0, T]$, an asset position in a market with stochastic price impact. We illustrate the application in the following subsections.

[2] Minimality here means that if (Y', Z') is another pair of processes satisfying (6) and (7), then $Y_t \leq Y'_t$, a.s.

3.1 Optimal closure of an asset position

Suppose that an agent has to close, up to time T, an initial position of x financial asset shares in an illiquid market. For simplicity let $x < 0$. Assume that the agent can choose among the position processes $X \in \mathcal{A}_0(x)$, with X_t representing the agent's asset position at time $t \in [0,T]$. Notice that the position satisfies $X_0 = x$ and $X_T = 0$. One can interpret $\dot{X}_t$ as the trading rate at time $t \in [0,T)$. A negative trading rate means selling. We suppose that trading at a rate $\dot{X}_t$ creates a linear temporary price impact of $\eta_t \dot{X}_t$ (cf. also with the seminal papers on optimal trade execution [5] and [1]). Let the process π denote the asset's uninfluenced mid-market price. Then the agent's expected costs from following a position strategy $X \in \mathcal{A}_0(x)$ sum up to

$$E\left[\int_0^T \left(\pi_t + \eta_t \dot{X}_t\right) \dot{X}_t dt\right].$$

The term $E\left[\int_0^T \gamma_t X_t^2 dt\right]$ in (5) can be interpreted as a measure of the risk due to fundamental market movements. Indeed, suppose for a moment that the asset price π is a Brownian motion with constant volatility $\sigma > 0$ and that the position process X is deterministic. Then the variance of $\int_0^T \pi_t \dot{X}_t dt = -X_0\pi_0 - \int_0^T X_t d\pi_t$ is given by $\int_0^T \sigma^2 X_t^2 dt$. One can interpret the functional $E\left[\int_0^T \gamma_t X_t^2 dt\right]$ also as an approximation of the value-at-risk with stochastic risk aversion; for details we refer to [3] and [4].

We remark that linear quadratic cost functionals in optimal targeting problems are used also by [8], [9].

Theorem 2.4 provides the position process minimizing the sum of expected execution costs and risk. Besides, the optimal trading rate $\dot{\widehat{X}}$ is given by Equation (10). Notice that the optimal trading rate decomposes into two parts. The first part, $-\frac{U}{\eta}$, exploits expected price returns over the trading period. Notice that it does *not* depend on the current position $\widehat{X}$. Consequently, it does not necessarily vanish if the current position is equal to zero. The second part of the optimal trading rate, $-\frac{Y}{\eta}\widehat{X}$, can be ascribed to the minimization of liquidity costs. It depends linearly on the current position $\widehat{X}$ and, thus, does not contribute to the trading rate if there is no open position in the asset. The singular terminal condition (7) ensures that the second part predominates in the trading rate as the liquidation horizon approaches.

The interpretation of the first part, $-\frac{U}{\eta}$, as an *investment component* of the trading rate is confirmed by considering the case where the price process is a martingale, and hence has zero returns. Indeed, if π is a martingale, then by the very definition we have $U = 0$, and hence the first part of the optimal trading rate vanishes. The optimal position is given by $\widehat{X} = xH$. Moreover, Theorem 2.4 implies that $v(x) = Y_0 x^2 - \pi_0 x$. Hence, the minimal costs $v(x)$ for closing the initial position x decomposes into the difference between the mark-to-market value of the initial position, $\pi_0 x$, and the term $Y_0 x^2$. We remark that, if $\gamma = 0$ and both processes π and η are martingales, then it is optimal to close the position at a constant rate, i.e. $\dot{\widehat{X}}_t = x\dot{H}_t = -\frac{x}{T}$ (see [2, Section 4]).

We now consider again the general case, where the price process possibly possesses a drift. By Theorem (2.4) the term $-v(0) = E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right]$ represents the maximal expected revenues if there is no initial position in the asset. By exploiting the knowledge about the drift of the price process π, the trader can generate nonnegative expected revenues equal to $E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right]$ by investing in the asset. In the case where π is a martingale we have $v(0) = 0$.

The marginal increase in expected revenues generated by an infinitesimal small initial position is given by $-v'(0)$. By Theorem 2.4 we have $-v'(0) = \pi_0 - 2U_0$. Performing an integration by parts yields $-v'(0) = -E\left[\int_0^T \pi_s dH_s\right]$. Thus, $-v'(0)$ equals the expected revenues from trading at the uninfluenced price π according to the strategy H. As mentioned above, the strategy H is optimal in the case without directional views. If η is a martingale and $\gamma = 0$ we have $dH_s = -1/T ds$ and hence $-v'(0) = E\left[\frac{1}{T}\int_0^T \pi_s ds\right]$, i.e. the marginal increase in expected revenue equals the expected average price over the liquidation period.

In the following example, we present an explicit solution in a Markovian framework.

Example 3.1. *(The price process evolves according to a geometric Brownian motion) Assume that $\gamma = 0$ and that η is a martingale satisfying the Assumption 2.1. The asset's uninfluenced price process is a geometric Brownian motion with drift $\mu \neq 0$ and volatility $\sigma > 0$, i.e. π satisfies the dynamics*

$$d\pi_t = \mu \pi_t dt + \sigma \pi_t dW_t,$$

with some positive initial value π_0. It follows from [2, Corollary 4.6] that the minimal solution to (6) and (7) is given by

$$Y_t = \frac{\eta_t}{T-t}.$$

In particular, we have that $H_t = 1 - \frac{t}{T}$. It follows from the Definition (8) of U that

$$\begin{aligned} U_t &= -\frac{1}{2}\mu \int_t^T \frac{T-s}{T-t} E[\pi_s|\mathcal{F}_t] ds = -\frac{1}{2}\mu\pi_t \int_t^T \frac{T-s}{T-t} e^{\mu(s-t)} ds \\ &= \frac{1}{2}\pi_t \left(1 - \frac{1}{\mu(T-t)}\left(e^{\mu(T-t)} - 1\right)\right). \end{aligned}$$

Therefore, we can express the optimal trading rate $\dot{\hat{X}}$ in feedback form depending on the current price impact η, the current price π and the current position size x as follows

$$\dot{\hat{X}}_t(\eta, \pi, x) = -\left(\frac{\pi}{2\eta}\left(1 - \frac{1}{\mu(T-t)}\left(e^{\mu(T-t)} - 1\right)\right) + \frac{x}{T-t}\right). \tag{11}$$

Let us take a closer look at Equation (11) in the case where $\mu > 0$. If we take the limit when T goes to infinity and all other parameters remain fixed, we have $\dot{\hat{X}} \geq 0$. If the liquidation period is long enough, it is optimal to buy the asset - no matter whether the *current position is short or long - to benefit from a potentially higher price when selling the asset at a later point in time. If we consider the limit $x \to \infty$ ($x \to -\infty$) in* (11), *we see that* $\dot{\widehat{X}} \leq 0$ ($\dot{\widehat{X}} \geq 0$). *If the current long (short) position in the asset is sufficiently large it is optimal to sell (buy) the asset, in order to meet the liquidation constraint* $\widehat{X}_T = 0$. *Finally consider the case where the current price impact gets arbitrarily large* ($\eta \to \infty$). *In this case we obtain* $\dot{\widehat{X}}_t \to -\frac{x}{T-t}$. *This means that if trading becomes very expensive, the investment part of the optimal trading rate vanishes and it is optimal to close the position in a cost-minimizing way.*

3.2 Targeting an uncertain position size

Consider an agent who has to build up an asset position up to time T, but only gradually learns the precise target position to be reached. Think, e.g. of an airline company buying on forward markets the kerosine it needs in two years. The *precise* amount of kerosine that the company needs in two years depends on ticket sales, airplane utilization rates, and other factors not known already today.

Formally, suppose that the target position is of the form $\int_0^T \lambda_t dt$, where λ is a progressively measurable process and $E\left(\int_0^T |\lambda_t| dt\right)^4 < \infty$. As in the previous subsection, we denote by X_t the agent's asset position size at time $t \leq T$.

We denote by $\mathcal{A}_\lambda(x)$ the set all processes in $\mathcal{A}(x)$ that meet the terminal state constraint $X_T = \int_0^T \lambda_t dt$, a.s. We now turn to the minimization problem

$$w(x) = \min_{X \in \mathcal{A}_\lambda(x)} J(X) = \min_{X \in \mathcal{A}_\lambda(x)} E\left[\int_0^T \left(\gamma_t X_t^2 + \pi_t \dot{X}_t + \eta_t \dot{X}_t^2\right) dt\right]. \tag{12}$$

We first show that we can reduce problem (12) to problem (5).

Lemma 3.2. *A control* $X^* \in \mathcal{A}_\lambda(x)$ *is optimal in* (12) *if and only if the control* $\tilde{X}^* = X^* - \int_0^\cdot \lambda_s ds \in \mathcal{A}_0(x)$ *is optimal in*

$$\tilde{v}(x) = \inf_{X \in \mathcal{A}_0(x)} \tilde{J}(X) = \min_{X \in \mathcal{A}_0(x)} E\left[\int_0^T \left(\gamma_t X_t^2 + \tilde{\pi}_t \dot{X}_t + \eta_t \dot{X}_t^2\right) dt\right]$$

with

$$\tilde{\pi}_t = \pi_t + 2\eta_t \lambda_t - \int_0^t \left(2\gamma_s \int_0^s \lambda_r dr\right) ds.$$

Proof. See Appendix. □

By the previous lemma we can again use Theorem 2.4 for deriving the optimal control for problem (12). As before, let (Y, Z) be the minimal solution of (6) and (7), and $H_t = \exp\left(-\int_0^t \frac{Y_s}{\eta_s} ds\right)$.

Corollary 3.3. *Assume that $\tilde{\pi}$ satisfies Property c) of Assumption 2.1. Let*

$$\tilde{U}_t = -\frac{1}{2} E\left[\int_t^T \frac{H_s}{H_t} d\tilde{\pi}_s \middle| \mathcal{F}_t\right],$$

and $\tilde{X}$ be the solution of the ODE (10) *with U replaced by $\tilde{U}$. Then $X_t^* = \tilde{X}_t + \int_0^t \lambda_t dt$ is the optimal control in* (12)*.*

Example 3.4. *Assume that the price impact is constant $\eta_t = \eta > 0$ for all $t \in [0, T]$. Moreover, suppose $\gamma = 0$ and that π is martingale. Assume that $\tilde{\pi} = \pi + 2\eta\lambda$ satisfies Assumption 2.1 c).*

By Corollary 3.3 the optimal trading rate in (12) *satisfies*

$$\dot{X}_t^* = \lambda_t - \frac{1}{\eta}\left(\tilde{U}_t + Y_t \tilde{X}_t\right).$$

It follows e.g. from [2, Corollary 4.6] that the minimal solution Y to the BSDE (6) *with singular terminal condition* (7) *is given by $Y_t = \frac{\eta}{T-t}$. In particular, we have $H_t = 1 - \frac{t}{T}$. Using integration by parts we compute*

$$\begin{aligned} \tilde{U}_t &= -\frac{1}{2H_t} E\left[\int_t^T H_s d\tilde{\pi}_s \middle| \mathcal{F}_t\right] = -\frac{\eta}{H_t} E\left[\int_t^T H_s d\lambda_s \middle| \mathcal{F}_t\right] \\ &= \eta\lambda_t - \frac{\eta}{T-t} E\left[\int_t^T \lambda_s ds \middle| \mathcal{F}_t\right]. \end{aligned}$$

This yields

$$\dot{X}_t^* = -\frac{1}{T-t}\left(\tilde{X}_t - E\left[\int_t^T \lambda_s ds \middle| \mathcal{F}_t\right]\right).$$

Since $\tilde{X}_t = X_t^* - \int_0^t \lambda_t dt$, we obtain

$$\dot{X}_t^* = -\frac{1}{T-t}\left(X_t^* - E\left[\int_0^T \lambda_s ds \middle| \mathcal{F}_t\right]\right). \tag{13}$$

Equation (13) *shows that the following extension of the "linear" trading strategy is also optimal in the presence of volume uncertainty: Based on the information available at time t the optimal trading rate is simply given by the ratio between the expected remaining open position and the remaining liquidation time.*

We remark that [10] and [11] also deal with stochastic position targets: the authors consider the problem of how to optimally follow a trading target in an illiquid market with a non-temporary price impact depending on order sizes. In contrast to our model, optimal controls are *singular* and are verified with BSDEs that have non-singular terminal conditions.

The next two sections are devoted to prove Theorem 2.4. In the following section we introduce a penalized version of our control problem and solve it by means of non-singular BSDEs. In Section 5 we finally show Theorem 2.4.

4 A penalized version of the problem

In this section we study the penalized problem

$$v^L(x) = \min_{X\in\mathcal{A}(x)} J^L(X) = \min_{X\in\mathcal{A}(x)} E\left[\int_0^T \left(\gamma_t X_t^2 + \pi_t \dot{X}_t + \eta_t \dot{X}_t^2\right) dt + LX_T^2\right] \qquad (14)$$

without terminal state constraint.

We first state a sufficient condition for optimality (maximum principle).

Proposition 4.1. *Let $X \in \mathcal{A}(x)$ be a strategy such that the process*

$$M_t^X = -\pi_t - 2\eta_t \dot{X}_t + \int_0^t 2\gamma_s X_s ds \qquad (15)$$

is a local martingale satisfying the following two conditions:

1. *For all $Y \in \mathcal{A}(x)$ the integral process $\int_0^\cdot Y_t dM_t^X$ is a strict martingale*
2. *$M_T^X = 2LX_T + 2\int_0^T \gamma_s X_s ds$.*

Then X is optimal in (14).

Proof. Let $K \in A(x)$. Set $\theta = X - K$. Using the convexity of $x \mapsto x^2$ we obtain

$$\begin{aligned}
&\int_0^T \pi_t(\dot{X}_t - \dot{K}_t) + \eta_t(\dot{X}_t^2 - \dot{K}_t^2)dt \le \int_0^T \left(\pi_t\dot{\theta}_t + 2\eta_t\dot{X}_t\dot{\theta}_t\right) dt \\
=& \int_0^T (\pi_t + 2\eta_t\dot{X}_t)d\theta_t = \int_0^T (-M_t^X + \int_0^t 2\gamma_s X_s ds)d\theta_t \\
=& \theta_T(-M_T^X + \int_0^T 2\gamma_s X_s ds) + \int_0^T \theta_t dM_t^X - 2\int_0^T \theta_t\gamma_t X_t dt \\
=& -2L\theta_T X_T + \int_0^T \theta_t dM_t^X - 2\int_0^T \theta_t \gamma_t X_t dt. \qquad (16)
\end{aligned}$$

By the very assumptions the process $\int_0^\cdot \theta_t dM_t^X$ is a martingale. We can take, therefore, expectations in (16), and we obtain

$$\begin{aligned}
E\left[\int_0^T \left(\pi_t(\dot{X}_t - \dot{K}_t) + \eta_t(\dot{X}_t^2 - \dot{K}_t^2)\right) dt\right] &= E\left[-2L\theta_T X_T - 2\int_0^T \theta_t\gamma_t X_t dt\right] \\
&\le E\left[-L(X_T^2 - K_T^2) - \int_0^T \gamma_t(X_t^2 - K_t^2)dt\right],
\end{aligned}$$

which yields the claim. □

Consider the coupled system of BSDEs

$$Y_t^L = L - \int_t^T Z_s dW_s - \int_t^T \left(\frac{(Y_s^L)^2}{\eta_s} - \gamma_s\right) ds \qquad (17)$$

$$U_t^L = \frac{\pi_T}{2} - \int_t^T \Phi_s dW_s - \frac{1}{2}\int_t^T d\pi_s - \int_t^T \frac{Y_s^L U_s^L}{\eta_s} ds \qquad (18)$$

Remark 4.2. The process U^L evolves according to similar dynamics as the process U on $[0,T)$. Equation (18) arises from Equation (9) by replacing Y by Y^L. The terminal value U^L_T, however, does *not* coincide with the terminal condition $\lim_{t\nearrow T} U_t = 0$.

We define a candidate optimal strategy X^L as the pathwise solution of the ODE

$$\dot{X}^L_t = -\frac{1}{\eta_t}\left(U^L_t + Y^L_t X^L_t\right), \tag{19}$$

with initial condition $X^L_0 = x$.

Existence and uniqueness of a solution to the BSDE (17) is verified in [2]. Moreover, we have that $Y^L_t \nearrow Y_t$ a.s. for every $t < T$. Observe that the pathwise ODE (19) and the linear BSDE (18) are explicitly solvable. We have

$$X^L_t = H^L_t\left(x - \int_0^t \frac{U^L_s}{\eta_s H^L_s} ds\right)$$

where $H^L_t = \exp\left(-\int_0^t \frac{Y^L_s}{\eta_s} ds\right)$ is the solution of the homogenous equation $\dot{H}_t = -\frac{Y_t}{\eta_t} H_t$. Moreover,

$$U^L_t = \frac{1}{2} E\left[\frac{H^L_T}{H^L_t}\pi_T - \int_t^T \frac{H^L_s}{H^L_t} d\pi_s | \mathcal{F}_t\right],$$

which is well-defined by the assumptions imposed on π and since H^L is positive and pathwise nonincreasing. Observe that the candidate optimal solution X^L is explicitly expressed in terms of Y^L.

The following Lemma provides some auxiliary moment estimates and convergence results. The proof is given in the Appendix.

Lemma 4.3. *The following expectations are finite:* $E[\sup_{t\in[0,T]} \pi_t^8]$, $E[\sup_{t\in[0,T]} (U^L_t)^2]$, $E[\sup_{t\in[0,T]} U_t^2]$, $\sup_{t\in[0,T]} E\left[(U^L_t)^8\right]$ *and* $\sup_{t\in[0,T]} E\left[U_t^8\right]$.

Moreover, we have $\lim_{L\to\infty} U^L_0 = U_0$ *and* $\lim_{L\to\infty} E\int_0^T \frac{(U^L_s)^2}{\eta_s} ds = E\int_0^T \frac{U_s^2}{\eta_s} ds$.

Next, we show admissibility of the candidate optimal strategy.

Lemma 4.4. *We have* $E[\sup_t |X^L_t|^4] < \infty$. *In particular,* $X^L \in \mathcal{A}(x)$.

Proof. For ease of notation we drop the superscript L and simply write $X = X^L$ in the proof. Since H_t is nonincreasing we have $|X_t| \le |x| + \int_0^t \left|\frac{U_s}{\eta_s}\right| ds$ and hence

$$E\left[\sup_{0\le t\le T} |X_t|^4\right] \;\le\; 2^4\left(|x|^4 + E\left[\left(\int_0^T \left|\frac{U_s}{\eta_s}\right| ds\right)^4\right]\right).$$

Moreover

$$E\left[\left(\int_0^T \left|\frac{U_s}{\eta_s}\right| ds\right)^4\right] \le \left(E\left[\int_0^T \frac{1}{\eta_s^2} ds\right]^4\right)^{\frac{1}{2}} \left(E\left[\int_0^T U_s^2 ds\right]^4\right)^{\frac{1}{2}}$$

Jensen's inequality and Lemma 4.3 imply

$$E\left(\int_0^T U_s^2 ds\right)^4 \le T^3 E\left[\int_0^T U_s^8 ds\right] < \infty.$$

All in all, the estimates show that $E\left[\sup_{0\le t\le T}|X_t|^4\right] < \infty$. □

Lemma 4.5. *The process* M^{X^L}*, defined as in* (15)*, satisfies the Conditions 1. and 2. from Lemma 4.1. In particular* X^L *is optimal in* (14)*.*

Proof. To simplify notation we omit, throughout the proof, the superscript L and write $X = X^L$, $U = U^L$, etc. Itô's formula implies

$$\begin{aligned} dM_t^X &= -d\pi_t + 2dU_t + 2X_t dY_t + 2Y_t dX_t + 2\gamma_t X_t dt \\ &= 2\frac{Y_t U_t}{\eta_t}dt + 2\Phi_t dW_t + 2X_t\left(\frac{Y_t^2}{\eta_t} - \gamma_t\right)dt + 2X_t Z_t dW_t - 2Y_t\frac{U_t + Y_t X_t}{\eta_t}dt + 2\gamma_t X_t dt \\ &= 2\Phi_t dW_t + 2X_t Z_t dW_t, \end{aligned}$$

and hence M^X is a local martingale. We need to verify that $\int_0^\cdot K_t dM_t^X$ is a martingale for every $K \in \mathcal{A}(x)$. To this end let $N_t = \frac{1}{2}\int_0^t K_s dM_s^X = \int_0^t K_s \Phi_s dW_s + \int_0^t K_s X_s Z_s dW_s$, $t \in [0,T]$. We show that $E[\langle N, N\rangle_T^{\frac{1}{2}}] < \infty$, which implies that N is a strict martingale on $[0,T]$.

Notice that $\sqrt{x+y} \le 1 + \sqrt{x} + \sqrt{y}$ for all $x, y \ge 0$. Therefore

$$\begin{aligned} E[\langle N, N\rangle_T^{\frac{1}{2}}] &= \sqrt{2}\left[E\left(\int_0^T K_t^2 \Phi_t^2 dt + \int_0^T K_t^2 X_t^2 Z_t^2 dt\right)^{\frac{1}{2}}\right] \\ &\le \sqrt{2}\left(1 + E\left[\left(\int_0^T K_t^2\Phi_t^2 dt\right)^{\frac{1}{2}}\right] + E\left[\left(\int_0^T K_t^2 X_t^2 Z_t^2 dt\right)^{\frac{1}{2}}\right]\right). \end{aligned}$$

Notice that

$$\begin{aligned} E\left[\left(\int_0^T K_t^2\Phi_t^2 dt\right)^{\frac{1}{2}}\right] &\le E\left[\left(\sup_{0\le t\le T}|K_t|\left(\int_0^T \Phi_t^2 dt\right)^{\frac{1}{2}}\right)\right] \\ &\le \left(E\left[\sup_{0\le t\le T}|K_t|^2\right]\right)^{\frac{1}{2}}\left(E\left[\int_0^T \Phi_t^2 dt\right]\right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Moreover

$$\begin{aligned} E\left[\left(\int_0^T K_t^2 X_t^2 Z_t^2 dt\right)^{\frac{1}{2}}\right] &\le E\left[\sup_{0\le t\le T}|K_t||X_t|\left(\int_0^T Z_t^2 dt\right)^{\frac{1}{2}}\right] \\ &\le \left(E\left[\sup_{0\le t\le T}|X_t|^4\right]\right)^{\frac{1}{4}}\left(E\left[\sup_{0\le t\le T}|K_t|^4\right]\right)^{\frac{1}{4}}\left(E\left[\int_0^T Z_t^2 dt\right]\right)^{\frac{1}{2}} < \infty. \end{aligned}$$

Finally, $U_T = \pi_T/2$ and $Y_T = L$ imply that $M_T^X = 2LX_T + \int_0^T 2\gamma_s X_s ds$. □

We can also give a closed form representation of the value function v^L.

Proposition 4.6. *We have*

$$v^L(x) = Y_0^L x^2 + (2U_0^L - \pi_0)x - E\left[\int_0^T \frac{(U_s^L)^2}{\eta_s} ds\right].$$

Proof. To simplify notation we omit again the superscript L. Set

$$V_t = Y_t X_t^2 + (2U_t - \pi_t)X_t - E\left[\int_t^T \frac{U_s^2}{\eta_s} ds \middle| \mathcal{F}_t\right]$$

and

$$N_t = E\left[\int_0^T \frac{U_s^2}{\eta_s} ds \middle| \mathcal{F}_t\right].$$

Notice that one can show, with similar estimates as in the proof of Lemma 4.4, that $E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right] < \infty$. In particular N is a martingale.

The dynamics of V satisfy

$$\begin{aligned}
dV_t &= X_t^2 dY_t + 2Y_t X_t \dot{X}_t dt + \dot{X}_t(2U_t - \pi_t)dt + X_t(2dU_t - d\pi_t) + \frac{U_t^2}{\eta_t} dt - dN_t \\
&= X_t^2\left(\frac{Y_t^2}{\eta_t} - \gamma_t\right) dt + X_t^2 Z_t dW_t - 2Y_t X_t \frac{U_t + Y_t X_t}{\eta_t} dt \\
&\quad - \frac{U_t + Y_t X_t}{\eta_t}(2U_t - \pi_t)dt + X_t db_t + 2X_t \frac{Y_t U_t}{\eta_t} dt + 2X_t \Phi_t dW_t \\
&\quad - X_t d\pi_t + \frac{U_t^2}{\eta_t} dt - dN_t \\
&= -(\gamma_t X_t^2 + \pi_t \dot{X}_t + \eta_t \dot{X}_t^2)dt + X_t^2 Z_t dW_t + 2X_t \Phi_t dW_t - X_t \psi_t dW_t - dN_t. \quad (20)
\end{aligned}$$

We next show that the integral processes with respect to the Brownian motion are strict martingales. Observe that

$$\begin{aligned}
E\left[\left(\int_0^T (X_t^2 Z_t)^2 dt\right)^{\frac{1}{2}}\right] &\leq E\left[\sup_{0\leq t\leq T} |X_t|^2 \left(\int_0^T Z_t^2 dt\right)^{\frac{1}{2}}\right] \\
&\leq \left(E\left[\sup_{0\leq t\leq T} |X_t|^4\right]\right)^{\frac{1}{2}} \left(E\left[\int_0^T Z_t^2 dt\right]\right)^{\frac{1}{2}} < \infty,
\end{aligned}$$

which shows that $\int_0^\cdot X_t^2 Z_t dW_t$ is a martingale. Similary, one can show that $\int_0^\cdot X_t \Phi_t dW_t$ and $\int_0^\cdot X_t \psi_t dW_t$ are martingales.

Taking expectations in Equation (20) yields

$$E[V_T] = V_0 - E\int_0^T (\gamma_t X_t^2 + \pi_t \dot{X}_t + \eta_t \dot{X}_t^2)dt.$$

Since $V_T = LX_T^2$, we have shown the claim. □

5 Proof of Theorem 2.4

In this section we prove that the strategy $\widehat{X}$ is indeed optimal in the control problem (5). For the reader's convenience we briefly recall some definitions. The candidate optimal strategy is given by

$$\dot{\widehat{X}}_t = -\frac{1}{\eta_t}\left(U_t + Y_t\widehat{X}_t\right),$$

where Y is the minimal solution of the BSDE

$$dY_t = \left((p-1)\frac{Y_t^2}{\eta_t} - \gamma_t\right)dt + Z_t dW_t, \qquad \lim_{t\nearrow T} Y_t = \infty$$

and U is given by

$$U_t = -\frac{1}{2}E\left[\int_t^T \frac{H_s}{H_t}d\pi_s \middle| \mathcal{F}_t\right]$$

with $H_t = e^{-\int_0^t \frac{Y_s}{\eta_s}ds}$.

We first verify that $\widehat{X}$ satisfies the integrability condition $E\left[\sup_{0\le t\le T}|\widehat{X}_t|^4\right] < \infty$ and the constraint $\widehat{X}_T = 0$.

Lemma 5.1. *We have $\widehat{X} \in \mathcal{A}_0(x)$.*

Proof. The proof of $E[\sup_t |\widehat{X}_t|^4] < \infty$ goes along the lines of the proof of Lemma 4.4. Next we show that $\lim_{t\nearrow T} H_t = 0$, a.s. First observe that the process $YH + \int_0^\cdot \gamma_s H_s ds$ is nonnegative local martingale. In particular, it is a nonnegative supermartingale and thus possesses a limit as $t \nearrow T$, a.s. Since $\lim_{t\nearrow T} Y_t = \infty$, it must hold true that $\lim_{t\nearrow T} H_t = 0$. (Confer also with the proof of Theorem 3.2. in [2]).

Notice that $\lim_{t\nearrow T}\frac{H_t}{H_s}\frac{U_s}{\eta_s} = 0$ and $\left|\frac{H_tU_s}{H_s\eta_s}\right| \le \left|\frac{U_s}{\eta_s}\right|$, for Lebesgue-a.a. $s \in [0,T)$. Moreover $E\left[\int_0^T \left|\frac{U_s}{\eta_s}\right| ds\right] < \infty$. Therefore, for P-almost all ω we have $\int_0^T \left|\frac{U_s}{\eta_s}\right| ds < \infty$. Dominated convergence, therefore, implies

$$\lim_{t\nearrow T}\int_0^t \frac{H_tU_s}{H_s\eta_s}ds = 0.$$

This shows that $\widehat{X}_T = 0$ P-a.s. □

Proof of Theorem 2.4. First notice that from the very definition we have $v^L(x) \le v(x)$. Next observe that

$$\lim_{L\to\infty} v^L(x) = Y_0x^2 + (2U_0 - \pi_0)x - E\left[\int_0^T \frac{U_s^2}{\eta_s}ds\right]. \tag{21}$$

Indeed, from Proposition 4.6 we know

$$v^L(x) = Y_0^L x^2 + (2U_0^L - \pi_0)x - E\left[\int_0^T \frac{(U_s^L)^2}{\eta_s} ds\right].$$

Moreover we have by construction $\lim_{L\to\infty} Y_0^L = Y_0$. Lemma 4.3 guarantees the convergence of the last two terms on the RHS, and hence (21).

We proceed by proving that

$$J(\widehat{X}) \leq Y_0 x^2 + (2U_0 - \pi_0)x - E\left[\int_0^T \frac{U_s^2}{\eta_s} ds\right]. \tag{22}$$

To this end we define

$$V_t = Y_t \widehat{X}_t^2 + (2U_t - \pi_t)\widehat{X}_t - E\left[\int_t^T \frac{U_s^2}{\eta_s} ds \middle| \mathcal{F}_t\right].$$

As in the proof of Proposition 4.6 one can show that

$$dV_t \;=\; -(\gamma_t \widehat{X}_t^2 + \pi_t \dot{\widehat{X}}_t + \eta_t \dot{\widehat{X}}_t^2)dt + dL_t, \tag{23}$$

where L is a strict martingale on $[0, T)$. Taking expectations yields, for every $t \in [0, T)$,

$$V_0 = E\left[\int_0^t (\gamma_s \widehat{X}_s^2 + \pi_s \dot{\widehat{X}}_s + \eta_s \dot{\widehat{X}}_s^2) ds\right] + E[V_t],$$

and hence

$$V_0 \geq E\left[\int_0^t (\gamma_s \widehat{X}_s^2 + \pi_s \dot{\widehat{X}}_s + \eta_s \dot{\widehat{X}}_s^2) ds\right] + E\left[(2U_t - \pi_t)\widehat{X}_t - \int_t^T \frac{U_s^2}{\eta_s} ds\right]. \tag{24}$$

By monotone convergence we have

$$\lim_{t\nearrow T} E\left[\int_0^t (\gamma_s \widehat{X}_s^2 + \eta_s \dot{\widehat{X}}_s^2) ds - \int_t^T \frac{U_s^2}{\eta_s} ds\right] = E\left[\int_0^T (\gamma_s \widehat{X}_s^2 + \eta_s \dot{\widehat{X}}_s^2) ds\right].$$

Moreover, Lemma 4.3 ensures that the random variable $\sup_{t\in[0,T]} |(2U_t - \pi_t)\widehat{X}_t|$ is integrable. Therefore dominated convergence implies

$$\lim_{t\nearrow T} E\left[(2U_t - \pi_t)\widehat{X}_t\right] = 0.$$

Notice that Lemma 4.3 implies that $E \sup_{t\in[0,T]} \left|\int_0^t \widehat{X}_s d\pi_s\right| < \infty$. Appealing to dominated convergence once more yields

$$E\left[\int_0^t \pi_s \dot{\widehat{X}}_s ds\right] = E\left[\widehat{X}_t \pi_t - \widehat{X}_0 \pi_0 + \int_0^t \widehat{X}_s d\pi_s\right] \to E\left[-\widehat{X}_0 \pi_0 + \int_0^T \widehat{X}_s d\pi_s\right] = E\left[\int_0^T \pi_s \dot{\widehat{X}}_s ds\right]$$

as $t \nearrow T$. All in all we obtain from Equation (24)

$$V_0 \geq E \int_0^T (\gamma_s \widehat{X}_s^2 + \pi_s \dot{\widehat{X}}_s + \eta_s \dot{\widehat{X}}_s^{\,2}) ds = J(\widehat{X}).$$

By putting everything together we get

$$v(x) \leq J(\widehat{X}) \leq V_0 = \lim_{L \to \infty} v^L(x) \leq v(x).$$

Every inequality is indeed an equality, and thus we have shown the theorem. □

6 Appendix

Proof of Lemma 3.2. Let $X \in \mathcal{A}_\lambda(x)$ and set $\tilde{X} = X - \int_0^\cdot \lambda_s ds \in \mathcal{A}_0(x)$. The result follows from

$$\tilde{J}(X) = J(\tilde{X}) + E\left[\int_0^T \left(\gamma_t \left(\int_0^t \lambda_s ds\right)^2 + \lambda_t \pi_t + \eta_t \lambda_t^2\right) dt\right]. \tag{25}$$

To prove Equation (25), first note that

$$\begin{aligned} \tilde{J}(X) &= \tilde{J}(\tilde{X} + \int_0^\cdot \lambda_s ds) \\ &= E\left[\int_0^T \left(\tilde{\alpha}_t \tilde{X}_t + \gamma_t \tilde{X}_t^2 + (\pi_t + 2\eta_t \lambda_t)\dot{\tilde{X}}_t + \eta_t \dot{\tilde{X}}_t^2\right) dt\right] \\ &\quad + E\left[\int_0^T \left(\gamma_t \left(\int_0^t \lambda_s ds\right)^2 + \lambda_t \pi_t + \eta_t \lambda_t^2\right) dt\right] \end{aligned}$$

with

$$\tilde{\alpha}_t = 2\gamma_t \int_0^t \lambda_s ds.$$

By integration by parts we obtain

$$\int_0^T \tilde{\alpha}_t \tilde{X}_t dt = -\int_0^T \left(\int_0^t \left(2\gamma_s \int_0^s \lambda_r dr\right) ds \ \dot{\tilde{X}}_t\right) dt,$$

which implies Equation (25). □

Proof of Lemma 4.3. First observe that the Burkholder-Davis-Gundy inequality and Assumption 2.1 imply $E[\sup_{t \in [0,T]} \pi_t^8] < \infty$.

Moreover we have

$$|U_t^L| \leq E\left[\left|\frac{H_T^L}{H_t^L}\pi_T\right| + \left|\int_t^T \frac{H_s^L}{H_t^L} db_s\right| \middle| \mathcal{F}_t\right] \leq E\left[|\pi_T| + \int_0^T |db_s| \middle| \mathcal{F}_t\right] \tag{26}$$

Then a further application of the Burkholder-Davis-Gundy inequality yields $E[\sup_{t\in[0,T]}(U_t^L)^2] < \infty$. Moreover, Equation (26) and Assumption 2.1 imply $E\left[(U_t^L)^8\right] < C$ for some constant C independent of $t \in [0,T]$. The claims $E[\sup_{t\in[0,T]} U_t^2] < \infty$ and $\sup_{t\in[0,T]} E\left[(U_t)^8\right] < \infty$ follow by similar arguments.

Next, fix $t < T$. We have $\left|\frac{H_T^L}{H_t^L}\pi_T + \int_t^T \frac{H_s^L}{H_t^L} db_s\right| \leq |\pi_T| + \int_0^T |db_s| \in L^1(P)$. Moreover, by monotone convergence $\lim_{L\to\infty} H_t^L = H_t$. Then dominated convergence for conditional expectations implies $\lim_{L\to\infty} U_t^L = U_t$.

For the last claim observe that

$$E\left[\int_0^T \frac{(U_s^L - U_s)^2}{\eta_s} ds\right] \leq \left(E\left[\int_0^T \frac{1}{\eta_s^2} ds\right]\right)^{\frac{1}{2}} \left(E\left[\int_0^T (U_s^L - U_s)^4 ds\right]^{\frac{1}{2}}\right).$$

Moreover, there exist constants C_1 and C_2, depending only on T, such that

$$|U_t^L - U_t|^4 \leq C_1 E\left[\left|\frac{H_T^L}{H_t^L}\right|^4 |\pi_T|^4 + \left(\int_t^T \left|\frac{H_s^L}{H_t^L} - \frac{H_s}{H_t}\right| |db_s|\right)^4 \middle| \mathcal{F}_t\right],$$

and

$$E\int_0^T |U_t^L - U_t|^4 dt \leq C_2 E\left[|\pi_T|^4 \int_0^T \left|\frac{H_T^L}{H_t^L}\right|^4 dt + \int_0^T \left(\int_t^T \left|\frac{H_s^L}{H_t^L} - \frac{H_s}{H_t}\right| |db_s|\right)^4 dt\right].$$

Notice that $\left|\frac{H_s^L}{H_t^L} - \frac{H_s}{H_t}\right| \leq 2$ and $\lim_{L\to\infty}\left|\frac{H_s^L}{H_t^L} - \frac{H_s}{H_t}\right| = 0$, for all $t \leq s < T$. Thus dominated convergence implies $\lim_{L\to\infty} E\int_0^T \frac{(U_s^L - U_s)^2}{\eta_s} ds = 0$, and hence the second result. □

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