# Mathematical Finance

Lectures by Professor *Stefan* Ankirchner

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Notes and $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX typesetting
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## Preliminary Remarks

These notes are a child of my efforts to gain a deeper understanding of professor Ankirchner's lectures on Mathematical Finance. A surely not the worst method of doing so is to go through one's own written notes, try to understand every argument and line of thought, and prepare a readable account of all these elaborations – in brief, to make a set of lecture notes and run the risk of presenting them to others. My wish is, of course, that they may be helpful to these others, so that in this way the trouble of composing them may, at least, yield some additional benefits.

Since these notes have not yet been revised by professor Ankirchner, every word in them are within my full responsibility. The more I thank him for his trust that they be reasonably meaningful and close to his lectures, and his readiness to make them available on the web.

I have tried to capture as far as possible the spirit and contents of the lectures and to accurately reflect what has been written on the blackboard, the way it was written on the blackboard, and (a much more difficult task) what has been spoken. This is what is set here in the notes in normal type. At various places I had the impression that some more elaborations and reminders could be helpful; these ramblings have been set in small type. So what is set in normal type here should be a best approximation of what has been presented in the lectures.

This version is surely not in final form, so for some time to come these notes will be updated every now and then. Therefore, everyone interested should have have a look at the web site from time to time:

`http://www.uni-bonn.de/~tkruse/teaching/Math_Finance_1213/MathFin.pdf`

Questions, suggestions. or critical annotations are welcome. Please mail

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CHAPTER 1

# The Value at Risk

Usually, financial institutions measure the risk of loss in their portfolios with the so–called *value–at–risk*:

**Definition.** *Let* $\alpha \in (0,1)$ *. The* value–at–risk at level $\alpha$ *of a portfolio, denoted* $\mathrm{V@R}_\alpha$ *, is the smallest* $c \in \mathbb{R}$ *such that the probability of the loss over a given time period* (typically 10 trading days) *to exceed* $c$ *is not larger than* $1-\alpha$ *.*

For this, we need to specify a probability space; the loss will then be a random variable w.r.t. this probability space. More precisely, then, let the loss be a random variable $L$ on a probability space $(\Omega, \mathcal{F}, P)$ . Then

$$\mathrm{V@R}_\alpha(L) := \inf \{\, c \in \mathbb{R} \mid \; P[L > c] \leqslant 1-\alpha \,\} \, .$$

Let $F_L(x) := P[L \leqslant x]$ be the distribution function of $L$ . Note that

$$\mathrm{V@R}_\alpha(L) = \inf \{\, c \in \mathbb{R} \mid \; F_L(c) \geq \alpha \,\}$$

and so is nothing else but the $\alpha$*–quantile* of $L$ , i.e. there is a percentage of $100\alpha\%$ cases in which the loss over the given time period is less than $c$ . Indeed, we have

$$P[L > c] = 1 - F_L(c) \leqslant 1-\alpha \quad \text{iff} \quad F(c) \geq \alpha \, .$$

Consider a portfolio consisting of just one financial asset with price dynamics given by the SDE

$$(*) \qquad dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\mu \in \mathbb{R}$ ("drift rate"), $\sigma > 0$ ("diffusion rate" or "volatility"), and $(W_t)_{t\geq 0}$ a Brownian Motion.

Recall that we can explicitely solve this SDE: If the initial condition is $S_0 > 0$ , the current price, then

$$S_t = S_0 \, \mathrm{e}^{\sigma W_t + \left(\mu - \sigma^2/2\right)t} \quad , \quad t \in \mathbb{R}_+$$

(often called *Geometric Brownian Motion*).

In order to refresh the needed prerequisites for these lectures, let us recapitulate how to do so. First note that the global Lipschitz conditions for our SDE are trivially fulfilled, so we know by the general theory that there exists a unique strong solution for any given initial condition. So it is only a matter of finding one.

For this, let $\alpha \in \mathbb{R}$ and define the process $X_t := \mathrm{e}^{-\alpha t} S_t$ ; then $S_t = \mathrm{e}^{\alpha t} X_t$ , and

$$dS_t = \alpha \, \mathrm{e}^{\alpha t} X_t dt + \mathrm{e}^{\alpha t} dX_t \, ,$$

so that the SDE $(*)$ becomes

$$\alpha\,\mathrm{e}^{\alpha t}\,X_t dt + \mathrm{e}^{\alpha t}\,dX_t = \mu\,\mathrm{e}^{\alpha t}\,X_t dt + \sigma\,\mathrm{e}^{\alpha t}\,X_t dW_t$$

or

$$dX_t = (\mu - \alpha)X_t dt + \sigma X_t dW_t$$

with initial condition $X_0 = S_0 > 0$. Thus putting $\alpha := \mu$, this SDE simplifies to

$$dX_t = \sigma X_t dW_t\,.$$

Naively, one would proceed as

$$\frac{dX_t}{X_t} = d\log X_t = \sigma dW_t = d(\sigma W_t)\,,$$

concluding

$$X_t = S_0\,\mathrm{e}^{\sigma W_t}\,,$$

but this conclusion is wrong. The reason is provided by the celebrated ITÔ *formula*: as soon as a process has non–vanishing quadratic variation, a further quadratic term creeps up into its differential introducing additional terms.

EXCURSION: THE WONDROUS WORLD OF THE ITÔ FORMULA

Recall the simplest instance of ITÔ's formula from Stochastic Analysis. Let $U \subseteq \mathbb{R}$ be open, $X = (X_t)_{t\geq 0}$ a continuous semi–martingale with values in $U$, and $f \in \mathcal{C}^2(U)$ a twice continuously differentiable function; then, for all $0 \leqslant s \leqslant t$

$$(**) \qquad f(X_t) - f(X_s) = \int_s^t f'(X_u)\,dX_u + \frac{1}{2}\int_s^t f''(X_u)\,d\langle X\rangle_u\,,$$

where the first integral on the RHS is ITÔ's celebrated stochastic integral, and the second one is a RIEMANN–STIELTJES–integral w.r.t. the quadratic variation process $\langle X\rangle$ of $X$ (which is pathwise of bounded variation thus making the RS–integral meaningful). It has become standard, and useful, to write such integral identities as identities between formally defined "stochastic (or ITÔ) differentials"; for a process $X$ its stochastic differential $dX$ is defined as a map on the set of all intervals $[s,t]$, $0 \leqslant s \leqslant t$, mapping the interval $[s,t]$ to the random variable $X_t - X_s$ (we are considering real valued stochastic processes with parameter space $[0,\infty)$). Thus

$$dX([s,t]) := X_t - X_s\,.$$

Before introducing the so–called ITÔ *Calculus* recall that the indefinite stochastic integral is itself a stochastic process which, in general, cannot be defined pathwise but requires a sophisticated involved closing construction which provides it at one stroke, so to speak, as a fully established stochastic process. A standard denotation for this process is $X \bullet Y$, where the process $X$ is called the *integrand* and $Y$ the *integrator* (see **[18]**, p. 179, Definition 4.31); the notation is meant to emphasize the bilinear character of the so–defined integral. The ITÔ *Calculus* emerges by introducing the following two rules: If $X, Y$ are processes, define

a) $XdY := d(X \bullet Y)$;
b) $dXdY := d\langle X, Y\rangle$.

To understand rule a), note the analogy with the calculus of differential forms and STOKES' Theorem (or the Fundamental Theorem of Calculus). For just this heuristic moment here we now write $X(t)$ in place of $X_t$ to keep the notation more symmetric. If we define a *null chain* to be a finite formal linear combination $c_0 := \sum_i c_i P_{t_i}$ with $c_i \in \mathbb{Z}$, $P_t$ an abstract point corresponding to $t \in \mathbb{R}_+$, and put $X(c_0) := \sum_i c_i X(t_i)$, then

$$dX([s,t]) := X(\partial[s,t]),$$

where we define the boundary $\partial$ of the interval $[s,t]$ as the null chain $\partial[s,t] := P_t - P_s$ (from now on we write $X_t$ again). If we therefore define an Itô *differential* to be a finite formal sum $D := \sum_i X^i dY^i$ and have an equation $D([s,t]) = Z_s - Z_t$ for a process $Z$, it is natural to call $Z$ the *integral of* $D$ and to write

$$Z =: \int D \quad , \quad Z_t - Z_s = \left(\int D\right)_t - \left(\int D\right)_s =: \int_s^t D\,.$$

This then leads to the notation (put $D := XdY$ and $Z := X \bullet Y$):

$$X \bullet Y =: \int X\,dY \quad , \quad (X \bullet Y)_t - (X \bullet Y)_s =: \int_s^t X\,dY$$

Thus our definition in a) of the Itô differential $XdY$ reads in this notation

$$XdY = d\int XdY$$

since both sides, interpreted as functions on intervals $[s,t]$, yield the same value $(X \bullet Y)_t - (X \bullet Y)_s$. In this notation, of course, this equation is not feasible for a definition of $XdY$, since the *definiens* on the RHS containes the *definiendum* on the LHS, but it nicely demonstrates that the operations $d$ and $\int$ are inverse to each other.

To summarize, there is a close formal analogy to the Exterior Calculus:

- in the Exterior Calculus, the proper objects to integrate are differential forms, not functions, and the differentiation operator $d$ – the exterior derivative – is precisely so defined as to make Stokes' Theorem true in the simple case of ntegrating over rectangular objects (and hence in general);
- in the Itô Calculus, the proper objects to integrate are Itô differentials, not stochastic processes, and the differentiation operator is precisely so defined as to make Stokes' Theorem true in the simple case of integrating over intervals (note that Itô differentials correspond to 1–forms, so we are in the 1–dimensional case, and Stokes' Theorem is just the Fundamental Theorem of Calculus).

All these elaborations are pretty formal and without real content, but turn out to establish an effective formalism, where effective formalism means that applying stolidly the rules leads to correct results without the need of thinking and along the way reduces the risk of error.

Unfortunately, more often than not one finds notation like

$$XdY([s,t]) = \int_s^t X\,dY = \int_s^t X_u\,dY_u\,,$$

where the RHS is, in principle, to be repudiated. This notation appears frequently for at least two reasons. The first is, that after the fact, i.e. after having established the stochastic integral, it is given, although not definable, as the limit of pathwise defined Riemann–Stieltjes sums, where the limit has to be taken *stochastically* (see [**18**], Satz 4.43):

$$(\spadesuit) \qquad \int_s^t X\,dY = st\text{-}\lim_{n\to\infty} \sum_{i=1}^n X_{s+(i-1)h}(Y_{s+ih} - Y_{s+(i-1)h}) \quad , \quad h := \frac{t-s}{n} \,,$$

and the notation with the subscript $u$ is a remnant of that. The second, more significant, reason is that one wants to keep track of the variables during calculations (e.g. when applying the chain rule) and to be able to write integrals like $\int X_t dt$ in place of $\int X d\,\mathrm{id}_{\mathbb{R}_+}$. All in all experience shows that keeping the variables explicit during computations supports clarity and narrows down the cluttering of notation. Therefore we too will stick to this habit.

In this way, one succeeds in introducing the notion of differential, even though most interesting stochastic processes have paths which are nowhwere differentiable almost surely, and with this formalism, ITÔ's formula $(**)$ then takes the concise form

$$(***) \qquad df(X)_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t \,.$$

There results a formalism based on ITÔ's formula which is known as ITÔ *Calulus*, which is the stochastic counterpart to classical Integral and Differential Calculus, and which lies at the foundation of Stochastic Analysis.

END OF THE EXCURSION

Now the above naive ansatz inspires to apply ITÔ's formula to $\log X_t$ :

$$d\log X_t = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}d\langle X\rangle_t \,.$$

(here one might object that we do not know, at this stage, if $X_t$ will always be positive. Regarding $X_t^2$ instead of $X_t$ reduces this objection to the question if $X_t$ does not vanish (since due to $\log X^2 = 2\log X$ the computations below are not affected). Although this is true, we do not know this at this stage, but it suffices that our ruminations will produce a candidate for the solution, which we will prove to be one by other means.)

We have $dX_t = \sigma X_t dW_t$ , hence $d\langle X\rangle_t = d\langle X, X\rangle_t = dX_t dX_t = \sigma^2 X_t^2 dW_t dW_t = \sigma^2 X_t^2 d\langle W, W\rangle_t$ $= \sigma^2 X_t^2 d\langle W\rangle_t = \sigma^2 X_t^2 dt$, and plugging all this into the last equation gives

$$d\log X_t = \sigma dW_t - \frac{1}{2}\sigma^2 dt = d\left(\sigma W_t - \frac{1}{2}\sigma^2 t\right) \,,$$

and so

$$X_t = S_0\, \mathrm{e}^{\sigma W_t - \sigma^2 t/2} \,,$$

or

$$S_t = S_0\, \mathrm{e}^{\sigma W_t + (\mu - \sigma^2/2)t} \,.$$

In this way, we have found how the solution should look like. To verify that it is indeed a solution, write

$$S_t = f(Y_t) \quad , \quad Y_t := \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \,,$$

where $f(y) := S_0\,\mathrm{e}^y$ . Applying ITÔ's formula to $f$ , there comes

$$dS_t = S_0\,\mathrm{e}^{Y_t}\,dY_t + \frac{1}{2}S_0\,\mathrm{e}^{Y_t}\,d\langle Y\rangle_t = S_t dY_t + \frac{1}{2}S_t d\langle Y\rangle_t \,.$$

Now $dY_t = (\mu - \sigma^2/2)dt + \sigma dW_t$ and $\langle Y\rangle_t = \sigma^2 t$ ; there comes

$$dS_t = \left(\mu - \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dW_t + \frac{1}{2}\sigma^2 S_t dt = \mu S_t dt + \sigma S_t dW_t$$

as desired.

Throughout the lecture, let

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-z^2/2}\, dz\,.$$

**Lemma 1.1.** *Let $L = -N(S_T - S_0)$ be the loss on a portfolio consisting of $N \geq 0$ asset shares, where $T > 0$* (typically $T = 10/252$, 10 trading days over 252 trading days a year)*. Then*

$$\text{V@R}_\alpha = -NS_0 \left( \mathrm{e}^{-a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T} - 1 \right)\,, \tag{1.1}$$

*where $a$ is such that $\Phi(a) = \alpha$, i.e. the $\alpha$–quantile of the standard normal distribution.*

PROOF. Let $q$ be the RHS of (1.1). Then

$$\begin{aligned}
&P\,[L\,>\,q] \\
&\quad = P\left[-N\left(S_0\,\mathrm{e}^{\sigma W_T + \left(\mu - \sigma^2/2\right)T} - S_0\right) \,>\, -N\left(S_0\,\mathrm{e}^{-a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T} - S_0\right)\right] \\
&\quad = P\left[\mathrm{e}^{\sigma W_T + \left(\mu - \sigma^2/2\right)T} \,<\, \mathrm{e}^{-a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T}\right] \\
&\quad = P\left[\sigma W_T \,<\, -a\sigma\sqrt{T}\right] \\
&\quad = P\left[W_T/\sqrt{T} \,<\, -a\right]
\end{aligned}$$

and so, since $W_T/\sqrt{T}$ is normally distributed around 0 with variance 1,

$$= \Phi(-a) = 1 - \Phi(a) = 1 - \alpha\,,$$

hence the result. QED

Now let us have a look at short positions.

**Lemma 1.2.** *Let $L = N(S_T - S_0)$ be the loss of a short position of $N \geq 0$ shares. Then*

$$\text{V@R}_\alpha = NS_0 \left( \mathrm{e}^{a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T} - 1 \right)\,.$$

The proof is analogous to the proof of Lemma 1.1.

**Remark**. Note that the V@R of a short position is not equal to the V@R of the corresponding long position. Reason: Geometric Brownian Motion is log–normal distributed, and this distribution is not symmetric around the median ("Skewness" of the log–normal distribution). □

Commonly, one uses an even simpler *linear approximation.* Typically

$$T = \frac{10}{252} \approx 0.004$$

and so is very small, so the Taylor development gives a good approximation:

$$\begin{aligned}
&\mathrm{e}^{-a\sigma\sqrt{T}+\left(\mu-\sigma^2/2\right)T} \\
&= 1 - a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T + \frac{1}{2}\left(-a\sigma\sqrt{T} + \left(\mu - \sigma^2/2\right)T\right)^2 + \cdots \\
&= 1 - a\sigma\sqrt{T} + \left(\mu - \sigma^2/2 + a^2\sigma^2\right)T + O(T^{3/2})
\end{aligned}$$

If we use only the $\sqrt{T}$–term, we obtain the following linear approximations.

For a long position:

$$\ell\mathrm{V@R}_\alpha(-N(S_T - S_0)) = NS_0 a\sigma\sqrt{T}\,.$$

For a short position:

$$\ell\mathrm{V@R}_\alpha(N(S_T - S_0)) = NS_0 a\sigma\sqrt{T}\,.$$

**Remark**. 1) Contrary to V@R, $\ell$V@R is the same for short and long positions.

2) Assume $L$ is Gaussian, i.e. $L = S_0\sigma W_T$ . Then

$$\mathrm{V@R}_\alpha(L) = \mathrm{V@R}_\alpha(S_0\sigma\sqrt{T}X) = S_0\sigma\sqrt{T}a = \ell\mathrm{V@R}_\alpha(\pm(S_T - S_0))$$

where $X$ is $\mathcal{N}(0,1)$ .

3) Let $\alpha, \beta \in (0,1)$ , $\Phi(a) = \alpha$ , $\Phi(b) = \beta$ , and $T, U > 0$ . Then

$$\ell\mathrm{V@R}_\alpha(\pm N(S_U - S_0)) = \frac{b}{a}\frac{\sqrt{U}}{\sqrt{T}}\,\ell\mathrm{V@R}_\alpha(\pm N(S_T - S_0))\,.$$

### V@R for financial derivatives

Portfolios can contain financial contracts that are not liquidly traded so that you do not have market prices for them. So you must determine their value by other means, i.e. from other asset prices (Example: Options).

Suppose that the value of a financial contract at time $t$ is given by $F(S_t)$ , where $S_t$ is the price of the underlying (= reference asset).

**Lemma 1.3.** *Let $F$ be strictly increasing and $L = -(F(S_T) - F(S_0))$ . Then*

$$\mathrm{V@R}_\alpha(L) = F(S_0) - F(S_0 - \mathrm{V@R}_\alpha(-(S_T - S_0)))\,. \tag{1.2}$$

Proof. Let $q$ be the RHS of (1.2). Then

$$\begin{aligned}
P[L > q] &= P[-(F(S_T) - F(S_0)) > q] \\
&= P[F(S_T) < F(S_0 - \mathrm{V@R}_\alpha(-(S_T - S_0)))] \\
&= P[S_T < S_0 - \mathrm{V@R}_\alpha(-(S_T - S_0))]
\end{aligned}$$

$$\begin{aligned}&= P\left[-(S_T - S_0)\ >\ \mathrm{V@R}_\alpha(-(S_T - S_0))\right]\\&= 1 - \alpha\end{aligned}$$

which implies $q = \mathrm{V@R}_\alpha(L)$. QED

Let us assume $F \in \mathcal{C}^\infty$ be strictly increasing. The TAYLOR approximation of $F$ around $S_0$ implies that

$$\begin{aligned}&\mathrm{V@R}_\alpha(\,-\,(F(S_T) - F(S_0)))\\&\qquad = F'(S_0)\,\mathrm{V@R}_\alpha(-(S_T - S_0)) + \frac{1}{2}F''(S_0)\,\mathrm{V@R}_\alpha(-(S_T - S_0))^2 + \cdots\end{aligned}$$

The first derivative of $F$ w.r.t. the underlying price $S$ is usually called *Delta*. We write $\Delta(S) = F'(S)$. With this notation, we have the $\Delta$–*approximation of the* V@R:

$$\mathrm{V@R}_\alpha(-(F(S_T) - F(S_0))) \approx \Delta(S_0)\,\mathrm{V@R}_\alpha(-(S_T - S_0))\,.$$

The approximation including the 2nd derivative is called $\Delta$–$\Gamma$–*approximation* .

## V@R **of a portfolio**

Usually a portfolio contains many (maybe hundreds) assets which are heavily correlated. How does one obtain a risk estimate for those?

Let $F(S_t^1, \ldots, S_t^d)$ be the value of a portfolio depending on $d$ risk factors. For simplicity we assume that $S := (S_t^1, \ldots, S_t^d)$ is Gaussian with mean $(S_0^1, \ldots, S_0^d)$ and covariance matrix $C = (c_{ij})$, i.e. $c_{ij} = \mathrm{cov}(S_T^1, \ldots, S_T^d)$.

Now we make a TAYLOR expansion; we get the $\Delta$–approximation

$$-(F(S_T) - F(S_0)) \approx -\sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(S_0^1, \ldots, S_0^d)(S_T^i - S_0^i) =: L\,.$$

Let $\Delta_i := \dfrac{\partial F}{\partial x_i}(S_0^1, \ldots, S_0^d)$. The sum of centered Gaussian distributions is again a centered Gaussian distribution. So $L$ is normally distributed with mean zero and variance

$$\sigma_L^2 = \begin{pmatrix}\Delta_1\\ \vdots\\ \Delta_d\end{pmatrix}^{\top} C \begin{pmatrix}\Delta_1\\ \vdots\\ \Delta_d\end{pmatrix}$$

which implies

$$\mathrm{V@R}_\alpha(L) = \sigma_L a$$

with $a$ the $\alpha$–quantile of the standard normal distribution.

This last formula comes about as follows. Let $L$ be a random variable with mean $\mu$ and variance $\sigma^2$. Then the random variable $X := (L - \mu)/\sigma$ has mean 0 and variance 1. Then

$$P\left[X > c\right] = P\left[L > \mu + \sigma c\right]$$

implies

$$\mathrm{V@R}_\alpha(L) = \mu + \sigma\, \mathrm{V@R}_\alpha(X)\,.$$

Now suppose the loss $L$ is $\mathcal{N}(\mu, \sigma^2)$–distributed. Then the random variable $X$ is $\mathcal{N}(0,1)$–distributed. Since the $\mathrm{V@R}_\alpha$ is nothing but the $\alpha$–quantile, we have $\mathrm{V@R}_\alpha(X) = \Phi^{-1}(\alpha) = a$. Hence

$$\mathrm{V@R}_\alpha(L) = \mu + \sigma a\,,$$

which explains the above formula.

**Remark**. As a practical remark one may say that this is a simple formula with high practical relevance; in particular, it applies to very large portfolios.

CHAPTER 2

# The One-Period (Asset Price) Model

We consider a financial market with $d+1$ assets (stocks, bonds, ...). Denote the price vector at time $t = 0$ as

$$\pi^0, \dots, \pi^d \geq 0$$

and the corresponding price system as

$$(\pi^0, \dots, \pi^d) \in \mathbb{R}_+^{d+1}.$$

The prices at time $t = 1$

$$S^0, \dots, S^d \geq 0$$

are not known before $t = 1$. We model them as non-negative random variables on a probabilitiy space $(\Omega, \mathcal{F}, P)$. Throughout we assume that $S^0$ is the price of a bond with fixed interest rate $r \geq 0$. To simplify notation we assume that

$$\pi^0 = 1 \quad \text{and} \quad S^0 = 1 + r.$$

Further, we assume $S^1, \dots, S^d$ will be risky assets depending on scenarios $\omega \in \Omega$. We employ the following *notation*:

$$\overline{\pi} = (1, \pi) = (\pi^0, \dots, \pi^d)$$
$$\overline{S} = (1, S) = (S^0, \dots, S^d).$$

At time $t = 0$ an investor chooses a *portfolio* (= a vector of positions)

$$\overline{\xi} = (\xi^0, \xi) = (\xi^0, \dots, \xi^d) \in \mathbb{R}_+^{d+1}$$

(which is *not* random). The interpretation is:

$$\xi^i \quad = \quad \text{number of shares of asset } i \text{ in the investor's portfolio.}$$

We allow the $\xi^i$ to be negative: If $\xi^0 < 0$, then the investor is taking a loan at $t = 0$, and has to pay back $|\xi^0|\,(1+r)$ at time $t = 1$. If $\xi^i < 0$ for some $1 \leqslant i \leqslant d$, then the investor is *short selling* asset $i$ at time $t = 0$.

We can define the *value* of a portfolio: The value of the portfolio $\overline{\xi}$ at time 0 is the scalar product

$$\overline{\xi} \cdot \overline{\pi} = \sum_{i=0}^{d} \xi^i \pi^i = \xi^0 + \xi \cdot \pi, \tag{2.1}$$

and the value of the portfolio $\overline{\xi}$ at time 1 is the scalar product

$$\overline{\xi} \cdot \overline{S} = \sum_{i=0}^{d} \xi^i \S^i = \xi^0(1+r) + \xi \cdot S\,. \tag{2.2}$$

We now make the following definition which is of paramount importance in what follows.

**Definition.** *A portfolio $\overline{\xi} \in \mathbb{R}^{d+1}$ is called an* arbitrage opportunity *iff it has the following three properties:*

*(i)* $\overline{\xi} \cdot \overline{\pi} \leqslant 0$ *;*
*(ii)* $\overline{\xi} \cdot \overline{S} \geq 0$ *$P$–a.s. ;*
*(iii)* $P\left[\overline{\xi} \cdot \overline{S} > 0\right] > 0$ *.*

**Lemma 2.1.** *The following statements are equivalent:*

*(1) There exist arbitrage opportunities.*
*(2) There exists a vector $\xi \in \mathbb{R}^d$ such that*

$$\xi \cdot S \geq (1+r)\xi \cdot \pi$$

*and*

$$P\left[\xi \cdot S > (1+r)\xi \cdot \pi\right] > 0\,.$$

*(3) Let $Y := \dfrac{S}{1+r} - \pi$ be the vector of "discounted gains" (an $\mathbb{R}^d$–valued random variable). Then there exists a vector $\xi \in \mathbb{R}^d$ such that*

$$\xi \cdot Y \geq 0 \ P\text{-a.s.}$$

*and*

$$P\left[\xi \cdot Y > 0\right] > 0\,.$$

Proof. *(1) $\Longrightarrow$ (2):* Let $\overline{\xi}$ be an arbitrage opportunity. Then, by (2.1) and (2.2)

$$\overline{\xi} \cdot \overline{\pi} = \xi^0 + \xi \cdot \pi \leqslant 0\,,$$
$$\overline{\xi} \cdot \overline{S} = \xi^0(1+r) + \xi \cdot S > 0 \quad P\text{–a.s.}$$

This implies that $\xi \cdot S \geq (1+r)\xi \cdot \pi$ $P$–a.s. Moreover, if $\overline{\xi} \cdot \overline{S} > 0$, then $\xi \cdot S > (1+r)\xi \cdot \pi$. Hence,

$$P\left[\xi \cdot S > (1+r)\xi \cdot \pi\right] > P\left[\overline{\xi} \cdot \overline{S} > 0\right] > 0\,.$$

*(2) $\Longrightarrow$ (3):* If (2) holds, then so does (3) with the same $\xi$ by definition of $Y$, since $\xi \cdot Y \geq (>)\, 0 \iff \xi \cdot S \geq (>)\, (1+r)\xi \cdot \pi$.

*(3) $\Longrightarrow$ (1):* Put $\xi^0 := -\xi \cdot \pi$ and $\overline{\xi} := (\xi^0, \xi)$. Then the claim is that $\overline{\xi}$ is an arbitrage opportunity. Indeed, we have

(i) $\overline{\xi} \cdot \overline{\pi} = \xi^0 + \xi \cdot \pi = 0$ because of (2.1) and the definition of $\xi^0$

$$
\begin{aligned}
\text{(ii)}\quad \overline{\xi}\cdot\overline{S} &= (1+r)\xi^0 + \xi\cdot S \quad \text{because of (2.2)}\\
&= (1+r)\xi^0 + (1+r)(\xi\cdot Y + \xi\cdot\pi) \quad \text{by the definition of } Y\\
&= (1+r)(\xi^0 + \xi\cdot Y + \xi\cdot\pi)\\
&= (1+r)(\xi\cdot Y + \overline{\xi}\cdot\overline{\pi})\\
&= (1+r)(\xi\cdot Y)\\
&\geq 0 \ P\text{–a.s.} \quad \text{by assumption,}
\end{aligned}
$$

and so, since, as we have just seen, $\overline{\xi}\cdot\overline{S} = (1+r)(\xi\cdot Y)$ for this special $\xi$,

$$\text{(iii)}\quad P\left[\overline{\xi}\cdot\overline{S} > 0\right] = P\left[\xi\cdot Y > 0\right] > 0 \quad \text{by assumption.}$$

In this way, we see that the properties (i) – (iii) of the definition of an arbitrage opportunity do indeed hold. QED

This lemma shows that absence of arbitrage is equivalent to the following property of the market: Any investment in risky assets yielding with positive probability a higher profit than an investment in the risk–free bond must entail a downside risk.

Our next aim: To show the equivalence of *absence of arbitrage* and existence of a so–called *risk–neutral measure* (this equivalence is known as FFToAP, the *First Fundamental Theorem of Asset Pricing*).

**Definition.** *A probability measure $P^*$ on $(\Omega, \mathcal{F})$ is called* risk–neutral *iff*

$$\mathbb{E}^*\left[\frac{S^i}{1+r}\right] = \pi^i \quad , \quad 0 \leqslant i \leqslant d$$

*(where $\mathbb{E}^*$ denotes expectation w.r.t. $P^*$).*

We define

$$\mathcal{P} := \{\, P^* \mid P^* \text{ is a risk–neutral probability measure with } P^* \sim P \,\} \,.$$

What does the equivalence "$\sim$" mean? Recall the definition of absolute continuity and equivalence of two probability measures on $(\Omega, \mathcal{F})$: One has

$$Q \ll P :\Longleftrightarrow \forall\, A \in \mathcal{F} \,:\, (P[A] = 0 \implies Q[A] = 0)$$

("$Q$ is absolutely continuous w.r.t. $P$"), and then

**Definition.** $Q \sim P :\Longleftrightarrow Q \ll P$ *and* $P \ll Q$

("$Q$ and $P$ are equivalent measures"). In other words, $Q$ and $P$ are equivalent measures iff they have the same null sets. Note that two equivalent probability measures define the same notion of arbitrage opportunities and so of arbitrage–freeness, a remark which will be important in the sequel.

There is the following famous result:

RADON–NIKODYM **Theorem.** *If $P \ll Q$, then $P$* has a density w.r.t. $Q$, *i.e. there exists a random variable $X \geq 0$ with $X \in L^1(Q)$ such that*

$$\forall\, A \in \mathcal{F} \,:\; P[A] = \int_A X \, dQ\,.$$

The density $X$ is usually denoted by $\dfrac{dP}{dQ}$ and called the RADON–NIKODYM *derivative of $P$ w.r.t. $Q$.*

**Theorem 2.2.** ("First Fundamental Theorem of Asset Pricing", FFToAP) *The (one–period) market model is arbitrage–free iff $\mathcal{P} \neq \emptyset$. In this case, there exists a $P^* \in \mathcal{P}$ with bounded density $\dfrac{dP^*}{dP}$.*

For the proof of this theorem we need the following version of the "Separating Hyperplane Theorem" in $\mathbb{R}^n$, $n \in \mathbb{N}$:

**Lemma 2.3.** *Let $C \subseteq \mathbb{R}^n$ be a non–empty convex set with $0 \notin C$. Then there exists a vector $\eta \in \mathbb{R}^n$ such that $\eta \cdot x \geq 0$ for all $x \in C$, and with $\eta \cdot x_0 > 0$ for at least one $x_0 \in C$. If $\inf_{x\in C} \|x\| > 0$, then there exists an $\eta \in \mathbb{R}^n$ such that $\inf_{x \in C} \eta \cdot x > 0$.*

I want to give some lines of explanation regarding the name for this result. Recall that a *hyperplane* in a vector space $V$ is a proper linear subspace of maximal dimension, i.e. of codimension 1. Alternatively, it is the kernel of a nontrivial linear form. If $V$ is a finite dimensional Euclidean vector space, the linear forms correspond in a one–to–one fashion to the vectors in $V$ by mapping $x \in V$ to the linear form $\varphi_x$ given by scalar multiplication with $x$, i.e. $\varphi_x(y) := x \cdot y$ for all $y \in V$; clearly, then, the non–trivial linear forms correspond to the nonzero vectors $x \in V \setminus \{0\}$, and the hyperplanes are just the subsets

$$H = H_x = \{\, y \in V \mid\; x \cdot y = 0 \,\}\,,$$

for some $x \in V \setminus \{0\}$, i.e. the orthogonal complements $\mathbb{R}x^{\perp}$ of lines. Here we may assume $\|x\| = 1$; then $x$ is called the *positive unit normal vector* corresponding to $H_x$.

If $H$ is a hyperplane, the complement $V \setminus H$ consists of two connected components, which are nonempty, open and convex. This can be seen as follows. Choose an $x \in V \setminus \{0\}$, where w.l.o.g. we may assume $\|x\| = 1$, such that $H = H_x = \{\, y \in V \mid\; x \cdot y = 0 \,\}$. Then $V = H_+ \cup H_-$ with

$$H_\pm := \{\, y \in V \mid\; x \cdot y \gtrless 0 \,\}$$

called the two *open half spaces defined by $H$*; note that no one is prefered over the other and that they can only be distinguished after an $x$ has been chosen. For any $y_0, y_1$ let $[y_0, y_1] := \{\, y_\lambda := \lambda y_0 + (1-\lambda) y_1 \mid\; \lambda \in [0,1] \,\}$ be the closed line segment joining $y_0$ and $y_1$; then it is immediate that for $y_0, y_1 \in H_\pm$ one has $[y_0, y_1] \subseteq H_\pm$, whence both $H_\pm$ are convex and thus connected. On the other hand, let $y_\pm \in H_\pm$; then, given any continuous path $\gamma : [0,1] \longrightarrow V$ joining $y_+$ and $y_-$, the continuous map $t \mapsto x \cdot \gamma(t)$ on $[0,1]$ has to vanish at some $t_0$, whence $\gamma(t_0) \in H$ and so $V \setminus H$ cannot be connected. By given $x$, $H_+$ is called the positive open half space with unit normal vector $x$, which then is said to point into the interior of $H_+$.

Similarly, there are the two *closed half spaces defined by $H$*,

$$H_{\geq 0(\leqslant 0)} := \overline{H}_{\pm} := \{\, y \in V \mid \ x \cdot y \geq 0(\leqslant 0) \,\} \ .$$

These clearly are closed, and, in fact, the closures of the corresponding open half spaces, and also connected and convex, but not disjoint. Note that any one of the four half spaces $H_{\pm}, \overline{H}_{\pm}$ determines the three others. Any of them determines, and is determined by, a unit normal vector $x$. In this way, open or closed half spaces correpond in a one–to–one fashion to points of the unit sphere in $V$.

The content of the lemma can then be interpreted geometrically by saying that any convex set not containing 0 is contained in a closed half space and so separated from the points of the opposite open half space by its boundary, which is a hyperplane, whence the name "Separating Hyperplane Theorem". The second statement then is if the convex set has positive distance from 0 it is, in fact, contained in an open half space.

PROOF. I first prove the second statement. Let $\overline{C}$ be the closure of $C$; note that $\overline{C}$ is convex, too (if $x, y \in \overline{C}$, choose sequences $(x_k)_{k\in\mathbb{N}}, (y_k)_{k\in\mathbb{N}}$ in $C$ with $x_k \to x, y_k \to y$; then, for any $\alpha \in [0,1]$, $\alpha x_k + (1-\alpha)y_k \to \alpha x + (1-\alpha)y$). Choose $y \in \overline{C}$ with $\|y\| = \inf_{x\in C} \|x\|$.

This is possible by the following reasoning. Take any $x_0 \in C$. Then $\|x_0\| > 0$. The set $B := \overline{\mathbb{B}}\,(0; \|x_0\|) \cap \overline{C}$ is nonempty and compact, and $\inf_{x\in C} \|x\| = \inf_{x\in\overline{C}} \|x\| = \inf_{x\in B} \|x\|$. Since $B$ is compact, there is $y \in B \subseteq \overline{C}$ with $\|y\| = \inf_{x\in B} \|x\|$.

Now, if $x \in C$, we have for all $\alpha \in [0,1]$ that $\alpha x + (1-\alpha)y$ is in $\overline{C}$, and so

$$\|y + \alpha(x-y)\|^2 = \|\alpha x + (1-\alpha)y\|^2 \geq \inf_{z\in\overline{C}} \|z\|^2 = \inf_{z\in C} \|z\|^2 = \|y\|^2$$

which implies

$$2\alpha y \cdot (x-y) + \alpha^2\|x-y\|^2 \geq 0\,.$$

Suppose $\alpha \neq 0$; dividing by $\alpha$ and letting $\alpha \to 0$ gives

$$y \cdot (x-y) \geq 0$$

i.e.

$$x \cdot y \geq \|y\|^2 > 0\,.$$

So $\eta := y$ does the job.

Regarding the first statement, a proof for this can be found in Appendix A1 of **[12]**. QED

The idea of proof for the first statement in *loc. cit.* consists of reducing it to the proven second statement by making slight shifts to $C$ as to make the distance to 0 positive. Then the second case applies, and each shift gives a separating hyperplane for the slightly shifted $C$. Taking a sequence of shifts approaching 0 gives a sequence of half spaces and so a sequence of unit vectors on the unit sphere; since this sphere is compact, there exists a convergent subsequence, and the limit unit vector is then the positive unit normal vector for a half space satisfying the requirements of the first statement of the lemma.

For reasons of promoting diversity, I want to give an alternative proof from scratch, hence not based on the second statement, which is more geometric in spirit. We proceed in several steps.

*Step 1.* (This is also the first step of the proof in *loc. cit.*) Recall that a subset of a vector space $V$ is called *affine* iff with any two different points it contains the line though these points. This is equivalent to saying that with any points $a_0, a_1, \ldots, a_n$ it contains the *affine combinations* $\sum_{i=0}^n \lambda_i a_i$, $\sum_{i=0}^n \lambda_i = 1$. For any subset $S \subseteq V$, the *affine hull* $\mathrm{Aff}(S)$ is the smallest affine subspace of $V$ containing $S$. Since an arbitrary intersection of affine subspaces is an affine subspace, this is the intersection of all affine subspaces containing $S$. Since any such subspace must contain all affine combinations on points of $S$, there follows

$$\mathrm{Aff}(S) = \{ \textstyle\sum_{i=0}^n \lambda_i a_i \mid n \geq 0 \ , \ a_0, \ldots, a_n \in S \ , \ \sum_{i=0}^n \lambda_i = 1 \} \ .$$

We call a convex set $C \subseteq V$ *regular* iff $\mathrm{Aff}(C) = V$ (this is not standard terminology); note that a regular convex set is, in particular, nonempty. It is then sufficient to prove the lemma for regular convex sets. For suppose $C$ is not regular and nonempty (otherwise there is nothing to prove). Then $C$ is regular in the affine subspace $\mathrm{Aff}(C)$ generated by $C$. By applying a suitable translation we may assume $\mathrm{Aff}(C) = \mathrm{Lin}(C)$, the linear hull of $C$, while keeping the condition $0 \notin C$.

Let now $V := \mathbb{R}^n$. By choosing appropriate coordinates we may further assume $\mathrm{Lin}(C) = \mathbb{R}^m \subseteq \mathbb{R}^n$. Then if the lemma holds for $C \subseteq \mathbb{R}^m$, it yields $\eta \in \mathbb{R}^m$ which then is also good for $\mathbb{R}^n$.

*Step 2.* We now show that it suffices to assume that $C$ is open. We begin by noting the equivalences

(i) $C$ is regular;
(ii) $C$ contains a nondegenerate simplex;
(iii) the interior $C^\circ$ of $C$ is not empty.

*(i) $\Longrightarrow$ (ii):* Recall that elements $a_0, a_1, \ldots, a_m$ in $V$ are called *affinely independent* if $\sum_{i=0}^m \lambda_i a_i = 0$ and $\sum_{i=0}^m \lambda_i = 0$ together imply that $\lambda_i = 0$, $i = 0, 1, \ldots, m$. A *nondegenerate $n$–simplex* $< a_0, a_1, \ldots, a_n >$ in $V$ is defined to be the convex hull of $n+1$ affinely independent points $a_0, a_1, \ldots, a_n$, where $n = \dim V$. Now if $C$ is convex and $\mathrm{Aff}(C) = V$, $C$ must contain $n+1$ affinely independent points, $n = \dim V$. But since $C$ is convex, it must contain their convex hull.

*(ii) $\Longrightarrow$ (iii):* Define the standard $n$–simplex $\Delta_n \subseteq \mathbb{R}^n$ as

$$\Delta_n := < e_0, e_1, \ldots, e_n > := \mathrm{Conv}\,\{e_0, e_1, \ldots, e_n\}$$

where $e_1, \ldots, e_n$ are the standard unit vectors, $e_0 := 0$, and Conv denotes the convex hull. This is a nondegenerate $n$–simplex. Given a nondegenerate $n$–simplex $\Sigma := < a_0, a_1, \ldots, a_n >$, there is a unique affine map sending $e_i$ to $a_i$, $i = 0, 1, \ldots, n$ and hence $\Delta_n$ to $\Sigma$. This map necessarily is an affine isomorphism and so, in particular, a homeomorphism under which interior points of $\Delta_n$ and $\Sigma$ correspond. But surely

$$\Delta_n^\circ = \{ x = (x_1, \ldots, x_n) \mid x_i > 0 \textit{ for all } i \textit{ and } \textstyle\sum_{i=1}^n x_i < 1 \} \neq \emptyset \,.$$

*(iii) $\Longrightarrow$ (i):* Let $x \in C^\circ$ be an interior point of $C$. By linearly translating $C$ we may assume $x = 0$. Let $\varepsilon > 0$ be such that $\mathbb{B}\,(0; \varepsilon) \subseteq C$ (the open ball taken w.r.t. any accomodating norm). Then the vectors $\varepsilon e_1/\|e_1\|, \ldots, \varepsilon e_n/\|e_n\|$ are in $C$, and they generate $\mathbb{R}^n$. So $C$ is regular, and the equivalence of (i) – (iii) is proved.

We now come to the crucial point, which allows the reduction of the proof of the separation theorem to open convex sets. First some notation. Choose any norm on $\mathbb{R}^n$ and let

$$\mathbb{B} := \mathbb{B}\,(0;1) := \{\, x \in \mathbb{R}^n \mid \|x\| < 1 \,\}$$

be the unit ball. One then has

$$\forall x \in \mathbb{R}^n \; \forall r \in \mathbb{R}_+ \;:\; \mathbb{B}\,(x;r) = x + r\mathbb{B}\,.$$

For any $y_0, y_1 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we put $y_\lambda := (1-\lambda)y_0 + \lambda y_1$. Define the *closed line segment* $[y_0, y_1]$ to be $\{\, y_\lambda \mid \lambda \in [0,1] \,\}$. The *half open and open line segments* $[y_0, y_1)$, $(y_0, y_1]$, and $(y_0, y_1)$ are defined in an analogous manner.

**Lemma.** *Let $C \subseteq \mathbb{R}^n$ be a regular convex set. Let $y_1 \in \overline{C}$. Then for any $y_0 \in C^\circ$ there holds $[y_0, y_1) \subseteq C^\circ$.*

Proof (*see* [**31**], Theorem 6.1). Given $\lambda \in [0,1)$ we want to find $\varepsilon > 0$ such that $\mathbb{B}\,(y_\lambda;\varepsilon) \subseteq C$. Note that for all $\varepsilon > 0$ we have $y_1 \in C + \varepsilon\mathbb{B}$ since $y_1 \in \overline{C}$. Then

$$\begin{aligned}\mathbb{B}\,(y_\lambda;\varepsilon) &= y_\lambda + \varepsilon\mathbb{B} = (1-\lambda)y_0 + \lambda y_1 + \varepsilon\mathbb{B} \subseteq (1-\lambda)y_0 + \lambda(C+\varepsilon\mathbb{B}) + \varepsilon\mathbb{B} \\ &= (1-\lambda)\left[y_0 + \varepsilon\frac{1+\lambda}{1-\lambda}\mathbb{B}\right] + \lambda C = (1-\lambda)\mathbb{B}\,(y_0;\varepsilon(1+\lambda)/(1-\lambda)) + \lambda C\,.\end{aligned}$$

We have $y_0 \in C^\circ$ by hypothesis, so there is $\delta > 0$ with $\mathbb{B}\,(y_0;\delta) \subseteq C$. So if we choose $\varepsilon := \delta(1-\lambda)/(1+\lambda)$, we do indeed have $\mathbb{B}\,(y_\lambda;\varepsilon) \subseteq (1-\lambda)\mathbb{B}\,(y_0;\delta) + \lambda C \subseteq (1-\lambda)C + \lambda C \subseteq C$. QED

**Corollary.** *Let $C \subseteq \mathbb{R}^n$ be regular convex. Then $C^\circ$ and $\overline{C}$ are regular convex.*

Proof. Let $y_0, y_1 \in C^\circ$. Then $y_1 \in \overline{C}$ and so $[y_0, y_1) \subseteq C^\circ$ by the lemma, hence $[y_0, y_1] \subseteq C^\circ$, which proves $C^\circ$ is convex. Since $C^\circ$ has a nonempty interior, as to say $C^\circ$, $C^\circ$ is regular convex.

That $\overline{C}$ is convex has been shown above. Alternatively, note that

$$\overline{C} = \bigcap_{\varepsilon>0}(C + \varepsilon\mathbb{B})\,.$$

Now if $C_1, C_2$ are convex, so is $C_1 + C_2$; this is immediate from the definition. Hence $\overline{C}$ is an intersection of convex sets and thus convex.

Since $\overline{C}^\circ \supseteq C^\circ$, it is regular convex. QED

**Corollary.** *Let $C \subseteq \mathbb{R}^n$ be regular convex. Then $\overline{C} \subseteq \overline{C^\circ}$ and so $\overline{C} = \overline{C^\circ}$. And $\overline{C}^\circ \subseteq C^\circ$ and so $\overline{C}^\circ = C^\circ$.*

Proof. Let $y_1 \in \overline{C}$. Since $C$ is regular convex, there exists an $y_0 \in C^\circ$. By the lemma, $[y_0, y_1) \subseteq C^\circ$, which clearly implies $y_1 \in \overline{C^\circ}$. So $\overline{C} \subseteq \overline{C^\circ}$. On the other hand, $C^\circ \subseteq C$, and so $\overline{C^\circ} \subseteq \overline{C}$.

For the second claim, let $y_1 \in \overline{C}^\circ$. Let again $y_0 \in C^\circ \subseteq \overline{C}^\circ$. Since $C$ is regular convex, so is $\overline{C}$ and therefore so is $\overline{C}^\circ$. Hence $[y_0.y_1] \subseteq \overline{C}^\circ$. Since $\overline{C}^\circ$ is open, $y_\lambda = (1-\lambda)y_0 + \lambda y_1$ is still in $\overline{C}^\circ$ for $\lambda > 1$ sufficiently close to 1 and so in $\overline{C}$. Then $y_1 = (1-\lambda^{-1})y_0 + \lambda^{-1}y_\lambda \in [y_0, y_\lambda)$ and so in $C^\circ$ by the last lemma. So indeed $\overline{C}^\circ \subseteq C^\circ$. On the other hand, $C \subseteq \overline{C}$, and so $C^\circ \subseteq \overline{C}^\circ$. QED

Now suppose Lemma 2.3 has been proved for *open* convex sets. Given any regular convex set $C$, its interior $C^\circ$ is, by what has been said above, regular, convex, and open. So we have

$\eta \in \mathbb{R}^n$ such that $\eta \cdot x \geq 0$ for all $x \in C^\circ$ and $\eta \cdot x_0 > 0$ for at least one $x_0 \in C^\circ$. But since the scalar product is continuous, we have $\eta \cdot x \geq 0$ for all $x \in \overline{C^\circ} = \overline{C}$ and so *a forteriori* for all $x \in C$. And again *a forteriori* we have $x_0 \in C^\circ \subseteq C$ with $\eta \cdot x_0 > 0$. The last claim follows because, again for reasons of continuity, we have $\inf_{x \in C^\circ} \|x\| = \inf_{x \in \overline{C^\circ} = \overline{C}} \|x\| = \inf_{x \in C} \|x\|$ and $\inf_{x \in C^\circ} \eta \cdot x = \inf_{x \in \overline{C^\circ} = \overline{C}} \eta \cdot x = \inf_{x \in C} \eta \cdot x$.

*Step 3.* Before presenting the main construction step for the Separating Hyperplane Theorem, we need the following preparatory fact.

**Lemma.** *Let $L \subseteq \mathbb{R}^n$ be a proper linear subspace which is not a hyperplane. Then $\mathbb{R}^n \setminus L$ is connected.*

PROOF. Let $x_0, x_1 \in \mathbb{R}^n \setminus L$. If $x_1 - x_0 \in L$, no $x_\lambda \in [x_0, x_1]$ can be in $L$, for otherwise

$$x_\lambda = (1-\lambda)x_0 + \lambda x_1 = x_0 + \lambda(x_1 - x_0) = (1-\lambda)(x_0 - x_1) + x_1 \in L$$

would imply $x_0, x_1 \in L$. Thus the line segment $[x_0, x_1]$ then is a path in $\mathbb{R}^n \setminus L$ joining $x_0$ and $x_1$.

If $x_1 - x_0 \notin L$, let $p : \mathbb{R}^n \longrightarrow L^\perp$ be the orthogonal projection onto the orthogonal complement of $L$; then $\eta := p(x_1 - x_0) \neq 0$, and $H := \{ x \in \mathbb{R}^n \mid \eta \cdot x = 0 \}$ is a hyperplane contaning $L$. Since $\mathbb{R}^n \setminus L$ is open there is $\delta > 0$ with $\mathbb{B}(x_i; \delta) \subseteq \mathbb{R}^n \setminus L$, $i = 0, 1$, and so we can vary $x_0, x_1$ within these open balls without leaving their connected components, whence we may assume $x_0, x_1 \notin H$. Now $H \setminus L \neq \emptyset$; take $y \in H \setminus L$, and then the two line segments $[x_0, y]$ and $[y, x_1]$ define a path in $\mathbb{R}^n \setminus L$ joining $x_0$ and $x_1$. QED.

The following result now is the main construction step for the Separating Hyperplane Theorem:

**Lemma.** *Let $C \subseteq \mathbb{R}^n$ be a nonempty open convex set. Let $L \subseteq \mathbb{R}^n$ be a linear subspace with $C \cap L = \emptyset$. Then either $L$ is a hyperplane, or there exists a vector $x \in \mathbb{R}^n \setminus L$ such that $C \cap (L + \mathbb{R}x) = \emptyset$.*

PROOF. Consider the set

$$D := L + \bigcup_{\lambda > 0} \lambda C = \bigcup_{y \in L, \lambda > 0} (y + \lambda C) = \bigcup_{\lambda > 0} \lambda (L + C).$$

(The geometric interpretation of this set is as follows. Since $L$ and $C$ are convex, so is $L + C$ ; in fact, $L + C$ is the union of the translates of $C$ with vectors in $L$. A *cone* in $\mathbb{R}^n$ is a subset closed under the multiplication with positive scalars, and a *convex cone* is a cone which is convex. With these notions $D$ is the convex cone generated by $L + C$, i.e. the smallest convex cone containing $L + C$ ([**31**], COROLLARY 2.6.1.))

Then

$$(\star) \qquad D \cap L = \emptyset = D \cap (-D).$$

For the first equality, suppose we have $x \in D \cap L$. Then $x = y + \lambda c$ with $y \in L$, $c \in C$, and $\lambda > 0$. But then $c = \lambda(x - y)^{-1} \in C \cap L$, contrary to the hypothesis. For the second equality, suppose we would have $x \in D \cap (-D)$. Then

$$x = y_+ + \lambda_+ c_+ = y_- - \lambda_- c_- \quad \text{with } y_\pm \in L,\ c_\pm \in C, \text{ and } \lambda_\pm > 0.$$

But then

$$c := \frac{\lambda_+}{\lambda_+ + \lambda_-} c_+ + \frac{\lambda_-}{\lambda_+ + \lambda_-} c_- = y_- - y_+ \in C \cap L$$

contrary to the hypothesis. So $(\star)$ holds true.

Now suppose $L$ is not a hyperplane. We have $D \cup (-D) \subseteq \mathbb{R}^n \setminus L$, since $D \cap L = \emptyset$. Both $D$ and $-D$ are nonempty and open as unions of open sets (here we use that $C$ is open). So, since

$\mathbb{R}^n \setminus L$ is connected and $D$, $-D$ are disjoint, we must have $\mathbb{R}^n \setminus L \neq D \cup (-D)$, and so there must be an $x \in \mathbb{R}^n \setminus L$ sucht that $\pm x \notin D$.

Suppose there would be $y \in L$ and $\lambda \in \mathbb{R}$ such that $c := y + \lambda x \in C$. We cannot have $\lambda = 0$. So $x = -\lambda^{-1} y + \lambda^{-1} c$ would be such that either $x \in D$ or $-x \in D$ which contradicts the choice of $x$. Hence $C \cap (L + \mathbb{R}x) = \emptyset$. QED

**Corollary.** (Separating Hyperplane Theorem) *Let $C \subseteq \mathbb{R}^n$ be a nonempty open convex set with $0 \notin C$. Then there is a hyperplane $H \subseteq \mathbb{R}^n$ with $C \cap H = \emptyset$, or equivalently an open half space $H_+ \subseteq \mathbb{R}^n$ with $C \subseteq H_+$.*

PROOF. The equivalence of both claimed properties are clear: if $H$ is a hyperplane with $C \cap H = \emptyset$, $C$ must be contained in one of the two connected components of $\mathbb{R}^n \setminus H$ since it is connected, and these components are open half spaces; conversely, if $C$ is contained in an open half space $H_+$, $H := \partial H_+$ is a hyperplane with $C \cap H = \emptyset$.

To find a hyperplane $H$ with $C \cap H = \emptyset$, put $L_0 := \{0\}$ and use the last lemma iteratively to build a flag $L_0 \subset L_1 \subset \cdots \subset L_i \subset \cdots \subset L_{n-1}$ of linear subspaces $L_i$ with $\dim L_i = i$ and $C \cap L_i = \emptyset$. Then $H := L_{n-1}$ provides the required properties. QED

**Remark**. The essence of the Separating Hyperplane Theorem is that if one orders the set of convex subsets of $\mathbb{R}^n$ not containing 0 by inclusion, the maximal elements of this partially ordered set are just the open half spaces. This continues to hold in infinite dimensional real vector spaces, where it can be proved similarly by using ZORN's Lemma, and then it constitutes one of the many manifestations of the HAHN–BANACH Theorem.

For the equivalence of the Separating Hyperplane Theorem with the HAHN–BANACH Theorem see [**4**].

*Step 4.*

**Corollary.** *Lemma 2.3 holds true.*

PROOF. We need only to prove the first statement; the second statement has been taken care of already above.

As explicated under Step 2 it suffices to prove Lemma 2.3 for $C$ nonepty open. By the last corollary (Separating Hyperplane Theorem) we can find an open half space $H_+$ with $C \subseteq H_+$. If $\eta$ is the positive unit normal vector cooresponding to $H_+$ we have $\eta \cdot x > 0$ for all $x \in C$ and therefore also $\inf_{x \in C} \eta \cdot x \geq 0$. This proves the first statement. QED

Now we can start the proof of Theorem 2.2.

PROOF *of Theorem 2.2.* Let us start with the simple direction "$\Longleftarrow$". Assume we have $P^* \in \mathcal{P}$. Let $\overline{\xi} \in \mathbb{R}^{d+1}$ be such that $\overline{\xi} \cdot \overline{S} \geq 0$ $P$–a.s. and $P\left[\overline{\xi} \cdot \overline{S} > 0\right] > 0$. Since $P^* \sim P$ by assumption, we have also $\overline{\xi} \cdot \overline{S} \geq 0$ $P^*$–a.s. and $P^*\left[\overline{\xi} \cdot \overline{S} > 0\right] > 0$. Hence, $\mathbb{E}^*\left[\overline{\xi} \cdot \overline{S}\right] > 0$. Now observe that

$$\overline{\xi} \cdot \overline{\pi} = \sum_{i=0}^{d} \xi^i \mathbb{E}^*\left[\frac{S^i}{1+r}\right] = \frac{1}{1+r} \mathbb{E}^*\left[\overline{\xi} \cdot \overline{S}\right] > 0 .$$

Thus, the third defining property of an arbitrage opportunity can never be satisfied.

The other direction " $\Longrightarrow$ " is a little more complicated. Let

$$Y^i := \frac{S^i}{1+r} - \pi^i \quad , \quad 1 \leqslant i \leqslant d$$

and set

$$Y := (Y^1, \dots, Y^d) \quad \text{("discounted gains").}$$

According to Lemma 2.1 (3) the market is arbitrage–free iff the following implication holds true:

$$\forall \xi \in \mathbb{R}^d : \ \xi \cdot Y \ \geq \ 0 \ P\text{–a.s.} \implies \xi \cdot Y = 0 \ P\text{–a.s.}. \tag{2.3}$$

Thus it is enough to show the implication

$$(2.3) \implies \exists\, P^* \in \mathcal{P} : \ \frac{dP^*}{dP} \text{ bounded.}$$

First, assume that the net gains $Y^i$ are integrable, i.e. $Y \in L^1(P)$, or, equivalently, $\mathbb{E}\left[\|Y\|\right] < \infty$. Let

$$\mathcal{Q} := \{\, Q \mid Q \textit{ is a probability measure with } Q \sim P \textit{ and } \frac{dQ}{dP} \textit{ bounded} \,\}.$$

Write

$$\mathbb{E}^Q\left[Y\right] := (\mathbb{E}^Q\left[Y^1\right], \dots, \mathbb{E}^Q\left[Y^d\right]) \in \mathbb{R}^d$$

which is defined for any $Q \in \mathcal{Q}$. Now introduce

$$C := \{\, \mathbb{E}^Q\left[Y\right] \ \mid Q \in \mathcal{Q} \,\}\,.$$

Then $C$ is a convex set in $\mathbb{R}^d$. Indeed, for $Q_1, Q_2 \in \mathcal{Q}$ and $\alpha \in [0,1]$ we have

$$\alpha \mathbb{E}^{Q_1}\left[Y\right] + (1-\alpha)\mathbb{E}^{Q_2}\left[Y\right] = \mathbb{E}^{Q_\alpha}\left[Y\right]$$

with $Q_\alpha := \alpha Q_1 + (1-\alpha) Q_2$. Notice that $Q_\alpha \in \mathcal{Q}$ as

$$\forall \alpha \in (0,1)\,,\ \forall\, A \in \mathcal{F} \ : \ Q_\alpha\left[A\right] = 0 \iff Q_1\left[A\right] = 0 \ \wedge \ Q_2\left[A\right] = 0\,,$$

and

$$\frac{dQ_\alpha}{dP} = \alpha \frac{dQ_1}{dP} + (1-\alpha)\frac{dQ_2}{dP}$$

is a bounded density for $Q_\alpha$ w.r.t. $P$. So $C$ is convex.

We want to show there exists a risk–neutral measure equivalent to $P$. Observe that the construction of $C$ is such that $0 \in C$ corresponds just to those measures $Q \in \mathcal{Q}$ which satisfy

$$\mathbb{E}^Q\left[Y^i\right] = \mathbb{E}^Q\left[\frac{S^i}{1+r}\right] - \mathbb{E}^Q\left[\pi^i\right] = \mathbb{E}^Q\left[\frac{S^i}{1+r}\right] - \pi^i = 0 \quad , \quad 1 = 1, \dots, d \tag{2.4}$$

and so to the risk-neutral measures with bounded densities (note that $\mathbb{E}^Q\left[\pi^i\right] = \pi^i$ since the $\pi^i$ are deterministic). We want to pin down such a measure and argue by contradiction; i.e. we want to show that the assumption that such a measure does not exist contradicts our hypothesis of arbitrage–freeness. So we assume $0 \notin C$ and will show that then (2.3) is not satisfied. By Lemma 2.3, there exists

$\xi \in \mathbb{R}^d$ such that $\xi \cdot x \geq 0$ for all $x \in C$ and $\xi \cdot x_0 > 0$ for at least one $x_0 \in C$. Thus, for this $\xi$:

$$\forall Q \in \mathcal{Q} : \ \xi \cdot \mathbb{E}^Q[Y] \geq 0 \tag{2.5}$$

and

$$\exists Q_0 \in \mathcal{Q} : \ \xi \cdot \mathbb{E}^{Q_0}[Y] > 0 . \tag{2.6}$$

But, for any $Q$ taking expectations is a linear operation, and so

$$\xi \cdot \mathbb{E}^Q[Y] = \sum_{i=1}^d \xi^i \mathbb{E}^Q\left[Y^i\right] = \sum_{i=1}^d \mathbb{E}^Q\left[\xi^i Y^i\right] = \mathbb{E}^Q\left[\sum_{i=1}^d \xi^i Y^i\right] = \mathbb{E}^Q[\xi \cdot Y] ,$$

and so $\mathbb{E}^{Q_0}[\xi \cdot Y] > 0$. Since $Q_0 \sim P$, this implies $P[\xi \cdot Y > 0] > 0$.

Next we show that $\xi \cdot Y \geq 0$ $P$–a.s.; this, together with $P[\xi \cdot Y > 0] > 0$, then shows that (2.3) is not satisfied. To this end, let $A := \{\xi \cdot Y < 0\}$; we are going to show that $P[A] = 0$. For $n \geq 2$ define

$$\varphi_n := \left(1 - \frac{1}{n}\right) \mathbb{1}_A + \frac{1}{n} \mathbb{1}_{A^c}$$

(where for any set $X$ the symbol $\mathbb{1}_X$ denotes its characteristic function), and let $Q_n$ be the measure with density (2.7). Now $0 < \varphi_n \leqslant 1$, and so $Q_n \in \mathcal{Q}$. Moreover, by the very choice of $\xi$ (see (2.5)),

$$0 \leqslant \xi \cdot \mathbb{E}^{Q_n}[Y] = \mathbb{E}^{Q_n}[\xi \cdot Y] = \frac{1}{\mathbb{E}[\varphi_n]} \mathbb{E}[\varphi_n(\xi \cdot Y)] . \tag{2.7}$$

Assume $P[A] > 0$, then the RHS of (2.7) converges, by LEBESGUE's Dominated Convergence Theorem (here we use $\mathbb{E}[\|Y\|] < \infty$), to

$$\frac{1}{P[A]} \mathbb{E}\left[\mathbb{1}_{\{\xi \cdot Y < 0\}} \xi \cdot Y\right] < 0$$

as $n \to \infty$, which contradicts (2.7). Therefore, $P[A] = 0$, so (2.3) cannot hold, and the market has arbitrage opportunities, contrary to our hypothesis.

Finally, we drop the assumption that $Y \in L^1(P)$. There exists a probability measure $Q \sim P$ with $Y \in L^1(Q)$; for example choose $Q$ so that

$$\frac{dQ}{dP} = \frac{c}{1 + \|Y\|} \quad \text{with} \quad c^{-1} := \mathbb{E}[1 + \|Y\|] .$$

By the first part (with $Q$ playing the part of $P$) there exists $P^* \sim Q$ with $\dfrac{dP^*}{dQ}$ bounded and $\mathbb{E}^*[Y^i] = 0$, $i = 1, \dots d$. What we need to check is that $\dfrac{dP^*}{dP}$ is also bounded; but

$$\frac{dP^*}{dP} = \frac{dP^*}{dQ} \frac{dQ}{dP}$$

is bounded as a product of bounded densities. QED

**Definition.** *The* return $R(\overline{\xi})$ *of a portfolio* $\overline{\xi} \in \mathbb{R}^{d+1}$ *is defined by*

$$R(\overline{\xi}) := \frac{\overline{\xi} \cdot \overline{S} - \overline{\xi} \cdot \overline{\pi}}{\overline{\xi} \cdot \overline{\pi}}.$$

The next lemma tells that, amongst other things, under a risk–neutral measure the expected return is equal to the interest rate:

**Lemma 2.4.** *Let the market model be arbitrage–free and* $\overline{\xi} \in \mathbb{R}^{d+1}$ *such that* $\overline{\xi} \cdot \overline{\pi} \neq 0$. *Let* $P^* \in \mathcal{P}$. *Then*

1) *Under any risk–neutral measure* $P^*$, *the expected return of a portfolio* $\overline{\xi}$ *equals the risk–free return:*

$$\mathbb{E}^*\left[R(\overline{\xi})\right] = r;$$

2) *Under any a probability measure* $Q \sim P$ *with* $\mathbb{E}^Q\left[\|S\|\right] < \infty$ *the expected return of a portfolio* $\overline{\xi}$ *is given by*

$$\mathbb{E}^Q\left[R(\overline{\xi})\right] = r - \mathrm{cov}_Q\left(\frac{dP^*}{dQ}, R(\overline{\xi})\right).$$

Proof. 1) One has

$$R(\overline{\xi}) = \frac{\overline{\xi} \cdot \overline{S} - \overline{\xi} \cdot \overline{\pi}}{\overline{\xi} \cdot \overline{\pi}} = \frac{\overline{\xi} \cdot \overline{S}}{\overline{\xi} \cdot \overline{\pi}} - 1 = \frac{1}{\overline{\xi} \cdot \overline{\pi}}\overline{\xi} \cdot \overline{S} - 1$$

and so, since $\overline{\xi} \cdot \overline{\pi}$ is deterministic,

$$\mathbb{E}^*\left[R(\overline{\xi})\right] = \frac{1}{\overline{\xi} \cdot \overline{\pi}}\mathbb{E}^*\left[\overline{\xi} \cdot \overline{S}\right] - 1 = (1+r) - 1 = r.$$

2) One has

$$r = \mathbb{E}^*\left[R(\overline{\xi})\right] = \mathbb{E}^Q\left[\frac{dP^*}{dQ}R(\overline{\xi})\right] = \mathrm{cov}_Q\left(\frac{dP^*}{dQ}, R(\overline{\xi})\right) + \mathbb{E}^Q\left[R(\overline{\xi})\right].$$

This finishes the proof. QED

We now turn to financial derivatives. Here is a list of examples.

*Forward contract* (or simply *forward*)

= agreement between two parties to buy or sell an asset at a specified future $T$ at a price $K$ agreed on today. Usually there is an asset–dependent cash flow only at $T$:

| | | |
|---|---|---|
| Buyer | : | receives the asset, pays $K$; |
| Seller | : | receives $K$, delivers the asset. |

Within our one–period model: If $T = 1$, then a forward on asset $i$ provides the random payoff $S^i - K$ at time $1$.

*Call option*

= the right, but not the obligation, to buy an asset at a future date $T$ at an agreed price $K$, the *strike price.*

Within our one–period model: If $T = 1$, then the call option provides the payoff $(S^i - K)^+$ (here $x^+ := \max(x, 0)$).

*Put option*

= the right, but not the obligation, to sell an asset at a future date $T$ for an agreed price $K$.

Within our one–period model: If $T = 1$, then the value of the put at time 1 is given by $(K - S^i)^+$.

*Basket option*

= option on a basket of assets. Example: Let $S^1, \ldots, S^{30}$ be the DAX stock prices. DAX at time $1 = \sum_{i=1}^{30} \alpha_i S^i$ ($\alpha_i$ = weights). Such a "call on the DAX" has the payoff $(\alpha \cdot S - K)^+$.

For other examples see [**21**].

Derivatives involve claims that can be made if certain outcomes occur; so the claims are random. We make a precise mathematical definition of such "contingent claims" (within our one–period model):

**Definition.** *A* contingent claim *is a random variable $C$ on $(\Omega, \mathcal{F})$ such that $0 \leqslant C < \infty$ $P$–a.s.*

**Remark**. We assume non–negativity for mathematical convenience only.

We extend our financial market by a contingent claim $C$. Let $\pi^C$ be the price for $C$ at time $0$. We set $\pi^{d+1} := \pi^C$ and $S^{d+1} := C$.

**Definition.** $\pi^C \in \mathbb{R}_+$ *is called an* arbitrage–free price *if the extended market model $((\pi^0, \ldots, \pi^{d+1}), (S^0, \ldots, S^{d+1}))$ is arbitrage–free.*

We denote the set of arbitrage–free prices by $\Pi(C)$.

**Theorem 2.5.** *Let the original market be arbitrage–free, i.e. $\mathcal{P} \neq \emptyset$ with $\mathcal{P}$ the set of risk–neutral measures for the original market model. Then $\Pi(C)$ is not empty and satisfies*

$$\Pi(C) = \left\{ \mathbb{E}^* \left[ \frac{C}{1+r} \right] \;\middle|\; P^* \in \mathcal{P} \text{ s.t. } \mathbb{E}^*[C] < \infty \right\}. \tag{2.8}$$

Proof. "$\subseteq$": Let $\pi^C$ be an arbitrage–free price. Then the extended market model is arbitrage–free, and by Theorem 2.2 (FFToAP) there exists a risk–neutral measure $P^* \sim P$ with $\pi^i = \mathbb{E}^* \left[ \frac{S^i}{1+r} \right]$ $i = 0, \ldots, d$ and $\mathbb{E}^* \left[ \frac{C}{1+r} \right] = \pi^C$. In particular, $P^* \in \mathcal{P}$, and so $\pi^C$ belongs to the RHS of (2.8).

"$\supseteq$": Let $C$ be such that $\mathbb{E}^*[C] < \infty$ and $\pi^C = \mathbb{E}^*\left[\frac{C}{1+r}\right]$ for some $P^* \in P$. Then $P^*$ is risk–neutral for the extended market model. Again by FFToAP the extended market model is arbitrage–free, and so, by definition, $\pi^C$ belongs to the LHS of (2.8).

So we have the equality (2.8), and there remains to show that $\Pi(C)$ is not empty. For this, we have, for a given contingent claim $C$, to exhibit a risk–neutral measure $P^*$ for the original market model such that $\mathbb{E}^*\left[\frac{C}{1+r}\right] < \infty$. For this, let $\widetilde{P} \sim P$ be a new probability measure with $\mathbb{E}^\sim[C] < \infty$ (where $\mathbb{E}^\sim$ denotes taking expectation under $\widetilde{P}$); for instance, one can take $\widetilde{P}$ with density

$$\frac{d\widetilde{P}}{dP} := \frac{c}{1+C} \quad \text{where} \quad c^{-1} := \mathbb{E}\left[\frac{1}{1+C}\right].$$

Since arbitrage–freeness is preserved under change to an equivalent measure, the original market model remains arbitrage–free under $\widetilde{P}$. By FFToAP (Theorem 2.2) there exists $P^* \in \mathcal{P}$ with $\frac{dP^*}{dP}$ bounded. Then

$$\mathbb{E}^*[C] = \mathbb{E}^\sim\left[\frac{dP^*}{dP}C\right] < \infty$$

since

$$\mathbb{E}^\sim[C] = \mathbb{E}\left[\frac{cC}{1+C}\right] = c\mathbb{E}\left[\frac{C}{1+C}\right] < \infty.$$

Therefore, $\pi^C := \mathbb{E}^*\left[\frac{C}{1+r}\right]$ is an element of $\Pi(C)$. QED

Let us have a closer look at bounds for the arbitrage–free prices. Define the *arbitrage bounds*

$$\pi_\downarrow(C) := \inf \Pi(X) \quad , \quad \pi^\uparrow(C) := \sup \Pi(C).$$

**Theorem 2.6.** *Let the original market be arbitrage–free, i.e. $\mathcal{P} \neq \emptyset$ with $\mathcal{P}$ the set of risk–neutral measures for the original market model. Then*

$$\begin{aligned} \pi_\downarrow(C) &= \inf_{P^*\in\mathcal{P}} \mathbb{E}^*\left[\frac{C}{1+r}\right] \\ &= \max\left\{ m \in [0,\infty) \;\middle|\; \exists \xi \in \mathbb{R}^d :\; m + \xi\cdot Y \leqslant \frac{C}{1+r}\ P\text{–a.s.}\right\} \end{aligned} \tag{2.9}$$

*and*

$$\begin{aligned} \pi^\uparrow(C) &= \sup_{P^*\in\mathcal{P}} \mathbb{E}^*\left[\frac{C}{1+r}\right] \\ &= \min\left\{ m \in [0,\infty] \;\middle|\; \exists \xi \in \mathbb{R}^d :\; m + \xi\cdot Y \geq \frac{C}{1+r}\ P\text{–a.s.}\right\}, \end{aligned} \tag{2.10}$$

*where $Y$ is the vector of "discounted net gains" in the original model.*

PROOF. (We prove (2.9) only; the proof of (2.10) is completely analogous and can be found in **[12]**, p. 18 ff. With the same argument they, in turn, omit the proof of (2.9), so our treatment here nicely complement theirs.)

Denote by $M$ the set of $m \in [0, \infty)$ for which there exists $\xi \in \mathbb{R}^d$ such that $m + \xi \cdot Y \leqslant \dfrac{C}{1+r}$ $P$–a.s. For $m \in M$ and $P^* \in \mathcal{P}$ we have, by taking expectations

$$m \leqslant \mathbb{E}^* \left[ \frac{C}{1+r} \right]$$

and so, in particular,

$$\sup M \leqslant \inf_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{1+r} \right] .$$

Thus we obtain

$$\sup M \leqslant \inf_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{1+r} \right] \leqslant \inf_{P^* \in \mathcal{P} : \mathbb{E}^*[C] < \infty} \mathbb{E}^* \left[ \frac{C}{1+r} \right] = \pi_{\downarrow}(C) . \tag{2.11}$$

What we want to show is that the equalities are, in fact, an equalities, and that $\sup M$ is indeed attained for some element of $M$.

Pick a number $m < \pi_{\downarrow}(C)$. Then $m$ is not an arbitrage–free price for $C$. So there exists an arbitrage opportunity in our extended market model, where $C$ is an additional tradable asset with price $m$, i.e. there exists, by Lemma 2.1, $(\xi, \xi^{d+1}) \in \mathbb{R}^{d+1}$ such that

$$\xi \cdot Y + \xi^{d+1} \left( \frac{C}{1+r} - m \right) \geq 0 \ P\text{–a.s.} \tag{2.12}$$

and

$$P \left[ \xi \cdot Y + \xi^{d+1} \left( \frac{C}{1+r} - m \right) > 0 \right] > 0 . \tag{2.13}$$

For $P^* \in \mathcal{P}$ with $\mathbb{E}^* [C] < \infty$ we have

$$\mathbb{E}^* [\xi \cdot Y] = \xi \cdot \mathbb{E}^* [Y] = 0$$

since $\mathbb{E}^* [Y] = 0$ because of (2.4). Hence, by taking expectations in (2.12):

$$\xi^{d+1} \left( \mathbb{E}^* \left[ \frac{C}{1+r} \right] - m \right) \geq 0 . \tag{2.14}$$

Suppose $\xi^{d+1} = 0$. Then (2.12) yields $\xi \cdot Y \geq 0$ $P$–a.s. Because the original model is arbitrage–free, this implies $\xi \cdot Y = 0$ $P$–a.s. (Lemma 2.1 (3)). But this contradicts (2.13). Therefore, $\xi^{d+1} \neq 0$. Since $\mathbb{E}^* \left[ \dfrac{C}{1+r} \right] - m > 0$, we conclude that strict inequality holds in (2.14) and that $\xi^{d+1} > 0$. The inequality (2.12) implies

$$\frac{1}{\xi^{d+1}}(\xi \cdot Y) \geq m - \frac{C}{1+r} \quad P\text{–a.s.}$$

Hence, by rearranging terms

$$m + \zeta \cdot Y \leqslant \frac{C}{1+r} \quad P\text{–a.s.}$$

where we have introduced the vector

$$\zeta := -\frac{1}{\xi^{d+1}}\xi \in \mathbb{R}^d .$$

Therefore $m \in M$. We thus have shown

$$m < \pi_{\downarrow}(C) \implies m \in M$$

and so

$$\pi_{\downarrow}(C) = \sup\{m < \pi_{\downarrow}(C)\} \leqslant \sup M$$

and so the inequalities in (2.11) are all equalities.

We finally have to show that $\sup M$ is indeed attained by some $m_\infty \in M$. The corresponding fact for $\pi^{\uparrow}(C)$ is proved in [**12**], p. 20. The proof in the case of $\pi_{\downarrow}(C)$ is analogous and goes through *mutatis mutandis.* QED

**Remark.** 1) For an interpretation of this theorem see the remark after Theorem 2.7 below.

2) The fact that $\sup M$ is attained by some $m_\infty \in M$ is, loosely speaking, due to the fact that membership of $M$ is defined by a non–strict inequality which is preserved under taking suprema. For a detailed proof that it is indeed the case, we need the notion of an *irredundant*, or *non–redundant*, market model:

**Definition.** *An arbitrage–free market model is called* irredundant *iff* $\overline{\xi} \cdot \overline{S} = 0$ *implies* $\overline{\xi} = 0$, *i.e. if the random variables* $S^0, \dots, S^d$ *are linearly independent.*

This is not a real restriction: If the market model is redundant, i.e. if there is a $\overline{S} \in \mathbb{R}^{d+1} \setminus \{0\}$ with $\overline{\xi} \cdot \overline{S} = 0$ I can express some $S^i$ linearly in terms of the other $S^j$ as $S^i = \sum_{j \neq i} (\xi^j/\xi^i) S^j$, and then, by taking expectations, $\pi^i = \sum_{j \neq i} (\xi^j/\xi^i)\pi^j$, and so I can pass to a reduced arbitrage–free market model where $S^i$ is a tradable contingent claim with unique arbitrage–free price $\pi^i$. Then one also has $Y^i = \sum_{j \neq i} (\xi^j/\xi^i) Y^j$. If we denote the quantities corresponding to the reduced model by a prime we have the two sets

$$M := \max\left\{ m \in [0,\infty) \;\middle|\; \exists \zeta \in \mathbb{R}^d : \; m + \zeta \cdot Y \leqslant \frac{C}{1+r} \; P\text{–}a.s. \right\}$$

and

$$M' := \max\left\{ m' \in [0,\infty) \;\middle|\; \exists \zeta' \in \mathbb{R}^{d-1} : \; m' + \zeta' \cdot Y' \leqslant \frac{C}{1+r} \; P\text{–}a.s. \right\}$$

with $Y' = (Y^1, \dots, Y^{i-1}, Y^{i+1}, \dots, Y^d)$. Then $M = M'$. Namely, suppose given $m \in M$; so we have $\zeta \in \mathbb{R}^d$ with $m + \zeta \cdot Y \leqslant C/(r+1)$ $P$–a.s. Then $\zeta \cdot Y = \zeta' \cdot Y'$ with $(\zeta')^j = \zeta^j + \zeta^i(\xi^j/\xi^i)$ for $j \neq i$; hence there is $\zeta' \in \mathbb{R}^{d-1}$ with $m + \zeta' \cdot Y' \leqslant C/(r+1)$ and so

$m \in M'$. Conversely, if $m' \in M'$, take $\zeta' \in \mathbb{R}^{d-1}$ with $m' + \zeta' \cdot Y' \leqslant C/(r+1)$ and put $\zeta := ((\zeta')^1, \dots, (\zeta')^{i-1}, 0, (\zeta')^i, \dots, (\zeta')^{d-1})$, then $\zeta' \cdot Y' = \zeta \cdot Y$ which shows $m' \in M$.

After finitely many steps I thus arrive at an irredundant equivalent arbitrage–free market model of which the given one is an arbitrage–free extension with the same set $M$.

So we may assume our market model is irredundant. Since $C \geq 0$, we have $-\infty < \sup M$. Let $(m_n)_{n\in\mathbb{N}}$ be a sequence in $M$ with $m_n \uparrow m_\infty := \sup M$. For each $n$ choose $\xi_n \in \mathbb{R}^d$ with

$$(*) \qquad m_n + \xi_n \cdot Y \leqslant \frac{C}{r+1} \quad P\text{–a.s.}$$

Then the sequence $(\xi_n)_{n\in\mathbb{N}}$ is bounded. For suppose it were not. After eventually passing to a subsequence, we may assume $\lim_{n\to\infty} \xi_n = \infty$ and $\|\xi_n\| \neq 0$ for all $n$. Consider the sequence $(\eta_n)_{n\in\mathbb{N}}$ with $\eta_n := \xi_n/\|\xi_n\|$; since this sequence is bounded, we may, after again eventually passing to a subsequence, assume that it converges to a vector $\eta$ which then has $\|\eta\| = 1$. On the other hand, by $(*)$ ,

$$\frac{m_n}{\xi_n} + \eta_n \cdot Y \leqslant \frac{\pi_\downarrow(C)}{\xi_n} + \eta_n \cdot Y \leqslant \frac{C}{\|\xi_n\|(r+1)} \quad P\text{–a.s.}$$

for all $n$. Passing to the limit $n \to \infty$ yields $\eta \cdot Y \leqslant 0$ or $(-\eta) \cdot Y \geq 0$, hence by arbitrage–freeness $(-\eta) \cdot Y = 0$ and so by irredundacy $-\eta = 0$, which contradicts $\|\eta\| = 1$. Therefore the sequence $(\xi_n)_{n\in\mathbb{N}}$ *is* bounded, and so we may, after eventually passing to a subsequence, assume it is convergent to a vector $\xi$, say. From $(*)$ we then get in the limit $n \to \infty$

$$m + \xi \cdot Y \leqslant \frac{C}{r+1} \quad P\text{–a.s.}$$

which shows that indeed $\sup M = m_\infty \in M$.

The FFToAP (Theorem 2.2) settled the question of existence of risk–neutral measures. Our next aim is to settle the question of uniqueness, which will lead to the SFToAP (Secound Fundamental Theorem of Asset Pricing).

**Definition.** *A contingent claim $C$ is* attainable *(or* replicable*) if there is a portfolio $\overline{\xi} \in \mathbb{R}^{d+1}$ such that $C = \overline{\xi} \cdot \overline{S}$ $P$–a.s. Then $\overline{\xi}$ is called a* replicating portfolio.

We next show that in arbitrage–free markets attainable claims have a unique arbitrage–free price.

**Theorem 2.7.** *Let the (original) market be arbitrage–free, $C$ a contingent claim. Then*

*(1) if $C$ is attainable, there is a unique arbitrage–free price; in fact,*

$$(2.15) \qquad \Pi(C) = \left\{\overline{\xi} \cdot \overline{\pi}\right\}$$

*where $\overline{\xi}$ is a replicating portfolio. In particular, $\pi_\downarrow(C) = \pi^\uparrow(C)$ is the unique arbitrage–free price for $C$ ;*

*(2) if $C$ is not attainable, then*

$$(2.16) \qquad \pi_\downarrow(C) < \pi^\uparrow(C)$$

*and*

$$\Pi(C) = (\pi_{\downarrow}(C), \pi^{\uparrow}(C)) \,. \tag{2.17}$$

*In particular, neither $\pi_{\downarrow}(C)$ nor $\pi^{\uparrow}(C)$ is an arbitrage–free price for $C$.*

Proof. (1) is in fact quite simple. Let $\overline{\xi}$ be a replicating portfolio for $C$, so that $C = \overline{\xi} \cdot \overline{S}$. Then for $P^* \in \mathcal{P}$

$$\mathbb{E}^*\left[\frac{C}{1+r}\right] = \mathbb{E}^*\left[\frac{\overline{\xi} \cdot \overline{S}}{1+r}\right] = \overline{\xi} \cdot \mathbb{E}^*\left[\frac{\overline{S}}{1+r}\right] = \overline{\xi} \cdot \overline{\pi}$$

independent of $\mathcal{P}^*$. (Note that, in particular, this result implies that the arbitrage–free price of an attainable claim $C$ is independent of the choice of a replicating portfolio. This is readily confirmed: Let $\overline{\zeta}$ be another replicating portfolio for $C$, so that $C = \overline{\zeta} \cdot \overline{S} = \overline{\xi} \cdot \overline{S}$ and so $(\overline{\xi} - \overline{\zeta}) \cdot \overline{S} = 0$. Then for $P^* \in \mathcal{P}$ by taking expectations

$$0 = \mathbb{E}^*\left[(\overline{\xi} - \overline{\zeta}) \cdot \overline{S}\right] = (\overline{\xi} - \overline{\zeta}) \cdot \mathbb{E}^*\left[\overline{S}\right] = (1+r)(\overline{\xi} - \overline{\zeta}) \cdot \overline{\pi}$$

and so $\overline{\xi} \cdot \overline{\pi} = \overline{\zeta} \cdot \overline{\pi}$.)

(2) is more involved. Notice that for any contingent claim the set $\Pi(C)$ is an interval (this follows from Theorem 2.5: Since convex combinations of risk–neutral measures are again risk–neutral measures, $\Pi(C)$ is convex). Now suppose $C$ is not attainable. We show that $\pi_{\downarrow} := \pi_{\downarrow}(C)$, $\pi^{\uparrow} := \pi^{\uparrow}(C) \notin \Pi(C)$; this then implies $\Pi(C) = (\pi_{\downarrow}, \pi^{\uparrow})$.

By Theorem 2.6 there exists $\xi \in \mathbb{R}^d$ such that

$$\pi_{\downarrow} + \xi \cdot Y \leqslant \frac{C}{1+r} \quad P\text{–a.s.}$$

Assume that

$$\pi_{\downarrow} + \xi \cdot Y = \frac{C}{1+r} \quad P\text{–a.s.}\,,$$

i.e. that

$$P\left[\pi_{\downarrow} + \xi \cdot Y = \frac{C}{1+r}\right] = 1\,;$$

we will derive a contradiction. Let

$$\overline{\xi} := (\pi_{\downarrow} - \xi \cdot \pi, \xi) \in \mathbb{R}^{d+1}\,,$$

then

$$\overline{\xi} \cdot \overline{S} = (\pi_{\downarrow} - \xi \cdot \pi)(1+r) + \xi \cdot S = (1+r)(\pi_{\downarrow} + \xi \cdot Y) = C \quad P\text{–a.s.}$$

which means that $C$ is attainable, in contradiction to the assumption. Hence

$$P\left[\pi_{\downarrow} + \xi \cdot Y < \frac{C}{1+r}\right] > 0\,. \tag{2.18}$$

But this allows to construct an arbitrage opportunity: Let

$$\zeta := (\xi \cdot \pi - \pi_{\downarrow}, -\xi, 1) \in \mathbb{R}^{d+2};$$

then $\zeta$ is an arbitrage opportunity in the extended market with $\pi^{d+1} := \pi_{\downarrow}$ and $S^{d+1} := C$. For this, let us check the three defining properties of an arbitrage opportunity:

$$\zeta \cdot (\overline{\pi}, \pi^{d+1}) = \xi \cdot \pi - \pi_{\downarrow} - \xi \cdot \pi + \pi_{\downarrow} = 0,$$

so the first property is OK. Further,

$$\begin{aligned}\zeta \cdot (\overline{S}, C) &= (\xi \cdot \pi - \pi_{\downarrow})(1+r) - \xi \cdot S + C \\ &= (1+r)\left(-\pi_{\downarrow} - \xi \cdot Y + \frac{C}{1+r}\right) \\ &\geq 0 \quad P\text{–a.s.}\end{aligned}$$

which accounts for the second property. Further

$$P\left[\zeta \cdot (\overline{S}, C) > 0\right] = (1+r)P\left[-\pi_{\downarrow} - \xi \cdot Y + \frac{C}{1+r}\right] > 0$$

because of (2.18). So everything is OK and $\zeta$ is an arbitrage opportunity, as claimed.

This shows $\pi_{\downarrow}$ is not an arbitrage–free price, i.e. $\pi_{\downarrow} \notin \Pi(C)$. Similarly, one shows that $\pi_{\downarrow} \notin \Pi(C)$. QED

**Remark**. With these definitions and results it is possible to throw some light on the meaning of Theorem 2.6. For given $m \in [0, \infty)$ consider $m + \xi \cdot Y$ as a contingent claim, the outcome of an initial investment $m$ plus the result of subsequent trading according to a strategy manifesting itself in the portfolio $\xi$. It has the unique arbitrage–free price $m$. The assertions of Theorem 2.6 are then described in [**12**] as follows (Remark 1.32):

> *[Theorem 2.6]* (numbering here and subsequently adapted to our text B.M.) *shows that $\pi^{\uparrow}(C)$ is the lowest possible price of a portfolio $\overline{\xi}$ with*
>
> $$\overline{\xi} \cdot \overline{S} \geq C \quad P\text{–a.s.}$$
>
> *Such a portfolio is often called a "superhedging strategy" or "superreplication" of C, and the identities for $\pi_{\downarrow}(C)$ and $\pi^{\uparrow}(C)$ obtained in [Theorem 2.6] are often called* superhedging duality relations. *When using $\overline{\xi}$, the* seller *of C would be protected against any possible future claims of the buyer of C. Thus, a natural goal for the seller would be to finance such a superhedging strategy from the proceeds of C. Conversely, the objective of the* buyer *would be to cover the price of C from the sale of a portfolio $\overline{\eta}$ with*
>
> $$\overline{\eta} \cdot \overline{S} \leqslant C \quad P\text{–a.s.}$$
>
> *which is possible if and only if $\overline{\pi} \cdot \overline{\eta} \leqslant \pi_{\downarrow}(C)$. Unless C is an attainable payoff, however, neither objective can be fulfilled by trading C at an arbitrage-free price, as shown in [Theorem 2.7 above]. Thus, any arbitrage-free price involves a trade-off between these two objectives.*

**Definition.** *An arbitrage–free market model is called* complete *iff every contingent claim is attainable.*

This is a very strong property, but we will see that it holds in some important models (binomial model, BLACK–SCHOLES model).

Observe that we always have for the set $\mathcal{V}$ of possible portfolio values:

$$\mathcal{V} = \left\{ \overline{\xi} \cdot \overline{S} \;\middle|\; \overline{\xi} \in \mathbb{R}^{d+1} \right\} \subseteq L^1(\Omega, \sigma(\overline{S}), P^*)) \subseteq L^1(\Omega, \mathcal{F}, P^*) = L^1(\Omega, \mathcal{F}, P)\,,$$

the last equality because of $P^* \sim P$. In general, the inclusions are strict, but we will see that in complete markets these are equalities. We need an auxiliary result.

**Definition.** *An event $A \in \mathcal{F}$ is called an* atom (in $\mathcal{F}$) *iff $P[A] > 0$, and for all $B \in \mathcal{F}$, $B \subseteq A$ either $P[B] = 0$ or $P[B] = P[A]$.*

**Lemma 2.8.** *For all $p \in [0, \infty]$*

$$\begin{aligned}&\dim L^p(\Omega, \mathcal{F}, P)\\ &\qquad = \sup \{ n \in \mathbb{N} \mid \exists A_1, \ldots, A_n \text{ partition of } \Omega \ \forall i : \ A_i \in \mathcal{F}\,,\ P[A_i] > 0 \}\,.\end{aligned}$$

PROOF. 1. Let $A_1, \ldots, A_n$ be a partition of $\Omega$ with $A_i \in \mathcal{F}$ $P[A_i] > 0$ for all $i$. Then the characteristic functions $\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}$ are linearly independent in $L^p(\Omega, \mathcal{F}, P)$ for all $p$, and hence $\dim L^p(\Omega, \mathcal{F}, P) \geq n$.

2. If the RHS $= \infty$ then there is nothing to show. So we may assume the RHS is finite, $n_0$, say. So $n_0$ is the supremum. Then there exists a partition $A_1, \ldots, A_{n_0}$ with $A_i \in \mathcal{F}$ and $P[A_i] > 0$ for all $i$. Since $n_0$ is maximal, all the $A_i$ must be atoms, otherwise the partition could be strictly refined. Moreover, any $Z \in L^p(\Omega, \mathcal{F}, P)$ is constant on $A_i$ $P$-a.s. for all $i$, hence

$$Z = \sum_{i=1}^{n_0} c_i \mathbb{1}_{A_i}$$

with $c_i$ the $P$–a.s. constant value of $Z|A_i$. This means that $\mathbb{1}_{A_1}, \ldots, \mathbb{1}_{A_n}$ is a basis of $L^p(\Omega, \mathcal{F}, P)$. QED

Now we are ready for

**Theorem 2.9.** ("Second Fundamental Theorem of Asset Pricing", SFToAP) *An arbitrage–free market model is complete iff there exists exactly one risk–neutral measure, i.e. iff $|\mathcal{P}| = 1$. In this case,* $\dim L^0(\Omega, \mathcal{F}, P) \leqslant d + 1$.

PROOF. " $\Longrightarrow$ ": Let $A \in \mathcal{F}$. Take $C := \mathbb{1}_A$. Since the market model is complete, $C$ is attainable, say with replicating strategy $\overline{\xi}$. Then, by Theorem 2.7, (1)

$$P^*[A] = \mathbb{E}^*[C] = (1 + r)\overline{\xi} \cdot \overline{\pi}$$

independent of $P^*$. So any two risk–neutral measures are equal.

" $\Longleftarrow$ ": Let $|\mathcal{P}| = 1$ . Let $C$ be a bounded contingent claim. Then $\Pi(C)$ contains only one element, and Theorem 2.7, (2) does not hold. Thus, $C$ is attainable. This implies

$$L^\infty(\Omega, \mathcal{F}, P) \subseteq \mathcal{V}$$

and so

$$\dim L^\infty(\Omega, \mathcal{F}, P) \leqslant d + 1 \,.$$

Lemma 2.8 implies that there are at most $d+1$ atoms in $\mathcal{F}$ , and consequently $L^\infty(\Omega, \mathcal{F}, P) = L^0(\Omega, \mathcal{F}, P)$ i.e. any random variable is bounded.

So any contingent claim is attainable. QED

**Remark** 2.10. Notice that in the proof of " $\Longrightarrow$ " we needed only that every *bounded* contingent claim is attainable. Therefore, a market is complete iff every *bounded* contingent claim is attainable. □

So this was the one–period model. Let us now generalize it to the multi–period model.

CHAPTER 3

# The Multi–Period (Asset Price) Model

Topics of this section are:

- fundamental theorems for the multi–period model;
- dynamic hedging of financial derivatives.

Literature: [**12**], Chapter 5.

First, we recall some definitions from Stochastics. We fix a time horizon $T \in \mathbb{N}$ and a probability space $(\Omega, \mathcal{F}, P)$.

**Definition.** *A family* $(\mathcal{F}_t)_{t\in\{0,1,\dots,T\}}$ *of* $\sigma$*–algebras in* $\mathcal{F}$ *is called a* filtration *if*

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_T \subseteq \mathcal{F},$$

*i.e. if each* $\mathcal{F}_t$ *is a* $\sigma$*–subalgebra of* $\mathcal{F}$ *for all* $t \in \{0,1,\dots,T\}$ *and* $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ *for all* $t \in \{0,1,\dots,T-1\}$. *In this case,* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ *is called a* filtered propbability space.

**Definition.** *Let* $(E, \mathcal{E})$ *be a measurable space. A family of random variables* $X = (X_t)_{t\in\{0,1,\dots,T\}}$ *with values in* $E$ *(i.e. measurable maps* $X_t : \Omega \longrightarrow E$*) is called a* stochastic process.

**Definition.** *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ *be a filtered propbability space. A stochastic process* $(X_t)_{t\in\{0,1,\dots,T\}}$ *is called* adapted *if for all* $t \in \{0,1,\dots,T\}$ $X_t$ *is* $\mathcal{F}_t$*–measurable.*

*A stochastic process* $(Y_t)_{t\in\{1,\dots,T\}}$ *is called* predictable *if* $Y_t$ *is* $\mathcal{F}_{t-1}$*–measurable for all* $t \in \{1,\dots,T\}$.

## The Multi-Period Model

We consider a financial market with one non–risky asset and $d$ risky assets. Trading times are $0,1,\dots,T$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a filtered propbability space. For all $1 \leqslant i \leqslant d$ and $t \in \{0,1,\dots,T\}$ let $S_t^i$ be a non–negative random variable on $(\Omega, \mathcal{F})$. We interpret $S_t^i$ as the price of asset $i$ at time $t$. The price of the non–risky asset is supposed to be

$$S_t^0 = (1+r)^t \quad , \quad t \in \{0,1,\dots,T\}$$

(which is a simplification, but a very convenient one), where $r$ is the interest rate.

The $\mathbb{R}^{d+1}$–valued stochastic process

$$\overline{S}_t = (S_t^0, S_t) = (S_t^0, \dots, S_t^d) \quad , \quad t \in \{0, 1, \dots, T\}$$

will be refered to as the *price process.* The filtration $(\mathcal{F}_t)$ is considered as the *information flow.* Throughout we assume that $(\overline{S}_t)$ is adapted to $(\mathcal{F}_t)$.

To simplify notation we further assume that $\mathcal{F}_0$ is trivial, i.e. for all $A \in \mathcal{F}_0$ we have $P[A] = 0$ or $P[A] = 1$.

By a *trading strategy* (or *portfolio strategy*) we mean any $\mathbb{R}^{d+1}$–valued predictable stochastic process

$$\overline{\xi}_t = (\xi_t^0, \xi_t) = (\xi_t^0, \dots, \xi_t^d) \quad , \quad t \in \{1, \dots, T\} \, .$$

*Interpretation*

| | | |
|---|---|---|
| $\xi_t^i$ | $=$ | # shares of asset $i$ in the investor's portfolio between $t-1$ and $t$; |
| $\xi_t^i S_{t-1}^i$ | $=$ | value of position in asset $i$ at time $t-1$ (after remixing the portfolio); |
| $\xi_t^i S_t^i$ | $=$ | value of position in asset $i$ at time $t$ (before remixing). |

Value of the portfolio $\overline{\xi}_t$

at time $t-1$ : $\overline{\xi}_t \cdot \overline{S}_{t-1} = \sum_{i=0}^d \xi_t^i S_{t-1}^i$ (after remixing);
at time $t$ : $\overline{\xi}_t \cdot \overline{S}_t = \sum_{i=0}^d \xi_t^i S_t^i$ (before remixing).

In other words, the value of the portfolio at time $t-1$ after all investments at time $t-1$ are done is $\overline{\xi}_t \cdot \overline{S}_{t-1}$, which changes in the period from $t-1$ to $t$ into $\overline{\xi}_t \cdot \overline{S}_t$, upon which the portfolio is rearranged at time $t$ with resulting value $\overline{\xi}_{t+1} \cdot \overline{S}_t$.

**Definition.** *A trading strategy* $\overline{\xi}$ *is called* self–financing *if*

$$\overline{\xi}_t \cdot \overline{S}_t = \overline{\xi}_{t+1} \cdot \overline{S}_t \quad , \quad 1 \leqslant t \leqslant T$$

*"Remix the portfolio without changing its value".*

*The* value process *of a trading strategy is defined by*

$$V_0 := \overline{\xi}_1 \cdot \overline{S}_0 \quad \text{and} \quad V_t := \overline{\xi}_t \cdot \overline{S}_t \, , \; 1 \leqslant t \leqslant T \, .$$

$V_0$ *is sometimes refered to as the* initial capital.

**Lemma 3.1.** *A self–financing strategy* $\overline{\xi} = (\xi^0, \xi)$ *with value process* $V = (V_t)$ *is uniquely determined by* $\xi$ *and the initial capital* $V_0$.

Proof. For any strategy $\overline{\xi}$ we have $V_0 = \overline{\xi}_1 \cdot \overline{S}_0$. You can rewrite this equation:

$$V_0 = \xi_1^0 + \xi_1 \cdot S_0$$

and so

$$\xi_1^0 = V_0 - \xi_1 \cdot S_0 \, .$$

Suppose $\overline{\xi}_{t-1}$, and hence $V_{t-1}$, are already determined. The self–financing condition mplies

$$V_{t-1} = \overline{\xi}_{t-1} \cdot \overline{S}_{t-1} = \overline{\xi}_t \cdot \overline{S}_{t-1} = \xi_t^0 (1+r)^{t-1} + \xi_{t-1} \cdot S_{t-1}$$

and so

$$\xi_t^0 = \frac{1}{(1+r)^{t-1}} (V_{t-1} - \xi_{t-1} \cdot S_{t-1}) .$$

QED

**Definition.** *A self–financing trading strategy is called an* arbitrage opportunity *if the associated value process $V = (V_t)$ satisfies*

*(i)* $V_0 \leqslant 0 \quad P$*–a.s.;*
*(ii)* $V_T \geq 0 \quad P$*–a.s.;*
*(iii)* $P[V_T > 0] > 0$.

Next we show that the market is arbitrage–free, i.e. there are no arbitrage opportunities in the sense of the above definition) iff there are no arbitrage opportunities for each single trading period.

**Lemma 3.2.** *The following statements are equivalent:*

*(a) there exists an arbitrage opportunity;*
*(b) there exists a time point $t$, $1 \leqslant t \leqslant T$ , and $\eta \in (L^0(\Omega, \mathcal{F}, (\mathcal{F}_t), P))^d$ such that*

$$\eta \cdot S_t \geq (1+r)\eta \cdot S_{t-1} \quad P\text{–a.s.}$$

*and*

$$P[\eta \cdot S_t > (1+r)\eta \cdot S_{t-1}] > 0;$$

*(c) same as (b) with $\eta \in (L^\infty(\Omega, \mathcal{F}, (\mathcal{F}_t), P))^d$*
*(d) there exists an arbitrage opportunity $\overline{\xi} = (\xi^0, \xi)$ with bounded $\xi$ and $V_0 = 0$.*

PROOF. *(a) $\Longrightarrow$ (b):* Let $\overline{\xi}$ an arbitrage opportunity. Then $V_0 \leqslant 0$, $V_T \geq 0$, and $P[V_t > 0] > 0$. Let

$$t := \min\{ k \mid V_k \geq \quad P\text{–a.s. and } P[V_k > 0] > 0 \} .$$

Observe that $t \in \{1, \ldots, T\}$ and that $V_{t-1} \leqslant 0 \quad P$–a.s. or $P[V_{t-1} < 0] > 0$.

*First case:* $V_{t-1} \leqslant 0 \quad P$–a.s. Then

$$\overline{\xi}_t \cdot \overline{S}_{t-1} = \overline{\xi}_{t-1} \cdot \overline{S}_{t-1} = V_{t-1} \leqslant 0 \quad P\text{–a.s.}$$

hence

$$\underbrace{\overline{\xi}_t \cdot \overline{S}_t}_{\geq 0} \geq (1+r) \underbrace{\overline{\xi}_t \cdot \overline{S}_{t-1}}_{\leqslant 0} \quad P\text{–a.s.}$$

and

$$P\left[\overline{\xi}_t \cdot \overline{S}_t > (1+r)\overline{\xi}_t \cdot \overline{S}_{t-1}\right] \,>\, 0\,.$$

Now you can forget bars: the first terms of $\overline{\xi}_t \cdot \overline{S}_t$ and $(1+r)\overline{\xi}_t \cdot \overline{S}_{t-1}$ coincide, since $S^0_t = (1+r)S_{t-1}$ . Thus, there comes

$$\xi_t \cdot S_t \geq (1+r)\xi_t \cdot S_{t-1} \quad P\text{–a.s.}$$

and

$$P\left[\xi_t \cdot S_t > (1+r)\xi_t \cdot S_{t-1}\right] \,>\, 0\,,$$

and so $\eta := \xi_t$ does the job.

*Second case:* $P\left[V_{t-1} < 0\right] \,>\, 0$ . Let $\overline{\eta} = (\eta^0, \eta) := \overline{\xi}_t \mathbb{1}_{\{V_{t-1}<0\}}$ .. Then

$$\begin{aligned}\overline{\eta} \cdot \overline{S}_{t-1} &= V_{t-1}\mathbb{1}_{\{V_{t-1}<0\}} \quad \text{by the self–financing condition}\\ &\leqslant 0 \quad P\text{–a.s.}\end{aligned}$$

and $P\left[\overline{\eta} \cdot \overline{S}_{t-1} < 0\right] \,>\, 0$ . We further observe

$$\overline{\eta} \cdot \overline{S}_t = V_t \mathbb{1}_{\{V_{t-1}<0\}} \geq 0$$

which implies

$$\overline{\eta} \cdot \overline{S}_t \geq (1+r)\overline{\eta} \cdot \overline{S}_{t-1} \quad P\text{–a.s.}$$

and

$$P\left[\overline{\eta} \cdot \overline{S}_t > (1+r)\overline{\eta} \cdot \overline{S}_{t-1}\right] \,>\, 0\,,$$

Again, one may forget about the bars, as above.

*(b)* $\implies$ *(c):* Let $\eta$ be as in (b), and define

$$\eta^{(n)} := \eta \mathbb{1}_{\{|\eta| \leqslant n\}} \quad , \quad n \in \mathbb{N}\,.$$

Then

$$\eta^{(n)} \cdot S_t \geq (1+r)\eta^{(n)} \cdot S_{t-1} \quad P\text{–a.s.}$$

Choosing $n$ large enough yields the result, since

$$\lim_{n\to\infty} P\left[\eta^{(n)} \cdot S_t > (1+r)\eta^{(n)} \cdot S_{t-1}\right] = P\left[\eta \cdot S_t > (1+r)\eta \cdot S_{t-1}\right]$$

simply by the $\sigma$–additivity of $P$ .

*(c)* $\implies$ *(d):* Let $t$ and $\eta$ be as in (c). Define a strategy $\xi$ by

$$\xi_t := \eta \quad , \quad \xi_k := 0 \quad \text{for } k \neq t\,.$$

By Lemma 3.1, $\xi$ and $V_0$ uniquely determine a self–financing strategy.$\overline{\xi} = (\xi^0, \xi)$ . It is straightforward to show that $\overline{\xi}$ is an arbitrage opportunity.

*(c)* $\implies$ *(a):* This is trivial QED

We now turn to risk–neutral measures; these will be those measures which turn the price–processes into *martingales.* Recall their definition:

**Definition.** *Let* $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ *be a filtered probabbility space. A stochastic process* $M = (M)_{t\in\{0,\ldots,T\}}$ *is called a* martingale (w.r.t. $Q$ and $(\mathcal{F}_t)$) *if*

*(1)* $M$ *is adapted;*
*(2)* $\mathbb{E}^Q[|M_t|] < \infty$*; for all* $t \in \{0, \dots, T\}$ *;*
*(3)* $\mathbb{E}^Q[M_t|\mathcal{F}_s] = M_s$ *for all* $0 \leqslant s \leqslant t \leqslant T$ *.*

**Remark** 3.3. In the definition of martingales, one can replace (3) with the property

*(3')* $\mathbb{E}^Q[M_t|\mathcal{F}_{t-1}] = M_{t-1}$ for all $1 \leqslant t \leqslant T$.

Proof. *(3)* $\implies$ *(3'):* clear.

*(3')* $\implies$ *(3):* Let $0 \leqslant s < t \leqslant T$. By the tower property of conditional expectations

$$\mathbb{E}^Q[M_t|\mathcal{F}_s] = \mathbb{E}^Q\left[\mathbb{E}^Q[M_t|\mathcal{F}_{t-1}]\,|\mathcal{F}_s\right] = \mathbb{E}^Q[M_{t-1}|\mathcal{F}_s]$$

and the claim follows by induction.

**Definition.** *A probability measure* $Q$ *on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ *is called a* risk–neutral (*or* martingale measure) *if it is trivial on* $\mathcal{F}_0$ *and if the discounted price processes* $S_t^i/S_t^0$, $1 \leqslant i \leqslant d$ *, are martingales w.r.t.* $Q$ *.*

*A martingale measure* $P^*$ *is called an* equivalent martingale measure (EMM) *if* $P^* \sim P$ *. The set of all EMMs will be denoted by* $\mathcal{P}$ *.*

We next study value processes under martingale measures. We start with the following

**Lemma 3.4.** *Let* $\overline{\xi} = (\xi^0, \xi)$ *be a self–financing trade strategy. Then the associated discounted value process* $D_t := V_t/S_t^0$ *satisfies*

$$D_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot \left(\frac{S_k}{S_k^0} - \frac{S_{k-1}}{S_{k-1}^0}\right) = D_{t-1} + \xi_t \cdot \left(\frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0}\right).$$

*In particular,* $D_t$ *depends only on* $\xi$ *and* $V_0$ *.*

Proof. Write $D_t$ as a telescoping sum

$$\begin{aligned} D_t &= (D_t - D_{t-1}) + (D_{t-1} - D_{t-2}) + \cdots + (D_1 - D_0) + D_0 \\ &= \left(\overline{\xi}_t \cdot \frac{S_t}{S_t^0} - \overline{\xi}_{t-1} \cdot \frac{S_{t-1}}{S_{t-1}^0}\right) + \cdots + V_0 \\ &= \left(\overline{\xi}_t \cdot \frac{S_t}{S_t^0} - \overline{\xi}_t \cdot \frac{S_{t-1}}{S_{t-1}^0}\right) + \cdots + V_0 \end{aligned}$$

because $\overline{\xi}$ is self–financing

$$= \overline{\xi}_t \cdot \left(\frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0}\right) + \cdots + V_0 .$$

which accounts for the first equality. The second equality is a direct consequence of the first one. QED

**Theorem 3.5.** *Let $Q$ be a probability measure, trivial on $\mathcal{F}_0$. Suppose that $\mathbb{E}^Q[|S_t|] < \infty$ for all $0 \leqslant t \leqslant T$. Then the following statements are equivalent:*

*(a) $Q$ is a martingale measure;*
*(b) for all self–financing strategies $\overline{\xi} = (\xi^0, \xi)$ with $\xi$ bounded we have that the associated discounted process $D_t$ is a $Q$–martingale;*
*(c) for all self–financing strategies $\overline{\xi} = (\xi^0, \xi)$ with $\xi$ bounded we have that $\mathbb{E}^Q[D_T] = V_0$.*

Proof. *(a) $\Longrightarrow$ (b):* Let $|\xi| \leqslant C$. By Lemma 3.4 we have

$$D_t = V_0 + \sum_{k=1}^{t} \xi_k \cdot \left( \frac{S_k}{S_k^0} - \frac{S_{k-1}}{S_{k-1}^0} \right) .$$

By the triangle inequality

$$|D_t| = |V_0| + |C| \sum_{k=1}^{t} \left( \frac{|S_k|}{S_k^0} + \frac{|S_{k-1}|}{S_{k-1}^0} \right) ,$$

so since all $|S_k| \in L^1(Q)$, we obtain $D_t \in L^1(Q)$.

There remains to show the martingale property. We have, again by Lemma 3.4

$$D_{t+1} = D_t + \xi_{t+1} \cdot \left( \frac{S_{t+1}}{S_{t+1}^0} - \frac{S_t}{S_t^0} \right) ;$$

hence by taking conditional expectations w.r.t. $\mathcal{F}_t$

$$\begin{aligned} \mathbb{E}^Q[\, D_{t+1} |\, \mathcal{F}_t \,] &= D_t + \xi_{t+1} \cdot \left( \frac{S_{t+1}}{S_{t+1}^0} - \frac{S_t}{S_t^0} \right) ; \\ &= \mathbb{E}^Q[\, D_t |\, \mathcal{F}_t \,] + \mathbb{E}^Q\left[ \xi_{t+1} \cdot \left( \frac{S_{t+1}}{S_{t+1}^0} - \frac{S_t}{S_t^0} \right) \middle|\, \mathcal{F}_t \right] \\ &= \mathbb{E}^Q[\, D_t |\, \mathcal{F}_t \,] + \xi_{t+1} \cdot \mathbb{E}^Q\left[ \left( \frac{S_{t+1}}{S_{t+1}^0} - \frac{S_t}{S_t^0} \right) \middle|\, \mathcal{F}_t \right] . \end{aligned}$$

But $D_t$ is $\mathcal{F}_t$–measurable, hence $\mathbb{E}^Q[\, D_t |\, \mathcal{F}_t \,] = D_t$. The discounted price processes $S_t^t / S_t^0$ are, by assumption, martingales under $Q$, the second term vansihes, and we are left with

$$\mathbb{E}^Q[\, D_{t+1} |\, \mathcal{F}_t \,] = D_t$$

whence $D_t$ is a martingale under $Q$.

*(b) $\Longrightarrow$ (c):* By assumption, $Q$ is trivial on $\mathcal{F}_0$:

$$D_0 = \mathbb{E}^Q[\, D_T |\, \mathcal{F}_0 \,] = \mathbb{E}^Q[D_T] .$$

Since $D_0 = V_0$, we are done.

*(c)* $\Longrightarrow$ *(a):* We have to show that

$$\mathbb{E}^Q\left[\left.\frac{S_t}{S_t^0}\right|\ \mathcal{F}_{t-1}\right] = \frac{S_{t-1}}{S_{t-1}^0} \quad \text{for } 1 \leqslant i \leqslant d \quad \text{and } 1 \leqslant t \leqslant T\,.$$

Fix any $i$ with $1 \leqslant i \leqslant d$. Let $\eta \in L^\infty(\Omega, \mathcal{F}_{t-1}, Q)$ and define a strategy $\xi$ via

$$V_0 := 0$$

$$\xi_k^j = \begin{cases} \eta : & j = i \text{ and } k = t\,; \\ 0 : & \text{for all } j \neq i \text{ or } k \neq t\,. \end{cases}$$

Let

$$D_t = \eta\left(\frac{S_t^i}{S_t^0} - \frac{S_{t-1}^i}{S_{t-1}^0}\right)$$

be the associated discounted value process. By (c)

$$\mathbb{E}^Q\left[\eta\left(\frac{S_t^i}{S_t^0} - \frac{S_{t-1}^i}{S_{t-1}^0}\right)\right] = V_0 = 0\,.$$

This shows that

$$\mathbb{E}^Q\left[\left.\frac{S_t^i}{S_t^0}\right|\ \mathcal{F}_{t-1}\right] = \frac{S_{t-1}^i}{S_{t-1}^0}$$

(choose $\eta := \mathbb{1}_B$ , $B \in \mathcal{F}_{t-1}$). QED

We can now come to the main result of this section:

**Theorem 3.6.** (First Fundamental Theorem of Asset Pricing, FFToAP) *The multi–period market model is arbitrage free iff $\mathcal{P} \neq \emptyset$ , i.e. there exists an equivalent risk–neutral measure. In this case, there exists a $P^* \in \mathcal{P}$ such that $dP^*/dP$ is bounded.*

Proof. " $\Longleftarrow$ " (the simple direction.) Let $P^* \in P$ be a risk–neutral measure and let $\overline{\xi}$ be a self–financing strategy with $\xi$ bounded. Suppose $V_0 = 0$ and $V_T \geq 0$ $P$–a.s. According to Lemma 3.2 it is enough to show that $V_T = 0$ $P$–a.s.

Start with the following observation: $V_T \geq 0$ $P^*$–a.s. as well, since $P^* \sim P$ and so both measures have the same null sets. By Theorem 3.5 (c) $\mathbb{E}^*[V_T] = (1+r)^T\mathbb{E}^*[D_T] = (1+r)^T V_0 = 0$. Hence $V_T = 0$ $P^*$–a.s. and so $V_T = 0$ $P$–a.s., again because $P^* \sim P$ .

" $\Longrightarrow$ " (the hard direction.) A general proof can be found in **[12]** (our problem here is that in our approach the multi–period model is not just a succession of one–period models, since in our one–period model the start prices are deterministic, whereas **[12]** treats the more general case). We are going to prove this direction under the assumption that for each time $t$ , $0 \leqslant t \leqslant T$ , there exists a finite partition $\mathfrak{A}_t = \{A_1, A_2, \ldots\}$ of $\Omega$ where every $A_i$ is an atom in $\mathcal{F}_t$ . Notice that on each atom $A$ in $\mathcal{F}_t$ the price $S_t$ is $P$–a.s. constant.

i) Let $A$ be an atom in $\mathcal{F}_t$ ($t$ fixed). Let $\pi = \pi^A \in \mathbb{R}^d$ be such that $S_t = \pi$ $P$–a.s. Due to Lemma 3.2, any $\xi \in \mathbb{R}^d$ satisfies

$$\mathbb{1}_A \xi \cdot S_{t+1} \geq (1+r)\mathbb{1}_A \xi \cdot S_t \quad P\text{–a.s.}$$

Consequently, with $P^A[\bullet] := P[\bullet|A] := \dfrac{P[\bullet \cap A]}{P[A]}$:

$$\xi \cdot S_{t+1} \geq (1+r)\xi \cdot S_t \quad P^A\text{–a.s.}$$

and so

$$\xi \cdot S_{t+1} = (1+r)\xi \cdot S_t \quad P^A\text{–a.s.}\,,$$

since, as remarked above, the prices are $P$–a.s. constant on each atom.

By Lemma 2.1 we obtain that there are no arbitrage opportunities in the single–period model with probabilitiy measure $P^A$ and prices $\pi$ and $S_{t+1}$.

ii) By Theorem 2.2 (the First Fundamental Theorem of Asset Pricing in the single–period case) there exists a probability measure $\widetilde{P}_t^A \sim P^A$ such that

$$\mathbb{E}^{\widetilde{P}_t^A}[S_{t+1}] = (1+r)\pi \quad \text{and} \quad \frac{d\widetilde{P}_t^A}{dP} \text{ bounded.}$$

Our task is now to glue all these local measures together. For this, define, for each $t$, a stochastic kernel $\mu_t : \Omega \times \mathcal{F}_{t+1} \longrightarrow [0,1]$ by

$$\mu_t(\omega, B) := \sum_{A \in \mathfrak{A}_t} \mathbb{1}_A(\omega) \widetilde{P}_t^A[B]\,.$$

With these kernels we perform an iterative construction of a risk–neutral measure $P^* \sim P$ as follows.

Let $P_0^* := P$ on $\mathcal{F}_0$.

Suppose that $P_t^*$ is defined on $(\Omega, \mathcal{F}_t)$ such that

- $P_t^* \sim P$ on $\mathcal{F}_t$ with bounded density $\dfrac{dP_t^*}{dP}$;
- the discounted price processes $\left(\dfrac{S_k^i}{S_0^i}\right)_{0 \leqslant k \leqslant t}$ are $P_t^*$–martingales for all $1 \leqslant i \leqslant d$.

Define for $B \in \mathcal{F}_{t+1}$

$$P_{t+1}^*[B] := \int_\Omega \mu_t(\omega, B)\, dP_t^*$$

Then we have

- $P_{t+1}^* = P_t^*$ on $\mathcal{F}_t$;
- $P_{t+1}^* \sim P$ on $\mathcal{F}_{t+1}$ with bounded density $\dfrac{dP_{t+1}^*}{dP}$;

- the discounted price processes $\left(\dfrac{S_k^i}{S_0^i}\right)_{0\leqslant k\leqslant t+1}$ are $P_{t+1}^*$–martingales for all $1 \leqslant i \leqslant d$.

For the first point, let $B \in \mathcal{F}_t$; then

$$\mu_t(\omega, B) := \sum_{A\in\mathfrak{A}_t} \mathbb{1}_A(\omega)\widetilde{P}_t^A[B] = \sum_{\substack{A\in\mathfrak{A}_t:\\ P[A\cap B]=P[A]}} \mathbb{1}_A(\omega) = \mathbb{1}_B(\omega) \quad P_t^*\text{–a.s.},$$

which implies $P_{t+1}^*[B] = P_t^*[B]$.

To elucidate this rather condensed derivation, first note that for all events $B$ we have $P^A[B] = P^A[A\cap B]$, which is equivalent to $B\setminus A$ being a $P^A$–null set; moreover, since $\widetilde{P}_t^A$ is equivalent to $P^A$, it has the same null events and hence the same a.s.–events as $P^A$, and so for all events $B$ we have that $B\setminus A$ is also a $\widetilde{P}_t^A$–null set, hence $\widetilde{P}_t^A[B] = \widetilde{P}_t^A[A\cap B]$. For the same reasons, if $A$ is an atom in $\mathcal{F}_t$ w.r.t $P$, $A$ is also an atom in $\mathcal{F}_t$ w.r.t $\widetilde{P}_t^A$. Now, if $A$ is an atom in $\mathcal{F}_t$ w.r.t $P$, it follows that $P^A$ is deterministic on $\mathcal{F}_t$, i.e. for all $B \in \mathcal{F}_t$ one has either $P^A[B] = 0$ or $P^A[B] = 1$, and so $\widetilde{P}_t^A$ is deterministic on $\mathcal{F}_t$, too, so for all $B \in \mathcal{F}_t$ we have either $\widetilde{P}_t^A[B] = 0$ or $\widetilde{P}_t^A[B] = 1$, and $\widetilde{P}_t^A[B] = 1$ iff $P^A[B] = P[B|A] = 1$ iff $P[A\cap B] = P[A]$. There follows the second equality (the first one is a mere definition).

For the last equality put

$$C := \bigcup \{\, A \in \mathfrak{A}_t \mid \ P[A\cap B] = P[A] \,\};$$

the claim is then $P_t^*[\mathbb{1}_B \neq \mathbb{1}_C] = 0$. Now

$$\{\mathbb{1}_B \neq \mathbb{1}_C\} = \{\mathbb{1}_B = 1,\, \mathbb{1}_C = 0\} \cup \{\mathbb{1}_B = 0,\, \mathbb{1}_C = 1\} = (B\setminus C) \cup (C\setminus B)$$

Since $P_t^* \sim P$ on $\mathcal{F}_t$ by the induction hypothesis, and $\mathcal{F}_t$ is a $\sigma$–algebra, we have that $B$, $C$, and hence $B\setminus C$ and $C\setminus B$ all belong to $\mathcal{F}_t$, and it suffices to show that $P[B\setminus C] = P[C\setminus B] = 0$. This is equivalent to $P[C] = P[B\cap C] = P[B]$.

For simplicity, put $\mathfrak{A}_t' := \{\, A \in \mathfrak{A}_t \mid \ P[A\cap B] = P[A] \,\}$. Note that, by hypothesis, the $A \in \mathfrak{A}_t$ are mutually disjoint. Then

$$\begin{aligned} P[C] &= P\left[\bigcup_{A\in\mathfrak{A}_t'} A\right] = \sum_{A\in\mathfrak{A}_t'} P[A] = \sum_{A\in\mathfrak{A}_t'} P[A\cap B] \\ &= P\left[\bigcup_{A\in\mathfrak{A}_t'} (A\cap B)\right] = P\left[\left(\bigcup_{A\in\mathfrak{A}_t'} A\right)\cap B\right] = P[C\cap B]\,. \end{aligned}$$

Further note that, since either $P[A\cap B] = P[A]$ or $P[A\cap B] = 0$, we have $\sum_{A\in\mathfrak{A}_t\setminus\mathfrak{A}_t'} P[A\cap B] = 0$ and so

$$\begin{aligned} P[C\cap B] &= P\left[\left(\bigcup_{A\in\mathfrak{A}_t'} A\right)\cap B\right] = P\left[\bigcup_{A\in\mathfrak{A}_t'} (A\cap B)\right] = \sum_{A\in\mathfrak{A}_t'} P[A\cap B] \\ &= \sum_{A\in\mathfrak{A}_t} P[A\cap B] = P\left[\bigcup_{A\in\mathfrak{A}_t} (A\cap B)\right] = P\left[\left(\bigcup_{A\in\mathfrak{A}_t} A\right)\cap B\right] = P[B] \end{aligned}$$

and we are done.

The second point follows from the induction hypothesis and the definition of $P^*_{t+1}$.

Again one may feel the need for some more explanation. Let $B \in \mathcal{F}_{t+1}$; we then have to show that $P^*_{t+1}[B] = 0$ iff $P[B] = 0$. Now

$$P[B] = \sum_{A \in \mathfrak{A}_t} P[A \cap B] = \sum_{A \in \mathfrak{A}_t} P^A[B]\, P[A]$$

Since $P[A] > 0$ for all $A \in \mathfrak{A}$ by definition of an atom, we have that $P[B] = 0$ iff $P^A[B] = 0$ for all $A \in \mathfrak{A}_t$.

On the other hand,

$$\begin{aligned} P^*_{t+1}[B] &:= \int_\Omega \mu_t(\omega, B)\, dP^*_t = \sum_{A \in \mathfrak{A}_t} \int_\Omega \mathbb{1}_A \widetilde{P}^A_t[B]\; dP^*_t \\ &= \sum_{A \in \mathfrak{A}_t} \widetilde{P}^A_t[B] \int_\Omega \mathbb{1}_A\, dP^*_t = \sum_{A \in \mathfrak{A}_t} \widetilde{P}^A_t[B]\, P^*_t[A] \end{aligned}$$

Since $P^*_t \sim P$ on $\mathcal{F}_t$ by the induction hypothesis, we have $P^*_t[A] > 0$ for all $A \in \mathfrak{A}_t$. Hence $P^*_{t+1}[B] = 0$ iff $\widetilde{P}^A_t[B] = 0$ for all $A \in \mathfrak{A}_t$.

But $\widetilde{P}^A_t$ was chosen to be equivalent to $P^A$. Thus indeed $P^*_{t+1}[B] = 0$ iff $P[B] = 0$.

Finally, we can write

$$\begin{aligned} P^*_{t+1}[B] &= \sum_{A \in \mathfrak{A}_t} \widetilde{P}^A_t[B]\, P^*_t[A] = \sum_{A \in \mathfrak{A}_t} P^*_t[A] \int_\Omega \mathbb{1}_B\, d\widetilde{P}^A_t \\ &= \sum_{A \in \mathfrak{A}_t} P^*_t[A] \int_\Omega \mathbb{1}_B \frac{d\widetilde{P}^A_t}{dP^A}\, dP^A = \int_\Omega \left\{ \sum_{A \in \mathfrak{A}_t} P^*_t[A] \frac{d\widetilde{P}^A_t}{dP} \cdot \frac{1}{P[A]} \right\} \mathbb{1}_B\, dP \end{aligned}$$

So

$$\frac{dP^*_{t+1}}{dP} = \sum_{A \in \mathfrak{A}_t} P^*_t[A] \frac{d\widetilde{P}^A_t}{dP} \frac{1}{P[A]}$$

and we have to show this function on $\Omega$ is bounded. But, by assumption, the $A \in \mathfrak{A}_t$ form a partition of $\Omega$, so each $\omega \in \Omega$ is contained in exactly one $A$, and so it suffices to show that there is a common bound to all summands. Now, by the induction hypothesis, $\dfrac{dP^*_t}{dP}$ is bounded on all of $\Omega$, so there is $\beta \in \mathbb{R}_+$ such that $P^*_t[C] \leqslant \beta P[C]$ for all events $C$. Therefore,

$$\forall\, A \in \mathfrak{A}_t \;:\; P^*_t[A] \frac{d\widetilde{P}^A_t}{dP} \frac{1}{P[A]} \leqslant \beta \frac{d\widetilde{P}^A_t}{dP}$$

Each Radon–Nikodym derivative $d\widetilde{P}^A_t/dP$, $A \in \mathfrak{A}_t$, is bounded, and since by assumption there are only finitely many of them, they have a common bound, and we are done.

Now to the third and last point. For $B \in \mathcal{F}_t$ we compute

$$\begin{aligned} \mathbb{E}^{P^*_{t+1}}\left[ \mathbb{1}_B \frac{S^i_{t+1}}{S^0_{t+1}} \right] &= \frac{1}{S^0_{t+1}} \int \mathbb{1}_B(\omega') S^i_{t+1}(\omega')\, P^*_{t+1}[d\omega'] \\ &= \frac{1}{S^0_{t+1}} \iint \mathbb{1}_B(\omega') S^i_{t+1}(\omega') \mu_t(\omega, d\omega')\, P^*_t[d\omega] \end{aligned}$$

$$= \frac{1}{S^0_{t+1}} \iint \left\{ \sum_{A \in \mathfrak{A}(\mathcal{F}_t)} \mathbb{1}_B(\omega') S^i_{t+1}(\omega') \mathbb{1}_A(\omega) \widetilde{P}^A_t [d\omega'] \right\} P^*_t [d\omega]$$

$$= \frac{1}{S^0_{t+1}} \int \left\{ \sum_{A \in \mathfrak{A}(\mathcal{F}_t)} \mathbb{1}_A(\omega) \int \mathbb{1}_B(\omega') S^i_{t+1}(\omega') \widetilde{P}^A_t [d\omega'] \right\} P^*_t [d\omega]$$

$$= \frac{1}{S^0_{t+1}} \int \mathbb{1}_B(\omega) \left\{ \sum_{A \in \mathfrak{A}(\mathcal{F}_t)} \mathbb{1}_A(\omega) \mathbb{E}^{\widetilde{P}^A_t} \left[ S^i_{t+1} \right] \right\} P^*_t [d\omega]$$

as $\mathbb{1}_B$ is $\mathcal{F}_t$–measurable and so $P = \widetilde{P}^A_t$–a.s. constant on the $\mathcal{F}_t$–atoms $A$; hence it can be pulled out of the $\widetilde{P}^A_t$–integral

$$= \frac{1}{S^0_t} \int \mathbb{1}_B(\omega) \left\{ \sum_{A \in \mathfrak{A}(\mathcal{F}_t)} \mathbb{1}_A(\omega) \pi^{A,i} \right\} P^*_t [d\omega]$$

since by ii) above $\mathbb{E}^{\widetilde{P}^A_t} \left[ S^i_{t+1} \right] = (1+r)\pi^{A,i}$ and $S^0_{t+1} = (1+r) S^0_t$

$$= \frac{1}{S^0_t} \mathbb{E}^{P^*_t} \left[ \mathbb{1}_B S^i_t \right]$$

which establishes the desired martingale properties.

Now that the above three points are settled, performing the iteration over $t$ until $T$ implies that $P^* := P^*_T$ is an EMM. QED

**Definition.** *A random variable $C \geq 0$ on $(\Omega, \mathcal{F}, P)$ is called a* contingent claim.

We next illustrate this notion by giving some common examples of contingent claims which are traded on the market.

*Payoffs of frequently traded options:*

1) *Call option* on asset $i$ with *strike price* $K \in \mathbb{R}_+$ and *maturity* $T$ *:*

$$C^{\text{call}} = (S^i_T - K)^+ .$$

2) *Put option* on asset $i$ with *strike price* $K \in \mathbb{R}_+$ and *maturity* $T$ *:*

$$C^{\text{put}} = (K - S^i_T)^+ .$$

3) The payoff of an *Asion option* depends on the average price of the underlying asset

$$S^i_{\text{av}} = \frac{1}{T+1} \sum_{k=0}^{T} S^i_k .$$

Asian options offer protection against decreasing resp. increasing average prices:

| | | |
|---|---|---|
| Examples: | Asian call | $(S^i_{\text{av}} - K)^+$ ; |
| | Asian put | $(K - S^i_{\text{av}})^+$ . |

4) *Barrier option* The right to exercise the option is linked to whether the underlying crosses a *barrier level* before maturity.

Barrier options
- offer protection against more specific events;
- are cheaper than options without barriers.

We have 4 main (basic) types of barier options:

*up–and–in:* price of the underlying moves up and crosses an upper barrier $\Longrightarrow$ option is activated;
*up–and–out:* price of the underlying moves up and crosses an upper barrier $\Longrightarrow$ option is knocked out;

Note that sum of up–and–in + up–and–out = call.

*down–and–in:* price of the underlying moves down and crosses a lower barrier $\Longrightarrow$ option is activated;
*down–and–in:* price of the underlying moves down and crosses a lower barrier $\Longrightarrow$ option is knocked out.

Example: payoff of an up–and–in call:

$$C = \begin{cases} (S^i_T - K)^+ & \text{if } \max_{0\leqslant t\leqslant T} S^i_T \geq B \\ \quad 0 & \text{else.} \end{cases}$$

**Definition.** *A contingent claim is* attainable *if there exists a self–financing trading strategy $\overline{\xi}$ with $\overline{\xi}_T \cdot \overline{S}_T = C$ $P$–a.s. In this case, $\overline{\xi}$ is said to* replicate $C$ .

In the remainder of this section it is assumed that $\mathcal{P} \neq \emptyset$ , i.e. our market model is arbitrage–free.

**Definition.** *Let $C$ be a contingent claim. An adapted stochastic process $S^{d+1} = (S^{d+1}_t)_{t\in[0,T]}$ is called an* arbitrage–free price process of $C$ *if $S^{d+1}_T = C$ and the extended model $(S^0, S^1, \dots, S^{d+1})$ admits no arbitrage.*

**Proposition 3.7.** *Let $C$ be an attainable contingent claim, and let $V_t$ be the value process associated to the replicating strategy $\overline{\xi}$ . Then $V_t$ is the only arbitrage–free price of $C$ at time $t$ , $0 \leqslant t \leqslant T$ .*

Proof. (The idea of the proof is very clear once you think of it in economic terms.) Let $(S^{d+1}_t)$ be an arbitrage–free price process of $C$ . Assume $P\left[S^{d+1}_t > V_t\right] > 0$ ; we will derive a contradiction from this. Define a portfolio as follows: For $k > t$ let

$$\zeta^0_k \quad := \left(\xi^0_k + \frac{1}{S^0_t}\left(S^{d+1}_t - V_t\right)\right) \mathbb{1}_{\left\{S^{d+1}_t > V_t\right\}} ;$$

$$\begin{aligned}\zeta_k \quad &:= \xi_k \mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}} ;\\ \zeta_k^{d+1} &:= -\mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}} .\end{aligned}$$

For $k \leqslant t$ let

$$(\overline{\zeta}_k, \zeta_k^{d+1}) := 0$$

(i.e. if the price of the contingent claim is too high, you sell it and put the value into the riskless asset). Observe that $(\overline{\zeta}, \zeta^{d+1})$ is self–financing:

$$\begin{aligned}\overline{\zeta}_{t+1} \cdot \overline{S}_t + \zeta_{t+1}^{d+1} \cdot S_t^{d+1} &= \Big(\underbrace{\overline{\xi}_{t+1} \cdot \overline{S}_t}_{=\overline{\xi}_t \cdot \overline{S}_t = V_t} + \left(S_t^{d+1} - V_t\right) - S_t^{d+1}\Big) \mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}}\\ &= \Big(V_t + (S_t^{d+1} - V_t) - S_:t^{d+1}\Big) \mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}}\\ &= 0\\ &= \overline{\zeta}_t \cdot \overline{S}_t + \zeta_t^{d+1} \cdot S_t^{d+1} .\end{aligned}$$

After time $t$, $\overline{\zeta}$ remains self–financing since $\zeta^0$ is not further modified then. Moreover,

$$\begin{aligned}\overline{\zeta}_T \cdot \overline{S}_T + \zeta_T^{d+1} \cdot S_T^{d+1} &= \Big(\underbrace{\overline{\xi}_T \cdot \overline{S}_T}_{=C} + \frac{S_T^0}{S_t^0}\left(S_t^{d+1} - V_t\right) - C\Big) \mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}}\\ &= (1+r)^{T-t}\left(S_t^{d+1} - V_t\right) \mathbb{1}_{\left\{S_t^{d+1} > V_t\right\}}\\ &\geq 0 \text{, and } > 0 \text{ with positive probability.}\end{aligned}$$

Thus $(\overline{\zeta}, \zeta^{d+1})$ is an arbitrage opportunity in the extended market, a contradiction. So $S_t^{d+1} \leqslant V_t$ $P$–a.s.

Now we have to show the reverse inequality. This time the price of the contingent claim is too low. Similarly what has been done above, you now buy the contingent claim and so set up again an arbitrage opportunity, hence $S_t^{d+1} \geq V_t$ $P$–a.s. The following Proposition, then, together with the FFToAP implies that $S_t^{d+1}$ is indeed an arbitrage–free price process of $C$. QED

Next we show that the discounted arbitrage–free value process of $V = (V_t)$ is a martingale w.r.t. any EMM.

**Proposition 3.8.** *Let $C$ be an attainable claim. Then $C$ is integrable w.r.t. any EMM $P^* \in \mathcal{P}$. Moreover, the value process associated to the replicating strategy $\overline{\xi}$ satisfies*

$$V_t = \frac{1}{(1+r)^{T-t}} \mathbb{E}^*\left[C \middle| \mathcal{F}_t\right] \quad , \quad 0 \leqslant t \leqslant T .$$

*In particular, the discounted value process $D_t := V_t / S_t^0$ is a nonnegative $P^*$–martingale.*

Proof. Let $\overline{\xi} = (\xi^0, \xi)$ replicate $C$

1. Show via backward induction: $D_t \geq 0$ $P$–a.s. for all $0 \leqslant t \leqslant T$. First, $D_t = C/S_T^0 \geq 0$ $P$–a.s. Suppose $D_t \geq 0$ $P$–a.s. Then, by Lemma 3.4,

$$D_{t-1} = D_t - \xi_t \cdot \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) \geq -\xi_t \cdot \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) .$$

The idea is to take conditional expectations. Since we do not know if $\xi_t$ is integrable, we cut it off: Let

$$\xi_t^n := \xi_t \mathbb{1}_{\{|\xi| \leqslant n\}} \quad , \quad n \in \mathbb{N} .$$

Then $\xi_t^n$ is bounded, and

$$D_{t-1}\mathbb{1}_{\{|\xi| \leqslant n\}} = \mathbb{E}^* \left[ D_{t-1}\mathbb{1}_{\{|\xi| \leqslant n\}} \middle| \mathcal{F}_{t-1} \right] \geq -\xi_t^n \cdot \underbrace{\mathbb{E}^* \left[ \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) \middle| \mathcal{F}_{t-1} \right]}_{= 0 \quad P\text{–a.s. by martingale property}} .$$

Letting $n \uparrow \infty$ yields $D_{t-1} \geq 0$ $P$–a.s.

2. Show for $0 \leqslant t \leqslant T$: $\mathbb{E}^* \left[ D_t \middle| \mathcal{F}_{t-1} \right] = D_{t-1}$.

(Notice: Conditional expectations can be defined for nonnegative random variables that are not necessarily integrable by cutting off and taking limits via monotone convergence.)

On the event $\{|\xi_t| \leqslant n\}$ we have

$$\begin{aligned}
&\mathbb{E}^* \left[ \mathbb{1}_{\{|\xi| \leqslant n\}} D_t \middle| \mathcal{F}_{t-1} \right] - \mathbb{1}_{\{|\xi| \leqslant n\}} D_{t-1} \\
&\qquad = \mathbb{E}^* \left[ \xi_t^n \cdot \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) \middle| \mathcal{F}_{t-1} \right] \\
&\qquad = \xi_t^n \cdot \mathbb{E}^* \left[ \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) \middle| \mathcal{F}_{t-1} \right] \\
&\qquad = 0 .
\end{aligned}$$

Again, by letting $n \uparrow \infty$, we get $\mathbb{E}^* \left[ D_t \middle| \mathcal{F}_{t-1} \right] = D_{t-1}$.

3. Show intgrability by a forward recursion. First, $D_0 \in \mathbb{R}_+$ is the value of the replicating portfolio at time $t = 0$, so $D_0$ is integrable.

Let $D_t$ be integrable for some $t$ with $0 \leqslant t \leqslant T - 1$. Then

$$\mathbb{E}^* \left[ D_{t+1} \right] = \mathbb{E}^* \left[ \mathbb{E}^* \left[ D_{t+1} \middle| \mathcal{F}_t \right] \right] = \mathbb{E}^* \left[ D_t \right] .$$

Now 1. – 3. prove Proposition 3.8. QED

We now turn to contingent claims that are not necessarily attainable.

*Notation:* Let $\Pi(C)$ be the set of all $t = 0$ components of arbitrage–free price processes of $C$.

Since $\mathcal{F}_0$ is trivial, we can identify $\Pi(C)$ with a subset of $\mathbb{R}_+$. As in the one–period case, we denote

$$\pi_{\downarrow}(C) := \inf \Pi(C) \quad , \quad \pi^{\uparrow}(C) := \sup \Pi(C) . \tag{3.1}$$

**Theorem 3.9.** *We have*

$$\Pi(C) = \left\{ \mathbb{E}^* \left[ \frac{C}{S_T^0} \right] \;\middle|\; P^* \in \mathcal{P} \text{ s.t. } \mathbb{E}^* [C] < \infty \right\} . \tag{3.2}$$

*Moreover,* $\Pi(C) \neq \emptyset$ *, and*

$$\pi_{\downarrow}(C) = \inf_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{S_T^0} \right] \quad , \quad \pi^{\uparrow}(C) = \sup_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{S_T^0} \right] .$$

Proof. 1. "$\subseteq$": Let $\pi^C \in \Pi(C)$ an arbitrage–free price at time $t = 0$, i.e. there exists an arbitrage–free process $(S_t^{d+1})$ of $C$ with $S_0^{d+1} = \pi^C$. By Theorem 3.6, there exits an EMM $P^*$ of the model $(S^0, \ldots, S^{d+1})$. In particular, $P^*$ is an EMM for the non–extended model $(S^0, \ldots, S^d)$, so $P^* \in \mathcal{P}$, and $\mathbb{E}^* [C/S_t^0] = \pi^C$.

"$\supseteq$": Let $S_t^{d+1} := (1+r)^{-(T-t)} \mathbb{E}^* [\, C | \, \mathcal{F}_t ]$ for $P^* \in \mathcal{P}$ with $\mathbb{E}^* [C] < \infty$. Then $S_0^{d+1}/S_t^0$ is a $P^*$–martingale. So $P^*$ is an EMM for the extended model, and by Theorem 3.6, $(S_0, \ldots, S^{d+1})$ is arbitrage–free. Therefore, $\mathbb{E}^* [C/S_T^0] = S_0^{d+1} \in \Pi(C)$.

2. We now show that $\Pi(C) \neq \emptyset$. Let

$$\frac{d\widetilde{P}}{dP} := c \cdot \frac{1}{1+C} \quad \text{with} \quad c^{-1} := \mathbb{E}^* \left[ \frac{1}{1+C} \right] .$$

The non–extended model is arbitrage–free w.r.t $\widetilde{P}$ . By Theorem 3.6 there exists $P^* \in \mathcal{P}$ such that $dP^*/dP$ is bounded (the set $\mathcal{P}$, originally defined by the initial measure $P$ is equal to $\mathcal{P}$ defined by the measure $\widetilde{P}$, since only the null sets matter). Note that

$$\mathbb{E}^* [C] = \mathbb{E}^{\widetilde{P}} \left[ \frac{dP^*}{d\widetilde{P}} C \right] = \mathbb{E}^P \left[ \frac{dP^*}{d\widetilde{P}} \frac{cC}{1+C} \right] < \infty .$$

Hence $\mathbb{E}^* [C/S_T^0] \in \Pi(C)$.

3. The relation

$$\pi_{\downarrow}(C) = \inf_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{S_T^0} \right]$$

follows from (3.1). The relation

$$\pi^{\uparrow}(C) = \sup_{P^* \in \mathcal{P}} \mathbb{E}^* \left[ \frac{C}{S_T^0} \right]$$

needs a bit of work. If $\mathbb{E}^* [C] < \infty$ for all $P^* \in \mathcal{P}$ the statement follows again from (3.1).

Suppose $\mathbb{E}^\infty [C] = \infty$ for some $P^\infty \in \mathcal{P}$. We then have to show that $\pi^{\uparrow}(C) = \infty$.

Let $b \in \mathbb{R}_+$ and choose $n \in \mathbb{N}$ such that

$$\mathbb{E}^\infty [C \wedge n] > (1+r)^T b\,.$$

Now $C \wedge n$ is a contingent claim, and we let $S_t^{d+1}$ be its price process:

$$S_t^{d+1} := \frac{1}{(1+r)^{T-t}} \mathbb{E}^\infty [\, C \wedge n |\, \mathcal{F}_t\,]\,.$$

The model $(S^0, \dots, S^{d+1})$ possesses an EMM, namely $P^\infty$, and hence is arbitrage–free. Let $\widehat{P} \in \mathcal{P}$ with $\widehat{\mathbb{E}}[C] < \infty$, which exists since $\Pi(C) \neq \emptyset$. Since $P^\infty, \widehat{P} \in \mathcal{P}$, we have $P^\infty \sim \widehat{P}$. Then, by Theorem 3.6, there exists $P^* \sim \widehat{P}$ with $dP^*/d\widehat{P}$ bounded and such that $P^*$ is an EMM for the extenden model $(S^0, \dots, S^{d+1})$. The boundedness of $dP^*/d\widehat{P}$ guarantees $\mathbb{E}^*[C] < \infty$. This implies that $\pi := \mathbb{E}^*[C/S_T^0]$ belongs to $\Pi(C)$. Finally,

$$\pi = \frac{1}{S_T^0}\mathbb{E}^*[C] > \frac{1}{S_T^0}\mathbb{E}^*[C \wedge n] = \frac{1}{S_T^0}\mathbb{E}^*\left[S_T^{d+1}\right] = S_0^{d+1} > b\,.$$

Since $b$ was arbitrary, this shows $\pi^\uparrow(C) = \infty$. QED

**Remark** 3.10. As for the one–period model one can show that if a contingent claim $C$ is not attainable then

$$\pi_\downarrow(C) < \pi^\uparrow(C) \quad \text{and} \quad \Pi(C) = (\pi_\downarrow(C), \pi^\uparrow(C))\,.$$

In particular, the boundaries are not arbitrage–free prices (see **[12]**, Thm. 5.32).

**Definition.** *An arbitrage–free model is called* complete *if every contingent claim is attainable.*

**Proposition 3.11.** *(a) For an arbitrage–free model to be complete it suffices that every* bounded *contingent claim is attainable.*

*(b) If the model is complete, there exists a partition of $\Omega$ into at most $(d+1)^T$ atoms in $\mathcal{F}$.*

Proof. 1. Suppose that every bounded contingent claim is attainable. Notice that any $\mathbb{1}_A$, $A \in \mathcal{F}$, can be replicated then. I.e., $\mathbb{1}_A$ can be written as a sum of $\mathcal{F}_T$–measurable random variables. Thus $\mathcal{F}_T = \mathcal{F}$.

We show the conclusion of (b) via induction on $T$:

*$T = 1$* : By Remark 2.10 the market is complete (one–period case) and Lemma 2.8 + Theorem 2.9 imply that there exists a partition of $\Omega$ into at most $d+1$ atoms.

*$T-1 \to T$* : Suppose that the conclusion in (b) holds true in the model up to time $T-1$. Besides, let $C \geq 0$ be bounded and $\mathcal{F}_T$–measurable, and let $\xi$ be a replicating strategy. Then $C = \overline{\xi}_T \cdot \overline{S}_T$. Note that $\overline{\xi}_T$ is $\mathcal{F}_T$–measurable and hence $P$–a.s. constant on any atom $A \in \mathcal{F}_{T-1}$. This implies

$$\dim L^\infty(\Omega, \mathcal{F}_T, P[\bullet|A]) \leqslant d+1\,.$$

since on $A$ one has essentially a one–period model. Lemma 2.8 implies that $(\Omega, \mathcal{F}_T, P[\bullet|A])$ has at most $d+1$ atoms. Applying the induction hypothesis yields that $(\Omega, \mathcal{F}_T, P)$has at most $(d+1)^T$ atoms.

2. Suppose that every bounded contingent claim is attainable. Then, by the 1. part, there are at most $(d+1)^T$ atoms in $\mathcal{F}_T$. Consequently, every random variable on $(\Omega, \mathcal{F}, P)$ is bounded, which further implies that any contingent claim is attainable, i.e. the market is complete. QED

**Theorem 3.12.** (Second Fundamental Theorem of Asset Pricing) *An arbitrage–free model is complete if and only if* $|\mathcal{P}| = 1$.

Proof. " $\Longrightarrow$ " Let $A \in \mathcal{F}$. Then $C := \mathbb{1}_A$ is attainable, say by the replicating strategy $\overline{\xi}$. Let $V_t$ be the associated value process. Then, by Proposition 3.7, $V_0$ is the only arbitrage–free price of $C$ at time $0$. Thus

$$P^*[A] = \mathbb{E}^*[\mathbb{1}_A] = (1+r)^T \mathbb{E}^*\left[\frac{C}{S_T^0}\right] = (1+r)^T V_0$$

by Theorem 3.2 (note that $\{V_0\} = \Pi(C) = \{\,\mathbb{E}^*[C/S_T^0] \mid P^+ \in \mathcal{P}\,\})$. So there is at most one risk–neutral measure, and since there exists al leat one, we conclude $|\mathcal{P}| = 1$.

" $\Longleftarrow$ " Let $C$ be a contingent claim Then $\Pi(C)$ contains exactly one element. By Remark 3.10, if $C$ is not attainable, then $\Pi(C) = (\pi_\downarrow(C), \pi^\uparrow(C))$, which cannot consist of a single element. So $C$ is attainable. QED

CHAPTER 4

# The Binomial Option Pricing Model (BOPM)

This is a model first proposed by [**5**] and therefore also called the Cox–Ross–Rubinstein model (CRR model), a discrete version of the celebrated continuous Black–Scholes model.

The BOPM is a more specific model than the general multi–period model in Section 3. “More specific” means there are more assumptions. These additional assumptions are:

- only one risky asset with price process denoted by $S_t = S_t^1$;
- between $t$ and $t+1$ the price moves up or down by a constant factor $u$ resp. $d$, where $0 < d < u$:

$$S_{t+1} = \begin{cases} uS_t & \text{with probability } p \\ dS_t & \text{with probability } 1-p\,. \end{cases}$$

The price process can be described by a *binomial tree*.

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & \cdots \\
 & & u^2S_0 \arrow[ur] \arrow[dr] & \\
 & uS_0 \arrow[ur] \arrow[dr] & & \cdots \\
S_0 \arrow[ur] \arrow[dr] & & udS_0 \arrow[ur] \arrow[dr] & \\
 & dS_0 \arrow[ur] \arrow[dr] & & \cdots \\
 & & d^2S_0 \arrow[ur] \arrow[dr] & \\
 & & & \cdots
\end{tikzcd}$$

At time $T$ there are $T+1$ nodes; the binomial tree is *recombining* ($T+1$ in place of $2T$ nodes, which makes things much more tractable).

Precise definition of the underlying probability space: Let

$$\Omega := \{u, d\}^T = \{\, \omega = (\omega_1, \ldots, \omega_T) \mid \omega_i \in \{u, d\}\,\}\,.$$

We denote by

$$Y_t(\omega) := \omega_t$$

the projection onto the $t$–coordinate, $1 \leqslant t \leqslant T$.

Let $S_0 \in \mathbb{R}_+$ the price at time $0$, and

$$S_t = S_0 \prod_{i=1}^{t} Y_i \quad \text{for } 1 \leqslant t \leqslant T\,.$$

Define $\mathcal{F}_t := \sigma(S_0, \ldots, S_t)$ $0 \leqslant t \leqslant T$, and observe that

$$\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\}\ ;\\
\mathcal{F}_t &= \sigma(Y_1, \ldots, Y_t) \quad \text{for } 1 \leqslant t \leqslant T\,;\\
\mathcal{F}_T &= \mathfrak{P}(\Omega)\ .
\end{aligned}$$

Throughout assume that $P$ is a probability measure on $\mathcal{F} = \mathcal{F}_T$ with $P[\{\omega\}] > 0$ for all $\omega \in \Omega$.

If $d < 1 + r < u$ we can define an EMM as follows. Let

$$p^* := \frac{(1+r) - d}{u - d}\,,$$

and for $\omega \in \Omega$

$$P^*[\{\omega\}] := (p^*)^k (1 - p^*)^{T-k}\,, \tag{4.1}$$

where $k$ is the number of coordinates in $\omega$ equal to $u$, hence $T - k$ the number of coordinates in $\omega$ equal to $d$.

Notice that $P^*$ is the $T$–fold product of the measure $\mu$ on $\{u, d\}$ with $\mu[\{u\}] = p^*$ and $\mu[\{d\}] = 1 - p^*$.

Moreover, $P^*$ is the unique measure for which $P^*[Y_t = u] = p^*$, $1 \leqslant t \leqslant T$, and $Y_1, \ldots, Y_T$ are independent.

Now the claim is that $P^*$ is indeed an EMM. In fact,

$$\begin{aligned}
\mathbb{E}^*\left[\left.\frac{S_{t+1}}{1+r}\right| \mathcal{F}_t\right] &= \frac{S_t}{1+r}\mathbb{E}^*\left[\left. Y_{t+1}\right| \mathcal{F}_t\right]\\
&= \frac{S_t}{1+r}\mathbb{E}^*[Y_{t+1}] \quad \text{since } Y_t \text{ is independent of } \mathcal{F}_t\\
&= \frac{S_t}{1+r}(p^* u + (1-p^*)d)\\
&= \frac{S_t}{1+r}\left(\frac{1+r-d}{u-d}u + \frac{u-1-r}{u-d}d\right)\\
&= S_t\,.
\end{aligned}$$

In particular, this shows that $d < 1 + r < u$ is sufficient for the BOPM to be arbitrage–free. Actually, this condition is also necessary:

**Theorem 4.1.** *The BOPM is arbitrage–free if and only if $d < 1+r < u$. In this case, the POBM is complete, and the only EMM is given by equation* (4.1).

PROOF. " $\Longleftarrow$ ": This has been shown above.

" $\Longrightarrow$ ": Let the BOPM be arbitrage–free. Then there exists an EMM, $Q$, say. The martingale property implies

$$\mathbb{E}^Q\left[\left.\frac{S_{t+1}}{1+r}\right|\mathcal{F}_t\right] = S_t \quad , \quad 0 \leqslant t \leqslant T-1 .$$

Besides, we have

$$\mathbb{E}^Q\left[\left.\frac{S_{t+1}}{1+r}\right|\mathcal{F}_t\right] = \frac{S_t}{1+r}\mathbb{E}^Q\left[\left.Y_{t+1}\right|\mathcal{F}_t\right]$$

and hence

$$\begin{aligned} 1+r &= \mathbb{E}^Q\left[\left.Y_{t+1}\right|\mathcal{F}_t\right] \\ &= \mathbb{E}^Q\left[\left.u\mathbb{1}_{\{Y_{t+1}=u\}} + d\mathbb{1}_{\{Y_{t+1}=d\}}\right|\mathcal{F}_t\right] \\ &= uQ\left[Y_{t+1}=u|\mathcal{F}_t\right] + d\left(1 - Q\left[Y_{t+1}=u|\mathcal{F}_t\right]\right) . \end{aligned}$$

Thus there must hold

$$Q\left[Y_{t+1}=u|\mathcal{F}_t\right] = \frac{1+r-d}{u-d} . \tag{4.2}$$

Since $Q$ is equivalent to $P$, we have $0 < Q\left[Y_{t+1}|\mathcal{F}_t\right] < 1$. This entails $d < 1+r < u$. This proves " $\Longrightarrow$ ".

It remains to show completeness if the model is arbitrage–free. To this end observe that (4.2) implies that $Y_{t+1}$ is independent of $\mathcal{F}_t$ w.r.t. $Q$. This implies that $Y_1, \ldots, Y_T$ are independent w.r.t $Q$, and so $Q$ must be the product measure, i.e. $Q = P^*$ with $P^*$ defined as in (4.1). There follows $|\mathcal{P}| = 1$, and so the BOPM is complete by the SFToAP. QED

In the remainder of this section we assume $d < 1+r < u$. Denote the EMM by $P^*$.

*Next:* Arbitrage–free pricing of contingent claims (defined in the sense of Section 3).

First notice that for any contingent claim $C$ there exists a function $h : \mathbb{R}_+^T \longrightarrow \mathbb{R}$ such that

$$C = h(S_0, \ldots, S_T) .$$

**Proposition 4.2.** *Let $C = h(S_0, \ldots, S_T)$ be a contingent claim. The value process $V_t$ of a strategy $\overline{\xi}$ replicating $C$ (or, equivalently, the only arbitrage–free price process of $C$) satisfies*

$$V_t = v_t(S_0, \ldots, S_t)$$

*where*

$$v_t(x_0, \ldots, x_t) = \frac{1}{(1+r)^{T-t}}\mathbb{E}^*\left[h(x_0, \ldots, x_t, x_tY_{t+1}, \ldots, x_tY_{t+1}\cdots Y_T)\right] .$$

For the proof, we need a lemma:

**Lemma 4.3.** *Let $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_n)$ be random vectors such that $\sigma(X_1, \dots, X_m)$ is independent of $\sigma(Y_1, \dots, Y_n)$. Let $\Phi : \mathbb{R}^{m+n} \longrightarrow \mathbb{R}$ be such that that $\mathbb{E}[\Phi(X, Y)] < \infty$ and suppose that $\varphi(x) := \mathbb{E}[\Phi(x, Y)]$ is defined for any $x \in \mathbb{R}^m$. Then*

$$\mathbb{E}[\, \Phi(X, Y) |\, \sigma(X_1, \dots, X_m)\,] = \varphi(X)\,.$$

Proof. Exercise. QED

Proof *of Proposition 4.2* By Proposition 3.8 we have for $0 \leqslant t \leqslant T$:

$$\begin{aligned} V_t &= \frac{1}{(1+r)^{T-t}} \mathbb{E}^* [\, h(S_0, \dots, S_T) |\, \mathcal{F}_t\,] \\ &= \frac{1}{(1+r)^{T-t}} \mathbb{E}^* [\, h(S_0, \dots, S_t, S_t Y_{t+1}, S_t Y_{t+1} Y_{t+2}, \dots, S_t Y_{t+1} \cdots Y_T) |\, \mathcal{F}_t\,]\,. \end{aligned}$$

Since $Y_{t+1}, Y_{t+2}, \dots Y_T$ are independent of $\mathcal{F}_t = \sigma(S_0, \dots, S_t)$, we have, by Lemma 4.3,

$$V_t = v_t(S_0, \dots, S_t)\,.$$

QED

**Proposition 4.4.** *Let $C = h(S_0, \dots, S_T)$ be a contingent claim. The deterministic functions $v_t$ defined in Proposition 4.2 satisfy the following backward recursion:*

$$v_T(x_0, \dots, x_T) = h(x_0, \dots, x_T)\,;$$

*and, for $t < T$:*

$$v_t(x_0, \dots, x_t) = \frac{1}{1+r} \left[p^* v_{t+1}(x_0, \dots, x_t, x_t u) + (1-p^*) v_{t+1}(x_0, \dots, x_t, x_t d)\right]\,.$$

Proof. The martingale property of the discounted value process of $V_t$ replicating $C$ implies

$$\begin{aligned} &\frac{1}{1+r} \mathbb{E}^* [\, V_{t+1} |\, S_0 = x_0, \dots, S_t = x_t\,] \\ &\qquad = \mathbb{E}^* [\, V_t |\, S_0 = x_0, \dots, S_t = x_t\,] \\ &\qquad = v_t(x_0, \dots, x_t)\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\mathbb{E}^* [\, V_{t+1} |\, S_0 = x_0, \dots, S_t = x_t\,] \\ &\quad = p^* v_{t+1}(x_0, \dots, x_t, x_t u) + (1-p^*) v_{t+1}(x_0, \dots, x_t, x_t d)\,, \end{aligned}$$

which yields the result. QED

Now let us have a look at a specific contingent claim: A call option.

**Example**. The value function of a call $C = (S_T - K)^+$ (European call) satisfies

$$\begin{aligned} v_t(x_0, \ldots, x_t) = v_t(x_t) &= \frac{1}{(1+r)^{T-t}} \mathbb{E}^* \left[ (x_t Y_{t+1} \cdots Y_T - K)^+ \right] \\ &= \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \underbrace{\binom{T-t}{k}}_{\#\text{ of up–movements}} (p^*)^k (1-p^*)^{T-k} (x_t u^k d^{T-k} - K)^+ \end{aligned}$$

*Simplification:* Let

$a$ := minimal number of up–movements such that $x_t u^a d^{T-t-a} > K$ ("the option is in the money").

Then the option value at time $t$ is given by

$$v_t(x_t) = \frac{1}{(1+r)^{T-t}} \sum_{k=0}^{T-t} \underbrace{\binom{T-t}{k}}_{\#\text{ of up–movements}} (p^*)^k (1-p^*)^{T-k} (x_t u^k d^{T-k} - K)^+$$

How to calculate replicating strategies? The answer is given by

**Theorem 4.5.** *Let $C = h(S_0, \ldots, S_T)$ be a contingent claim. The strategy $\overline{\xi} = (\xi^0, \xi)$ replicating $C$ satisfies*

$$\xi_t = \Delta(S_0, \ldots, S_{t-1})$$

*where*

$$\Delta(x_0, \ldots, x_{t-1}) = \frac{v_t(x_0, \ldots, x_{t-1}, ux_{t-1}) - v_t(x_0, \ldots, x_{t-1}, dx_{t-1})}{ux_{t-1} - dx_{t-1}}.$$

Proof. Recall that

$$\overline{\xi}_t \cdot \left( \overline{S}_t - \overline{S}_{t-1} \right) = V_t - V_{t-1}$$

since $\overline{\xi}$ is self–financing. If $Y_t = u$, this means

$$\xi_t^0 \left( S_t^0 - S_{t-1}^0 \right) + \xi_t \left( uS_{t-1} - S_{t-1} \right) = v_t(S_0, \ldots, S_{t-1}, uS_{t-1}) - v_{t-1}(S_0, \ldots, S_{t-1}),$$

and if $Y_t = d$,

$$\xi_t^0 \left( S_t^0 - S_{t-1}^0 \right) + \xi_t \left( dS_{t-1} - S_{t-1} \right) = v_t(S_0, \ldots, S_{t-1}, dS_{t-1}) - v_{t-1}(S_0, \ldots, S_{t-1}).$$

Subtracting the second equation from the first gives

$$\xi_t(uS_{t-1} - dS_{t-1}) = v_t(S_0, \ldots, S_{t-1}, uS_{t-1}) - v_t(S_0, \ldots, S_{t-1}, dS_{t-1})$$

and hence the result. QED

**Remark**. 1) $\Delta(x_0, \ldots, x_{t-1})$ is called the *Delta* of the contingent claim $C$ at time $t$. Notice that the Delta is a difference quotient. In the continuous case ("Black–Scholes model") this will go over into a differential quotient.

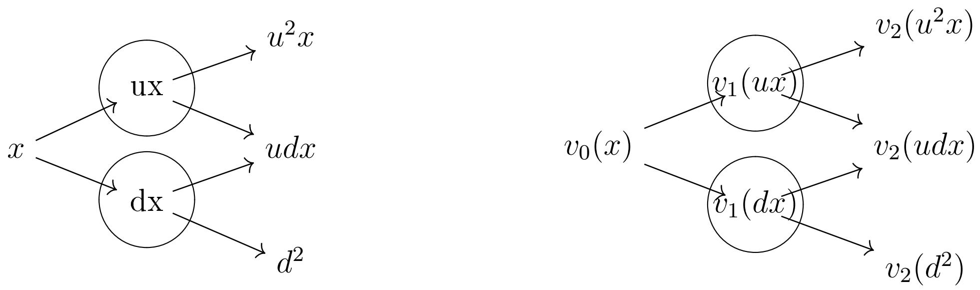
Interpretation: Sensitivity of the option's value w.r.t. the price of the underlying.

2) $V_0 = v_0(S_0)$ and $\xi_t = \Delta_t(S_0, \ldots, S_{t-1})$ uniquely determine a self–financing strategy called *delta hedge*. It is the only strategy replicating $C$. □

**Corollary 4.6.** *Let $C = (S_T - K)^+$, and let $v_t(x)$ be the arbitrage–free value of $C$ at time $t$ conditioned on $S_t = x$. Then the Delta satisfies*

$$\Delta_t(x) = \frac{v_t(ux) - v_t(dx}{(u-d)x}.$$

Proof. Immediate fro Thm. 4.5. QED




The encircled quantities are needed for the calculation of $\Delta_1$.

Pseudo–algorithm for calculating the value of Delta of a call at time t = 0:

$x$ = price at time 0

*# price vector at time $T$*

$S(T, 0) = xd^T$
**for** $j = 1 : T$
$\quad S(T, j) = S(T, j-1) * u/d$
**end**

*# option value at time $T$*

**for** $j = 0 : T$
$\quad C(T, j) = \max(S(T, j) - K, 0)$
**end**

*# recursion*

**for** $t = T - 1 : -1 : 0$
$\quad$**for** $j = 0 : t$
$\quad\quad C(T, j) = [p^*(t+1, j+1) + (1-p^*)C(t+1, j))]/(1+r)$

$S(t,j) = S(t+1,j)/d$
**end**
**if** $t = 1$ **then**
delta $= (C(1,1) - C(1,0))/(S(1,1) - S(1,0))$
**end**
**end**

*# results*

$C(0,0) =$ option value
delta $=$ the Delta of $C$

**Remark**. At first sight, it does not seem necessary to develop computer programs for pricing European options, since we have explicit formulas. However, the full power of this algorithm is revealed when studying *American options*, i.e. options that can be exercises at any time before maturity. In this case, there are no closed formulas, but a few simple changes in the above code make it work as well. □

**Remark**. For the use of the binomial model in practice, one has to estimate the parameters $u$ and $d$. This is done by putting

$$u := \exp\left(\sigma\sqrt{\frac{T}{N}}\right) \quad , \quad d := \frac{1}{u} = \exp\left(-\sigma\sqrt{\frac{T}{N}}\right)$$

where $\sigma$ is taken from statistical data of standard options, which are liquidly traded so that one has market prices; the choice of $\sigma$ is then made such that $C^{\text{model}}(\sigma) = C^{\text{market}}$, i.e. that when I plug $\sigma$ into the option–pricing formula of the model I get the observed market price.

The above choices of $u$ and $d$ are motivated by what they become in the continuous limit model by taking shorter and shorter trading periods; as we will see later, this will be the Black–Scholes model. □

*Next* (and last) *aim* of this section: Explicit price formulas for barrier options. We first introduce some auxiliary random variables:

$$C_i(\omega) := \begin{cases} +1 & \text{if } \omega_i = u \\ -1 & \text{if } \omega_i = d \end{cases} \quad , \quad 1 \leqslant i \leqslant T\,;$$

$$Z_t := \sum_{i=1}^{t} C_i \quad , \quad 1 \leqslant t \leqslant T\,.$$

From now on assume $P$ is given as

$$P\left[\{\omega\}\right] = \frac{1}{|\Omega|} = 2^{-T} \quad , \quad \omega \in \Omega\,,$$

i.e. is such that $Z = (Z_t)$ is the simple random walk in $\mathbb{Z}$ under $P$.
In addition, we assume $d = u^{-1}$. Then the price satisfies

$$S_t(\omega) = S_0 \prod_{i=1}^{t} Y_i(\omega) = S_0 u^{\sum_{i=1}^{t} C_i(\omega)} = S_0 u^{Z_t(\omega)}.$$

Observe that there holds

$$\frac{Z_t + t}{2} = \#\ \textit{up movements until time } t\,,$$

and that this random variable is binomially distributed, and hence

$$P\left[Z_t = k\right] = P\left[\frac{Z_t + t}{2} = \frac{k+t}{2}\right] = \begin{cases} \binom{t}{(k+t)/2} 2^{-t} & k + t \text{ even} \\ \\ 0 & k + t \text{ odd}. \end{cases}$$

**Lemma 4.7.** (Reflection Principle) *For all $k \in \mathbb{N}$ and $l \geq 0$ we have*

*(a)* $P\left[\max_{0 \leqslant k \leqslant T} Z_t \geq k\,,\ Z_T = k - l\right] = P\left[Z_T = k + l\right];$

*(b)* $P\left[\max_{0 \leqslant k \leqslant T} Z_t = k\,,\ Z_T = k - l\right] = \dfrac{2(k+l+1)}{T+1} P\left[Z_T = k + l + 1\right].$

PROOF. (a): We have, for $k, l \in \mathbb{N}$

$$P\left[Z_T = k + l\right] = P\left[\frac{Z_T + T}{2} = \frac{T + k + l}{2}\right];$$

$$P\left[Z_{T+1} = k + l + 1\right] = P\left[\frac{Z_{T+1} + T + 1}{2} = \frac{T + 2 + k + l}{2}\right].$$

For $k \in \mathbb{N}$ let

$$\tau(\omega) := \inf\{\, t \geq 0 \mid \ Z_t(\omega) = k\,\} \wedge T\,.$$

For any $\omega$, let $\Phi(\omega) := (\omega_1, \dots, \omega_{\tau(\omega)}, \frac{1}{\omega_{\tau(\omega)+1}}, \dots, \frac{1}{\omega_T})$. Then $\Phi$ is a bijection from

$$A_{k,l} := \left\{ \omega \in \Omega \;\middle|\; \max_{0 \leqslant t \leqslant T} Z_t(\omega) \geq k\,,\ Z_T(\omega) = k - l \right\}$$

onto

$$B_{k,l} := \left\{ \omega \in \Omega \mid \ \max_{0 \leqslant t \leqslant T} Z_t(\omega) > k\,,\ Z_T(\omega) = k + l \right\} = \{Z_T = k + l\}\,,$$

hence $P\left[A_{k,l}\right] = P\left[B_{k,l}\right]$, and so (a).

(b): $1^{st}$ *case* – $T + k + l$ *not* even. Then both sides of (b) are zero.

$2^{nd}$ *case* – $T+k+l$ even. Then, for $j := \dfrac{T+k+l}{2}$ :

$$\begin{aligned}
&P\left[\max_{0\leqslant t\leqslant T} Z_t = k\,,\ Z_T = k-l\right]\\
&\qquad = P\left[\max_{0\leqslant t\leqslant T} Z_t \geq k\right] - P\left[\max_{0\leqslant t\leqslant T} Z_t \geq k+1\right]\\
&\qquad = P\left[Z_T = k+l\right] - P\left[Z_T = (k+1)+(l+1)\right]\\
&\qquad = 2^{-T}\binom{T}{j} + 2^{-t}\binom{T}{j+1}\\
&\qquad = 2^{-T}\left[\frac{T!}{(T-j)!j!} - \frac{T!}{(T-j-1)!(j+1)!}\right]\\
&\qquad = 2^{-T}\frac{T!}{(T-j-1)!j!}\left[\frac{1}{T-j} - \frac{1}{j+1}\right]\\
&\qquad = 2^{-T}\frac{T!}{(T-j-1)!j!}\frac{2j+1-T}{(T-j)(j+1)}\\
&\qquad = 2^{-T}\frac{(T+1)!}{(T+j)!(j+1)!}\frac{2j+1-T}{T+1}\\
&\qquad = 2P\left[\frac{Z_{T+1}+T+1}{2} = j+1\right]\frac{k+l+1}{T+1}\\
&\qquad = P\left[Z_{T+1} = k+l+1\right]\frac{2(k+l+1)}{T+1}\,.
\end{aligned}$$

QED

**Remark** 4.8. Notice that

$$\frac{dP^*}{dP}(\omega) = \frac{P^*\left[\{\omega\}\right]}{P\left[\{\omega\}\right]} = 2^T(p^*)^{(Z_T(\omega)+T)/2}(1-p^*)^{(-Z_T(\omega)+T)/2}\,.$$

We will need this formula in the sequel. □

**Lemma 4.9.** (Value of an up–and–in call) *Consider the up–and–in call*

$$C = \begin{cases} (S_T-K)^+ & \text{if } \max\limits_{0\leqslant t\leqslant T} S_t \geq B\,;\\ 0 & \text{else}\,, \end{cases}$$

*where* $B = S_0u^k > S_0 \vee K$ *for some* $k \geq 1$ *. Let*

$$l_k := \sup\{\, l \mid 2l-T \leqslant k\,\}\,.$$

*Then the arbitrage–free price* $\mathbb{E}^*\left[C/S_T^0\right]$ *satisfies*

$$\mathbb{E}^*\left[\frac{C}{S_T^0}\right] = \frac{1}{(1+r)^T}\left[\sum_{l=k}^{l_k}(S_0u^{2l-T}-K)^+(p^*)^l(1-p^*)^{T-l}\binom{T}{k-l+T}\right.$$

$$+\sum_{l=l_k+1}^{T}(S_0u^{2l-T}-K)^+(p^*)^l(1-p^*)^{T-l}\binom{T}{l}\Bigg].$$

Proof. Using $S_t = S_0u^{Z_t} \geq S_0u^k$ and Remark 4.8, we can write

$$\begin{aligned}\mathbb{E}^*[C] &= \mathbb{E}^*\left[(S_T-K)^+\mathbb{1}_{\{\max S_t \geq B\}}\right]\\ &= \sum_{l=0}^{T} P\left[\max Z_t \geq k\,,\ Z_T = 2l-T\right]2^T(p^*)^l(1-p^*)^{T-l}(S_0u^{2l-T}-K)^+\end{aligned}$$

For $l \leqslant l_k$, we have

$$\begin{aligned}&P[\max Z_t \geq k\,,\ Z_T = \underbrace{2l-T}_{k-(k-2l+T)}\ ]\\ &\qquad= P\left[Z_T = k+(k-2l+T)\right] \quad \text{(Reflection Principle)}\\ &\qquad= P\left[Z_T = 2k-2l+T\right]\\ &\qquad= P\left[\frac{Z_T+T}{2} = k-l+T\right]\\ &\qquad= \begin{cases} 2^{-T}\binom{T}{k-l+T} & \text{if } k \leqslant l\,;\\ \\ 0 & \text{else.}\end{cases}\end{aligned}$$

For $l > l_k$, we have

$$P[\underbrace{\max Z_t \geq k}_{\text{redundant}},\ Z_T = 2l-T] = P\left[Z_T = 2l-T\right] = 2^{-T}\binom{T}{l}.$$

The result follows QED

Yet another one:

**Lemma 4.10.** (Value of a lookback put with floating strike) *Let*

$$C := \max_{0\leqslant t\leqslant T} S_t - S_T\,.$$

*Then*

$$\begin{aligned}&\mathbb{E}^*\left[\frac{C}{S_T^0}\right] = -S_0+\\ &+\frac{S_0}{(1+r)^T}\sum_{k=0}^{T}u^k\left\{\sum_{l=k}^{l_k}(p^*)^l(1-p^*)^{T-l}\frac{2(k-l)+T+1}{T+1}\binom{T+1}{k-l+T+1}\right\},\end{aligned}$$

*where* $l_k := \sup\{\, l \mid 2l-T \leqslant k\,\}$.

PROOF. The proof is not difficult once one has found an intelligent way to implement the Reflection Principle. One has

$$\mathbb{E}^*\left[\frac{C}{S_T^0}\right] = \frac{1}{(1+r)^T}\mathbb{E}^*\left[\max_{0\leqslant t\leqslant T} S_t\right] - S_0$$

and

$$\mathbb{E}^*\left[\max_{0\leqslant t\leqslant T} S_t\right] = \sum_{k=0}^{T} S_0 u^k P^*\left[\max_{0\leqslant t\leqslant T} Z_t = k\right] .$$

Note that

$$\begin{aligned}
P^*\left[\max_{0\leqslant t\leqslant T} Z_t = k\right] &= \sum_{l=0}^{T} P^*\left[\max_{0\leqslant t\leqslant T} Z_t = k\,,\; Z_T = 2l-T\right] \\
&= \sum_{l=0}^{T} 2^T (p^*)^l (1-p^*)^{T-l} P\left[\max_{0\leqslant t\leqslant T} Z_t = k\,,\; Z_T = 2l-T\right]
\end{aligned}$$

by Remark 4.8.

Now for $l \leqslant l_k$ (using Part (b) of the Reflection Principle)

$$\begin{aligned}
&P[\max_{0\leqslant t\leqslant T} Z_t = k\,,\; Z_T = \underbrace{2l-T}_{k-(k-2l+T)}\;] \\
&= \frac{2(k+1+k-(2l-T))}{T+1} P\left[Z_{T+1} = k+1+k-(2l-T)\right] \\
&= 2\frac{2(k+1+k-(2l-T))}{T+1} P\left[\frac{Z_{T+1}+T+1}{T+1} = \frac{2k-2l+2T+2}{2}\right] \\
&= \begin{cases} 2\dfrac{2(k-l)+T+1}{T+1}\dbinom{T+1}{k-l+T+1}2^{-T-1} & : \; k \leqslant l\,; \\ \\ 0 & : \; \text{else.} \end{cases}
\end{aligned}$$

This implies the statement of the Lemma QED

This is the last example for an exotic option within the BOPM which I have prepared. Now we turn to American Options.

CHAPTER 5

# Pricing and Hedging American Options in the BOPM

Throughout let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1\ldots,T}, P)$ be a filtered probability space. First recall some results from martingale theory.

**Proposition and Definition 5.1.** (DOOB decomposition) *Let $Y = (Y_t)$ be an adapted stochastic process with $Y_t \in L^1(P)$, $0 \leqslant t \leqslant T$. Then there eists a unique decomposition*

$$Y_t = M_t - A_t \tag{5.1}$$

*such that $M = (M_t)$ is a $P$–martingale, $A = (A_t)$ is predictable and $A_0 = 0$. This decomposition is called the* DOOB decomposition *of $(Y_t)$.*

PROOF. *1. Uniqueness:* $A_0 := 0$ is unique by definition. Suppose $A_0, \ldots, A_{t-1}$ are unique for some $0 \leqslant t < T$. Taking conditional expectation w.r.t $\mathcal{F}_{t-1}$ in (5.1) yields

$$\mathbb{E}^* [\, Y_t | \, \mathcal{F}_{t-1} \,] = \mathbb{E}^* [\, M_t | \, \mathcal{F}_{t-1} \,] - \mathbb{E}^* [\, A_t | \, \mathcal{F}_{t-1} \,] = M_{t-1} - A_t$$

since $(M_t)$ is a martingale and $(A_t)$ predictable. On the other hand,

$$Y_{t-1} = M_{t-1} - A_{t-1} \,.$$

Subtracting from this equation its predecessor yields

$$Y_{t-1} - \mathbb{E}^* [\, Y_t | \, \mathcal{F}_{t-1} \,] = A_t - A_{t-1}$$

or

$$A_t = A_{t-1} - \mathbb{E}^* [\, Y_t - Y_{t-1} | \, \mathcal{F}_{t-1} \,]$$

is given uniquely by $Y$ and $A_{t-1}$. Therefore, by induction, the $A_t$ are unique, and so are the $M_t = Y_t + A_t$.

*2. Existence:* One has no choice but to define $A_t$ recursively as

$$\begin{aligned} A_0 &:= 0\,; \\ A_t &:= A_{t-1} - \mathbb{E}^* [\, Y_t - Y_{t-1} | \, \mathcal{F}_{t-1} \,] \quad , \quad 1 \leqslant t \leqslant T \,. \end{aligned}$$

Then put

$$M_t := Y_t + A_t \quad , \quad 0 \leqslant t \leqslant T \,.$$

It is then a straightforward induction to verify that $(A_t)$ is predictable and $(M_t)$ a $P$–martingale. QED

**Definition.** *A stochastic process* $X = (X_t)_{t=0,\dots,T}$ *is called a* submartingale *(resp. a* supermartingale*) if*

*(1)* $X$ *is adapted;*
*(2)* $X_t \in L^1(P)$, $0 \leqslant t \leqslant T$ *;*
*(3) for all* $0 \leqslant s \leqslant t \leqslant T$ *:* $\mathbb{E}[\, X_t |\, \mathcal{F}_s \,] \geq X_s$ *(resp.* $\leqslant X_s$*).*

(As mnemonic: A supermartingale implements a downward trend; "there is nothing super about a supermartingale".)

**Lemma 5.2.** *Let* $X = (X_t)$ *be an adapted process with* $X_t \in L^1(P)$, $0 \leqslant t \leqslant T$. *Let* $X_t = M_t - A_t$ *its* DOOB *decomposition. Then*

*(a)* $X_t$ *is a supermartingale iff* $A_t$ *is non–decreasing;*
*(b)* $X_t$ *is a submartingale iff* $A_t$ *is non–increasing.*

PROOF. (a): $\mathbb{E}[\, X_t - X_{t-1} |\, \mathcal{F}_{t-1} \,] = -\mathbb{E}[\, A_t - A_{t-1} |\, \mathcal{F}_{t-1} \,] = -A_t + A_{t-1} \leqslant 0$ iff $A_t$ is non–decreasing.

(b): is completely analogous. QED

**Definition.** *An* American contingent claim *is a nonnegative adapted process* $C = (C_t)_{t=0,\dots,T}$.

**Example**. American put $C_t = (K - S_t)^+$.

Now let us have a closer look at American options within the binomial model. In the BOPM: for every American contingent claim $(C_t)$ we can find measurable functions $f_t : \mathbb{R}_+^t \longrightarrow \mathbb{R}_+$ such that

$$C_t = f_t(S_1, \dots, S_t).$$

To simplify notation, we only consider payoff functions of the form $f_t(x_1, \dots, x_t) = f_t(x_t)$, i.e. which do not depend on the past.

*Question* : How do we price an American option $(f_t(S_t))_{t=0,\dots,T}$ within the BOPM?

Let $V_t :=$ minimal capital required at time $t$ to replicate the option.

Then, at time $T$:

$$V_T = f_T(S_T).$$

At time $T-1$:

$$V_T = f_{T-1}(S_{T-1}) \vee \mathbb{E}^* \left[ \left. \frac{f_T(S_T)}{1+r} \right| \; \mathcal{F}_{T-1} \right].$$

At time $t < T$:

$$V_t = f_t(S_t) \vee \mathbb{E}^* \left[ \left. \frac{V_{t+1}}{1+r} \right| \; \mathcal{F}_t \right].$$

Notice that $V_t = v_t(S_t)$ with $v_t : \mathbb{R} \longrightarrow \mathbb{R}$ a deterministic function satisfying the recursion

$$v_T(x_T) = f_T(x_T)$$

and

$$v_t(x_t) = f_t(x_t) \vee \frac{1}{1+r}\left[p^* v_{t+1}(ux_t) + (1-p^*)v_{t+1}(dx_t)\right] \quad , \quad 0 \leqslant t \leqslant T\,.$$

The discounted value process $(V_t/S_t^0)$ is the so–called SNELL *envelope* of $(f_t(S_t)/S_t^0)$ under $P^*$.

So what is a SNELL envelope? You can define it for any adapted intgrable process:

**Definition.** *Let* $Y = (Y_t)$ *be an adapted process such that* $Y_t \in L^1(P)$, $0 \leqslant t \leqslant T$. *The* SNELL envelope $U = (U_t)$ of $Y$ under $P$ *is recursively defined by*

$$U_T := Y_T\,;$$

*and, for* $0 \leqslant t < T$ :

$$U_t := Y_t \vee \mathbb{E}\left[\, U_{t+1} \middle|\, \mathcal{F}_t \right]\,.$$

**Proposition 5.3.** *Let* $Y = (Y_t)$ *be an adapted process and* $Y_t \in L^1(P)$ *for all* $0 \leqslant t \leqslant T$. *The* SNELL *envelope* $U = (U_t)$ *of* $Y$ *is the smallest supermartingale dominating* $Y$ *(i.e.* $U$ *is a supermartingale with* $U_t \geq Y_t$, $0 \leqslant t \leqslant T$, *and if* $\widetilde{U} = (\widetilde{U}_t)$ *is a supermartingale with* $\widetilde{U}_t \geq Y_t$, $0 \leqslant t \leqslant T$, *then* $\widetilde{U}_t \geq U_t$, $0 \leqslant t \leqslant T$*).*

PROOF. 1. $U_{t-1} = Y_{t-1} \vee \mathbb{E}[\,U_t|\, \mathcal{F}_{t-1}\,] \geq \mathbb{E}[\,U_t|\, \mathcal{F}_{t-1}\,]$, and so obviously $U$ is a supermartingale dominating $Y$.

2. Let $\widetilde{U} = (\widetilde{U}_t)$ be a supermartingale with $\widetilde{U}_t \geq Y_t$, $0 \leqslant t \leqslant T$. Perform backward induction on $t$:

$t = T$: $\widetilde{U}_Z \geq Y_T = U_T$.

$t \to t-1$: Assume that $\widetilde{U}_t \geq U_t$. Then

$$\begin{aligned} \widetilde{U}_{t-1} &\geq \mathbb{E}\left[\,\widetilde{U}_t \middle|\, \mathcal{F}_{t-1}\right] \quad \text{since } \widetilde{U} \text{ is a supermartingale} \\ &\geq \mathbb{E}[\,U_t|\, \mathcal{F}_{t-1}\,] \quad \text{by the induction hypothesis.} \end{aligned}$$

Moreover, $\widetilde{U}_{t-1} \geq Y_{t-1}$ by hypothesis, and so

$$\widetilde{U}_{t-1} \geq Y_{t-1} \vee \mathbb{E}[\,U_t|\, \mathcal{F}_{t-1}\,] = U_{t-1}\,.$$

QED

The SNELL envelope is then the answer to how to price American contingent claims based on replication.

*Question* : How to hedge an American option $(f_t(S_t))_{t=0,\ldots,T}$ in the BOPM?

**Lemma 5.4.** *Consider the BOPM. Let*

$$\Delta_t(x_{t-1}) := \frac{v_t(ux_{t-1} - v_t(dx_{t-1}))}{(u-d)x_{t-1}} \quad , \quad 1 \leqslant t \leqslant T\,.$$

*Let $\overline{\xi} = (\xi^0, \xi)$ be the unique self–financing strategy with initial capital $v_0(S_0)$ and $\xi_t = \Delta_t(S_{t-1})$. Then the discounted value process $(V_t^{\xi}/S_t^0)$ associated to $\overline{\xi}$ is the martingale in the* Doob *decomposition of the discounted value process $(v_t(S_t)/S_t^0)$ w.r.t. $P^*$.*

Proof. Let

$$w_t(x_t) := \frac{1}{1+r}\left[p^* v_{t+1}(ux_t) + (1-p^+)v_{t+1}(dx_{t+1})\right]$$

(= the value of the American option at time $t$ provided it is not exercised at time $t$).

We explicitely compute the Doob decomposition of the value process associated to an American option: Let

$$A_0 := 0$$

and

$$A_t := A_{t-1} + \frac{[f_{t-1}(S_t) - w_{t-1}(S_{t-1})]^+}{S_t^0}$$

for $1 \leqslant t \leqslant T$ (note the numerator is just what you loose when not exercising the option at time $t-1$). Then $(A_t)$ is predictable and non–decreasing. Moreover, let

$$M_t := v_0(S_0) + \sum_{k=1}^{t} \left\{ \frac{v_k(S_k)}{S_k^0} - \frac{w_{k-1}(S_{k-1})}{S_{k-1}^0} \right\} .$$

I claim $M = (M_t)$ is a martingale. For this, note that

$$\mathbb{E}\left[ \frac{v_k(S_k)}{1+r} \,\middle|\, \mathcal{F}_{k-1} \right] = w_{k-1}(S_{k-1})$$

which already shows that $(M_t)$ is a $P^*$–martingale.

We now want to show that $M_t - A_t$ is indeed the Doob decomposition of $(v_t(S_t)/S_t^0)$. Observe that $v_0(S_0) = M_0 - A_0$, and

$$\begin{aligned} M_t - M_{t-1} - A_t + A_{t-1} &= \frac{v_t(S_t)}{S_t^0} - \frac{w_{t-1}(S_{t-1})}{S_{t-1}^0} - \frac{[f_{t-1}(S_{t-1}) - w_{t-1}(S_{t-1})]^+}{S_{t-1}^0} \\ &= \frac{v_t(S_t)}{S_t^0} - \frac{w_{t-1}(S_{t-1}) \vee f_{t-1}(S_{t-1})}{S_{t-1}^0} \\ &= \frac{v_t(S_t)}{S_t^0} - \frac{v_{t-1}(S_{t-1})}{S_{t-1}^0} . \end{aligned}$$

By induction, then, this yields

$$\frac{v_t(S_t)}{S_t^0} = M_t - A_t$$

is the DOOB decomposition of the discounted value process of $(S_t)$.

It remains to show that

$$\frac{V_t^\xi}{S_t^0} = M_t \,.$$

This is again done by induction on $t$.

$t = 0$: $M_0 = v_0(S_0) = V_0^\xi$.

$t - 1 \to t$: By Lemma 3.4

$$\frac{V_t^\xi}{S_t^0} - \frac{V_{t-1}^\xi}{S_{t-1}^0} = \Delta_t(S_{t-1}) \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) .$$

Now we distinguish 2 cases:

*1. case:* $S_t = uS_{t-1}$. Then

$$\begin{aligned}
\Delta_t(S_{t-1}) \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) &= \Delta_t(S_{t-1})(u - (1+r)) \frac{S_{t-1}}{S_t^0} \\
&= \frac{1}{S_t^0} \underbrace{\frac{u-(1+r)}{u-d}}_{1-p^*} (v_t(uS_{t-1}) - v_t(dS_{t-1})) \quad \text{(Definition of } \Delta_t) \\
&= \frac{v_t(uS_{t-1})}{S_t^0} - \frac{1}{S_t^0}(p^* v_t(uS_{t-1}) + (1 - p*) v_t(dS_{t-1})) \\
&= \frac{v_t(uS_{t-1})}{S_t^0} - \frac{w_{t-1}(S_{t-1})}{S_{t-1}^0} .
\end{aligned}$$

*2. case:* $S_t = S_{t-1}$ . Then similarly

$$\Delta_t(S_{t-1}) \left( \frac{S_t}{S_t^0} - \frac{S_{t-1}}{S_{t-1}^0} \right) = \frac{v_t(dS_{t-1})}{S_t^0} - \frac{w_{t-1}(S_{t-1})}{S_{t-1}^0} .$$

There follows

$$\frac{V_t^\xi}{S_t^0} - \frac{V_{t-1}^\xi}{S_{t-1}^0} = M_t - M_{t-1}$$

whick yields the result by induction. QED

What is the use of this lemma? It shows what is the minimal capital required to replicate an American option in the BOPM:

**Proposition 5.5.** *Consider the BOPM. Let* $\Delta$, $\xi$ *and* $V_t^\xi$ *be defined as in Lemma 5.4. Then, for all* $0 \leqslant t \leqslant T$, $V_t^\xi \geq f_t(S_t)$.

*Moreover, if* $\overline{\xi}$ *is a self–financing strategy such that the asociated value process* $V^\xi = (V_t^\xi)$ *satisfies* $V_t^\xi \geq f_t(S_t)$ *for* $0 \leqslant t \leqslant T$ *, then* $V_0^\xi \geq v_0(S_0)$ *, i.e.* $v_0(S_0)$ *is the minimal capital necessary at* $t = 0$ *to hedge the American option.*

Proof. Lemma 5.4 implies $V_t^\xi \geq f_t(S_t)$ (since $v_t(S_t)/S_t^0 \leqslant V_t^\xi / S_t^0$). This is the first statement.

For the second statement, let $\overline{\xi}$ be the self–financing strategy with $V_t^\xi \geq f_t(S_t)$. By Proposition 5.3:

$$\frac{V_t^\xi}{S_t^0} \geq \frac{v_t(S_t)}{S_t^0} ,$$

since the RHS is the Snell envelope of $f_t(S_t)$, and hence $V_0^\xi \geq v_0(S_0)$. QED

We now change perspective from the seller to the buyer.

*Question* : What is the best time to exercise an American option?

**Definition.** *A map* $\tau : \Omega \longrightarrow \{0, 1, \ldots, T\} \cup \{+\infty\}$ *is called a* stopping time *iff* $\{T = t\} \in \mathcal{F}_t$ *for all* $0 \leqslant t \leqslant T$ *.*

**Remark**. 1. There is an alternative definition of stopping times: $\tau : \Omega \longrightarrow \{0, 1, \ldots, T\} \cup \{+\infty\}$ is a stopping time iff $\{T \leqslant t\} \in \mathcal{F}_t$ for all $0 \leqslant t \leqslant T$.

Proof. $\{\tau \leqslant t\} = \bigcup_{k=0}^{t} \{\tau = k\}$.

2. If $\sigma$ and $\tau$ are stopping times, so are $\sigma \wedge \tau$ and $\sigma \vee \tau$.

Proof.
$$\begin{aligned} \{\sigma \wedge \tau \leqslant t\} &= \{\sigma \leqslant t\} \cup \{\tau \leqslant t\} \in \mathcal{F}_t ; \\ \{\sigma \vee \tau \leqslant t\} &= \{\sigma \leqslant t\} \cap \{\tau \leqslant t\} \in \mathcal{F}_t . \end{aligned}$$

*Notation:* Let $Y = (Y_t)$ be a stochastic process. We write $Y^\tau = (Y_t^\tau)$ for the *stopped process*, defined by

$$Y_t^\tau(\omega) := Y_{t \wedge \tau(\omega)}(\omega) \quad , \quad 0 \leqslant t \leqslant T .$$

**Lemma 5.6.** (Optional Sampling Theorem). *Let* $M = (M_t)$ *be an adapted process such that* $M_t \in L^1(P)$ *,* $0 \leqslant t \leqslant T$ *. Then we have equivalences between*

*(a)* $M$ *is a* $P$*–martingale;*
*(b)* $M^\tau$ *is a* $P$*–martingale for any stopping time* $\tau$*;*
*(c)* $\mathbb{E}[M_\tau] = \mathbb{E}[M_0]$ *for all stopping times* $\tau \leqslant T$ *(here* $M_\tau$ *is the random variable defined by* $M_\tau(\omega) := M_{\tau(\omega)}(\omega)$*).*

Proof. *(a)* $\Longrightarrow$ *(b):* For $0 \leqslant t \leqslant T - 1$ we have

$$M_{t+1}^\tau - M_t^\tau = \mathbb{1}_{\{\tau > t\}} (M_{t+1} - M_t) .$$

What is enough to show is

$$\mathbb{E}\left[\, M_{t+1}^\tau - M_t^\tau \,\middle|\, \mathcal{F}_t \,\right] = 0 .$$

But

$$\begin{aligned}\mathbb{E}\left[\, M^{\tau}_{t+1} - M^{\tau}_t \middle|\, \mathcal{F}_t\,\right] &= \mathbb{E}\left[\, \mathbb{1}_{\{\tau>t\}}(M_{t+1} - M_t) \middle|\, \mathcal{F}_t\,\right] \\ &= \mathbb{1}_{\{\tau>t\}}\mathbb{E}\left[\, M_{t+1} - M_t \middle|\, \mathcal{F}_t\,\right] \quad \text{since } \mathbb{1}_{\{\tau>t\}} \text{ is } \mathcal{F}_t\text{–measurable} \\ &= 0\,.\end{aligned}$$

*(b) $\Longrightarrow$ (c):* For a martingale $M$, we have, for any $0 \leqslant s \leqslant t \leqslant T$

$$\mathbb{E}\left[M_s\right] = \mathbb{E}\left[\, M_s \middle|\, \mathcal{F}_0\,\right] = \mathbb{E}\left[\,\mathbb{E}\left[\, M_t \middle|\, \mathcal{F}_s\,\right]\middle|\, \mathcal{F}_0\,\right] = \mathbb{E}\left[\, M_t \middle|\, \mathcal{F}_0\,\right] = \mathbb{E}\left[M_t\right]\,,$$

and so, in particular, $\mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_T\right]$. Now, if $\tau$ is any stopping time, we have, by hypothesis, that $M^\tau$ is a martingale, hence $\mathbb{E}\left[M^\tau_0\right] = \mathbb{E}\left[M^\tau_T\right]$. But $M^\tau_0 = M_0$ and $M^\tau_T = M_\tau$.

*(c) $\Longrightarrow$ (a):* We have to show: if $t < T$ and $A \in \mathcal{F}_t$, then $\mathbb{E}\left[\mathbb{1}_A M_T\right] = \mathbb{E}\left[\mathbb{1}_A M_t\right]$.

So let $A \in \mathcal{F}_t$ and define

$$\tau(\omega) := \begin{cases} t & \omega \in A\,; \\ T & \omega \notin A\,. \end{cases}$$

Then $\tau$ is a stopping time. Note that

$$\mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_\tau\right] = \mathbb{E}\left[\mathbb{1}_A M_t\right] + \mathbb{E}\left[\mathbb{1}_{A^c} M_T\right]\,.$$

Besides,

$$\mathbb{E}\left[M_0\right] = \mathbb{E}\left[M_T\right] = \mathbb{E}\left[\mathbb{1}_A M_T\right] + \mathbb{E}\left[\mathbb{1}_{A^c} M_T\right]\,,$$

and so $\mathbb{E}\left[\mathbb{1}_A M_T\right] = \mathbb{E}\left[\mathbb{1}_A M_t\right]$, which is what we wanted to show. QED

**Corollary 5.7.** *Let $U = (U_t)$ be an adapted process, and $U_t \in L^1(P)$, $0 \leqslant t \leqslant T$. Then $U$ is a supermartingale iff $U^\tau$ is a supermartingale for all stopping times $\tau$.*

Proof. " $\Longleftarrow$ ": Take $\tau := T$.

" $\Longrightarrow$ ": Employ the Doob decomposition and write $U = M - A$ with $M$ a martingale and $A$ non–decreasing. Then $U^\tau = M^\tau - A^\tau$. By Lemma 5.6, $M^\tau$ is a martingale, $A^\tau$ is still non–decreasing, so this gives the Doob decomposition of $U^\tau$, which thus tells us that $U^\tau$ is a supermartingale. QED

Now we return to American options. In the remainder of this chapter we consider the BOPM only. Let $C_t = f_t(C_t)$, $0 \leqslant t \leqslant T$ be the payoff of an American option, and let $V_t = v_t(S_t)$ be the minimal capital needed at time $t$ to replicate $(C_s)_{t\leqslant s\leqslant T}$.

Some more notation. Let

$$\mathcal{T} := \{\, \tau \mid \ \tau \textit{ is a stopping time with } \tau \leqslant T\,\}\,.$$

Think of $\mathcal{T}$ as the set of the holder's possible exercise dates. The *optimal stopping problem* for the option holder is

$$\textit{Maximize } \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] \textit{ among all } \tau \in \mathcal{T}\,.$$

We will see that

$$\tau^* := \inf\{\, t \geq 0 \mid V_t = C_t \,\}$$

is one solution of the problem. Notice that $\tau^* \leqslant T$ , since $V_T = C_T$ .

**Theorem 5.8.** *$\tau^*$ solves the optimal stopping problem. More precisely,*

$$V_0 = \mathbb{E}^*\left[\frac{C_{\tau^*}}{S^0_{\tau^*}}\right] = \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] .$$

Proof. Recall that $U_t := V_t/S^0_t \;\; 0 \leqslant t \leqslant T$, is the Snell envelope of $(C_t/S^0_t)$. Let $\tau \in \mathcal{T}$. Then, by Proposition 5.3 and Corollary 5.7, $U^\tau$ is a $P^*$–supermartingale, and hence

$$V_0 = U_0 \geq \mathbb{E}^*\left[U^\tau_T\right] = \mathbb{E}^*\left[U_\tau\right] \geq \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right]$$

which implies

$$V_0 \geq \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] .$$

To prove the statement it is therefore enough to show that $\mathbb{E}^*\left[\frac{C_{\tau^*}}{S^0_{\tau^*}}\right] = V_0$.

Let $\widehat{U}_t := U^{\tau^*}_t$ . Then $(\widehat{U}_t)$ is a $P^*$–martingale:

$$\begin{aligned}
\mathbb{E}^*\left[\widehat{U}_{t+1}\,\middle|\,\mathcal{F}_t\right] &= \mathbb{E}^*\left[\mathbb{1}_{\{\tau^*>t\}}\widehat{U}_{t+1} + \mathbb{1}_{\{\tau^*\leqslant t\}}\widehat{U}_{t+1}\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau^*>t\}}\mathbb{E}^*\left[\widehat{U}_{t+1}\,\middle|\,\mathcal{F}_t\right] + \mathbb{1}_{\{\tau^*\leqslant t\}}\mathbb{E}^*\left[\widehat{U}_{\tau^*}\,\middle|\,\mathcal{F}_t\right] \\
&= \mathbb{1}_{\{\tau^*>t\}}U_t + \mathbb{1}_{\{\tau^*\leqslant t\}}U_{\tau^*} \\
&= U_{t\wedge\tau^*} = \widehat{U}_t\,.
\end{aligned}$$

Therefore, since at $\tau^*$ , $V_{\tau^*} = C_{\tau^*}$ ,

$$\mathbb{E}^*\left[\frac{C_{\tau^*}}{S^0_{\tau^*}}\right] = \mathbb{E}^*\left[U_{\tau^*}\right] = \mathbb{E}^*\left[\widehat{U}_T\right] = \mathbb{E}^*\left[\widehat{U}_0\right] = V_0\,.$$

QED

**Remark**. $\tau^*$ is not necessarily the only stopping time attaining $\sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[C_\tau/S^0_\tau\right]$.

Example: Let $C_t := KS^0_t$ . Then $\mathbb{E}^*\left[C_\tau/S^0_\tau\right] = K$ for all $\tau \in \mathcal{T}$ .

**Definition.** *A stopping time $\widehat{\tau} \in \mathcal{T}$ is called* optimal *if*

$$\mathbb{E}^*\left[\frac{C_{\widehat{\tau}}}{S^0_{\widehat{\tau}}}\right] = \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] .$$

**Proposition 5.9.** *$\widehat{\tau} \in \mathcal{T}$ is optimal iff $C_{\widehat{\tau}} = V_{\widehat{\tau}}$ and $U := V_{\bullet\wedge\widehat{\tau}}/S^0_{\bullet\wedge\widehat{\tau}}$ is a $P^*$–martingale. In particular, $\tau^*$ is the minimal optimal stopping time.*

Proof. " $\Longleftarrow$ ": Let $\widehat{\tau} \in \mathcal{T}$ be such that $C_{\widehat{\tau}} = V_{\widehat{\tau}}$ and $U$ is a $P^*$–martingale. By Theorem 5.8

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] = V_0 = U_0 = \mathbb{E}^*\left[U_T\right] = \mathbb{E}^*\left[\frac{V_{\widehat{\tau}}}{S^0_{\widehat{\tau}}}\right] = \mathbb{E}^*\left[\frac{C_{\widehat{\tau}}}{S^0_{\widehat{\tau}}}\right] ,$$

and so $\widehat{\tau}$ is optimal.

" $\Longrightarrow$ ": If $\widehat{\tau}$ is optimal, then

$$V_0 = \sup_{\tau\in\mathcal{T}} \mathbb{E}^*\left[\frac{C_\tau}{S^0_\tau}\right] = \mathbb{E}^*\left[\frac{C_{\widehat{\tau}}}{S^0_{\widehat{\tau}}}\right] \leqslant \mathbb{E}^*\left[\frac{V_{\widehat{\tau}}}{S^0_{\widehat{\tau}}}\right] \leqslant V_0 .$$

Since $C_{\widehat{\tau}} \leqslant V_{\widehat{\tau}}$ , we have $C_{\widehat{\tau}} = V_{\widehat{\tau}}$ $P^*$–a.s. Moreover, for all $\tau \in \mathcal{T}$:

$$U_0 \geq \mathbb{E}^*\left[U_\tau\right] \geq \mathbb{E}^*\left[U_T\right] = V_0 = U_0 .$$

Hence $\mathbb{E}^*\left[U_\tau\right] = U_0$ . By Lemma 5.6, $U$ is a martingale. QED

CHAPTER 6

# Convergence of Discrete Market Models to the Black–Scholes Model

Literature: **[12]**, 5.7, pp. 259-276.

Notation: $\mathcal{N}(\mu, \sigma^2)$ = normal distribution with mean $\mu$ and standard deviation $\sigma$.

**Definition.** *A stochastic process $W = (W_t)_{t \in \mathbb{R}_+}$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a (standard)* Brownian Motion *(BM) if*

*(1)* $W_0 = 0$;
*(2) its paths $t \mapsto W_t(\omega)$ are continuous for $P$–a.e. $\omega \in \Omega$;*
*(3) for any partition $0 = t_0 < t_1 < \cdots < t_n$ the increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}$, ..., $W_{t_n} - W_{t_{n-1}}$ are independent, and $W_{t_k} - W_{t_{k-1}} \sim \mathcal{N}(0, t_k - t_{k-1})$ for $k = 1, \dots, n$.*

In the Black–Scholes model the price process of a risky asset is modeled as

$$S_t = S_0 \, \mathrm{e}^{\sigma W_t + (\mu + r - \sigma^2/2)t} \quad , \quad t \geq 0$$

where

$\sigma$ : is called the *volatility*, i.e. the standard deviation of the log–returns
$$\log\left(\frac{S_{t+1}}{S_t}\right) =_{\text{in distribution}} \mathcal{N}(\mu + r - \sigma^2/2, \sigma^2);$$

$\mu$ : is called the *drift rate*;
$r$ : the continuously compounded *interest rate.*

In the discrete model: if $F$ is the interest rate per period, then the value of 1 € after $n$ periods is equal to $(1+F)^n$ €.

In the BS model: The future value of $S_0^0$ =1 € is described by the ODE

$$dS_t^0 = r S_t^0 \, dt \, .$$

Observe that $S_t^0 = \mathrm{e}^{rt}$. Continuous compounding corresponds to making the compounding period arbitrary small:

$$\lim_{n \to \infty} \left(1 + \frac{rt}{n}\right)^n = \mathrm{e}^{rt} \, ,$$

historically one of the routes to *define* the RHS already known to Euler and his predecessors, and, conversely, if this definition has been accomplished by other means, easily confirmed, e.g. by l'Hospital's Rule (apply it to $\log(1+rtx)/x$ for $x \to 0^+$ and then consider the case $x = 1/n$).

### Approximating the BS model with discrete market models

Consider a financial market with one risky and one non–risky aset. We fix a time horizon $T$. For every $N \in \mathbb{N}$ we choose a discrete time model with $N$ trading periods of length $T/N$. Let $r_N \geq 0$ be the interest rate per period $T/N$. (The main example to keep in mind: BOPM). Let

$(S_k^{(N)})_{0\leqslant k\leqslant N}$ the prices of the risky asset in the various $T/N$–models.

We define the (relative) returns in the model $N$ as

$$R_k^{(N)} := \frac{S_k^{(N)} - S_{k-1}^{(N)}}{S_{k-1}^{(N)}} \quad , \quad 1 \leqslant k \leqslant N .$$

We assume that for every $N$ the family of returns $(R_k^{(N)})_{1\leqslant k\leqslant N}$ is independent w.r.t an EMM $P_N^*$.

**Remark**. By considering an appropriate product space we may define all the models on a single measure space and we may assume that $P_N^* = P^*$, one fixed measure independent of $N$. □

We say that *condition (C) is satisfied* if

(1) $r_N$ is such that $\lim_{N\to\infty}(1+r_N)^N = \mathrm{e}^{rT}$;
(2) there exists a number $S_0 > 0$ such that $S_0^{(N)} = S_0$ for all $N$, and there exist sequences $(\alpha_N), (\beta_N)$ such that

$$0 < \alpha_N \leqslant R_k^{(N)} \leqslant \beta_N$$

and

$$\lim_{N\to\infty} \alpha_N = \lim_{N\to\infty} \beta_N = 0 ;$$

(3) for $\sigma_N^2 = \frac{1}{T}\sum_{K=1}^N \mathrm{var}_{P^*}(R_K^{(N)})$ we have $\lim \sigma_N^2 = \sigma^2 \in (0,\infty)$.

**Remark**. Note that (1) is equivalent to

(1') $\lim_{N\to\infty} N r_N = rT$.

Namely clearly $\lim_{N\to\infty}(1+r_N)^N = \mathrm{e}^{rT}$ iff $\lim_{N\to\infty} N\log(1+r_N) = rT$, but by l'Hospital's Rule $\lim_{N\to\infty}\log(1+r_N)/R_N = 1$.

Then we have

**Theorem 6.1.** *If condition (C) is satisfied, then the distribution of* $S_N^{(N)}$ *under* $P^*$ *converges weakly to the distribution of*

$$S_T = S_0\, \mathrm{e}^{\sigma W_T + (r-\sigma^2/2)T}$$

*(no drift since we are working under an EMM), i.e. to the log–normal distribution with parameters* $\log(S_0) + (r - \sigma^2/2)T$ *and* $\sigma^2 T$ *.*

(Look at this as a kind of multiplicative Central Limit Theorem.)

Before embarking upon the proof of this theorem, we split some preparations into several lemmas. First, we need the following variant of the Central Limit Theorem CLT:

**Lemma 6.2.** *For each natural number* $N \geq 1$ *let* $V_1^{(N)}, \dots, V_N^{(N)}$ *be* $N$ *independent random variables on a probability space* $(\Omega, \mathcal{F}, P)$ *. Suppose that*

*(i) there exist* $\gamma_N$ *with* $\gamma_N \to 0$ *for* $N \to \infty$ *and for all* $N$ $\left|V_k^{(N)}\right| \leqslant \gamma_N$ $P$*–a.s. for all* $1 \leqslant k \leqslant N$ *;*

*(ii)* $\sum_{k=1}^N \mathbb{E}\left[V_k^{(N)}\right] \longrightarrow m$ *as* $N \to \infty$ *;*

*(iii)* $\sum_{k=1}^N \operatorname{var} V_k^{(N)} \longrightarrow \sigma^2 > 0$ *as* $N \to \infty$ *.*

*Then* $Z_N := \sum_{k=1}^N V_k^{(N)} \xrightarrow{\mathrm{d}} \mathcal{N}(m, \sigma^2)$ *as* $N \to \infty$ *(where* $\xrightarrow{\mathrm{d}}$ *denotes convergence in distribution).*

Proof. Stochastics Lectures. QED

Moreover, we need

**Lemma 6.3.** (Slutzky) *Let* $(X_n)_{n\geq 1}$ *,* $(Y_n)_{n\geq 1}$ *be sequences of random variables such that* $X_n \xrightarrow{\mathrm{d}} X$ *and* $Y_n \longrightarrow 0$ *in probability. Then* $X_n + Y_n \xrightarrow{\mathrm{d}} X$ *.*

Proof *(Sketch).* Recall that $X_n \xrightarrow{\mathrm{d}} X$ iff $\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)]$ for all $f \in \mathcal{C}_b^0(\mathbb{R})$ (the bounded continuous functions). One can show that $X_n \xrightarrow{\mathrm{d}} X$ iff $\mathbb{E}[f(X_n)] \longrightarrow \mathbb{E}[f(X)]$ for all $f \in \mathcal{C}_b^0(\mathbb{R})$ which are uniformly continuous (Exercise).

So let $f \in \mathcal{C}_b^0(\mathbb{R})$ be uniformly continuous, and for $\varepsilon > 0$ let $\delta > 0$ be such that $|x - y| \leqslant \delta$ implies $|f(x) - f(y)| \leqslant \varepsilon$ . Then

$$\begin{aligned}
&|\mathbb{E}[f(X_n + Y_n) - f(X_n)]| \\
&\qquad \leqslant \mathbb{E}\left[|f(X_n + Y_n) - f(X_n)|\, \mathbb{1}_{|Y_n|>\delta}\right] + \mathbb{E}\left[|f(X_n + Y_n) - f(X_n)|\, \mathbb{1}_{|Y_n|\leqslant\delta}\right] \\
&\qquad \leqslant 2\|f\|_\infty P\left[|Y_n| > \delta\right] + \varepsilon \\
&\qquad \longrightarrow \varepsilon \quad \text{for } n \to \infty\,.
\end{aligned}$$

But $\varepsilon$ being arbitrary this yields

$$\lim_n \mathbb{E}\left[f(X_n+Y_n)\right] = \lim_n \mathbb{E}\left[f(X_n)\right] = \mathbb{E}\left[f(X)\right] ,$$

and hence the result. QED

Proof *of Theorem 6.1.* 1. We first show that

$$\log S_N^{(N)} = \log S_0^{(N)} + \sum_{k=1}^{N}\left(R_k^{(N)} - \frac{1}{2}\left(R_k^{(N)}\right)^2\right) + \Delta_N ,$$

with $|\Delta_N| \leqslant \delta(\alpha_N,\beta_N)\sum_{k=1}^N \left(R_k^{(N)}\right)^2$ and $\delta(\alpha,\beta) \longrightarrow 0$ as $\alpha,\beta \to 0$.

Start by writing

$$S_N^{(N)} = S_0^{(N)} \prod_{k=1}^{N} \frac{S_k^{(N)}}{S_{k-1}^{(N)}} = S_0^{(N)} \prod_{k=1}^{N}(1+R_k^{(N)})$$

and so

$$\log\left(S_N^{(N)}\right) = \log\left(S_0^{(N)}\right) + \sum_{k=1}^{N} \log\left(1+R_k^{(N)}\right) \tag{6.1}$$

We now try to estimate the $\log\left(1+R_k^{(N)}\right)$ on the basis of (2) of condition (C). To this end, we Taylor expand $f(x) := \log(1+x)$ around $x=0$ up to third order:

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(\vartheta x)x^3 \quad , \quad 0 \leqslant \vartheta \leqslant 1 ,$$

with

$$f'(x) = \frac{1}{1+x} \quad , \quad f''(x) = -\frac{1}{(1+x)^2} \quad , \quad f'''(x) = \frac{2}{(1+x)^3}$$

and so

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}\frac{1}{(1+\vartheta x)^3} .$$

Notice that for $-1 < \alpha \leqslant x \leqslant \beta$

$$\left|\frac{1}{3}\frac{1}{(1+\vartheta x)^3}\right| \leqslant \frac{1}{3}\frac{1}{(1+\min(0,\alpha))^3}(|\alpha| \vee |\beta|) =: \delta(\alpha,\beta)$$

and so one can write

$$\log(1+x) = x - \frac{1}{2}x^2 + R(x) \quad , \quad |R(x)| \leqslant \delta(\alpha,\beta)x^2 .$$

Applying this estimate to the $\log\left(1+R_k^{(N)}\right)$ in (6.1) clearly establishes the claim at the start of this paragraph.

2. $\Delta_N \longrightarrow 0$ in probability: One has, by taking expectations, and using $\mathbb{E}^*[X^2] = \operatorname{var}_{P^*}(X) + (\mathbb{E}^*[X])^2$ for any random variable $X$:

$$\begin{aligned}
\mathbb{E}^*[\Delta_N] &\leqslant \delta(\alpha_N, \beta_N) \sum_{k=1}^{N} \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] \\
&= \delta(\alpha_N, \beta_N) \sum_{k=1}^{N} \left\{\operatorname{var}_{P^*}\left(R_k^{(N)}\right) + \left(\mathbb{E}^*\left[R_k^{(N)}\right]\right)^2\right\} \\
&= \delta(\alpha_N, \beta_N) \left\{\sigma_N^2 T + N r_N^2\right\}
\end{aligned}$$

(because of (3) of condition (C) and (1) of Lemma 2.4)

$$\longrightarrow 0 \quad \text{for } N \to \infty\,,$$

the last conclusion because $\delta(\alpha_N, \beta_N) \longrightarrow 0$, $\sigma_N^2 T \longrightarrow \sigma^2 T$ and $N r_N \longrightarrow rT$ for $N \to \infty$. So $\Delta_N$ converges indeed to 0 in probability.

In particular, we note for the record that we have computed

$$(6.2)\quad \sum_{k=1}^{N} \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] = \sum_{k=1}^{N} \left\{\operatorname{var}_{P^*}\left(R_k^{(N)}\right) + \left(\mathbb{E}^*\left[R_k^{(N)}\right]\right)^2\right\} = \sigma_N^2 T + N r_N^2\,.$$

3. By Lemma 6.3 it is enough to show that the distribution of

$$Z^{(N)} := \sum_{k=1}^{N} \left(R_k^{(N)} - \frac{1}{2}\left(R_k^{(N)}\right)^2\right)$$

converges to $\mathcal{N}(rT - \sigma^2 T/2, \sigma^2 T)$. We do so by appealing to Lemma 6.2. So we have to verify the three properties (i) – (iii).

Let $V_k^{(N)} := R_k^{(N)} - \frac{1}{2}\left(R_k^{(N)}\right)^2$. Then

(i): $\left|V_k^{(N)}\right| \leqslant \gamma_N + \frac{1}{2}\gamma_N^2$ with $\gamma_N := |\alpha_N| \vee |\beta_N|$. This is clear by point (2) of condition (C).

(ii): $\sum_{k=1}^{N} \mathbb{E}^*\left[V_k^{(N)}\right] = N r_N - \frac{1}{2}(\sigma_N^2 + N r_N^2) \longrightarrow rT - \frac{1}{2}\sigma^2 T$. This follows from

$$\sum_{k=1}^{N} \mathbb{E}^*\left[V_k^{(N)}\right] = \sum_{k=1}^{N} \mathbb{E}^*\left[R_k^{(N)}\right] - \frac{1}{2}\sum_{k=1}^{N} \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right].$$

Now, as already remarked above, $\mathbb{E}^*\left[R_k^{(N)}\right] = r_N$ in view of (1) of Lemma 2.4, and so $\sum_{k=1}^{N} \mathbb{E}^*\left[R_k^{(N)}\right] = N r_N$. The second sum has just been computed in (6.2).

(iii) $\sum_{k=1}^N \mathrm{var}_{P^*}(V_k^{(N)}) \longrightarrow \sigma^2 T$ for $N \to \infty$. This is a little bit more involved. Observe that for any random variable $X$ we have

$$\begin{aligned}\mathrm{var}_{P^*}\left(X - \frac{1}{2}X^2\right) &= \mathbb{E}^*\left[\left(X - \frac{1}{2}X^2\right)^2\right] - \mathbb{E}^*\left[X - \frac{1}{2}X^2\right]^2 \\ &= \mathbb{E}^*\left[X^2 - X^3 + \frac{1}{4}X^4\right] - \mathbb{E}^*\left[X\right]^2 + \mathbb{E}^*\left[X\right]\mathbb{E}^*\left[X\right]^2 - \frac{1}{4}\mathbb{E}^*\left[X^2\right]^2 \\ &= \mathrm{var}_{P^*}(X) - \mathbb{E}^*\left[X^3\right] + \frac{1}{4}\mathbb{E}^*\left[X^4\right] + \mathbb{E}^*\left[X\right]\mathbb{E}^*\left[X\right]^2 - \frac{1}{4}\mathbb{E}^*\left[X^2\right]^2\end{aligned}$$

and so

$$\begin{aligned}\sum_{k=1}^N \mathrm{var}_{P^*}(V_k^{(N)}) = \sum_{k=1}^N \mathrm{var}_{P^*}(R_k^{(N)}) - \sum_{k=1}^N \mathbb{E}^*\left[\left(R_k^{(N)}\right)^3\right] + \frac{1}{4}\sum_{k=1}^N \mathbb{E}^*\left[\left(R_k^{(N)}\right)^4\right] \\ + \sum_{k=1}^N \mathbb{E}^*\left[R_k^{(N)}\right]\mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] + \sum_{k=1}^N \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right]^2\end{aligned}$$

We will now see that all the sums involving higher and mixed terms will vanish: For $p \geq 3$ we have

$$\sum_{k=1}^N \mathbb{E}^*\left[\left|R_k^{(N)}\right|^p\right] \leqslant \gamma_N^{p-2} \underbrace{\sum_{k=1}^N \mathbb{E}^*\left[\left|R_k^{(N)}\right|^2\right]}_{\text{bounded by (6.2)}} \longrightarrow 0 \quad \text{for } N \to \infty\,.$$

Moreover,

$$\sum_{k=1}^N \mathbb{E}^*\left[R_k^{(N)}\right]\mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] = r_N(\sigma_N^2 T + N r_N^2) \longrightarrow 0 \quad \text{for } N \to \infty$$

and, finally, by Jensen's inequality,

$$\sum_{k=1}^N \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right]^2 \leqslant \sum_{k=1}^N \mathbb{E}^*\left[\left(R_k^{(N)}\right)^4\right] \longrightarrow 0 \quad \text{for } N \to \infty\,.$$

Thus,

$$\lim_{N\to\infty} \sum_{k=1}^N \mathrm{var}_{P^*}(V_k^{(N)}) = \lim_{N\to\infty} \sum_{k=1}^N \mathrm{var}_{P^*}(R_k^{(N)}) = \sigma^2 T\,,$$

which is the end of the proof. QED

The next aim is to show one can apply this scaling procedure to the BOPM.

**Example**. Suppose that the market in the $N$th stage is a BOPM with interest rate $r_N := rT/N$. The up and down factors are supposed to be

$$u_N := \mathrm{e}^{\sigma\sqrt{\frac{T}{N}}} \quad , \quad d_N := \frac{1}{u_N} = \mathrm{e}^{-\sigma\sqrt{\frac{T}{N}}}$$

for a given fixed $\sigma > 0$. These data determine the $N$th model for each $N$. We have to verify the three criteria constituting condition (C).

(1) We have

$$(1 + r_N)^N = \left(1 + \frac{rT}{N}\right)^N \longrightarrow \mathrm{e}^{rT} \quad \text{for } N \to \infty\,,$$

so (1) holds trivially by construction. Or, even simpler, (1') is satisfied directly.

(2): By construction of the BOPM, we have that the returns $R_k^{(N)}$ take either the value $u_N - 1$ or $d_N - 1$. Now observer that $u_N \to 1$ and $d_N \to 1$ and that hence (2) is satisfied.

(3): Under the EMM $P^*$ the probability for an up movement has to be equal to

$$p_N^* = \frac{1 + r_N - d_N}{u_N - d_N}\,;$$

the probability for a down movement then is $1 - p_N^*$. The variance of the returns then satisfies

$$\begin{aligned} \mathrm{var}_{P^*}\left(R_k^{(N)}\right) &= \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] - \mathbb{E}^*\left[R_k^{(N)}\right]^2 = \mathbb{E}^*\left[\left(R_k^{(N)}\right)^2\right] - r_N^2 \\ &= p_N^*(u_N - 1)^2 + (1 - p_N^*)(1 - d_N)^2 - r_N^2 \end{aligned}$$

independent of $k$; hence

$$\text{(6.3)} \qquad \sum_{k=1}^{N} \mathrm{var}_{P^*}\left(R_k^{(N)}\right) = p_N^* N(u_N - 1)^2 + (1 - p_N^*)N(d_N - 1)^2 - N r_N^2\,.$$

Now notice that $\lim_{N\to\infty} p_N^* = 1/2$. This follows from L'HOSPITAL's Rule: Write

$$p_N = \frac{1 + rT/N - \mathrm{e}^{-\sigma\sqrt{T/N}}}{\mathrm{e}^{\sigma\sqrt{T/N}} - \mathrm{e}^{-\sigma\sqrt{T/N}}} = \frac{1 + rTx^2 - \mathrm{e}^{-\sigma\sqrt{T}x}}{\mathrm{e}^{\sigma\sqrt{T}x} - \mathrm{e}^{-\sigma\sqrt{T}x}} \quad , \quad x^2 := \frac{1}{N}\,.$$

By L'HOSPITAL, then,

$$\lim_{x\downarrow 0} \frac{1 + rTx^2 - \mathrm{e}^{-\sigma\sqrt{T}x}}{\mathrm{e}^{\sigma\sqrt{T}x} - \mathrm{e}^{-\sigma\sqrt{T}x}} = \lim_{x\downarrow 0} \frac{2rTx + \sigma\sqrt{T}\,\mathrm{e}^{-\sigma\sqrt{T}x}}{\sigma\sqrt{T}(\mathrm{e}^{\sigma\sqrt{T}x} + \mathrm{e}^{-\sigma\sqrt{T}x})} = \frac{1}{2}\,.$$

Finally, notice that $\lim_{N\to\infty} N(u_N - 1)^2 = \lim_{N\to\infty} N(d_N - 1)^2 = \sigma^2 T$, again by L'HOSPITAL's Rule: Write

$$N(u_N - 1)^2 = N\left(\mathrm{e}^{\sigma\sqrt{T/N}} - 1\right)^2 = \frac{\left(\mathrm{e}^{\sigma\sqrt{T/N}} - 1\right)^2}{(1/\sqrt{N})^2} = \frac{\left(\mathrm{e}^{\sigma\sqrt{T}x} - 1\right)^2}{x^2}\,,\ x^2 := \frac{1}{N}\,.$$

and analogously

$$N(d_N - 1)^2 = \frac{\left(\mathrm{e}^{-\sigma\sqrt{T}x} - 1\right)^2}{x^2}.$$

Now one has to apply L'Hospital twice, i.e. one has to differentiate numerator and denominator twice, to conclude

$$\lim_{x\downarrow 0} \frac{\left(\mathrm{e}^{\pm\sigma\sqrt{T}x} - 1\right)^2}{x^2} = \lim_{x\downarrow 0} \frac{2(\pm 1)^2\sigma^2 T\,\mathrm{e}^{\pm\sigma\sqrt{T}x}}{2} = \sigma^2 T$$

so that

$$\lim_{N\to\infty} N(u_N - 1)^2 = \lim_{N\to\infty} N(d_N - 1)^2 = \lim_{x\downarrow 0} \frac{\left(\mathrm{e}^{\pm\sigma\sqrt{T}x} - 1\right)^2}{x^2} = \sigma^2 T$$

as claimed. We so finally get from (6.3)

$$\lim_{N\to\infty} \sum_{k=1}^{N} \mathrm{var}_{P^*}\left(R_k^{(N)}\right) = \frac{1}{2}\sigma^2 T + \frac{1}{2}\sigma^2 T - 0 = \sigma^2 T,$$

and so (3) also holds.

Hence our scaled familiy of BOPMs satisfies condition (C) and Theorem 6.1 is applicable.

It follows that the prices $S_N^{(N)}$ in the BOPM converge to $S_T = S_0\,\mathrm{e}^{\sigma W_T + (r-\sigma^2/2)}$ in distribution under $P^*$, where $W_T \sim \mathcal{N}(0,T)$.

Do we have convergence of option prices? Consider a European option with payoff $f \in \mathcal{C}_b^0(\mathbb{R})$. Convergence in distribution implies $\lim_N \mathbb{E}^*\left[f(S_N^{(N)})\right] = \mathbb{E}^*\left[f(S_T)\right]$. Therefore, the arbitrage–free prices $(1+r_N)^{-N}\mathbb{E}^*\left[f(S_N^{(N)})\right]$ converge to $\mathrm{e}^{-rT} f(S_T)$, what we will define as the Black–Scholes price of the option with payoff $f$.

**Remark**. 1) Let $S_t = S_0\,\mathrm{e}^{\sigma W_t + (r-\sigma^2/2)t}$. The discounted price process $\mathrm{e}^{-rt} S_t = S_0\,\mathrm{e}^{\sigma W_t - \sigma^2 t/2}$ can be shown to be a martingale (e.g. by Itô's formula). Then the BS price is the expectation of the option payoff under a measure that turns the discounted risky asset price process into a martingale.

2) The payoff of a put option $f(x) = (K-x)^+$ belongs to $\mathcal{C}_b^0(\mathbb{R})$. The payoff of a call is unbounded. Nevertheless one can show that its arbitrage–free prices in the discrete models converge to $\mathrm{e}^{-rT}\,\mathbb{E}^*\left[(S_T - K)^+\right]$ (see **[12]**).

!! *From now on, time will be* $\mathbb{R}_+$ !!

CHAPTER 7

# The Black–Scholes Model

Literature: **[24]**

Throughout, let $W$ be a BM defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t^0 := \sigma(W_s : s \leqslant t)$, the $\sigma$–algebra generated by $W$ up to time $t$. Let

$$\mathcal{N} := \{ B \mid \exists A \in \mathcal{F} : B \subseteq A, P[A] = 0 \}$$

be the (completion of the) set of null sets of $P$. Recall that the filtration $\mathcal{F}_t := \mathcal{F}_t^0 \vee \mathcal{N}$ satisfies the "usual conditions" (right continuous, containing $\mathcal{N}$).

## Self–financing Portfolios

Recall the Black–Scholes assumptions. There are two assets: A risk–free one and a risky one. The price $S^0$ of the non–risky asset serves as reference value and is set to 1. It grows with constant interest rate $r > 0$ and hence satisfies the ODE

$$dS_t^0 = rS_t^0 \, dt \quad , \quad S_0^0 = 1 \, .$$

The price $S_t$ of the risky asset is assumed to satisfy the SDE

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t \quad , \quad S_0 = x > 0 \, ,$$

where $\mu$ is the *drift rate* and $\sigma$ the *volatitlity.*

We fix a *time horizon* $T > 0$ (think of the expiration date of an option). A *(trading) strategy* is a pair of $(\mathcal{F}_t)$–adapted stochastic processes $\overline{\xi} = (\xi^0, \xi)$ on $[0, T]$ such that

$$\int_0^T |\xi_t^0| \, dt < \infty \quad \text{and} \int_0^T \xi_t^2 \, dt < \infty \quad P\text{–a.s.}$$

Note that the integral processes

$$\int_0^t \xi_u^0 \, dS_u^0 = \int_0^t \xi_u^0 r \, \mathrm{e}^{ru} \, du$$

and

$$\int_0^t \xi_u \, dS_u^0 = \int_0^t \xi_u \mu S_u \, du + \int_0^t \xi_u \sigma S_u \, dW_u$$

are well–defined on $[0,T]$ e.g. the $dW$–integral is well–defined because we have $\int_0^t \xi_u^2\sigma^2 S_u^2\,du < \infty$ $P$–a.s.

*Interpretation*

$$\begin{aligned} \xi_t^0 &= \#\text{ shares of the non–risky asset in the investor's portfolio at time } t;\\ \xi_t^i &= \#\text{ shares of the risky asset in the investor's portfolio at time } t. \end{aligned}$$

As in the discrete case, we consider only self–financing strategies which we define next.

**Definition.** *Let $\overline{\xi} = (\xi^0, \xi)$ be a strategy. The associated* value process *is defined by*

$$V_t := \xi_t^0 S_t^0 + \xi_t S_t\,. \tag{7.1}$$

**Definition.** *A strategy $\overline{\xi} = (\xi^0, \xi)$ is called* self–financing *if the associated value process $V = (V_t)_{t\in[0,T]}$ satisfies the SDE*

$$dV_t = \xi_t^0\, dS_t^0 + \xi_t\, dS_t\,. \tag{7.2}$$

**Remark**. Recall that in the discrete model of Section 3 a strategy is defined to be self–financing if

$$\underbrace{\overline{\xi}_{t+1}\cdot \overline{S}_t}_{\text{after rebalancing the portfolio}} = \underbrace{\overline{\xi}_t\cdot \overline{S}_t}_{\text{before rebalancing the portfolio}} \qquad \text{for} \quad t \in \{0,1,\dots,T-1\}\,.$$

In this case, this can be reformulated as

$$V_t - V_{t-1} = \xi^0(S_t^0 - S_{t-1}^0) + \xi_t(S_t - S_{t-1}) \quad , \quad 1 \leqslant t \leqslant T\,.$$

In other notation (where $\Delta$ denotes the difference operator: $\Delta X_t = X_t - X_{t-1}$ )

$$\Delta V_t = \xi^0 \Delta S_t^0 + \xi_t \Delta S_t \quad , \quad 1 \leqslant t \leqslant T\,.$$

Letting $\Delta t \downarrow 0$ we get (7.2).

**Lemma 7.1.** *Let $\overline{\xi} = (\xi^0, \xi)$ be a strategy and $V = (V_t)_{t\in[0,T]}$ the associated value process. Then $\overline{\xi}$ is self–financing iff $V$ is a solution of the SDE*

$$dX_t = \xi_t\, dS_t + (X_t - \xi_t S_t) r\, dt \quad , \quad X_0 = V_0\,. \tag{7.3}$$

PROOF. Notice that $\xi_t^0 = \frac{V_t - \xi_t S_t}{S_t^0}$ by the very definition of $V_t$. Therefore, the self–financing condition is equivalent to

$$dV_t = \xi_t\, dS_t + \frac{V_t - \xi_t S_t}{S_t^0}\, dS_t^0 = \xi_t\, dS_t + (V_t - \xi_t S_t)\, r dt$$

because of $dS_t^0 = r S_0^t\, dt$. QED

**Corollary 7.2.** *A self–financing strategy* $\overline{\xi} = (\xi^0, \xi)$ *with value process* $V$ *is uniquely determined by* $\xi$ *and* $V_0$ *.*

PROOF. The SDE for $V$ given by the last lemma is linear of first order and therefore has a unique solution fo a given initial value $V_0$ . So if $X$ is this unique solution, $\xi^0$ is given by $\xi^0_t := \frac{X_t - \xi_t S_t}{S^0_t}$ . QED

In the following we will often work with discounted processes. We define the *discounted price process* $\widetilde{S}$ by $\widetilde{S}_t := \mathrm{e}^{-rt} S_t \,, t \in [0, T]$ .

**Lemma 7.3.** *Let* $\overline{\xi} = (\xi^0, \xi)$ *be a self–financing strategy with value process* $V$ *. The* discounted value process $D := \mathrm{e}^{-rt} V_t$ *satisfies*

$$dD_t = \xi_t \, d\widetilde{S}_t, .$$

PROOF. This will be a consequence of the product rule of Stochastic Analysis. This rule and Lemma 7.1 imply

$$\begin{aligned} \mathrm{e}^{-rt} V_t &= V_0 + \int_0^t V_u \, d\,\mathrm{e}^{-ru} + \int_0^t \mathrm{e}^{-ru} \, dV_u \\ &= V_0 - \int_0^t V_u r \,\mathrm{e}^{-ru} \, du + \int_0^t \mathrm{e}^{-ru} \xi_u \, dS_u + \int_0^t \mathrm{e}^{-ru} (V_u - \xi_u S_u) r \, du \\ &= V_0 + \int_0^t \mathrm{e}^{-ru} \xi_u \, dS_u + \int_0^t r \,\mathrm{e}^{-ru} \xi_u S_u \, du \,. \end{aligned}$$

Note that

$$d\widetilde{S}_t = d(\mathrm{e}^{-rt} S_t) = -r \,\mathrm{e}^{-rt} S_t \, dt + \mathrm{e}^{-rt} \, dS_t$$

and hence

$$\mathrm{e}^{-rt} V_t = V_0 + \int_0^t \xi_u \, d\widetilde{S}_u \,,$$

hence the result, which is but another way of writing this. QED

## Existence of an EMM

We will prove the existence of an EMM (Equivalent Martingale Measure) by a classical result of Stochastic Analysis, GIRSANOV's Theorem, which we now recall:

**Proposition 7.4.** (GIRSANOV Theorem) *Let $(\alpha_t)_{0\leqslant t\leqslant T}$ be an adapted process satisfying $\int_0^T \alpha_s^2\,ds < \infty$ $P$–a.s. and such that*

$$M_t := \exp\left(-\int_0^t \alpha_s\,dW_s - \frac{1}{2}\int_0^t \alpha_s^2\,ds\right)$$

*is a martingale on $[0,T]$ (sufficient conditions for this to be the case are supplied by* NOVIKOV's Theorem*, e.g. it suffices that $\alpha$ is bounded). Let $Q$ be the measure on $\mathcal{F}_T$ defined by*

$$Q\,[A] := \int \mathbb{1}_A M_T\,dP \quad , \quad A \in \mathcal{F}_T\,. \tag{7.4}$$

*Then $Q$ is a probability measure with $Q \sim P$, and for any $P$–martingale $X$ the process*

$$\widetilde{X}_t := X_t - \langle M, X\rangle_t = X_t + \int_0^t \alpha_s\,d\langle W_s, X\rangle \tag{7.5}$$

*is a $Q$–martingale with $\langle \widetilde{X}\rangle = \langle X\rangle$. In particular, $\widetilde{W}_t := W_t + \int_0^t \alpha_s\,ds$ is a Brownian Motion under $Q$.*

PROOF. The proof is a standard topic in Stochastic Analysis lectures. QED

To throw some more light upon the mechanisms behind this theorem it might be helpul to reformulate it in terms of the *stochastic exponential*. First, suppose $X$ is a deterministic variable, i.e. a function $X : U \longrightarrow \mathbb{R}$ defined on some open $U \subseteq \mathbb{R}$. Then the function $Y := \exp(X)$ satisfies the ODE $dY = Y dX$ and is the unique solution of this ODE with initial value $\exp(X_0)$.

Now let $X$ be a continuous semimartingale and consider the process $Y := \exp(X)$. By ITÔ's formula, $Y$ satisfies

$$dY = Y dX + \frac{1}{2} d\langle X\rangle$$

and we see that, as soon as $X$ has nonvanishing quadratic variation, the usual exponential does not have the property of obeying the simple SDE $dY = Y dX$. This leads to the following definition. The *stochastic*, or DOLÉANS–DADE*, exponential* $\mathcal{E}(X)$ of a continuous semimartingale $X$ is defined to be the unique solution $Y$ of the SDE $dY = Y dX$ with initial value $Y_0 = \exp(X_0)$.

For this definition to make sense one must, of course, show that such a solution does indeed exist and is unique. To find a candidate for the solution, we start with some heuristics. Given such a solution $Y$, apply ITÔ's formula to $\log Y$:

$$d\log Y = \frac{1}{Y} dY - \frac{1}{2Y^2} d\langle Y\rangle$$

(this only makes sense if $Y$ is strictly positive; although it will turn out that this is always the case, we do not know this yet, and it is here where the heuristics comes into the play). Now

$dY = YdX$ yields $d\langle Y\rangle = dYdY = Y^2 dXdX = Y^2 d\langle X\rangle$, and plugging all this stuff into the last equation gives

$$d\log Y = dX - \frac{1}{2}d\langle X\rangle = d(X - \frac{1}{2}\langle X\rangle)$$

leading to

$$Y = \exp\left(X - \frac{1}{2}\langle X\rangle\right)$$

establishing our candidate. So we put

$$\mathcal{E}(X) := Y := \exp\left(X - \frac{1}{2}\langle X\rangle\right)$$

and have to show that $Y$ is indeed a solution satisfying the requirements. Applying again ITO's formula, this time to $\exp(Z)$ with $Z := X - \frac{1}{2}\langle X\rangle$ (note that $Y = \exp(Z)$):

$$dY = \exp(Z)\,dZ + \frac{1}{2}\exp(Z)\,d\langle Z\rangle = YdZ + \frac{1}{2}Yd\langle Z\rangle\,.$$

But $dZ = dX - \frac{1}{2}d\langle X\rangle$ and $\langle Z\rangle = \langle X - \frac{1}{2}\langle X\rangle\rangle = \langle X\rangle - \frac{1}{2}\langle\langle X\rangle\rangle = \langle X\rangle$, since $\langle\langle X\rangle\rangle = 0$. Therefore

$$dY = YdX - \frac{1}{2}Yd\langle X\rangle + \frac{1}{2}Yd\langle X\rangle = YdX\,.$$

Since, in addition,

$$Y_0 = \exp\left(X_0 - \frac{1}{2}\langle X\rangle_0\right) = \exp(X_0)$$

because $\langle X\rangle_0 = 0$, this establishes existence.

Uniqueness now is simple but somehow a bit tricky. Let $Y$ be a solution of $dY = YdX$ with $Y_0 = \mathrm{e}^{X_0}$. Consider the process $Y\mathcal{E}(-X)$ and compute its differential by the product formula (this is the first trick):

$$\begin{aligned} d(Y\mathcal{E}(-X)) &= dY\mathcal{E}(-X) + Yd\mathcal{E}(-X) + dYd\mathcal{E}(-X) \\ &= Y\mathcal{E}(-X)\,dX - Y\mathcal{E}(-X)\,dX - Y\mathcal{E}(-X)\,dXdX \\ &= -Y\mathcal{E}(-X)\,d\langle X\rangle\,, \end{aligned}$$

since $dY = YdX$ and $d\mathcal{E}(-X) = -\mathcal{E}(-X)\,dX$. Hence the process $Z := Y\mathcal{E}(-X)$ satisfies $dZ = -ZdX$. On the other hand, a solution of this SDE for $Z$ is given by $\mathcal{E}(-\langle X\rangle) = \mathrm{e}^{-\langle X\rangle}$. Therefore we consider (this is the second trick):

$$\begin{aligned} d(Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}) &= d(Y\mathcal{E}(-X))\,\mathrm{e}^{\langle X\rangle} + Y\mathcal{E}(-X)\,d\,\mathrm{e}^{\langle X\rangle} + d(Y\mathcal{E}(-X))d\,\mathrm{e}^{\langle X\rangle} \\ &= -Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}\,d\langle X\rangle + Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}\,dX + Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}\,d\langle X\rangle d\langle X\rangle \\ &= Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}\,d\langle\langle X\rangle\rangle \\ &= 0\,, \end{aligned}$$

since $\langle X\rangle$ is of bounded variation. So $Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle}$ is a constant, and putting $t = 0$ shows this constant is 1. Hence $Y\mathcal{E}(-X)\,\mathrm{e}^{\langle X\rangle} = Y\,\mathrm{e}^{-X-\frac{1}{2}\langle X\rangle}\,\mathrm{e}^{\langle X\rangle} = Y\,\mathrm{e}^{-X+\frac{1}{2}\langle X\rangle} = 1$, whence $Y = \mathrm{e}^{X-\frac{1}{2}\langle X\rangle} = \mathcal{E}(X)$. This establishes uniqueness.

After this trickery a little pondering reveals, of course, what has happened. Since $\mathcal{E}(X) = \mathrm{e}^{X-\frac{1}{2}\langle X\rangle}$, its inverse is $\mathcal{E}(-X + \langle X\rangle) = \mathrm{e}^{-X+\frac{1}{2}\langle X\rangle}$ and not just $\mathcal{E}(-X)$ as in the deterministic case. So what we have computed is $d(Y/\mathcal{E}(X)) = 1$, and we see what we have been going through was just the same proof of uniqueness as in the classical Calculus, camouflaged by the more complicatd ITÔ Calculus.

In the same vein, one can define the *stochastic logarithm.* If $X$ is a strictly positive continuous semimartingale, its *stochastic logarithm* $\mathcal{L}(X)$ is defined to be the unique solution of the SDE

$$dY = \frac{1}{X} dX \quad , \quad Y_0 = \log X_0 \, .$$

This time, existence and uniqueness are no issue, since this SDE merely asserts, applying the semantics of the Itô Calculus

$$Y_t - Y_0 = \int_0^t \frac{1}{X} \, dX \quad , \quad Y_0 = \log X_0 \, ,$$

and so merely *defines* $\mathcal{L}(X)$ via

$$\mathcal{L}(X)_t := \log X_0 + \int_0^t \frac{1}{X_s} \, dX_s \, .$$

More interesting is the relation of the stochastic logarithm $\mathcal{L}$ to the stochastic exponential $\mathcal{E}$. If $Y = \mathcal{L}(X)$, then $dY = dX/X$ with $Y_0 = \log X_0$, hence $dX = XdY$ with $X_0 = \mathrm{e}^{Y_0}$ and so $X = \mathcal{E}(Y)$. We thus have $\mathcal{E}(\mathcal{L}(X)) = X$. Moreover,

$$X = \mathcal{E}(Y) = \mathrm{e}^{Y - \frac{1}{2}\langle Y \rangle} = \mathrm{e}^{Y} \, \mathrm{e}^{-\frac{1}{2}\langle Y \rangle}$$

and

$$d\langle Y \rangle = dY dY = \frac{1}{X^2} dX dX = \frac{1}{X^2} d\langle X \rangle$$

and so

$$\mathrm{e}^{Y} = X \, \mathrm{e}^{\frac{1}{2}\langle Y \rangle}$$

or

$$Y = \log X + \int \frac{1}{2X^2} d\langle X \rangle \, .$$

In other words, we get

$$\mathcal{L}(X)_t = \log X_t + \int_0^t \frac{1}{2X_t^2} d\langle X \rangle_t \ ,$$

a much nicer formula than the defining formula above, since we no longer have a stochastic integral here, but a nice well–behaved Riemann–Stieltjes–integral. The compatibility with the above defining integral is an immediate application of Itô's formula. Finally, also $\mathcal{L}(\mathcal{E}(Y)) = Y$ since $dX = XdY$ and $X_0 = \mathrm{e}^{Y_0}$ iff $dY = dX/X$ and $Y_0 = \log X_0$. So $\mathcal{E}$ and $\mathcal{L}$ are inverse to one another.

The stochastic exponential has the nice property of being a local martingale as soon as $X$ is, a property also not shared by the usual exponential. The same holds true for the stochastic logarithm. This is because the Itô integral has the following decisive feature:

*If $X$ is a local martingale with values in the open set $U \subseteq \mathbb{R}$ and $f \in \mathcal{C}^1(U)$, then the* Itô *integral*

$$M_t := \int_0^t f(X_u) \, dX_u$$

*is a local martingale.*

This is the key result motivating Itô's ansatz to the Stochastic Calculus via his integral and making his integral superior to other notions of a stochastic integral. The reason for this favourable behaviour of the ITÔ integral can be seen from the formula (♠) on page 4: In the usual approximating sums of the RIEMANN–STIELTJES integral the integrand may be evaluated at any point of the subdivision intervals; in the approximating sums of the ITÔ integral, on the contrary, the integrand *must* be evaluated at the left endpoints of the subdivision intervals, a prescription known as the *non–anticipating character* of the ITÔ integral; its computation does not lean on future knowledge, thus keeping the game fair. This feature makes the ITÔ integral a perfect fit to the needs of Mathematical Finance in describing the value process of a hedging strategy.

Since the SDE for $X$ can be written

$$Y_t = \exp(X_0) + \int_0^t \exp\left(X_t - \frac{1}{2}\langle X\rangle_t\right) d\left(X - \frac{1}{2}\langle X\rangle\right)_t$$

the above described key feature of the ITÔ integral implies that $Y$ is a local martingale as soon as $X$, hence $X - \frac{1}{2}\langle X\rangle$ , is a local martingale.

This raises the question when it will be a martingale. This is equivalent to $\mathcal{E}(X)_T$ being the density of a probability measure, i.e. to $\mathbb{E}[\mathcal{E}(X)_T] = 1$, a question which often can be surprisingly difficult to decide. A suffiecient, but by no means necessary, criterion is provided by the famous

NOVIKOV **Condition**: *If $X$ is a martingale with $\mathbb{E}\left[\frac{1}{2}\langle X\rangle_T\right] < \infty$ , then $\mathcal{E}(X)$ is a martingale.*

With these preliminaries out of the way one may try to make the mechanics behind the GIRSANOV Theorem a little bit more transparent. The first thing to note is that in the formulation above there is a hidden stochastic exponential: If we put $X_t := -\int_0^t \alpha_s\, dW_s$, we have $\langle X\rangle_t = \int_0^t \alpha_s^2\, ds$ by the ITÔ formula and so we recognize $M_t = \mathcal{E}(X)$ as a stochastic exponential, and as such it is a local martingale since $X$ is ($X$ is even a genuine martingale). The measure $Q$ can be alternatively characterized as being defined by the density $\frac{dQ}{dP} := \mathcal{E}(X)_T$ . Since this density is clearly strictly positive, it is immediate that the measure $Q$ is equivalent to $P$.

The next thing that arises is the question why the density considered to define the new measure is to come from a stochastic exponential and required to be a martingale. In fact, there is a very general scheme behind this.

Suppose one is given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and wants to analyze the transition to an equivalent measure $Q \sim P$. This equivalent measure $Q$ then induces for all $t$ measures $Q_t := Q|\mathcal{F}_t$ which are equivalent to $P_t := P|\mathcal{F}_t$ and so we get a family of random variables $D_t$, i.e. a process $D = (D_t)_{t\in[0,T]}$, by

$$D_t := \frac{dQ_t}{dP_t} \quad , \quad \text{the density of } Q_t \text{ w.r.t. } P_t\,.$$

The first clou here is that $D$ *is a martingale*, since by construction for all $s \leqslant t$ we have $Q_t[A_s] = Q_s[A_s]$ for all $A_s \in \mathcal{F}_s$ and so $\mathbb{E}[\,D_t|\,\mathcal{F}_s\,] = D_s$ by the construction of the conditional expectation. This martingale is strictly positive.

Conversely, suppose one is given a strictly positive martingale $D$ on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, then it defines a whole family of probability measures $Q_t$ via prescribing their densities to be $\frac{dQ_t}{dP} := D_t$. The fact that $D$ is a martingale implies that $Q_t \sim P|\mathcal{F}_t$ for all $t$ and that all these measures are compatible in the sense that $Q_t|\mathcal{F}_s = Q_s$ for $s \leqslant t$, hence defining a terminal measure $Q := Q_T$ on the terminal $\sigma$–algebra $\mathcal{F}_T = \sigma\left(\bigcup_t \mathcal{F}_t\right)$ which is equivalent to $P|\mathcal{F}_T$. So we suppose $\mathcal{F}_T = \mathcal{F}$ from now on.

So a measure change from $P$ to an equivalent measure $Q$ is the same as the giving of a strictly positive martingale $D$ w.r.t. to the filtration $(\mathcal{F}_t)$. The next clou is then that a semimartingale $Y$ under $P$ stays a semimartingale under $Q$, because the property of being a semimartingale is stable under a change to an equivalent measure (see e.g. [**18**], beginning of 5.6, pp.249-250); this is a highly nontrivial result, which shows amongst other things the feasibility of the semimartingales as a class of integrators for stochastic integration. Therefore, the only change that the measure change from $P$ to $Q$ can make to $Y$ is to its DOOB–MEYER decomposition

$$Y_t = M_t^P - A_t^P = M_t^Q - A_t^Q \,,$$

and so one has

$$M_t^Q = M_t^P - B_t \quad , \quad A_t^Q = A_t^P - B_t$$

where $B_t = A_t^Q - A_t^P$, hence a process of bounded variation, and so $\langle M^Q\rangle = \langle M^P\rangle = \langle Y\rangle$; a change to an equivalent measure can change the martingale part of a semimartingale only by a process of bounded variation – a so–called drift part – and so cannot change the quadratic variation of the semimartingale. Thus $P$– semimartingales stay $Q$–semimartingales, but $P$–martingales get transformed into $Q$–martingales by shifting them by a drift and vice versa. This is the gist of the GIRSANOV Theorem.

The final clou of the GIRSANOV theorem is that it succeeds in identifying this shift. In the most general situation, if $M^P$ is a local $P$–martingale, it states that the shift $B$ is given as

$$B = \int \frac{1}{D}\, d\langle Y, D\rangle = \langle Y, \mathcal{L}(D)\rangle = \langle M^P, \mathcal{L}(D)\rangle \,,$$

In particular, if $M^P$ is a local $P$–martingale, then

$$M^Q := M^P - \langle \mathcal{L}(D), M^P\rangle$$

is a local $Q$–martingale with $\langle M^Q\rangle = \langle M^P\rangle$ which will be a $Q$–martingale if $M^P$ was a $P$–martingale. Written with differentials this reads

$$dM^Q = dM^P - \frac{1}{D} d\langle M^P, D\rangle \quad \text{equivalently} \quad dM^P = dM^Q + \frac{1}{D} d\langle M^P, D\rangle \,.$$

If $M^P$ is a $P$–martingale, $M^Q$ is a $Q$–martingale, and so upon taking expectations w.r.t. $Q$, the $M^Q$–term drops out and we get that the term $\frac{1}{D} d\langle M^P, D\rangle$ desribes the infinitesimal change of the expectation $\mathbb{E}^Q\left[M^P\right]$ w.r.t. $Q$, hence the name *drift term*.

The core of the argument actually is rather straightforward, but the full chain of arguments is cluttered with technicalities. To give a bare outline, one starts with the plausible equivalence

$$M \text{ (local) } Q\text{–martingale} \iff D \cdot M \text{ (local) } P\text{–martingale}$$

by establishing the BAYES *formula*: if $X$ is a $Q$–integrable random variable, then for all $0 \leqslant t \leqslant T$

$$D_t \mathbb{E}^Q\left[\, X \mid \mathcal{F}_t\,\right] = \mathbb{E}^P\left[\, D_t X \mid \mathcal{F}_t\,\right] \,.$$

After that, the claim is just the product rule: we have,

$$\begin{aligned}
d(D \cdot M^Q) &= dD \cdot M^Q + D \cdot dM^Q + d\langle D, M^Q\rangle \\
&= dD \cdot M^Q + D \cdot \left(dM^P - \frac{1}{D}\langle D, M^P\rangle\right) + d\langle D, M^Q\rangle \\
&= dD \cdot M^Q + D \cdot dM^P - d\langle D, M^P\rangle + d\langle D, M^Q\rangle \\
&= dD \cdot M^Q + D \cdot dM^P
\end{aligned}$$

since $d\langle D, M^Q\rangle = d\langle D, M^P\rangle$ as $M^P$ and $M^Q$ differ only by a process with locally bounded variation. But $d(D \cdot M^Q) = dD \cdot M^Q + D \cdot dM^P$ shows that $D \cdot M^Q$ is a (local) $P$–martingale. So $M^Q$ is a (local) $Q$–martingale.

This has been so far the story from the perspective of a given probability measure $Q$ equivalent to $P$. In many applications it is just the other way round; one is given a semimartingale $Y$ and wants to construct an equivalent measure which transforms $Y$ in a specific way, e.g. into a solution of a specific SDE. Since the transformation of $Y$ is determined by $\mathcal{L}(D)$ where $D$ is the density process of the sought–for measure $Q$ w.r.t. the start measure $P$, one begins with a process $G$ – which we will call the GIRSANOV process – and requires $G = \mathcal{L}(D)$, i.e. $D := \mathcal{E}(G)$. Since the process $D$ should be a martingale in the end, the starting process $G$ should at least be a local martingale. One then has usually to verify the following two basic properties:

- The process $\mathcal{E}(G)$ should be a true martingale. This is equivalent with $Q$ defined by the density $\mathcal{E}(G)_T$ being a probability measure, which, in turn, is equivalent to $\mathbb{E}\left[\mathcal{E}(G)_T\right] = 1$. This can be surprisingly difficult to verify.
- The transformed process $Y - \langle G\rangle Y$ should have the desired properties w.r.t. $Q$, e.g. be a martingale.

In this way, on a filtered probability space, measure change processes $D$ induced by a probability measure $Q$ equivalent to a given probability measure $P$ and local $P$–martingales $G$ such that the stochastic exponential $\mathcal{E}(G)$ is a martingale correspond under $G := \mathcal{L}(D)$ and $D := \mathcal{E}(G)$. In particular, this explains the presence of the stochastic exponential in the usual formulations of the GIRSANOV Theorem.

Therefore, the setup of the GIRSANOV Theorem, which might look hermetic at first sight, reveals itself as being the generic scenario for the change to an equivalent measure. In particular, if one manages, given a semimartingale $Y$ under a measure $P$, to find a $G$ such that $\langle G, Y\rangle$ matches the drift part of $Y$, this drift part is annihilated under the new measure generated by $G$ via the density $\mathcal{E}(G)$, and $Y$ turns into a martingale under the changed measure. This explains the predominant role the GIRSANOV theorem plays in constructing martingale measures.

Finally, the above formulation in Proposition 7.4 describes the measure changes corresponding to local martingales $G$ under a BM. The structure of those local martingales is described in the famous ITÔ *Representation Theorem* (see Proposition 7.8 and the subsequent comments with the Propositions 7.8A. and 7.8B. below). From these result one sees that the required form of $M_t$ above is general, and I hope that this quite long elaborations have shed some light on the question why the formulation of Proposition 7.4 is as given.

In the following let

$$\vartheta := \frac{\mu - r}{\sigma} \quad \text{(ususally called the “market price of risk”)}$$

$$M_t := \mathrm{e}^{-\vartheta W_t - \frac{1}{2}\vartheta^2 t} \quad , \quad 0 \leqslant t \leqslant T\,.$$

Let $P^*$ be defined by

$$P^*[A] := \int \mathbb{1}_A M_T \, dP \quad , \quad A \in \mathcal{F}_T$$

and put

$$B_t := W_t + \vartheta t\,.$$

By Proposition 7.4 $B = (B_t)_{t\in[0,T]}$ is a BM under $P^*$.

**Proposition 7.5.** *The discounted price $\widetilde{S}$ is a $P^*$–martingale, and $d\widetilde{S}_t = \sigma \widetilde{S}_t dB_t$.*

Proof. One computes

$$\begin{aligned} d\widetilde{S}_t &:= d(\mathrm{e}^{-rt} S_t) \\ &= -r\,\mathrm{e}^{-rt} S_t dt + \mathrm{e}^{-rt} dS_t \\ &= -r\,\mathrm{e}^{-rt} S_t dt + \mathrm{e}^{-rt} \mu S_t dt + \mathrm{e}^{-rt} \sigma S_t dW_t \\ &= (\mu - r)\,\mathrm{e}^{-rt} S_t dt + \sigma\,\mathrm{e}^{-rt} S_t dW_t \\ &= \vartheta\sigma\,\mathrm{e}^{-rt} S_t dt + \sigma\,\mathrm{e}^{-rt} S_t dW_t \\ &= \sigma\,\mathrm{e}^{-rt} S_t d(\vartheta t + W_t) \\ &= \sigma \widetilde{S}_t dB_t\,, \end{aligned}$$

and so the second claim holds true. It implies

$$\widetilde{S}_t = S_0\,\mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t}\,,$$

a GBM (Geometric Brownian Motion), and hence a (strict, not only local) martingale under $P^*$. QED

The choice of $\vartheta$ might appear to come out of the blue, but is, in fact, quite direct and natural once one follows consequently the Girsanov setup as described in the comments immediately following Proposition 7.4. Since the filtration on our probability space is the one generated by the presupposed BM $W$, we start with the Girsanov-martingale as a martinale w.r.t. the given BM $W$; by the Martingale Representation Theorem (Proposition 7.8 below) it must be of the form $G_t = \int_0^t \gamma_s\, dW_s$ and should have the properties

- the associated densitiy process
$$M_t := \mathcal{E}\,(G)_t = \exp\left(\int_0^t \gamma_s\, dW_s - \frac{1}{2}\int_0^t \gamma_s^2\, ds\right)\,,$$
which is a local martingale, should be a true martingale, so that $P^*$ with $dP^*/dP := M_T$ is a probability measure;
- under $P^*$, the discounted price $\widetilde{S}_t$ should be a martingale.

Now we have seen above that $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$ satisfies the SDE

$$\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t\,.$$

Under the Girsanov transformation with $G_t$ the process $W$, which is a BM under $P$, gets transformed into

$$B_t = W_t - \langle G, W\rangle_t = W_t - \int_0^t \gamma_s\, ds\,.$$

Define $P^*$ by $dP^*/dP := M_T$. Under this measure $P^*$, the SDE for $\widetilde{S}_t$ becomes

$$d\widetilde{S}_t = (\mu - r)\widetilde{S}_t dt + \sigma \widetilde{S}_t d\left(W_t + \int_0^t \gamma_s \, ds\right) = (\mu - r + \sigma\gamma_t)\widetilde{S}_t dt + \sigma \widetilde{S}_t dW_t$$

so that we see that, if we choose

$$\gamma_t := -\frac{\mu - r}{\sigma} = -\vartheta \quad \text{i.e.} \quad G_t := -\vartheta W_t \,,$$

we get $d\widetilde{S} = \sigma \widetilde{S} dB = \widetilde{S} d(\sigma B)$, hence $\widetilde{S} = \mathcal{E}\,(\sigma B)$. In order to finish, we have to show

- $M := \mathcal{E}\,(G) = \mathcal{E}\,(-\vartheta W)$ is a $P$–martingale: Since $W$ is a local $P$–martingale, $M$ is a local $P$–martingale, and NOVIKOV's Condition (see page 85) is trivially fulfilled, so $M$ is indeed a $P$–martingale;
- $\widetilde{S} = \mathcal{E}\,(\sigma B)$ is a $P^*$–martingale: since $M$ is a $P$–martingale, $B$ is a BM under $P^*$ and so $\widetilde{S}$ is a local $P^*$–martingale. But again NOVIKOV's Condition is trivially fulfilled, and so $\widetilde{S}$ is a $P^*$–martingale.

Hence $P^*$ is an EMM.

As a byproduct, we get the solution given above:

$$\widetilde{S}_t = \mathcal{E}\,(\sigma B)_t = \mathrm{e}^{\sigma B_t - \frac{1}{2}\sigma^2 t} = \mathrm{e}^{\sigma W_t + (\sigma\vartheta - \frac{1}{2}\sigma^2)t} \,,$$

which is compliant with the solution for $S$ given at the beginnng of the course (see page 1).

Note that this discussion reveals that the risk–neutral measure is unique. In the discrete case this implies that the model under consideration is complete, so it suggests at least that the BS model is complete; this will be the next topic of the course after we will have discussed the issue of arbitrage–freeness.

It is, by the way, interesting to note that the BLACK–SCHOLES SDE retains it form under the new measure $P^*$:

$$dS_t = rS_t dt + \sigma S_t dB_t$$

which is immediately verified on the basis of $B_t = W_t + \vartheta t$. The difference to the SDE under the real–world measure $P$ is that the drift rate $\mu$, which is unobservable anyway, has been transformed away by the GIRSANOV transformation and been replaced by the constant interest rate $r$. This makes one think about the roles of a real–world measure and of an EMM in modelling financial processes, and some helpful remarks concerning this can be found in [**13**], Section 4, pp. 7-9.

Thus we have constructed an EMM. In the discrete case this would allow us to conclude that the model is arbitrage–free, since in this case the existence of an EMM and arbitrage–freeness are equivalent notions. In the continuous case this is not quite so, and we are now going to see that we have to modify the notion of being arbitrage–free, albeit in a way which is quite meaningful economically.

**Definition.** *A self–financing strategy $\overline{\xi}$ is called* admissible *if the associated value process $V$ is bounded from below, i.e. if there exists $c \in \mathbb{R}$ such that $V_t \geq c$, $P$–a.s. for all $t \in [0, T]$.*

**Lemma 7.6.** *Let $\overline{\xi}$ be an admissible strategy with value process $V = (V_t)_{t\in[0,T]}$. Then the discounted value process $D = (D_t)_{t\in[0,T]}$, $D_t := \mathrm{e}^{-rt} V_t$, is a $P^*$–supermartingale.*

PROOF. Note that by Lemma 7.3

$$dD_t = \xi_t \, d\widetilde{S}_t = \xi_t \sigma \widetilde{S}_t \, dB_t$$

which immediately implies that $D$ is a local martingale under $P^*$. Recall from Stochastic Analysis that any non–negative local martingale with $M_0 \in \mathbb{R}$ (i.e. $M_0$ a deterministic point) is a supermartingale (this is a direct application of FATOU's Lemma; if $M_0$ is allowed to be random, counterexamples can be constructed). Since $D$ is bounded from below by assumption, it becomes non–negative by adding a suitable constant; this shows that $D$ is a $P^*$–supermartingale. QED

**Definition.** *A self–financing strategy $\overline{\xi} = (\xi^0, \xi)$ is called an* arbitrage opportunity *if the associated value process $V$ satisfies*

(i) $V_0 \leqslant 0$;
(ii) $V_T \geq 0 \quad P$–*a.s.*;
(iii) $P[V_T > 0] > 0$.

**Proposition 7.7.** *There are no admissible strategies that are arbitrage opportunities.*

PROOF.. Let $\overline{\xi}$ be an admissible strategy. Assume that its associated value process $V$ satisfies $V_T \geq 0$ $P$–a.s. and $P[V_T > 0] > 0$. By Lemma 7.6 the discounted value process $D_t = \mathrm{e}^{-rt} V_t$ is a $P^*$–supermartingale. Hence

$$V_0 = D_0 \geq \mathbb{E}^* [D_T] > 0 \,,$$

which shows that $\overline{\xi}$ cannot be an arbitrage opportunity. QED

We next give an example for a self–financing strategy that is an arbitrage opportunity (and thus cannot be bounded from below).

**Example**. Let $V_0 := 0$ and $\zeta_t := \dfrac{1}{T-t}\dfrac{1}{\sigma \widetilde{S}_t}$, $t \in [0, T)$. Define a process on $[0, T)$ via

$$X_t := \int_0^t \zeta_u \, d\widetilde{S}_u = \int_0^t \frac{1}{T-u} \, dB_u \quad , \quad t \in [0, T)$$

(note that the last integrand is square–integrable over any interval $[0, t]$ with $t < T$, but not over $[0, T)$). One can interpret $X_t$ as a time–changed BM: Let

$$A(t) := \langle X \rangle_t = \int_0^t \frac{1}{(T-u)^2} \, du \quad , \quad t \in [0, T)$$

the (pathwise defined) quadratic variation of $X$, and let $a(t) := A^{-1}(t)$ for $t \in \mathbb{R}_+$. We put $\widetilde{W}_t := X_{a(t)}$ and claim this is a BM. For this, use LÉVY's Theorem: If a local martingale $M$ is such that $\langle M \rangle_t = t$, then $M$ is already a BM. Observe that $\langle \widetilde{W} \rangle_t = \langle X \rangle_{a(t)} = A(a(t)) = t$ for all $t \in \mathbb{R}_+$, and so LÉVY's Theorem implies that $\widetilde{W}$ is indeed a BM under $P^*$. Let

$$\tau := \inf \{ t \geq 0 \mid X_t = 1 \}$$

and

$$\gamma := \inf \left\{ t \geq 0 \mid \widetilde{W}_t = 1 \right\} .$$

Recall that $P^*[\gamma < \infty] = 1$ and $\mathbb{E}^*[\gamma] = \infty$. Since $a(\gamma) = \tau$, we have $P^*[\tau < T] = 1$.

Now we can define our arbitrage strategy. Let

$$\xi_t := \zeta_t \mathbb{1}_{[0,\tau]}(t) .$$

Note that $\int_0^T \xi_u^2 \, du < \infty$ $P$–a.s. The quantities $\xi$ and $V_0 = 0$ uniquely define a self–financing strategy $\overline{\xi} = (\xi^0, \xi)$ by Corollary 7.2. Let $V$ be the associated value process. By Lemma 7.3

$$\mathrm{e}^{-rT} V_T = \int_0^T \xi_t \, d\widetilde{S}_t = \int_0^\tau \frac{1}{T-u} \, dB_u = X_\tau = 1$$

$P$–a.s., and so $P^*$–a.s., which implies that $\xi$ is an arbitrage opportunity.

This kind of arbitrage possibility is a continuous cousin of a family of strategies in discrete models known classically, in the case of gambling strategies, as *martingale strategies*, or *martingales* in brief. Originally, the term *martingale* referred to the doubling strategy in a fair coin–tossing game, but then was applied to any strategy which runs higher and higher risks in order to compensate for all former losses (the origin of the term *martingale* has been a subject of much debate over the years, without conclusive results, see **[26]**). It is the classical way into ruin, but nevertheless again and again through the years people fall prey to one or the other variant of this strategy, as did the guy who ruined the Barings Bank in 1995.

Proposition 7.7 implies that $\overline{\xi}$ is not admissible (it is an exercise to check this directly). In light of the above discussion the notion of admissibility is economically meaningful; the ruin probability for these martingale strategies is 1, since the case of arbitrary large resource needs will occur almost surely, and no one has limitless resources. It makes therefore sense to consider only admissible strategies. □

As said above, in the continuous case the existence of an EMM is not equivalent to arbitrage–freeness as defined in the discrete case. This observation initiated the search for a notion replacing arbitrage–freeness which would be equivalent to the existence of an EMM on one hand and economically plausible on the other, and which would be valid in maximal generality. This was the begin of a long and convoluted, but interesting, story, which ended for the time being with the quite technical notion of "no free lunch with vanishing risk" of Delbaen and Schachermayer see [**6**], [**7**] and [**37**]. A clear and lucid survey of this story can be found in [**33**]; see also [**8**].

### Completeness of the BM Model

Recall the *Martingale Representation Theorem:*

**Proposition 7.8.** (Martingale Representation Theorem for square integrable Brownian martingales) *Let $W_t$ be a Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ being generated by $W$. Let $M = (M_t)_{t\in[0,T]}$ be a square–integrable $P$–martingale w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$. Then there exists an adapted process $H = (H_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T H_s^2\, ds\right] < \infty$ and*

$$M_t = M_0 + \int_0^t H_s\, dW_s \quad . \quad t \in [0,T]\,.$$

Proof. Stochastic Analysis lectures (e.g. [**32**], Theorem 2.5.2) or any generic book on Stochastics (e.g. [**28**], p. 186, Theorem 43, or [**18**], Satz 5.37). QED

There are a couple or remarks to make here which will be useful in future comments.

1) This representation is *unique* in the sense that, if one has

$$M_t = M_0 + \int_0^t H_s\, dW_s = M_0 + \int_0^t \widetilde{H}_s\, dW_s\,,$$

then $H = \widetilde{H}$ $P \otimes \lambda$–a.s., where $H, \widetilde{H}$ are considered as maps $\Omega \times [0,T] \to \mathbb{R}$ and $\lambda$ denotes Lebesgue measure.

This is a special case of the following simple result:

**Lemma.** *Let $W_t$ be a Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ being generated by $W$. Let $\alpha = (\alpha_t)_{t\in[0,T]}$ be an adapted process with $\mathbb{E}\left[\int_0^T \alpha_s^2\, ds\right] < \infty$. If the integral process $\int \alpha\, dW$ vanishes $P \otimes \lambda$–a.s., then so does $\alpha$.*

For the proof, just observe that the integral process has quadratic variation $\int \alpha_s^2\, ds$.

From this, uniqueness above follows easily since we have $\int_0^t (H_s - \widetilde{H}_s)\, dW_s = 0$ $P \otimes \lambda$–a.s.

2) There are two succesive generalizations which will be of importance in later comments. The first one is that the Martingale Representation Theorem holds as above for general local $P$–martingales:

**Proposition. 7.8A.** (Representation Theorem for local Brownian martingales) *Let $W_t$ be a Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ , the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ being generated by $W$. Let $M = (M_t)_{t\in[0,T]}$ be a local $P$–martingale w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$ . Then $M$ is continuous, and there exists a $P \otimes \lambda$–almost unique adapted process $H = (H_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T H_s^2\, ds\right] < \infty$ and*

$$M_t = M_0 + \int\limits_0^t H_s\, dW_s \quad . \quad t \in [0,T]\,.$$

The references for the proof are as above.

In particular, since any ITÔ integral w.r.t. a BM is a local martingale (see ), this result characterizes the local martingales under a BM.

Secondly, this generalizes to $d$–dimensional BMs:

**Proposition. 7.8B.** (Representation Theorem for local $d$–dimensional Brownian martingales) *Let $W_t = (W_t^1, \dots, W_t^d)$ be a $d$–dimensional Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ , the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ being generated by $W$. Let $M = (M_t)_{t\in[0,T]}$ be a local $P$–martingale w.r.t. $(\mathcal{F}_t)_{t\in[0,T]}$ . Then $M$ is continuous, and there exist a $d$–dimensional $P \otimes \lambda$–almost unique adapted processes $H_t = (H_t)_{t\in[0,T]}$ such that $\mathbb{E}\left[\int_0^T \|H\|^2\, ds\right] < \infty$ and*

$$M_t = M_0 + \int\limits_0^t H_s \cdot dW_s\ = M_0 + \sum_{i=1}^d \int\limits_0^t H_s{}^i dW_s^i \quad . \quad t \in [0,T]\,,$$

*where $H := (H^1, \dots, H^d)$ .*

Again, the references for the proof are as above.

**Definition.** *In this section, by a* contingent claim *we mean an $\mathcal{F}_T$–measurable random variable $C \geq 0$ that is square–integrable w.r.t. $P^*$ .*

**Theorem 7.9.** *For any contingent claim $C$ there exists an admissible strategy $\overline{\xi} = (\xi^0, \xi)$ such that the associated value process is given by*

$$V_t = \mathrm{e}^{-r(T-t)}\, \mathbb{E}^* \left[C|\mathcal{F}_t\right]\,.$$

*In particular, $V_T = C$ , i.e. $\overline{\xi}$ replicates $C$ .*

PROOF. The Theorem is essentially a consequence of the Martingale Representation Theorem. Let $M_t := \mathrm{e}^{-r(T-t)}\, \mathbb{E}^* \left[C|\mathcal{F}_t\right]$. Then $M$ is a martingale, in fact a martingale which is square–integrable w.r.t $P^*$. That $M$ is a martingale is immediate from the tower property of conditional expectations. That $M$ is square–integrable w.r.t. $P^*$ follows since $C$ is square–integrable w.r.t $P^*$. The

Martingale Representation Theorem then implies that there exists an adapted process $H$ with $\mathbb{E}^*\left[\int_0^T H_s^2\,ds\right] < \infty$ and $M_t = M_0 + \int_0^t H_s\,dB_s\;\; t \in [0,T]$.

Let
$$\xi_t := \frac{H_t}{\sigma \widetilde{S}_t}\,.$$
Moreover,let $\overline{\xi} = (\xi^0, \xi)$ be the self–financing strategy determined by $V_0 := M_0$ and $\xi$. Then the discounted value process $D_t = \mathrm{e}^{-rt}\, V_t$ associated to $\overline{\xi}$ satisfies

$$\begin{aligned}
D_t &= D_0 + \int_0^t \xi_u\, d\widetilde{S}_u \quad \text{by Lemma 7.3}\\
&= D_0 + \int_0^t \xi_u \sigma \widetilde{S}_u\, dB_u \quad \text{by Proposition 7.5}\\
&= M_0 + \int_0^t H_u\, dB_u\\
&= M_t\,.
\end{aligned}$$

This implies
$$V_t = \mathrm{e}^{rt}\, M_t = \mathrm{e}^{-r(T-r)}\, \mathbb{E}^*\left[C|\mathcal{F}_t\right]\,.$$
Why is $\xi$ admissible? This is just one simple observation: since $C \geq 0$ $P$–a.s. and so $P^*$–a.s., we have $V_t \geq 0$ $P^*$–a.s. and so $P$–a.s. QED

Next, I want to argue that the value of the replicating portfolio is the only arbitrage–free price. So let $C$ be a contingent claim and $\overline{\xi} = (\xi^0, \xi)$ the replicating process with value process $V_t = \mathrm{e}^{rt}\, M_t = \mathrm{e}^{-r(T-r)}\, \mathbb{E}^*\left[C|\mathcal{F}_t\right]$. Suppose that the claim is traded at a market price $M_t \neq V_t$ at time $t$. Then we are now going to show there exists an arbitrage opportunity. The general philosophy behind the construction of such an arbitrage opportunity is, of course, to sell the claim when the market price is too high (i.e. larger than $V_t$) and to buy when it is to low (i.e. smaller than $V_t$).

*1$^{st}$ case:* $M_t < V_t$. Buy the claim at time $t$ and sell the replicating portfolio. So you hold

$$\begin{aligned}
\zeta_u^0 &= -\xi_u^0 + \frac{1}{S_t^0}(V_t - M_t) && \text{non–risky asset shares}\\
\zeta_u &= -\xi_u && \text{risky asset shares}
\end{aligned}$$

at $u \in [t,T]$. Then

Portfolio value at $t$:
$$\zeta_t^0 S_t^0 + \zeta_t S_t + M_t = (-\xi_t^0 S_t^0 + V_t - M_t) + (-\xi_t S_t) + M_t$$

$$\begin{aligned}
&= -\xi_t^0 S_t^0 - \xi_t S_t + V_t \\
&= 0\,.
\end{aligned}$$

Portfolio value at $T$:

$$\zeta_T^0 S_T^0 + \zeta_t S_T + M_T = \zeta_T^0 S_T^0 + \zeta_t S_T + C$$

(since any other value of $M_T$ would lead to an immediate arbitrage opportunity modifying the market price)

$$\begin{aligned}
&= \left(-\xi_T^0 S_T^0 + \frac{S_T^0}{S_t^0}(V_t - M_t)\right) + (-\xi_T S_T) + C \\
&= -\xi_T^0 S_T^0 - \xi_T S_T + C + \frac{S_T^0}{S_t^0}(V_t - M_t) \\
&= \frac{S_T^0}{S_t^0}(V_t - M_t) \\
&> 0 \quad P\text{–a.s.}
\end{aligned}$$

and so we have arbitrage.

*2*$^{nd}$ *case:* $M_t > V_t$. Sell the claim at time $t$ and buy the replicating portfolio. So you hold

$$\begin{aligned}
\zeta_u^0 &= \xi_u^0 + \frac{1}{S_t^0}(M_t - V_t) && \text{non–risky asset shares} \\
\zeta_u &= \xi_u && \text{risky asset shares}
\end{aligned}$$

at $u \in [t, T]$. Similar calculations as above show:

Portfolio value at $t$: $0$.

Portfolio value at $T$: $\dfrac{S_T^0}{S_t^0}(M_t - V_t) > 0$.

and so again we have arbitrage.

The upshot is that the value process $V_t = \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[C|\mathcal{F}_t\right]$ is the only candidate for an arbitrage–free price process for $C$. And it is indeed an arbitrage–free price process in the sense that in the extended market model $(S^0, S, V)$ there are no arbitrage opportunities. I will not give a formal proof of this statement which is sufficiently parallel to the discrete case. The philosophy should be clear: one sets up the extended market model, chooses an EMM $P^*$ for the original model; then $P^*$ will be an EMM for the extended model iff the discounted value process of a replicating portfolio is a $P^*$–martingale.

As a final remark, note that the fact that $P^*$ is an EMM for the extended model $(S^0, S, V)$ implies that there are no admissible strategies in the extended model that are arbitrage opportunities (in the discrete case this was the easy part of the FFToAP).

**Definition.** *The* BS–price (BLACK–SCHOLES–price) *of a contingent claim $C$ at time $t$ is defined by* $\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[C|\mathcal{F}_t\right]$.

Now the time has come for the famous BLACK–SCHOLES–formula.

**Proposition 7.10.** *Let $C=(S_T-K)^+$, the payoff of a European call. Then the BS–price of $C$ at time $t=0$ is*

$$\text{BS–call}(S_0,K,T,r,\sigma)=S_0\Phi(d_1)-\mathrm{e}^{-rT}K\Phi(d_2)$$

*where $\Phi$ is the distribution function of $\mathcal{N}(0,1)$ and*

$$\begin{aligned} d_1 &:= \frac{\log(S_0/K)+(r+\sigma^2/2)T}{\sigma\sqrt{T}} \\ d_2 &:= \frac{\log(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}} \\ &= d_1-\sigma\sqrt{T}\,. \end{aligned}$$

PROOF. Note that

$$S_T=S_0\,\mathrm{e}^{\sigma B_T+(r-\sigma^2/2)T}\;.$$

Now recall that, if $X:\Omega\longrightarrow\mathbb{R}$ is a random variable on a probability space $(\Omega,\mathcal{F},P)$ and $f:\mathbb{R}\longrightarrow\mathbb{R}$ a measurable function, the expectation value $\mathbb{E}\left[f(X)\right]$ of the random variable $f(X)$ can be computed as an integral over $\mathbb{R}$:

$$\mathbb{E}\left[f(X)\right]=\int f\,dP_X$$

where $P_X$ is the probability measure on the BOREL measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ given by

$$\forall\, A\in\mathcal{B}(\mathbb{R}):\ P_X\left[A\right]:=P\left[X^{-1}(A)\right]=P\left[X\in A\right]\,.$$

In particular, if $P_X$ is absolute continuous w.r.t. the LEBESGUE measure on $\mathbb{R}$ with densitiy function $\varphi_X$ , one has

$$\mathbb{E}\left[f(X)\right]=\int f(x)\varphi_X(x)\,dx\,.$$

In our case we have $(S_T-K)^+=f(X)^+$ with

$$f(x)=S_0\,\mathrm{e}^{\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K$$

and

$$X:=\frac{B_T}{\sqrt{T}}$$

an $\mathcal{N}(0,1)$–distributed random variable.

Hence

$$\mathbb{E}^*\left[(S_T-K)^+\right]=\int\left(S_0\,\mathrm{e}^{\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K\right)^+\varphi(x)dx$$

where $\varphi(x) = (1/\sqrt{2\pi})\,\mathrm{e}^{-x^2/2}$. Now

$$S_0\,\mathrm{e}^{\sigma\sqrt{T}x+(r-\sigma^2/2)T} \geq K \iff x \geq -\frac{\log(S_0/K)+(r-\sigma^2/2)T}{\sigma\sqrt{T}} = -d_2\,.$$

Therefore

$$\begin{aligned}
\mathbb{E}^*\left[(S_T-K)^+\right] &= \int_{-d_2}^{\infty}\left(S_0\,\mathrm{e}^{\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K\right)\varphi(x)dx\\
&= \int_{-\infty}^{d_2}\left(S_0\,\mathrm{e}^{-\sigma\sqrt{T}x+(r-\sigma^2/2)T}-K\right)\varphi(x)dx\\
&= S_0\,\mathrm{e}^{rT}\int_{-\infty}^{d_2}\mathrm{e}^{-\sigma\sqrt{T}x-\sigma^2T/2}\;\mathrm{e}^{-x^2/2}\;\frac{1}{\sqrt{2\pi}}dx - K\int_{-\infty}^{d_2}\mathrm{e}^{-x^2/2}\;\frac{1}{\sqrt{2\pi}}dx\\
&= S_0\,\mathrm{e}^{rT}\int_{-\infty}^{d_2}\mathrm{e}^{-(x+\sigma\sqrt{T})^2/2}\;\frac{1}{\sqrt{2\pi}}dx - K\int_{-\infty}^{d_2}\mathrm{e}^{-x^2/2}\;\frac{1}{\sqrt{2\pi}}dx\\
&= S_0\,\mathrm{e}^{rT}\int_{-\infty}^{d_2+\sigma\sqrt{T}}\mathrm{e}^{-x^2/2}\;\frac{1}{\sqrt{2\pi}}dx - K\int_{-\infty}^{d_2}\mathrm{e}^{-x^2/2}\;\frac{1}{\sqrt{2\pi}}dx\,,
\end{aligned}$$

whence the result. QED

**Proposition 7.11.** *Let $P = (K-S_T)^+$, the payoff of a European put. Then the BS–price of $P$ at time $t=0$ is*

$$\text{BS–put}(S_0,K,T,r,\sigma) = -S_0\Phi(-d_1)+\mathrm{e}^{-rT}\,K\Phi(-d_2)\,.$$

*(Mnemonic: Just reverse all the signs.)*

PROOF. Amounts to a calculation entirely analogous to the one performed in the proof of Proposition 7.10. QED

Another method of proof appeals to

**Proposition 7.12.** (Put–Call–Parity)

$$\text{BS–put}(S_0,K,T,r,\sigma) = \text{BS–call}(S_0,K,T,r,\sigma) - S_0 + \mathrm{e}^{-rT}\,K\,.$$

PROOF. Note that for any real number $x$ one has $x^+ = \dfrac{|x|+x}{2}$. Hence

$$(-x)^+ = \frac{|-x|-x}{2} = \frac{|x|-x}{2} = \frac{|x|+x}{2} - x = x^+ - x\,.$$

So we have

$$(K-S_T)^+ = (S_T-K)^+ - S_t + K\,,$$

and hence, upon taking expectations,

$$\begin{aligned}
e^{-rT}\,\mathbb{E}^*\left[(K-S_T)^+\right] &= e^{-rT}\,\mathbb{E}^*\left[(S_T-K)^+\right] - e^{-rT}\,\mathbb{E}^*\left[S_T\right] + e^{-rT}\,K \\
&= \text{BS–call}(S_0,K,T,r,\sigma) - S_0 + e^{-rT}\,K\,.
\end{aligned}$$

QED

**Remark**. The Put–Call–Parity holds true not only in the BS–model but also in other models which are based on the same general principles (e.g. pricing based on the no–arbitrage paradigm). It holds in such models, since otherwise there would be – what else? – arbitrage opportunities.

## The Black-Scholes-PDE

Let $h:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ be Borel–measurable. Suppose that there exists $c,p>0$ such that $h(x)\leqslant c(1+x^p)$. Note that $h(S_T)$ is square–integrable w.r.t. $P^*$ and hence a contingent claim. Let

$$V_t = e^{-r(T-t)}\,\mathbb{E}^*\left[h(S_T)|\mathcal{F}_t\right]$$

be the BS–price of $h(S_T)$ at time $t$.

Now, for any $u,v\in[0,T]$ such that $u+v\in[0,T]$ there holds

$$\begin{aligned}
S_{u+v} &= S_0\,e^{\sigma B_{u+v}+(r-\sigma^2/2)(u+v)} \\
&= S_0\,e^{\sigma(B_u)+(r-\sigma^2/2)u+\sigma(B_{u+v}-B_u)+(r-\sigma^2/2)v} \\
&= S_0\,e^{\sigma(B_u)+(r-\sigma^2/2)u}\;e^{\sigma(B_{u+v}-B_u)+(r-\sigma^2/2)v} \\
&= S_u\,e^{\sigma(B_{u+v}-B_u)+(r-\sigma^2/2)v}\,.
\end{aligned}$$

In particular, we have for any $t\in[0,T]$ (putting $u:=t$ and $v:=T-t$:)

$$S_T = S_t\,e^{\sigma(B_T-B_t)+(r-\sigma^2/2)(T-t)}\,.$$

Since the BM $B$ has independent increments, Lemma 4.3 yields

$$\begin{aligned}
V_t &= e^{-r(T-t)}\,\mathbb{E}^*\left[h\left(S_t\,e^{\sigma(B_T-B_t)+(r-\sigma^2/2)(T-t)}\right)\Big|\,\mathcal{F}_t\right] \\
&= v(t,S_t)
\end{aligned}$$

where

$$v(t,x) := e^{-r(T-t)}\,\mathbb{E}^*\left[h\left(x\,e^{\sigma(B_T-B_t)+(r-\sigma^2/2)(T-t)}\right)\right]$$

a deterministic function. This deterministic function satisfies the following PDE:

**Proposition 7.13.** *$v$ belongs to $\mathcal{C}^{1,2}([0,T)\times(0,\infty))\cap\mathcal{C}^{0,2}([0,T]\times(0,\infty))$ and solves the PDE*

$$v_t(t,x)+rxv_x(t,x)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x)-rv(t,x)=0 \qquad (7.6)$$

*with boundary condition*

$$v(T,x) = h(x)\,.$$

(This PDE is sometimes referred to as the Black–Scholes PDE.)

Proof. The random variable

$$Z_t := x\,\mathrm{e}^{\sigma(B_T - B_t) + (r-\sigma^2/2)(T-t)}$$

has densitiy $\psi_{t,x}$ with

$$\psi_{t,x}(z) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\,\frac{1}{z}\,\mathrm{e}^{-\frac{(\log(z)-m-\log(x))^2}{2\sigma^2(T-t)}}$$

where $m = (r-\sigma^2/2)(T-t)$.

To see this, write

$$Z_t = \mathrm{e}^{\sigma(B_T-B_t)+(r-\sigma^2/2)(T-t)+\log(x)}$$

to conclude that $Z_t$ is $\log\mathcal{N}(\mu,\tau^2)$–distributed, where

$$\mu = (r-\sigma^2/2)(T-t)+\log(x) \quad , \quad \tau = \sigma\sqrt{T-t}\,.$$

Therefore, $Y := \log Z$ has the distribution density

$$\varphi_Y(y) = \mathcal{N}(\mu,\tau^2)(y) = \frac{1}{\sqrt{2\pi\tau^2}}\,\mathrm{e}^{-(y-\mu)^2/(2\tau^2)}\;.$$

Writing $Z_t = e^Y$, the distribution function $\Phi_{Z_t}$ turns out as

$$\Phi_{Z_t}(z) = P^*\left[e^Y \leqslant z\right] = P^*\left[Y \leqslant \log(z)\right] = \int_0^{\log(z)} \varphi_Y(y)\,dy = \int_0^z \frac{1}{x}\varphi_Y(\log(x))\,dx$$

hence $Z_t$ has the distribution density

$$\psi_{t,x}(z) := \varphi_{Z_t}(z) = \frac{1}{z}\varphi_Y(\log(z))$$

as claimed.

Note that

$$\begin{aligned} u(t,x) &= \int_0^\infty h(z)\frac{\partial}{\partial x}\psi_{t,x}(z)\,dz \\ &= -\frac{1}{x}\int_0^\infty h(z)\frac{(\log(z)-m-\log(x))^2}{2\sigma^2(T-t)}\psi_{t,x}(z)\,dz \end{aligned}$$

is defined and finite for all $(t,x) \in [0,T) \times (0,\infty)$. This implies that we can change differentiation and integration and so obtain $v_x(t,x) = u(t,x)$.

Putting $t := T$ in the definition of $v(t,x)$ immediately shows the boundary condition $v(T,x) = h(x)$. One can then show $v(t,x) \in \mathcal{C}^{0,2}([0,T]\times(0,\infty))$.

The last claim follows from the fact that one can write $v(t,x) = \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[h(Z_t)\right]$ and that the family $(h(Z_t))_{t\in[0,T]}$ is uniformly integrable due to the bound $c(1+x^p)$ on $h(x)$. The LEBESGUE Dominated Convergence Theorem then implies $\lim_{t\uparrow T}\mathbb{E}^*\left[h(Z_t)\right] = \mathbb{E}^*\left[h(Z_T)\right] = h(x)$ and so $\lim_{t\uparrow T} v(t,x) = v(T,x)$.

In many cases where $h$ is given concretely, explicit computations may show directly that $v(t,x)$ has the stated continuity and differentiability properties. For this, one can use the expression of $v(t,x)$ as an expectation value given above and use the fact that $B_T - B_t$ is $\mathcal{N}(0,T-t)$–distributed:

$$v(t,x) = \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[h(Z_t)\right] = \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[f(Y_t)\right]$$

with

$$f(y) = h\left(x\,\mathrm{e}^{\sigma\sqrt{T-t}y+(r-\sigma^2/2)(T-t)}\right)$$

and

$$Y_t = \frac{B_T - B_t}{\sqrt{T-t}}$$

an $\mathcal{N}(0,1)$–distributed random variable. Hence

$$\mathrm{e}^{r(T-t)}\,v(t,x) = \mathbb{E}^*\left[f(Y_t)\right] = \int h\left(x\,\mathrm{e}^{\sigma\sqrt{T-t}y+(r-\sigma^2/2)(T-t)}\right)\,\varphi(y)dy$$

where $\varphi(y) = (1/\sqrt{2\pi})\,\mathrm{e}^{-y^2/2}$.

Now we are in business. Let us consider the important example of a European call, i.e. $h(x) := (x-K)^+$. The computation of this integral then is virtually identical with the computation done in the case of the BS-price at time $t=0$ with $S_0$ replaced by $x$ and $T$ replaced by $T-t$. The outcome is

$$\mathrm{e}^{r(T-t)}\,v(t,x) = x\,\mathrm{e}^{r(T-t)}\int_{-\infty}^{d_1}\mathrm{e}^{-y^2/2}\,\frac{1}{\sqrt{2\pi}}dy - K\int_{-\infty}^{d_2}\mathrm{e}^{-y^2/2}\,\frac{1}{\sqrt{2\pi}}dy\,,$$

with

$$d_1 := \frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\begin{aligned} d_2 &:= \frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned}$$

So there comes

$$\begin{aligned} v(t,x) = x\Phi&\left(\frac{\log(x/K) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- \mathrm{e}^{r(T-t)}\,K\Phi\left(\frac{\log(x/K) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\,. \end{aligned}$$

which shows that $v(t,x) \in \mathcal{C}^{1,2}([0,T)\times(0,\infty))$.

As an extra bonus we note that, because $v(t,S_t)$ is the BS–price of a European call at time $t$, we get the following generalization of the BLACK–SCHOLES formula:

**Corollary.** *Let $C = (S_T - K)^+$, the payoff of a European call. Then the BS–price of $C$ at time $t \in [0, T)$ is*

$$\text{BS–call}(S_t, K, T-t, r, \sigma) = S_t \Phi(d_1) - \mathrm{e}^{-r(T-t)}\, K \Phi(d_2)$$

*where $\Phi$ is the distribution function of $\mathcal{N}(0,1)$ and*

$$d_1 := \frac{\log(S_t/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

*and*

$$\begin{aligned} d_2 &:= \frac{\log(S_t/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \\ &= d_1 - \sigma\sqrt{T-t}\,. \end{aligned}$$

□

That $v_{xx}$ and $v_t$ are defined and continuous on $[0, T) \times (0, \infty)$ can be shown in a similar manner.

It remains to show that $v$ satisfies (7.6). The ITÔ–formula for $v(t, S_t)$ reads

$$v(t, S_t) = v(0, S_0) + \int_0^t v_t(u, S_u)\, du + \int_0^t v_x(u, S_u)\, dS_u + \frac{1}{2} \int_0^t v_{xx}(u, S_u)\, d\langle S\rangle_u$$

with $\langle S\rangle$ the quadratic variation process of $S$. Now the SDE defining $S$ is

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

and by construction, $B_t = W_t + \vartheta t$, whence $dW_t = dB_t - \vartheta dt$. Therefore

$$dS_t = \mu S_t\, dt + \sigma S_t\, (dB_t - \vartheta dt) = (\mu - \sigma\vartheta) S_t dt + \sigma S_t dB_t$$

and so, in terms of $B$, the SDE for $S$ reads

$$dS_t = r S_t dt + \sigma S_t dB_t\,.$$

The formal ITÔ Calculus gives

$$\begin{aligned} d\langle S\rangle_t = dS_t\, dS_t &= r^2 S_t^2 dt\, dt + 2r\sigma S_t^2 dt\, dB_t + \sigma^2 S_t^2 dB_t\, dB_t \\ &= r^2 S_t^2 d\langle t\rangle + 2r\sigma S_t^2 d\langle t, B\rangle_t + \sigma^2 d\langle B\rangle_t \end{aligned}$$

and so

$$d\langle S\rangle_t = \sigma^2 S_t^2 dt$$

since $\langle t\rangle = \langle t, B\rangle_t = 0$, and $\langle B\rangle_t = t$, a famous result of LÉVY. Plugging these formulas for $dS_t$ and $d\langle S\rangle_t$ into ITÔ's formula yields

$$v(t, S_t) = v(0, S_0) + \int_0^t v_t(u, S_u)\, du + \int_0^t v_x(u, S_u)\, (r S_u du + \sigma S_u dB_u) +$$

$$+\frac{1}{2}\int_0^t v_{xx}(u,S_u)\sigma^2 S_u^2\,du$$

$$= v(0,S_0) + \int_0^t \left\{ v_t(u,S_u) + rS_u v_x(u,S_u) + \frac{1}{2}\sigma^2 S_u^2 v_{xx}(u,S_u)\right\} du +$$

$$+\int_0^t \sigma S_u v_x(u,S_u)\,dB_u$$

Let $D_t := \mathrm{e}^{-rt}\,v(t,S_t)$. Then $D$ is a $P^*$–martingale. The Product Formula of ITÔ Calculus yields

$$D_t = \mathrm{e}^{-rt}\,v(t,S_t) = v(0,S_0) + \int_0^t a_u\,du + \int_0^t \mathrm{e}^{-rt}\,\sigma S_u v_x(u,S_u)\,dB_u \tag{7.7}$$

with

$$a_u = \mathrm{e}^{-rt}\left\{ v_t(u,S_u) + rS_u v_x(u,S_u) + \frac{1}{2}\sigma^2 S_u^2 v_{xx}(u,S_u) - rv(u,S_u)\right\}$$

We have

$$\int_0^t a_u\,du = D_t - v(0,S_0) - \int_0^t \mathrm{e}^{-rt}\,\sigma S_u v_x(u,S_u)\,dB_u\,.$$

On the left hand side we have a process of bounded variation and hence with vanishing quadratic variation. On the right hand side we have a local martingale; this martingale, then, is to have vanishing quadratic variation, hence it has $P^*$-a.s. constant paths, so the process $\int_0^\bullet a_u\,du$ has $P^*$–a.s. constant paths, and taking $t = 0$ it follows that this process has to vanish $P^*$–a.s. This entails

$$\mathbb{E}^*\left[\int_0^T \mathbb{1}_{\{a_u\neq 0\}}\,du\right] = 0\,. \tag{7.8}$$

FUBINI's Theorem then implies that for LEBESGUE–a.a. $u \in [0,T)$ we have $a_u = 0$ $P$–a.s.

To see this, first note that the $P^*$-a.s. vanishing of $\int_0^\bullet a_u\,du$ means $P^*[A] = 1$ where $A$ is the event

$$A := \left\{ \forall\, t\,:\,\int_0^t a_u\,du = 0 \right\}.$$

Now consider the event

$$B := \left\{ \int_0^T a_u \, du \neq 0 \right\} .$$

Then surely $A \subseteq B^c$ and so $B \subseteq A^c$, hence $B$ is a null set w.r.t $P^*$. Then for any non–negative bounded random variable $X$ with $\{X \neq 0\} \subseteq B$ we have $\mathbb{E}^*[X] = 0$ and (7.8) follows.

FUBINI now entails

$$0 = \mathbb{E}^* \left[ \int_0^T \mathbb{1}_{\{a_u \neq 0\}} \, du \right] = \int_0^T \mathbb{E}^* \left[ \mathbb{1}_{\{a_u \neq 0\}} \right] du = \int_0^T P^* [a_u \neq 0] \, du$$

and so for LEBESGUE–a.a. $u \in [0, T)$ there holds $P^* [a_u \neq 0] = 0$, i.e. $a_u = 0$ $P^*$–a.s.

Since the distribution of $S_t$ is equivalent to the LEBESGUE measure on $(0, \infty)$ and $v \in \mathcal{C}^{1,2}$, there follows that $v$ must satisfy the equation (7.6) on $[0, T) \times (0, \infty)$.

To put some more flesh onto this meagre argumentative bone, define, for any given $u \in [0, T)$ a function $\beta_u$ as

$$\beta_u(x) = v_t(u, x) + rxv_x(u, x) + \frac{1}{2}\sigma^2 S_t^2 v_{xx}(u, x) - rv(u, x) .$$

We then have $a_u = \mathrm{e}^{-rt} \beta_u(S_u)$ and so $\beta_u(S_u) = 0$ $P^*$–a.s. Since $S_u$ is log–normally distributed, its distribution is equivalent to LEBESGUE measure. But if $X$ is a random variable whose distribution is equivalent to LEBESGUE measure, it must have dense image. Otherwise, there would be a nonempty set $U \subseteq \mathbb{R}$ such that $\{X \in U\} = \emptyset$ and so $P^* [X \in U] = 0$ contradicting the fact that $U$ has nonzero LEBESGUE measure. Therefore $\beta_u = 0$ on a dense subset of $\mathbb{R}$, and since $\beta_u$ is continuous, we have $\beta_u = 0$ everywhere. As this holds for all $u$, we get equation (7.6)

QED

**Proposition 7.14.** *Let $\overline{\xi} = (\xi^0, \xi)$ be the admissible strategy that replicates $C = h(S_T)$ and so with value process $V_t = \mathrm{e}^{-r(T-t)} \mathbb{E}^* [ h(S_T) | \mathcal{F}_t ]$. Then*

$$\xi_t = v_x(t, S_t) \quad \lambda \otimes P\text{–}a.s.$$

*with $\lambda$ the* LEBESGUE *measure on* $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

PROOF. Recall that the discounted value process $D_t := \mathrm{e}^{-rt} V_t$ satisfies

$$D_t = D_0 + \int_0^t \xi_u \sigma \widetilde{S}_u \, dB_u .$$

by Lemma 7.3 and Proposition 7.5 and so, in particular, is a $P^*$–martingale as an ITÔ–integral process w.r.t. a BM. From (7.7) in the proof of Proposition 7.13 above we get, because $a_u = 0$:

$$D_t = D_0 + \int_0^t \mathrm{e}^{-rt} \sigma S_u v_x(u, S_u) \, dB_u = D_0 + \int_0^t v_x(u, S_u) \sigma \widetilde{S}_u \, dB_u .$$

This implies

$$\int_0^t \xi_u \sigma \widetilde{S}_u \, dB_u = \int_0^t v_x(u, S_u) \sigma \widetilde{S}_u \, dB_u \, .$$

But when two integral processes coincide, their integrands coincide; this follows directly from the Lemma of the discussion on page 92 above. Hence the result. QED

**Remark**. The quantity $v_x(u, S_u)$ is called the *delta* of the contingent claim at time $t$. Confer with the delta from Section 4: In the BOPM, the delta was a difference quotient, namely

$$\Delta_t(x) = \frac{v_t(ux) - v_t(dx)}{ux - dx} \, .$$

Since the BOPM approximates the BS–model and converges to it in the limit when the time intervals go to zero, this difference quotient becomes a derivative, and things fit together nicely.

### Implied volatility

Let $C$ be a contingent claim with BS–price at $t = 0$ given by $\mathrm{e}^{-rT} \, \mathbb{E}^* \left[ C \right]$. For the calculation of the BS–price one needs to specify the parameters $r$ and $\sigma$. The interest rate $r$ can be derived from bond markets and so is not a big problem in general (which is not quite true as it stands, but we leave it at that here).

How to choose $\sigma$? There are two approaches to this problem, one goes under the name *historical vola* (where *vola* is a common abbrevation of the term *volatility*), and the other under the name *implied vola*.

**1. Historical vola.** The vola can be estimated from historical data. E.g. calculate the empirical std ( = standard deviation) of the log returns $\log(S_{t_1}/S_{t_0})$, $\log(S_{t_2}/S_{t_1})$, $\ldots$, where $t_0, t_1, t_2, \ldots$ are trading days. Note that historical volatility is based on price movements of the underlying.

But in practice, one refrains from proceeding this way. Instead, one uses

**2. Implied vola.** The vola $\sigma$ is derived (one also says *calibrated*) from current market prices of liquidly traded *plain vanilla options* (this notion refers to standard common options like calls and puts and the like, which are traded by exchange, so you have a market price at any moment in time, in contrast to more individual, complex derivatives, which usually are traded off–the–counter (OTC), the *exotic options*). With these volatilities one can price the more exotic options which are not traded by exchange. So implied volatility is, in contrast to historical volatility, not based on price movements of the underlying, but of other derivatives on the market.

For a given plain vanilla option there exists only one value of $\sigma$ such that the BS price coincides with the market price. Why is this the case?

**Lemma 7.15.** (*Vega* of a call or put) *Let* $c(\sigma) :=$ BS–call$(S_0, K, T, r, \sigma)$ *and* $p(\sigma) := BS_{\text{put}}(S_0, K, T, r, \sigma)$. *Then*

$$\frac{\partial c}{\partial \sigma} = \frac{\partial p}{\partial \sigma} = S_0 \Phi(d_1)\sqrt{T}$$

*where*

$$d_1 = \frac{\log(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} .$$

PROOF. Straightforward. QED

(In particular, the lemma shows that the derivatives are positive, which means that $c$ and $p$ grow with rising volatility. The economic *rationale* behind this result is as follows. Options can be regarded as an insurance against capital losses. Increase in volatility means higher risk, and as a consequence higher value of the insurances $c$ and $p$.)

The derivative of the option value w.r.t. the volatility is called *vega* (a fictitious greek letter, meant to be a cousin of gamma, delta, ..., and member of the family known in finance as *The Greeks*). Note that the vega is positive.

**Definition.** *Let* $M$ *be the market price of a call (or put) with maturity* $T$ *and strike* $K$. *A real* $\sigma^{\text{imp}} \in \mathbb{R}_+$ *is called* implied volatility *if*

$$\text{BS–call}(S_0, K, T, r, \sigma^{\text{imp}}) = M .$$

**Remark**. a) Notice that Lemma 7.15 implies that the implied vola is unique.

b) There is no closed form expression for $\sigma^{\text{imp}}$. The implied vola has to be approximated numerically, e.g. with NEWTON's method (which is very fast; cue: "quadratic convergence"). □

Suppose that for a fixed maturity $T$ several calls are traded with strikes $K_1 < K_2 < \cdots < K_n$ at market prices $M(T, K_j)$ $1 \leqslant j \leqslant n$. For any strike $K_j$ we can calculate the implied volatility $\sigma_j^{\text{imp}}$, $1 \leqslant j \leqslant n$. In practice, the function $K_j \mapsto \sigma_j^{\text{imp}}$ usually is not constant. One observes frequently one of two following phenomena:

**1) Volatility smile.** Wikipedia's explanation is as follows:

(source `http://en.wikipedia.org/wiki/Volatility_smile`)

> In finance, the volatility smile is the pattern in which in- and out-of-the-money options are observed to have higher implied volatilities than at-the-money options. A graph of implied volatility vs. strike price for a given expiry will form a upturned curve similar to the shape of a smile. The pattern displays different characteristics for different markets and is

> believed to result from risk averse traders' valuations of the probability of extreme price movements in the underlying instrument. Equity options traded in American markets did not show a volatility smile before the Crash of 1987 but began showing one afterwards.[1] The phenomenon is not fully understood, and modeling the volatility smile is an active area of research in quantitative finance ...
>
> References
>
> [1] John C. Hull, Options, Futures and Other Derivatives, 5th edition, page 335

As a complement the following definition of the web site *Dogs of the Dow*:

(source `http://www.investorglossary.com/volatility-smile.htm`)

> The options phenomena known as the volatility smile occurs when an at-the-money option (ATM) exhibits a lower implied volatility than either the in-the-money (ITM) or out-of-the-money (OTM) options. On a chart plotting implied volatility on the vertical axis and strike price on the horizontal axis, a u-shaped 'volatility smile' is formed. The volatility smile is graphed for options with the same expiration date. The volatility smile became more noticeable in equity and index options after the crash of 1987. Prior to observing the post-1987 volatility smile, it was assumed that there existed a constant and independent relationship between implied volatility and the strike price of options; the volatility smile was therefore a direct contradiction to one of the main assumptions in the Black Scholes Option Pricing Model. The presence of a volatility smile generally infers that there is more demand by option traders for in-the-money and/or out-of-the-money options rather than at-the-money options. The presence of a volatility smile then implies that the extrinsic values of the ITM and OTM options are greater than that of the ATM option. The volatility smile is generally a result of an anticipated increase in market volatility. To hedge against this expected volatility, traders are more likely to purchase and sell OTM and ITM options rather than ATM options; this excess demand is expressed by the shape of the volatility smile.

**2) Volatility skew:** Here, Wikipedia has to say the following *(loc. cit.)*:

> When implied volatility is plotted against strike price, the resulting graph is typically downward sloping for equity markets, or valley-shaped for currency markets. For markets where the graph is downward sloping, such as for equity options, the term "volatility skew" is often used. For other markets, such as FX options or equity index options, where the typical graph turns up at either end, the more familiar term "volatility smile" is used. For example, the implied volatility for upside (i.e. high strike) equity options is typically lower than for at-the-money equity options. However, the implied volatilities of options on foreign exchange contracts tend to rise in both the downside and upside directions. In equity markets, a small tilted smile is often observed near the money as a kink in the general downward sloping implicit volatility graph. Sometimes the term "smirk" is used to describe a skewed smile.

One also frequently observes that $\sigma^{\text{imp}}$ varies with the time to maturity $T$.

Let $M(T,K)$ be the market price of a call with maturity $T$ and strike $K$. The mapping $(T,K) \mapsto \sigma^{\text{imp}}(T,K)$ is called *volatility surface*. This is *the* crucial object for option traders.

Since the vola surface usually is not constant, practitioners and researchers have come up with models more general than the BS model (most of them containing the BS model aa a special case). Within these more general models the option price paradigm remains true: an arbitrage–free price of an option is equal to the discounted expectation of its payoff under an EMM.

CHAPTER 8

# The Local Volatility Model

The *local volatility (LV) model* is a generalization of the BS model. Again it will be a complete model (I will not be going to prove this, I just state it), so there exists exactly one EMM $P^*$. The risky asset price process is assumed to satisy an SDE of the form

$$dS_t = rS_t dt + S_t\sigma(t, S_t)dB_t \quad , \quad S_0 \in (0, \infty) . \tag{8.1}$$

Again, $r > 0$ is the interest rate and $B$ a BM w.r.t. $P^*$. The function $\sigma : [0, T] \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, called *local volatitlity function*, is assumed to be measurable and bounded. Moreover, we suppose that $x \mapsto \widetilde{\sigma}(t, x) := x\sigma(t, x)$ is LIPSCHITZ–continuous and satisfies a growth condition, so that the SDE (8.1) has a unique strong solution with initial condition $S_0$ (cf. Stochastic Analysis lectures). Note that $\sigma$ is given by a *deterministic function* and has no independent stochastics of its own; the only stochastics also in this model enters only through the price process $S$.

**Lemma 8.1.** *The price process $S$ is positive. Moreover, $\mathbb{E}^*\left[\sup_{t\in[0,T]} S_t^2\right] < \infty$.*

PROOF. Note that the solution $S$ to (8.1) satisfies

$$S_t = S_0 \exp\left(\int_0^t \sigma(u, S_u)\, dB_u + \int_0^t r - \frac{\sigma^2(u, S_u)}{2}\, du\right) ,$$

(semiexplicit solution formula for linear SDEs) and the RHS of this is positive for all $t \in [0, T]$.

In order to prove the second statement, put

$$M_t := \exp\left(\int_0^t \sigma(u, S_u)\, dB_u - \frac{1}{2}\int_0^t \sigma^2(u, S_u)\, du\right) \quad , \quad t \in [0, T] .$$

As a stochastic exponential, it is *a priori* a local martingale, but we can use NOVIKOV's Criterion to deduce it is, in fact, a martingale.

The intergral process $X_t = \int_0^t \sigma(u, S_u)\, dB_u$ is, as an ITÔ integral, a local martingale, and has, by the ITÔ formula, quadratic variation $\langle X\rangle = \int_0^t \sigma^2(u, S_u)\, du$. Due to the growth conditions imposed on $\sigma$, NOVIKOV's Condition (see page 85) is satisfied and so indeed $M_t$ as given above, is a $P^*$–martingale on $[0, T]$.

Now DOOB's $L^2$–inequality (an extremely strong and surprising result!) )implies

$$\mathbb{E}^*\left[\sup_{t\in[0,T]} M_t^2\right] \leqslant 4\mathbb{E}^*\left[M_T^2\right] .$$

Therefore,

$$\mathbb{E}^*\left[\sup_{t\in[0,T]} S_t^2\right] \leqslant S_0^2\,\mathrm{e}^{2rT}\,\mathbb{E}^*\left[\sup_{t\in[0,T]} M_t^2\right] \leqslant 4S_0^2\,\mathrm{e}^{2rT}\,\mathbb{E}^*\left[M_T^2\right] .$$

QED

What is our aim? Our aim is to price options within the local volatility model.

Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be the payoff function of a European option. Since there is only one EMM, there is only one arbitrage–free price. This arbitrage–free price of the option at $t \in [0, T]$ is given by

$$\mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\, h(S_T)|\, \mathcal{F}_t\,\right]$$

with $\mathcal{F}_t = \mathcal{F}_t^B \vee \mathcal{N}$ , where $\mathcal{F}_t^B$ is the filtration generated by the BM $B$ completed by the null sets $\mathcal{N}$ of $P^*$ .

Next step: You want to characterize the arbitrage–free price in terms of a PDE as in the BS model.

**Theorem 8.2.** *Let $v \in \mathcal{C}^{1,2}([0,T) \times (0,\infty)) \cap \mathcal{C}^0([0,T] \times (0,\infty))$ . Suppose that $v_x$ is bounded on $[0,T) \times (0,\infty)$ , and that $v$ satisfies*

$$\begin{cases} v_t + rxv_x + \frac{1}{2}\sigma^2(t,x)x^2 v_{xx} - rv = 0 \quad , \quad \forall (t,x) \in [0,T) \times (0,\infty)\,; \\ \qquad\qquad v(T,x) = h(x) \quad \forall x \in (0,\infty) \qquad \textit{(terminal condition).} \end{cases} \tag{8.2}$$

*Then $v(t, S_t) = \mathrm{e}^{-r(T-t)}\,\mathbb{E}^*\left[\, h(S_T)|\, \mathcal{F}_T\,\right]$ .*

PROOF. The proof is very similar to the corresponding proof for the BS PDE in the BS model, so I give only the steps.

The ITÔ formula and (8.2) imply

$$v(t, S_t) = v(0, S_0) + \int_0^t v_x(u, S_u)\sigma(u, S_u)S_u\, dB_u + \int_0^t rv(u, S_u)\, du \quad \text{for } t \in [0,T)\,.$$

With the product formula, we further get

$$\mathrm{e}^{-rt}\, v(t, S_t) = v(0, S_0) + M_t\,,$$

where

$$M_t = \int_0^t v_x(u, S_u)\sigma(u, S_u)S_u\,\mathrm{e}^{-ru}\; dB_u \quad \text{for } t \in [0,T)\,.$$

With Lemma 8.1 one can show that

$$\mathbb{E}^* \left[ \int_0^T v_x^2(u, S_u)\sigma^2(u, S_u)\, \mathrm{e}^{-2ru}\, S_u^2\, du \right] < \infty$$

and so $M_t$ is a martingale on $[0, T)$.

*A priori* $M_T$ does not have a meaning, but since $M_t$ is an $L^2$–bounded martingale, it is uniformly integrable, and LEBESGUE's Dominated Convergence Theorem shows that

$$M_T := \lim_{t \uparrow T} M_t$$

exists and that $(M_t)_{t\in[0,T]}$ is an $L^2$–martingale.

In particular, $\mathrm{e}^{-rt}\, v(t, S_t)$ is an $L^2$–martingale on $[0, T]$ (here we use continuity of $v$ in $T$). This implies

$$\mathrm{e}^{-rt}\, v(t, S_t) = \mathbb{E}^* \left[ \mathrm{e}^{-rT}\, v(T, S_T) \middle| \mathcal{F}_t \right] = \mathbb{E}^* \left[ \mathrm{e}^{-rT}\, h(S_T) \middle| \mathcal{F}_t \right] .$$

QED

If one wants, as illustrated by the last theorem, to apply the model in practice and price options by numerically solving such PDEs, the next task before being able to do so is to extract the form of the local volatility function $\sigma$ from empirical data, or, as one says, to *calibrate* the local volatility function. How to do so belongs to the best kept secrets of the trade. So one can be sure that each bank involved in the business has his own well–guarded method of calibration, and each time a paper appears whose authorship can be traced back to a bank or some other institution and which describes one method of calibration or the other, one can rest assured that this institution meanwhile has switched to some other method and the method described is no longer in use. Therefore, one can only give some hints. Nevertheless, I will sketch a method which became popular twenty years ago, just to give a taste of the matter. In the literature it goes under the heading DUPIRE*'s formula* and runs as follows. As in the case of implied volatility one looks at plain vanilla options which are liquidly traded and hence have market prices to extract from them the local volatility function which then, in turn, is used to price more exotic options. So suppose there is a market call price $C(T, K)$ for any maturity $T \geq 0$ and strike $K > 0$. One can show that if $C$ is "nice" (e.g. sufficiently smooth), then

$$\sigma(T, K) = \sqrt{2\, \frac{\dfrac{\partial C}{\partial T} + rK \dfrac{\partial C}{\partial K}}{K^2 \dfrac{\partial^2 C}{\partial K^2}}} \tag{8.3}$$

and this is DUPIRE's formula. It has been first derived by *Bruno* DUPIRE in 1994 (see **[9]**) from the FOKKER–PLANCK and KOLMOGOROV equations, but under somewhat blurry premises. The idea here is to determine $C(T, K)$ from empirical data and then to use DUPIRE's formula to obtain the local volatility function. To determine, or calibrate, $C(T,K)$ can e.g. be done by a parametric ansatz: Model $C(T,K)$ by a polynomial of low degree and then adjust its coefficients as to get the optimal fit reproducing the empirical data. This, however, does not seem to be the method exercised in practice; here one uses a link between the local volatility function and the volatility surface describing BS–implied volatility; see *Jim* GATHERAL's book [**17**] and his lecture notes [**15**], [**16**].

All in all, this calibration business is a very delicate and difficult affair and requires financial engineers combining the theoretician's knowledge with the practitioner's finesse. Anyway, we now suppose the calibration done and now we are going to use it.

Once we have calibrated $\sigma(t,x)$ to current market prices of calls (and puts), we can use the model for pricing exotic options (which are OTC). As an example, consider an up–and–out call option with payoff

$$C = \mathbb{1}_{\{\forall\, u\in[0,T]\,:\, S_u<a\}}(S_T-K)^+\,,$$

where $a > \max\{S_0, K\}$.

To simplify the analysis, we assume in the following that

$$(*)\qquad \{\forall\, u\in[0,T]\;:\; S_u<a\} = \{\forall\, u\in[0,T)\;:\; S_u<a\}\quad P\text{–a.s.}\,.$$

We can obtain the arbitrage–free price of $C$ by solving the following PDE:

$$(8.4)\qquad \begin{cases} v_t + rxv_x + \frac{1}{2}\sigma^2(t,x)xv_{xx} - rv = 0 \quad \forall (t,x)\in[0,T)\times(0,\infty)\,;\\ \hfill v(T,x) = (x-K)^+ \quad \forall x\in(0,a)\,;\\ \hfill v(t,0) = v(t,a) = 0 \quad \forall t\in[0,T]\,. \end{cases}$$

**Theorem 8.3.** *Let $v\in\mathcal{C}^{1,2}([0,T)\times[0,a])\cap\mathcal{C}^0([0,T]\times(0,a))$. Let $v$ be bounded on $[0,T]\times[0,a]$ and $v_x$ be bounded on $[0,T)\times[0,a]$. If $v$ satisfies (8.4), then $v(0,S_0) = \mathrm{e}^{-rT}\,\mathbb{E}^*[C]$.*

PROOF. Introduce a stopping time

$$\tau_n := \left(T-\frac{1}{n}\right)\wedge\inf\{\,t\geq 0 \mid\; S_t = a\,\}\,.$$

Let $D_t := \mathrm{e}^{-rt}\,v(t,S_t)$. With ITÔ's formula one can show that $D_{t\wedge\tau_n}$ is a $P^*$–martingale. Hence

$$\begin{aligned} v(0,S_0) = D_0 &= \mathbb{E}^*[D_{\tau_n}]\\ &= \mathbb{E}^*\left[\mathbb{1}_{\left\{\tau_n<T-\frac{1}{n}\right\}}D_{\tau_n} + \mathbb{1}_{\left\{\tau_n=T-\frac{1}{n}\right\}}D_{T-\frac{1}{n}}\right]\\ &= \mathbb{E}^*\left[\mathbb{1}_{\left\{\tau_n<T-\frac{1}{n}\right\}}\,\mathrm{e}^{-r\tau_n}\,v(\tau_n,a) + \mathbb{1}_{\left\{\tau_n=T-\frac{1}{n}\right\}}\,\mathrm{e}^{-r(T-\frac{1}{n})}\,v(T-\frac{1}{n},S_{T-\frac{1}{n}})\right]\,. \end{aligned}$$

The first term is 0 because of the last formula in (8.4), and we are left with

$$v(0, S_0) = \mathbb{1}_{\left\{\tau_n = T-\frac{1}{n}\right\}} \, \mathrm{e}^{-r(T-\frac{1}{n})} \, v(T - \frac{1}{n}, S_{T-\frac{1}{n}}) \, .$$

We have

$$\begin{aligned} \left\{\tau_n = T - \frac{1}{n}\right\} &= \left\{\forall\, t < T - \frac{1}{n} \ : \ S_t < a\right\} \\ &\overset{n\uparrow\infty}{\longrightarrow} \{\forall\, t < T \ : \ S_t < a\} \\ &= \left\{\sup_{t\in[0,T]} S_t < a\right\} \,, \end{aligned}$$

where we have made use of the assumption $(*)$. Dominated Convergence then implies

$$v(0, S_0) = \mathbb{E}^* \left[\mathbb{1}_{\left\{\sup_{t\in[0,T]} S_t < a\right\}} \, \mathrm{e}^{-rT} \, v(T, S_T)\right] = \mathrm{e}^{-rT} \, \mathbb{E}^* \left[C\right] \, .$$

QED

**Remark**. 1) In general, there is no closed form solution of the PDE (8.4). To solve it one has to resort to numerical procedures, e.g. finite difference methods.

2) A solution of (8.4) cannot be continuous in $(T, a)$. Indeed, the boundary conditions imply

$$\begin{aligned} &\lim_{t\uparrow T} v(t, a) = 0 \quad \text{and} \\ &\lim_{x\uparrow a} v(T, x) = (a - K)^+ \, . \end{aligned}$$

□

CHAPTER 9

# Affine Processes

Literature: **[10]**

We are on our way of describing further refined stochastic volatility models, in particular the HESTON model, and for this we first enter a more general discussion in an abstract setting and describe the so–called *affine models*.

Let $d \in \mathbb{N}$ and $W$ a $d$–dimensional BM on a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{N}$ with $\mathcal{F}_t^W = \sigma(W_s : s \leqslant t)$ and $\mathcal{N}$ the collection of $P$–null sets.

Let $\mathcal{X} \subseteq \mathbb{R}^d$ be a closed set with non–empty interior; consider this as a state space. Assume that $b : \mathcal{X} \longrightarrow \mathbb{R}^d$ and $\varrho : \mathcal{X} \longrightarrow \mathbb{R}^{d\times d}$ are continuous functions. Consider the SDE

$$dX_t = b(X_t)dt + \varrho(X_t)dW_t \quad , \quad X_0 = x \in \mathcal{X} \,. \tag{9.1}$$

We assume that for every $x \in \mathcal{X}$ there exists a unique strong solution $X$ of (9.1), and $X_t \in \mathcal{X}$ for all $t \geq 0$. If we want to stress the dependence on $x$ we write $X_t^x = X_t$. Throughout we set $a(x) := \varrho(x)\varrho(x)^\top$.

**Definition.** *The process $X$ uniquely solving* (9.1) *is called* affine *if there exist functions $\phi : \mathbb{R}_+ \times \mathrm{i}\,\mathbb{R}^d \longrightarrow \mathbb{C}$ and $\psi : \mathbb{R}_+ \times \mathrm{i}\,\mathbb{R}^d \longrightarrow \mathbb{C}^d$ such that*

*a) $\phi$ and $\psi$ are continuously differentiable in $t$;*
*b) for all $x \in \mathcal{X}$, $0 \leqslant t < T$, and $u \in \mathbb{R}^d$ we have*

$$\mathbb{E}\left[ \mathrm{e}^{\mathrm{i}\,u^\top X_T^x} \middle| \mathcal{F}_t \right] = \mathrm{e}^{\phi(T-t,\mathrm{i}\,u)+\psi(T-t,\mathrm{i}\,u)^\top X_t^x} \,. \tag{9.2}$$

Here we consider $d$–dimensional vectors as colum vectors, i.e. as $d \times 1$–matrices and so the matrix product $x^\top y$ is a $1 \times 1$–matrix, i.e. a number, in fact the scalar product $x \cdot y = \sum_{j=1}^d x_j y_j$.

**Remark**. 1) Note that $\left|\mathrm{e}^{\mathrm{i}\,u^\top X_T^x}\right| = 1$, where $|z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2} = \sqrt{x^2+y^2}$ is the *absolute value* or *modulus* of $z = x + \mathrm{i}\,y \in \mathbb{C}$ with $x, y \in \mathbb{R}$. By JENSEN's inequality, the modulus of the LHS of (9.2) then is bounded by 1, hence so is the modulus of the RHS, i.e. it is also bounded by 1, and thus $\mathrm{Re}(\phi(T-t,\mathrm{i}\,u) + \psi(T-t,\mathrm{i}\,u)^\top X_t^x) \leqslant 0$.

To see this, recall the famous EULER formula:

$$\mathrm{e}^{\mathrm{i}\,u} = \cos(u) + \mathrm{i}\sin(u) \quad \text{for all } u \in \mathbb{R}\,,$$

hence $\left|e^{\mathrm{i}\,u}\right| = \sqrt{\cos^2(u) + \sin^2(u)} = 1$ for all $u \in \mathbb{R}$.

Further, we have the complex conjugation mapping $z = x + \mathrm{i}\,y \mapsto \overline{z} := x - \mathrm{i}\,y$. As is easily computed, this is an involutive automorphism of the field $\mathbb{C}$ over $\mathbb{R}$, i.e. one has

(i) $\overline{z + z'} = \overline{z} + \overline{z'}$ ;
(ii) $\overline{zz'} = \overline{z}\,\overline{z'}$ ;
(iii) $\overline{\overline{z}} = z$ ;
(iv) $\overline{z} = z \iff z \in \mathbb{R}$ .

Clearly $z\overline{z} = |z|^2$ . It is then an easy matter to derive that the modulus is multiplicative; just compute

$$|zz'|^2 = zz'\overline{zz'} = zz'\overline{z}\,\overline{z'} = z\overline{z}z'\overline{z'} = |z|^2\,|z'|^2 ,$$

and since the modulus is nonnegative by definition there results $|zz'| = |z|\,|z'|$.

Now, if $z = x + \mathrm{i}\,y \in \mathbb{C}$, we have

$$e^z = e^{x+\mathrm{i}\,y} = e^x\, e^{\mathrm{i}\,y}$$

and so

$$|e^z| = \left|e^x\, e^{\mathrm{i}\,y}\right| = |e^x|\left|e^{\mathrm{i}\,y}\right| = |e^x| = e^x = e^{\mathrm{Re}(z)} .$$

There follows

$$|e^z| \leqslant 1 \iff e^{\mathrm{Re}(z)} \leqslant 1 \iff \mathrm{Re}(z) \leqslant 0$$

as desired.

2) We always assume that $\phi(0, \mathrm{i}\,u) = 0$ and $\psi(0, \mathrm{i}\,u) = \mathrm{i}\,u$. Then $\phi$ and $\psi$ are uniquely determined.

The first main result on affine models is the following lengthy theorem.

**Theorem 9.1.** *Suppose that $X$ is an affine process. Then the matrix $a(x)$ and the drift $b(x)$ are affine in $x$, i.e.*

$$\begin{cases} a(x) = a + \sum\limits_{j=1}^{d} x_j \alpha_j \\ b(x) = b + \sum\limits_{j=1}^{d} x_j \beta_j \end{cases} \tag{9.3}$$

*for $a, \alpha_j \in \mathbb{R}^{d\times d}$ and $b, \beta_j \in \mathbb{R}^d$.*

*Moreover, $\phi$ and $\psi$ solve the* RICCATI *equations*

$$\begin{cases} \phi_t(t, \mathrm{i}\,u) = \frac{1}{2}\psi(t, \mathrm{i}\,u)^\top a \psi(t, \mathrm{i}\,u) + b^\top \psi(t, \mathrm{i}\,u) ; \\ \qquad \phi(0, \mathrm{i}\,u) = 0 ; \\ \\ \psi_t^j(t, \mathrm{i}\,u) = \frac{1}{2}\psi(t, \mathrm{i}\,u)^\top \alpha_j \psi(t, \mathrm{i}\,u) + {\beta_j}^\top \psi(t, \mathrm{i}\,u) \quad , \quad j = 1, \ldots, d ; \\ \qquad \psi(0, \mathrm{i}\,u) = \mathrm{i}\,u . \end{cases} \tag{9.4}$$

*In particular,*

$$\phi(t, \mathrm{i}\,u) = \int\limits_0^t \left\{ \frac{1}{2}\psi(s, \mathrm{i}\,u)^\top a \psi(s, \mathrm{i}\,u) + b^\top \psi(s, \mathrm{i}\,u) \right\} ds .$$

*Conversely, suppose that $a(x)$ and $b(x)$ are affine of the form* (9.3)*, and suppose that there exists a solution $(\phi, \psi)$ of the* RICCATI *equations* (9.4) *such that* $\mathrm{Re}(\phi(t, \mathrm{i}\,u) + \psi(t, \mathrm{i}\,u)^\top x) \leqslant 0$ *for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^d$, and $x \in \mathcal{X}$. Then $X$ is an affine process.*

PROOF. Suppose that $X$ is affine. For fixed $T > 0$ and $u \in \mathbb{R}^d$ we consider the function $f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{C}$ defined by

$$f(t, x) := \mathrm{e}^{\phi(T-t, \mathrm{i}\,u) + \psi(T-t, \mathrm{i}\,u)^\top x} .$$

Equation (9.2) implies (is equivalent with the fact) that

$$M_t := f(t, X_t) = \mathrm{e}^{\phi(T-t, \mathrm{i}\,u) + \psi(T-t, \mathrm{i}\,u)^\top X_t}$$

is a $\mathbb{C}$–valued martingale on $[0, T]$ (i.e. $M_t = X_t + \mathrm{i}\,Y_t$ with $X_t, Y_t$ $\mathbb{R}$–valued martingales).

Indeed, if we denote the process on the LHS of (9.2) by $N$, we have by the tower property of conditional expectation for all $0 \leqslant s \leqslant t \leqslant T$

$$\mathbb{E}\left[\, N_t \middle|\, \mathbb{F}_s \right] = \mathbb{E}\left[ \mathbb{E}\left[ \mathrm{e}^{\mathrm{i}\,u^\top X_T^x} \middle| \mathcal{F}_t \right] \middle| \mathbb{F}_s \right] = \mathbb{E}\left[ \mathrm{e}^{\mathrm{i}\,u^\top X_T^x} \middle| \mathcal{F}_s \right] = N_s \,,$$

and so $N$ is a martingale by construction. Hence if (9.2) holds, $M = N$ is a martingale. Conversely, if $M$ is a martingale, we have

$$\mathbb{E}\left[\, M_T \middle|\, \mathbb{F}_t \right] = M_t \,,$$

but $M_T = \mathrm{e}^{\phi(0, \mathrm{i}\,u) + \psi(0, \mathrm{i}\,u)^\top X_T^x} = \mathrm{e}^{\mathrm{i}\,u^\top X_T^x}$, so this is just (9.2).

Applying ITÔ's formula to the real and imaginary part of $f$ yields

$$\begin{aligned} dM_t = {} & M_t\Big(-\phi_t(T-t, \mathrm{i}\,u) - \psi_t(T-t, \mathrm{i}\,u)^\top X_t\Big)dt + M_t\psi(T-t, \mathrm{i}\,u)^\top dX_t \\ & + \frac{1}{2} M_t \psi(T-t, \mathrm{i}\,u)^\top a(X_t)\psi(T-t, \mathrm{i}\,u)dt \end{aligned}$$

which implies, by replacing $dX_t$ using (9.1):

$$(*) \qquad dM_t = M_t I_t dt + M_t \psi(T-t, \mathrm{i}\,u)^\top \varrho(X_t) dW_t \,,$$

where

$$\begin{aligned} I_t = {} & -\phi_t(T-t, \mathrm{i}\,u) - \psi_t(T-t, \mathrm{i}\,u)^\top X_t + \psi(T-t, \mathrm{i}\,u)^\top b(X_t) \\ & + \frac{1}{2}\psi(T-t, \mathrm{i}\,u)^\top a(X_t)\psi(T-t, \mathrm{i}\,u) \,. \end{aligned}$$

Recall the multidimendional ITÔ formula. Let $X = (X^1, \ldots, X^d)$ be a $d$–dimensional continuous semimartingale with values in the open set $U \subseteq \mathbb{R}^d$ and $F \in \mathcal{C}^2(U)$. Then ITÔ's formula is

$$(\dagger) \qquad dF(X)_t = F'(X_t)dX_t + \frac{1}{2}F''(X)dX_t dX_t$$

with

$$(\dagger\dagger) \qquad F'(X_t)dX_t := \sum_{j=1}^d F_{x^j}(X_t)dX_t^j \quad \text{and} \quad F''(X_t)dX_t dX_t := \sum_{j,k=1}^d F_{x^j x^k}(X_t) d\langle X^j, X^k \rangle_t \,.$$

Needless to say that these are formal stochastic differentials and that equations like ITÔ's formula have to be interpreted properly by relations between stochastic integrals, which will not be repeated here. ITÔ's formula first holds for real–valued $F$, but since it is linear in $F$, it holds, when $F$ is complex–valued, for $\mathrm{Re}(F)$ as well as for $\mathrm{Im}(F)$ and so recombines linearly into an ITÔ's formula for $F$ which takes just the same form.

ITÔ's formula takes a particularly interesting form when we replace $d$ by $d+1$ and write points in $\mathbb{R}^{d+1}$ as $(t,x) = (t, x^1, \dots, x^d)$, interpreting $t$ as time, and consider $d+1$–dimensional processes $\overline{X}$ of the form $\overline{X}_t = (t, X_t)$ with $X$ a $d$–dimensional process. ITÔ's formula then specializes to its "time–dependent" form:

$$(\dagger\dagger\dagger)\qquad df(t,X_t) = f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2}f_{xx}(t,X_t)dX_t dX_t$$

With $f(t,x)$ defined as above, one computes

$$\begin{aligned}
f_t(t,x) &= f(t,x)(-\phi_t(T-t.\mathrm{i}\,u) - T-t, \mathrm{i}\,u^\top x);\\
f_{x^j}(t,x) &= f(t,x)\psi_j(T-t,\mathrm{i}\,u);\\
f_{x^j x^k}(t,x) &= f(t,x)\psi_j(T-t,\mathrm{i}\,u)\psi_k(T-t,\mathrm{i}\,u).
\end{aligned}$$

The terms in the time–dependent ITÔ formula for $dM_t$ then come out as

$$\begin{aligned}
f_t(t,X_t)dt &= -M_t\Big(\phi_t(T-t.\mathrm{i}\,u) - \psi_t(T-t,\mathrm{i}\,u)^\top X_t\Big)dt;\\
f_x(t,X_t)dX_t &= M_t\sum_{j=1}^d \psi_j(T-t,\mathrm{i}\,u)dX_t^j = M_t\psi(T-t,\mathrm{i}\,u)^\top dX_t;\\
f_{xx}(t,X_t)dX_t dX_t &= M_t\sum_{j,k=1}^d \psi_j(T-t,\mathrm{i}\,u)\psi_k(T-t,\mathrm{i}\,u)dX_t^j dX_t^k\\
&= M_t\sum_{j,k,p=1}^d \psi_j(T-t,\mathrm{i}\,u)\psi_k(T-t,\mathrm{i}\,u)\varrho_p^j(X_t)\varrho_p^k(X_t)dt\\
&= M_t\psi(T-t,\mathrm{i}\,u)^\top\varrho(X_t)\varrho(X_t)^\top\psi(T-t,\mathrm{i}\,u)dt.
\end{aligned}$$

In the third formula we have made use of (9.1), which in component form yields

$$dX_t^j = b^j(X_t)dt + \sum_{p=1}^d \varrho_p^j(X_t)dW_t^p \quad,\quad j=1,\dots,d,$$

and so

$$\begin{aligned}
dX_t^j dX_t^k &= \left(b^j(X_t)dt + \sum_{p=1}^d \varrho_p^j(X_t)dW_t^p\right)\left(b^k(X_t)dt + \sum_{q=1}^d \varrho_q^k(X_t)dW_t^q\right)\\
&= \sum_{p,q=1}^d \varrho_p^j\varrho_q^k d\langle W^p, W^q\rangle_t\\
&= \sum_{p,q=1}^d \varrho_p^j\varrho_q^k\delta^{pq}dt,
\end{aligned}$$

since the cross variations of $t$ with any other process vanish. It is now clear that they combine into the formula for $dM_t$ given above, Since $M$ is a martingale, we must have that the $dt$–term vanishes, i.e. $M_t I_t = 0$ $P$–a.s. Since $M_t \neq 0$ $P$–a.s., it must hold true that $I_t = 0$ for $P \otimes \lambda$–a.a. $(\omega, t)$ (with $\lambda$ denoting LEBESGUE measure). Since $I_t$ is continuous in $t$, there exists a $P$–null set $N$ such that for all $\omega \in N^c$ and all $t \in [0,T]$ we have $I_t = 0$.

This is the same kind of argumentation as in the proof of the BLACK–SCHOLES PDE. Writing $(*)$ properly as an equation between integrals gives

$$M_t - M_0 - \int_0^t M_s \psi(T-t, \mathrm{i}\,u)^\top \varrho(X_s)\, dW_s = \int_0^t M_s I_s \, ds\,.$$

On the LHS we have a martingale, on the RHS an integral process which therefore is of bounded variation. Hence both sides must vanish. So the integral process on the RHS has constant paths, which must be zero, and then the FUBINI argument as in *loc.cit.* provides the statements above.

In particular, $I_0 = 0$, i.e.

$$\phi_t(T, \mathrm{i}\,u) + \psi_t(T, \mathrm{i}\,u)^\top x = \psi(T, \mathrm{i}\,u)^\top b(x) + \frac{1}{2}\psi(T, \mathrm{i}\,u)^\top a(x) \psi(T, \mathrm{i}\,u) \tag{9.5}$$

for all $x \in \mathcal{X}$, $u \in \mathbb{R}^d$, and $T \geq 0$. Since $\phi$ and $\psi$ are continuously differentiable in $t$, letting $T \downarrow 0$ yields

$$\phi_t(0, \mathrm{i}\,u) + \psi_t(0, \mathrm{i}\,u)^\top x = \mathrm{i}\,u^\top b(x) + \frac{1}{2}\,\mathrm{i}\,u^\top a(x)(\mathrm{i}\,u)\,. \tag{$\star$}$$

I claim this equation already implies that $a$ and $b$ are affine, i.e. of the form (9.3): In particular, for $u := e_j$, where $e_j$ is the $j$–th unit vector in $\mathbb{R}^d$, we have

$$\phi_t(0, \mathrm{i}\,e_j) + \psi_t(0, \mathrm{i}\,e_j) x = \mathrm{i}\,b_j(x) - \frac{1}{2} a_{jj}\,. \tag{$\star1$}$$

Choosing $u := 2e_j$ leads to

$$\phi_t(0, \mathrm{i}\,2e_j) + \psi_t(0, \mathrm{i}\,2e_j) x = \mathrm{i}\,2b_j(x) - 2a_{jj}\,. \tag{$\star2$}$$

Let

$$A := \begin{pmatrix} \mathrm{i} & -\frac{1}{2} \\ 2\,\mathrm{i} & -2 \end{pmatrix}$$

Then $A$ is invertible with

$$A^{-1} = \begin{pmatrix} -2\,\mathrm{i} & -\frac{\mathrm{i}}{2} \\ 2 & -1 \end{pmatrix}$$

($\star1$) and ($\star2$) mean

$$A \begin{pmatrix} b_j(x) \\ a_{jj}(x) \end{pmatrix} = \begin{pmatrix} \phi_t(0, \mathrm{i}\,e_j) + \psi_t(0, \mathrm{i}\,e_j) x \\ \phi_t(0, \mathrm{i}\,2e_j) + \psi_t(0, \mathrm{i}\,2e_j) x \end{pmatrix} =: \gamma(x)\,,$$

and so

$$\begin{pmatrix} b_j(x) \\ \\ a_{jj}(x) \end{pmatrix} = A^{-1}\gamma(x)\,.$$

Now the RHS is affine in $x$, hence the $b_j(x)$ and $a_{jj}(x)$ are affine in $x$. From equation ($\star$) we can now further derive that the $a_{jk}(x)$, $j \neq k$, are affine in $x$ (choose $u := e_j + e_k$). So $a(x)$ and $b(x)$ are affine functions in $x$.

The coefficients $a$, $\alpha_j$ $b$, $\beta_j$ in (9.3) are at first complex numbers, why are they real? Notice that $a(x)$ and $b(x)$ are real functions by assumption. Choose an $x$ in the interior of $\mathcal{X}$, which is non–empty by assumption. Now suppose one entry of $a$ or $\alpha_i$ would have nonzero imaginary part. This must be cancelled exactly by the contribution of the imaginary parts of the other summands in (9.3) as to achieve that the corresponding entry in $a(x)$ has vanishing imaginary part. But by conveniently varying $x$ (note that $\mathcal{X}$ has non–empty interior) I can destroy this balance and the corresponding entry of $a(x)$ would acquire a nontrivial imaginary part in contradiction to its being real. The same argument applies to $b$ and the $\beta_j$. So $a$, $\alpha_j$ $b$, $\beta_j$ in (9.3) are all real.

The Riccati equations (9.4) are now a straight consequence of (9.5).

We finally prove the reverse direction. Suppose that $a(x)$ and $b(x)$ are of the form (9.3). Let $(\phi, \psi)$ be a solution of (9.4) such that $\mathrm{Re}(\phi(T-t, \mathrm{i}\,u) + \psi(T-t, \mathrm{i}\,u)^\top x) \leqslant 0$ for all $t \geq 0$, $u \in \mathbb{R}^d$, $x \in \mathcal{X}$. Then $M_t := f(t, X_t)$ with $f$ defined as above is a local martingale, since if you write down Itô's formula as above thus obtaining ($*$), the bounded variation term $M_t I_t dt$ in ($*$) drops out due to the Riccati equations (9.4). Moreover, $|M_t| \leqslant 1$ due to the assumption $\mathrm{Re}(\phi(T-t, \mathrm{i}\,u) + \psi(T-t, \mathrm{i}\,u)^\top x) \leqslant 0$. Hence $M$ is a bounded local martingale, hence a martingale, and then

$$\mathbb{E}\left[\, M_T \middle|\, \mathcal{F}_t \,\right] = M_t\,,$$

which is just (9.2). QED

I think now it is time for an example.

**Example**. Let $d := 1$ and

$$dX_t = \mu dt + \sigma dW_t\,,$$

where $\mu \in \mathbb{R}$ , $\sigma > 0$.

Here, $\mathcal{X} = \mathbb{R}$ (the only natural choice in this case, because we can solve the SDE for any initial value, and $W_t$ reaches any $x \in \mathbb{R}$ with probability 1).

Notice that $\quad a(x) = a = \sigma^2$

and $\quad b(x) = b = \mu\,.$

The RICCATI equations in this case read

$$\begin{aligned}\phi_t(t, \mathrm{i}\, u) &= \frac{1}{2}\sigma^2\psi^2(t, \mathrm{i}\, u) + \mu\psi(t, \mathrm{i}\, u)\,;\\ \phi(0, \mathrm{i}\, u) &= 0\,.\end{aligned}$$

$$\begin{aligned}\psi_t(t, \mathrm{i}\, u) &= 0\,;\\ \psi(0, \mathrm{i}\, u) &= \mathrm{i}\, u\,.\end{aligned}$$

The solutions are:

$$\psi(t, \mathrm{i}\, u) = \mathrm{i}\, u$$

and

$$\begin{aligned}\phi(t, \mathrm{i}\, u) &= \int_0^t \left\{\frac{1}{2}\sigma^2(\mathrm{i}\, u)^2 + \mu\, \mathrm{i}\, u\right\} ds\\ &= \left\{\mathrm{i}\, \mu u - \frac{1}{2}\sigma^2\mu^2\right\} t\,.\end{aligned}$$

Note that $\mathrm{Re}(\phi(t, \mathrm{i}\, u) + \psi(t, \mathrm{i}\, u)x) = -\sigma^2u^2t/2 \leqslant 0$. So all assumptions for the reverse conclusion of Theorem 9.1 are satisfied, so it implies that $X$ is affine. In particular, if $X_0 = 0$,

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\, uX_t}\right] = \mathrm{e}^{\mathrm{i}\, \mu tu - \frac{1}{2}\sigma^2tu^2}\,,$$

which indeed is the characteristic function of $\mathcal{N}(\mu t, \sigma^2t)$, a result we know already, but in this way we have checked it is consistent with the new results here on affine processes. □

We next ask

Question: *Are there conditions just on the coefficients of the SDE* (9.1) *which guarantee that $X$ is affine?*

The answer will be: *yes*; there are such conditions which will be necessary and sufficient, but they will work only if the state space will have a *canonical form.*

So our next aim is: refinement of Theorem 9.1 when the state space $\mathcal{X}$ has the following canonical form

$$\mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^n\,,$$

where $m.n \in \mathbb{Z}_+$ with $m + n = d$.

We first need some auxiliary results.

**Lemma 9.2.** *Suppose $a(x)$ and $b(x)$ admit continuous extensions to $\mathbb{R}^d$. Let $u \in \mathbb{R}^d\backslash\{0\}$ and $H := \left\{ x \in \mathbb{R}^d \;\middle|\; u^\top x \geq 0 \right\}$. We write $H^\circ := \left\{ x \in \mathbb{R}^d \;\middle|\; u^\top x > 0 \right\}$ for the interior and $\partial H := \left\{ x \in \mathbb{R}^d \;\middle|\; u^\top x = 0 \right\}$ for the boundary of $H$. Let $x \in \partial H$ and $X = X^x$ be a solution of (9.1).*

*If $X_t \in H$ for all $t \geq 0$ then*

$$u^\top a(x) u = 0$$

*and*

$$u^\top b(x) u \geq 0\,.$$

To interpret these results geometrically, note that

$$u^\top a(x) u = u^\top \varrho(x)\varrho(x)^\top u = u^\top \varrho(x)(u^\top \varrho(x))^\top = \|u^\top \varrho(x)\|^2$$

so the first equation $u^\top a(x)u = 0$ is equivalent to $\varrho(x)^\top u = 0$ , i.e. the diffusion driving $X$ stays parallel to the boundary $\partial H$. The second equation $u^\top b(x) u \geq 0$ says that the drift either stays parallel to the boundary $\partial H$ or is "pointing inward", i.e. into the direction leading into the interior $H^\circ$.

Proof *(of Lemma 9.2).* We first prove the second statement. Note that, writing (9.1) as an integral equation and multiplying it on the left with $u^\top$, one may pull $u^\top$ through the integrals due to their being linear, and hence

$$u^\top X_t = \int_0^t u^\top b(X_s)\,ds + \int_0^t u^\top \varrho(X_s)\,dW_s\,.$$

Since $a(x)$ and $b(x)$ are continuous, it is possible to choose a constant $K \in \mathbb{R}_+$ and a stopping time $\tau_1 > 0$ $P$–a.s. such that $\left|u^\top b(X_{t\wedge\tau_1})\right| \leqslant K$ and $\|u^\top \varrho(X_{t\wedge\tau_1})\|^2 = u^\top a(X_{t\wedge\tau_1})u \leqslant K$ for all $t \geq 0$. Then $\int_0^{t\wedge\tau_1} u^\top \varrho(X_s)\,dW_s$ is an $L^2$–local martingale, hence a true martingale. Therefore, upon taking expectations in the last equation, this martingale part drops out and leaves us with

$$\mathbb{E}\left[u^\top X_{t\wedge\tau_1}\right] = \mathbb{E}\int_0^{t\wedge\tau_1} u^\top b(X_s)\,ds$$

for all $t \geq 0$.

We now argue by contradiction. Suppose $u^\top b(x) =: \kappa < 0$. Choose a further stopping time $\tau_2$ as

$$\tau_2 := \tau_1 \wedge \inf\left\{ t \geq 0 \;\middle|\; u^\top b(X_t) \geq \kappa/2 \right\}\,.$$

Then $\tau_2 > 0$ $P$–a.s., and

$$\mathbb{E}\left[u^\top X_{t\wedge\tau_2}\right] = \mathbb{E}\int_0^{t\wedge\tau_2} u^\top b(X_s)\,ds < 0$$

contradicting $X_t \in H$ for all $t \geq 0$. Hence it must hold true that $u^\top b(x) \geq 0$, which is the second statement.

For the first statement, take $C > 0$ and define a process

$$\begin{aligned}
\mathcal{Z}_t : &= \mathcal{E}\left(-C \int_0^t u^\top \varrho(X_s)\, dW_s\right) \\
&= \exp\left(-C \int_0^t u^\top \varrho(X_s)\, dW_s - \frac{C^2}{2} \int_0^t u^\top a(X_s) u\, ds\right),
\end{aligned}$$

the stochastic exponential of the process $-C \int_0^t u^\top \varrho(X_s)\, dW_s$. The product formula from ITÔ Calculus yields

$$\begin{aligned}
u^\top X_t \mathcal{Z}_t &= \int_0^t u^\top X_s\, d\mathcal{Z}_s + \int_0^t \mathcal{Z}_s u^\top\, dX_s + \langle u^\top X, \mathcal{Z}\rangle_t \\
&= M_t + \int_0^t \mathcal{Z}_s u^\top\, b(X_s) ds + \int_0^t u^\top \varrho(X_s) \mathcal{Z}_s (-C u^\top \varrho(X_s))^\top\, ds \\
&= M_t + \int_0^t \mathcal{Z}_s \left(u^\top b(X_s) - C u^\top a(X_s) u\right)\, ds
\end{aligned}$$

with

$$M_t : = \int_0^t u^\top X_s \mathcal{Z}_s \left(-C u^\top \varrho(X_s)\right)\, dW_s + \int_0^t \mathcal{Z}_s u^\top \varrho(X_s)\, dW_s$$

a local martingale.

We now again argue by contradiction. So assume $u^\top a(x) u \geq \varepsilon > 0$. Let $\tau_3$ be a localizing stopping time for $M$, i.e. $M^{\tau_3}$ is a martingale. Moreover, set

$$\tau_4 := 1 \wedge \tau_1 \wedge \tau_3 \wedge \inf\left\{ t \geq 0 \mid\ u^\top a(X_t) u \leqslant \varepsilon/2 \right\}.$$

Then $\tau_4 > 0$ $P$–a.s. (since $a(x)$ is continuous). If now $C$ is so chosen that $C > 2K/\varepsilon$, we have

$$\begin{aligned}
\mathbb{E}\left[u^\top X_{\tau_4} \mathcal{Z}_{\tau_4}\right] &= \mathbb{E} \int_0^{\tau_4} \mathcal{Z}_s \underbrace{\Big(\underbrace{u^\top b(X_s)}_{\leqslant K} - \underbrace{C u^\top a(X_s) u}_{\geq C\varepsilon/2 > K}\Big)}_{<0} ds \\
&< 0.
\end{aligned}$$

$\mathcal{Z}$ is a stochastic exponential, so is positive. Hence $u^\top X_t$ must be negative at some $t$ between 0 and $\tau_4$, which contradicts $X_t \in H$ for all $t \geq 0$. Hence $u^\top a(x) u = 0$, which is the first statement. QED

The following technical result is an exercise in Linear Algebra:

**Lemma 9.3.** *Let $A = (a_{ij}) \in \mathbb{R}^{d\times d}$ be a positive semi–definite matrix* (i.e. $A$ symmetric, $\forall\, x \in \mathbb{R}^d\ :\ x^\top A x \geq 0$).

(a) *If $a_{ii} = 0$ for all $i \in \{1, \dots, d\}$, then $A = 0$.*
(b) *Let $m, n \in \mathbb{N}$ with $m + n = d$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times n}$ and suppose*

$$A = \begin{pmatrix} 0 & B \\ B^\top & C \end{pmatrix}.$$

*Then $B = 0$.*

Since this result will be used in the proof of the following crucial result, we give a proof for convenience. We stick to the convention that when a matrix is called positive semi–definite it is automatically symmetric and real.

PROOF. We will now prove the following statement:

*Let $A$ be a positive semi–definite $d\times d$–matrix. Let $0 \leqslant i \leqslant d$ be an index such that the diagonal element $a_{ii} = 0$. Then $a_{ij} = a_{ji} = 0$ for $1 \leqslant j \leqslant d$.*

It should be clear that this statement immediately implies both (a) and (b). To prove it let $\beta$ be the symmetric bilinear form defined by $A$:

$$\beta(u, v) := u^\top A v \quad \text{for} \quad u, v \in \mathbb{R}^d,$$

and let

$$q(u) := \beta(u, u) = u^\top A u \quad \text{for} \quad u \in \mathbb{R}^d$$

be the associated quadratic form. This is a map $q : \mathbb{R}^d \longrightarrow \mathbb{R}$, and the property of $A$ being positive semi–definite is equivalent to $q(u) \geq 0$ for all $u \in \mathbb{R}^d$. Let $e_1, \dots, e_d \in \mathbb{R}^d$ be the canonical unit vectors. We have

$$a_{ij} = \beta(e_i, e_j) \quad , \quad a_{ii} = \beta(e_i, e_i) = q(e_i)$$

for all $1 \leqslant i, j \leqslant d$. Choose any pair $(i, j)$ with $1 \leqslant i, j \leqslant d$ and consider the map $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\psi(t) := q(te_i + e_j).$$

We have $\psi(t) \geq 0$ for all $t \in \mathbb{R}$. On the other hand, one computes easily

$$q(te_i + e_j) = t^2 q(e_i) + 2t\beta(e_i, e_j) + q(e_j) = a_{ii}t^2 + 2a_{ij}t + a_{jj}.$$

Now let $i$ be such that $a_{ii} = 0$. Then

$$\psi(t) = 2a_{ij}t + a_{jj} \geq 0 \quad \text{for all } t \in \mathbb{R}.$$

But this clearly entails $a_{ij} = 0$. QED

For the statement of the next fundamental result we need some notation. Recall that $\mathcal{X} = \mathbb{R}^m_+ \times \mathbb{R}^n$, $m + n = d$. Define $I := \{1, \dots, m\}$ and $J := \{m + 1, \dots, m + n\}$. For a vector $\mu \in \mathbb{R}^d$ and $M \subseteq \{1, \dots, d\}$ we write $\mu_M :=$

$(\mu_i)_{\in M}$. We use a similar notation for matrices: for $\nu \in \mathbb{R}^{d\times d}$ and $N \subseteq \{1,\dots,d\}$ we define

$$\nu_{MN} := (\nu_{ij})_{i\in M, j\in N}\,.$$

Finally, let

$$\mathcal{B} := (\beta_t,\dots,\beta_d) \in \mathbb{R}^{d\times d}\,.$$

**Proposition 9.4.** *Let $X$ be an affine process on the canonical space $\mathcal{X} = \mathbb{R}_+^m \times \mathbb{R}^n$. Then*

*(1) $a$, $\alpha_j$ are positiv semi–definite;*
*(2) $\alpha_j = 0$ for all $j \in J$;*
*(3) $a_{II} = 0$ (and thus $a_{IJ} = a_{JI} = 0$);*
*(4) $\alpha_{i,kl} = \alpha_{i,lk} = 0$ for $k \in I \setminus \{i\}$ and $1 \leqslant i, l \leqslant d$;*
*(5) $b \in \mathbb{R}_+^m \times \mathbb{R}^n$;*
*(6) $\mathcal{B}_{IJ} = 0$;*
*(7) $\mathcal{B}_{II}$ has non–negative off–diagonal entries.*

Proof. *Ad (1):* Since $a(x) = \varrho(x)\varrho(x)^\top$, $a(x)$ is symmetric and positive semi–definite for all $x \in \mathcal{X}$; the claim then follows from (9.3).

*Ad (2):* Let $e_1,\dots,e_d$ be the canonical unit vectors. If $j \in J$, we have $x := \xi e_j \in \mathcal{X}$ for any $\xi \in \mathbb{R}$. Then, by (9.3),

$$a(x) = a + \xi\alpha_j$$

and so

$$u^\top a(x)u = u^\top au + \xi u^\top \alpha_j u \geq 0 \quad \text{for all } \xi \in \mathbb{R} \text{ and } u \in \mathbb{R}^d\,.$$

But this clearly enforces $\alpha_j = 0$ .

(Note that this argument does not work for $j \in I$, since then $\xi e_j \notin \mathcal{X}$ for all $\xi \neq 0$.)

*Ad (3) & (4):* This proof is somewhat more involved. For any real number $y$ let $y^+ := \max\{y, 0\}$ and consider the following continuous extensions of $b(x)$, $\varrho(x)$ and $a(x)$ to $\mathbb{R}^d$:

$$b(x) := b + \sum_{i\in I} x_i^+\beta_i + \sum_{j\in J} x_j\beta_j \quad , \quad \varrho(x) := \varrho(x_1^+,\dots,x_m^+, x_{m+1},\dots,x_{m+n})\,;$$

then for $x \in \mathbb{R}^d$

$$a(x) = a + \sum_{i\in I} x_i^+\alpha_i\,.$$

Let $x$ be a boundary point of $\mathcal{X}$. Then $x_k = 0$ for some $k \in I$.

To see why this is so, use the relation $\partial(M\times N) = (\partial M \times \overline{N}) \cup (\overline{M} \times \partial N)$ for subsets $M \subseteq Y$, $N \subseteq Z$ of topological spaces $Y$, $Z$. This comes about as follows. Recall the definition of boundary: If $T$ is a topological space, $A \subseteq T$, then $\partial A := \overline{A} \cap \overline{A^c}$. Thus $\partial(M\times N) = \overline{M\times N} \cap$

$\overline{(M\times N)^{\mathrm{c}}}$. Now $(y,z)\notin M\times N$ iff $y\notin M$ or $z\notin N$, hence $(M\times N)^{\mathrm{c}} = (M^{\mathrm{c}}\times Z)\cup(Y\times N^{\mathrm{c}})$, and so

$$\begin{aligned}
\partial(M\times N) &= \overline{M\times N}\cap\overline{(M\times N)^{\mathrm{c}}}\\
&= \overline{M\times N}\cap\overline{(M^{\mathrm{c}}\times Z\cup Y\times N^{\mathrm{c}})}\\
&= \overline{M\times N}\cap\big(\overline{M^{\mathrm{c}}\times Z}\cup\overline{Y\times N^{\mathrm{c}}}\big)\\
&= \big(\overline{M}\times\overline{N}\big)\cap\big(\overline{M^{\mathrm{c}}}\times Z\cup Y\times\overline{N^{\mathrm{c}}}\big)\\
&= \big(\overline{M}\times\overline{N}\cap\overline{M^{\mathrm{c}}}\times Z\big)\cup\big(\overline{M}\times\overline{N}\cap Y\times\overline{N^{\mathrm{c}}}\big)\\
&= (\partial M\times\overline{N})\cup(\overline{M}\times\partial N)
\end{aligned}$$

as desired. This being so, we now put $Y:=\mathbb{R}^m$, $Z:=\mathbb{R}^n$, and $M:=\mathbb{R}^m_+$, $N:=\mathbb{R}^n$, so that $M\times N=\mathcal{X}$. We then have, as $\partial\mathbb{R}^n=\emptyset$, $\partial\mathcal{X}=(\partial\mathbb{R}^m_+)\times\mathbb{R}^n$. Further, by the same formula, $\partial\mathbb{R}^m_+=\partial(\mathbb{R}_+\times\mathbb{R}^{m-1}_+)=(\partial\mathbb{R}_+\times\mathbb{R}^{m-1}_+)\cup(\mathbb{R}_+\times\partial\mathbb{R}^{m-1}_+)$. Iterating this and using $\partial\mathbb{R}_+=\{0\}$ we end up with

$$\partial\mathcal{X}=\bigcup_{i=0}^{m-1}(\mathbb{R}^i_+\times\{0\}\times\mathbb{R}^{m-i-1}_+\times\mathbb{R}^n)=\left(\bigcup_{i=0}^{m-1}\mathbb{R}^i_+\times\{0\}\times\mathbb{R}^{m-i-1}_+\right)\times\mathbb{R}^n\,.$$

By Lemma 9.2 we have ${e_k}^\top a(x)e_k=0$, i.e.

$${e_k}^\top\left(a+\sum_{i\in I\setminus\{k\}}x_i\alpha_i\right)e_k=0\,.$$

In particular, choose all $x_i=0$ for $i\in\{1,\dots,d\}$, then $a_{kk}={e_k}^\top a(x)e_k=0$; this holds for all $k\in I$. By Lemma 9.3 (a), $a_{II}=0$, which is (3).

Similarly, by putting all $x_i=0$ except one, we get $a_{i,kk}=0$ for $k\in M:=I\setminus\{i\}$, and by Lemma 9.3 (a) again we get $a_{i,MM}=0$. With $N:=\{1,\dots,d\}$, Lemma 9.3 (b) implies $\alpha_{i,MN}=0$, which is (4).

*Ad (5), (6) & (7):* Let $x\in\mathcal{X}$ be again a boundary point, i.e. $x_k=0$ for some $k\in I$. Lemma 9.2 implies

$$(\clubsuit)\qquad {e_k}^\top\left(b+\sum_{i\in I\setminus\{k\}}x_i^+\beta_i+\sum_{j\in J}x_j\beta_j\right)\geq 0\,.$$

Choosing all $x_i=0$ and $x_j=0$ we get $b_k\geq 0$ for all $k\in I$ which is (5). Moreover, ($\clubsuit$) implies

$${e_k}^\top\beta_i\geq 0\quad\text{for } i\in I\setminus\{k\}$$

and

$${e_k}^\top\beta_j=0\quad\text{for } j\in J$$

(argue as above in both cases by taking $\pm x_i$, $\pm x_j$ large enough ). Thus we obtain (6) and (7). QED

**Definition.** *We say that* $a, \alpha_j, b, \beta_j$ *are* admissible parameters *if the properties (1) – (7) from Proposition 9.4 are satisfied.*

We now have a reverse statement to Proposition 9.4 (a really strong result):

**Proposition 9.5.** *Let* $a(x)$ *and* $b(x)$ *be affine functions with admissible parameters* $a, \alpha_j, b, \beta_j \dot{}$*Then the process* $X$ *determined by them via the SDE* (9.1) *is an affine process.*

Proof. The proof involves very cumbersome notation, so I will only give a rough sketch; the case of the Heston model presented below will be treated paradigmatically in more detail. A rigorous, albeit terse, proof can be found in **[10]**, pp. 148–150.

Let $\mathbb{C}_-$ be the set of complex numbers wth non–positive real part. Consider the system of ODEs (cf. with (9.4))

$$
\begin{aligned}
&\text{(a)} && \phi_t(t, \mathrm{i}\, v) = \frac{1}{2}\psi_J(t, \mathrm{i}\, v) a_{JJ} \psi_J(t, \mathrm{i}\, v) + b^\top \psi(t, \mathrm{i}\, v)\,, \\
& && \phi(0, \mathrm{i}\, v) = 0\,; \\
&\text{(b)} && \psi_{k,t}(t, \mathrm{i}\, v) = \frac{1}{2}\psi(t, \mathrm{i}\, v)^\top \alpha_k \psi(t, \mathrm{i}\, v) + {\beta_k}^\top \psi(t, \mathrm{i}\, v) \quad , \quad k \in I\,; \\
&\text{(c)} && \psi_{j,t}(t, \mathrm{i}\, v) = {\beta_j}^\top \psi(t, \mathrm{i}\, v) \quad , \quad j \in J\,, \\
& && \psi(0, \mathrm{i}\, v) = \mathrm{i}\, v
\end{aligned}
$$

for $v \in \mathbb{R}^d$.

Since (c) is a linear ODE, one can write down immediately its solution, namely

$$
\psi_J(t, \mathrm{i}\, v) = \mathrm{i}\, \mathrm{e}^{t\mathcal{B}_{JJ}^\top} v_J\,. \tag{9.6}
$$

One can show that (b) has a solution on $[0, \infty)$ with $\psi_I(t, \mathrm{i}\, v) \in \mathbb{C}_-^m$. Moreover, one can show that $\phi(t, \mathrm{i}\, v) \in \mathbb{C}_-$ (use e.g. property (5)); see **[10]**. Therefore, $\mathrm{Re}(\phi(t, \mathrm{i}\, v) + \psi(t, \mathrm{i}\, v)^\top x) \leqslant 0$ for all $x \in \mathbb{R}_+^m \times \mathbb{R}^n$. By the last statement of Theorem 9.1, $X$ is affine. QED

This is what I wanted to tell you on affine models in general; now we turn to the Heston model.

CHAPTER 10

# The HESTON Model

The HESTON Model, first introduced in [**20**], is a generalization of the BLACK–SCHOLES Model, intended to improve on some of the shortcomings, but retaining the benefits, of the latter. One of the main weaknesses of the BLACK–SCHOLES Model is the assumption of constant volatility which is not in agreement with the observed behaviour of markets where phenomena like volatility smile and skew occur. So the main feature of the HESTON model is the introduction of a stochastic volatility. At the same time it also allows for closed formulas for pricing European calls and so keeps the numeric tractability of the BLACK-SCHOLES Model. Numeric tractability is one of the main reasons to render a model useful for practitioners since it makes the model easy to calibrate, whereas models giving abstract or general answers which require extensive numeric simulations like Monte Carlo techniques for solution may be mathematically beautiful, but are not feasible for practical purposes. Therefore, the HESTON Model has been widely accepted also in practice and has come into common use. It belongs to the class of affine models, which is interesting for just the same reason, namely to be numerically tractable and so easy to calibrate, and consequently it has been extensively studied the last ten years.

The HESTON Model is an option pricing model with one non–risky asset with value $S_t^0 = \mathrm{e}^{rt}$ at time $t \in \mathbb{R}_+$ and one risky asset, the underlying of the option. It will have two processes driven by stochastics, the price process and the volatility process, and the stochastics will be a BM in both cases which however will be correlated, which is what is observed in practice.

So the first step is to obtain two correlated BMs. We start with a 2–dimensional BM $W = (W^1, W^2)$ on a probability space $(\Omega, \mathcal{F}, P^*)$ with filtration $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{N}$. Then $W^1$ and $W^2$ are two 1–dimensional BMs which are independent. Let $\rho \in [-1, 1]$ and define

$$W^Z := \rho W^1 + \sqrt{1-\rho^2} W^2 .$$

Then $W^Z$ is again a BM and has correlation coefficent $\rho$ with $W^1$. Let us see why. That $W^Z$ is a BM is most easily seen via LÉVY's Theorem by computing the quadratic variation of $W^Z$; since $\langle W^1, W^2 \rangle = 0$ there comes

$$\langle W^Z \rangle_t = \langle \rho W^1 + \sqrt{1-\rho^2} W^2 \rangle_t = \rho^2 \langle W^1 \rangle_t + (1-\rho^2) \langle W^2 \rangle_t = \rho^2 t + (1-\rho^2) t = t ,$$

and Lévy's Theorem immediately tells us that $W^Z$ is a BM, but it would also be possible to check directly the defining properties of a BM, without using Lévy's heavy gun.

Since $W^1$, $W^2$, and $W^Z$ are standard BMs, the random variables $W_t^1$, $W_t^2$, and $W_t^Z$ have mean 0 and variance $t$, hence

$$\operatorname{var}(W_t^j) = \mathbb{E}\left[(W_t^j)^2\right] = t \quad , \quad j = 1, 2, Z\,.$$

For the covariance between $W_t^1$ and $W_t^Z$ there comes

$$\begin{aligned}\operatorname{cov}(W_t^1, W_t^Z) &= \operatorname{cov}(W_t^1, \rho W_t^1 + \sqrt{1-\rho^2} W_t^2) \\ &= \rho \operatorname{cov}(W_t^1, W_t^1) + \sqrt{1-\rho^2} \operatorname{cov}(W_t^1, W_t^2) \\ &= \rho \operatorname{var}(W_t^1) = \rho t\,,\end{aligned}$$

because $\operatorname{cov}(W_t^1, W_t^2) = 0$ as $W^1$ and $W^2$ are independent. So the correlation coefficient between $W_t^1$ and $W_t^Z$ comes out as

$$\operatorname{corr}(W_t^1, W_t^Z) = \frac{\operatorname{cov}(W_t^1, W_t^Z)}{\sqrt{\operatorname{var}(W_t^1)}\sqrt{\operatorname{var}(W_t^Z)}} = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho\,,$$

i.e. $W^1$ and $W^Z$ are correlated BMs with correlation parameter $\rho$. This is a wanted effect, because this allows, in particular, to implement a negative correlation coefficient, which is what one observes in reality, where falling prices enhance the nervousness in the market and thus fuel the volatility, so that the price process and the volatility process develop opposite trends.

The volatility process of the risky asset is described by its variance process which is supposed to satisfy the SDE

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^1 \quad , \quad V_0 > 0\,, \tag{10.1a}$$

where $\kappa \geq 0$, $\theta \geq 0$, and $\sigma \geq 0$; one should think of $\sqrt{V_t}$ as being the volatitlity process. From this we see

$$\begin{aligned} V_t > \theta \quad &: \qquad \text{drift} < 0 \\ V_t < \theta \quad &: \qquad \text{drift} > 0\,;. \end{aligned}$$

so $V_t$ is a process fluctuating around $\theta$, a behaviour known as being *mean reverting.*

As soon as $V_t$, and hence the diffusion term, approaches 0, the drift term becomes positive and dominating, driving the process off 0, so the diffusion term has the effect that the process stays above 0.

In more detail, the following possible behaviour of the volatility process is claimed in the literature (citation from **[22]**:

> The Feller classification of boundaries for one-dimensional diffusions (see [47, Chapter 15, Section 6]) implies the following:
> (i) if $2\kappa\theta \geq \sigma^2$, then the origin is unattainable;

(ii) if $2\kappa\theta < \sigma^2$, then the origin is a regular, attainable and reflecting boundary; this means that the variance process can touch 0 in finite time, but does not spend time there;

(iii) infinity is a natural boundary, i.e. it can not be attained in finite time and the process can not be started there.

As a rule, therefore, when considering the HESTON model the condition $2\kappa\theta \geq \sigma^2$ is assumed (FELLER *condition*).

**Remark**. Concerning the solvability of (10.1a) note that this SDE has a diffusion coefficient that is *not* LIPSCHITZ continuous (the square root function has a vertical tangent at 0) , so that the standard ITÔ theory of existence and uniquenss of solutions of SDEs does not apply. One can, however, show that there exists a unique strong solution, but the proof is quite involved. A rough sketch of the steps to take:

1) Show first that there exists a weak solution.
2) Then show that there exists at most one strong solution (= pathwise uniqueness).
3) Now use the funny result of YAMABE & WATANABE: Existence of a weak solution + pathwise uniqueness imply existence of a strong solution. For details see e.g. [**23**] , Chapter 5, 5.3.D.

The risky asset price is supposed to evolve according to the SDE

$$dS_t = rS_t dt + \sqrt{V_t} S_t dW_t^Z \quad , \quad S_0 \in (0, \infty) \,. \tag{10.1b}$$

Note that $V_t = \theta$ , $\sigma = 0$ is possible in (10.1a), which turns this SDE for $S_t$ into

$$dS_t = rS_t dt + \sqrt{\theta} S_t dW_t^Z \quad , \quad S_0 \in (0, \infty) \,,$$

the BLACK–SCHOLES SDE for the price process; hence, the BLACK–SCHOLES model is a special case of the HESTON model, namely the case of constant volatility.

One should emphasize that the equations (10.1a) and (10.1b) are supposed to hold under a given martingale measure $P^*$, not under the real–world measure $P$. Such a supposition is, somewhat chivalrously, quite often made, with some handwaving into the direction that it is "essentially" equivalent to the no–arbitrage condition. This point, however, can be quite subtle and has led in some cases to serious difficulties; see the discussion in [**36**], as in the case of continuous time "essentially" means that the naive "no– arbitrage" condition has to be replaced by the subtle and technical "no free lunch with vanishing risk" condition, which may be difficult to verify (all this, and much more, is very nicely explained in [**33**], see also [**8**]). Fortunate for us, [**36**] has checked that, under some mild assumptions, martingale measures do exist in the HESTON model; however, there is not a unique one, but they form a family (see *loc.cit*, THEOREM 3.5, THEOREM 3.6). So the HESTON model is known to be incomplete, there is no unique such measure, and so we expect that the measure will show up explicitely in the option pricing formula, in contrast to the BS–formula.

We now give an outline of the construction of a risk–neutral measure $P^*$, starting from the formulation of the HESTON model under a real world measure $P$. For this, one starts with a pair $W = (W^1, W^2)$ of uncorrelated BMs and postulates the equations

$$dV_t = \kappa(\theta - V_t)dt + \sigma\sqrt{V}_t dW_t^1$$
$$dS_t = \mu S_t dt + \sqrt{V}_t S_t dW_t^Z\ ,$$

with parameters $\kappa, \theta, \mu, \sigma$, where at least $\kappa, \theta$ might *a priori* differ in value from the model under $P^*$, which we therefore just for the discussion at hand denote by $\kappa^*, \theta^*$. The parameter $\mu$ does not appear in the model under $P^*$ and corresponds to an unknown drift, as in the BS model. Additionally, we assume as before a constant interest rate $r$ for the risk–free bond. Finally, we have a correlation coefficient $\rho \in [-1, 1]$ such that $W^Z = \rho W^1 + \sqrt{1-\rho^2}W^2$. Thus the BM $W^1$ drives the volatility process $V$ and the BM $W^Z$ drives the price process $S$, so that $V$ and $S$ are correlated.

To make the measure change from $P$ to a risk–neutral measure, i.e. a measure $P^*$ that turns the discounted price process $\widetilde{S}_t := \mathrm{e}^{-rt} S_t$ into a $P^*$–martingale, we proceed along the lines of our action in the BS case. Since this time we start with an underlying two–dimensional BM $W = (W^1, W^2)$ which generates the filtration, we start with a GIRSANOV process $G$ which is supposed to be a local $W$–martingale and which therefore has the form

$$G_t = \int_0^t \gamma_s^1 dW_s^1\, ds + \int_0^t \gamma_s^2 dW_s^1\, ds =: -\int_0^t \vartheta_s^1 dW_s^1\, ds - \int_0^t \vartheta_s^2 dW_s^1\, ds$$

by the Representation Theorem for Local Martingales 7.8B (see page 93). Here, $\vartheta^1, \vartheta^2$ are adapted processes satisfying $\mathbb{E}\left[\int_0^T (\vartheta^1)^2\, dt\right] < \infty$, $\mathbb{E}\left[\int_0^T (\vartheta^2)^2\, dt\right] < \infty$. The process $\vartheta^1$ is called the *market price of volatility risk.* We want to find out for which $\vartheta^1, \vartheta^2$ we have

- $D := \mathcal{E}(G)$ is a martingale (and hence the measure $P^*$ defined by $dP^*/dP := D_T$ is a probability measure);
- if this is the case, then the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt} S_t$ is a $P^*$–martingale.

First we have to make further assumptions since we do not yet have enough conditions to fix $\vartheta^1$ and $\vartheta^2$ (since we have a nontradable asset – the volatility process – we expect the model to be incomplete anyway). We begin by computing the change of the BMs under the GIRSANOV transformation. If we denote the transformed processes by a tilde, we get

$$\widetilde{W}_t^1 = W_t^1 - \langle G, W^1\rangle_t = W_t^1 + \int_0^t \vartheta_s^1\, ds\,;$$

$$\widetilde{W}_t^2 = W_t^2 - \langle G, W^2\rangle_t = W_t^2 + \int_0^t \vartheta_s^2\, ds\,;$$

$$\widetilde{W}_t^Z = W_t^Z - \langle G, W^Z\rangle_t = W_t^Z + \int_0^t \left(\rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_s^2\right) ds\,,$$

or in differential formulation

$$d\widetilde{W}_t^1 = dW_t^1 + \vartheta_t^1 dt \qquad \text{or} \quad dW_t^1 = d\widetilde{W}_t^1 - \vartheta_t^1 dt\,;$$
$$d\widetilde{W}_t^2 = dW_t^2 + \vartheta_t^2 dt \qquad \text{or} \quad dW_t^2 = d\widetilde{W}_t^2 - \vartheta_t^2 dt\,;$$

$$d\widetilde{W}_t^Z = dW_t^Z + \left(\rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_t^2\right) dt \quad \text{or} \quad dW_t^Z = d\widetilde{W}_t^Z - \left(\rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_t^2\right) dt\,.$$

Under the new measure $P^*$ defined by $dP^*/dP := D_T = \mathcal{E}\,(G)_T$ the HESTON SDEs take the form

$$\begin{aligned} dV_t &= \left(\kappa(\theta - V_t) - \sigma\sqrt{V}_t\vartheta_t^1\right) dt + \sigma\sqrt{V}_t d\widetilde{W}_t^1 \\ dS_t &= \left(\mu - \sqrt{V}_t\left(\rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_t^2\right)\right) S_t dt + \sqrt{V}_t S_t d\widetilde{W}_t^Z\,. \end{aligned}$$

For the discounted price process $\widetilde{S}_t = \mathrm{e}^{-rt}\,S_t$ we have $d\widetilde{S}_t = -r\,\mathrm{e}^{-rt}\,S_t + \mathrm{e}^{-rt}\,dS_t$ and so

$$d\widetilde{S}_t = \left(\mu - r - \sqrt{V}_t\left(\rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_t^2\right)\right) \widetilde{S}_t dt + \sqrt{V}_t\widetilde{S}_t d\widetilde{W}_t^Z$$

and so we get as a necessary condition for $\widetilde{S}_t$ to be at least a local martingale that the coefficient of $dt$ is to vanish, hence

$$\frac{\mu - r}{\sqrt{V}_t} = \rho\vartheta_s^1 + \sqrt{1-\rho^2}\vartheta_t^2$$

a quantity called the *market price of stock risk*, for then we will have $d\widetilde{S}_t = \sqrt{V}_t\widetilde{S}_t d\widetilde{W}_t^Z$ and so

$$\widetilde{S} = \mathcal{E}\left(\int \sqrt{V}\,d\widetilde{W}^Z\right)\,.$$

In particular, we see that under this condition the equation for $S$ under $P^*$ simplifies to

$$dS_t = rS_t dt + \sqrt{V}_t S_t d\widetilde{W}_t^Z\,.$$

So, just as in the BS model, under a risk neutral measure, the unknown trend $\mu$ has been transformed away and been replaced by the constant interest rate $r$.

This does not fix $\vartheta^1, \vartheta^2$. In [**20**], HESTON makes the additional proposal to take $\vartheta^1$ proportional to $\sqrt{V}$ and defends this choice by referring to some places in the literature on finance, but notwithstanding those economical arguments this is a clever thing to do mathematically, since this causes nice simplifications: If we put

$$\vartheta^1 := \frac{\lambda}{\sigma}\sqrt{V}$$

with a real parameter $\lambda$, the SDEs above simplify to

$$\begin{aligned} dV_t &= (\kappa(\theta - V_t) - \lambda V_t)\,dt + \sigma\sqrt{V}_t d\widetilde{W}_t^1 \\ dS_t &= rS_t dt + \sqrt{V}_t S_t d\widetilde{W}_t^Z\,. \end{aligned}$$

We thus see that they retain their form: they can be written

$$\begin{aligned} dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V}_t d\widetilde{W}_t^1 \\ dS_t &= rS_t dt + \sqrt{V}_t S_t d\widetilde{W}_t^Z\,, \end{aligned}$$

with

$$(\star) \qquad \kappa^* := \kappa + \lambda \quad , \quad \theta^* := \frac{\kappa\theta}{\kappa + \lambda}\,,$$

the form we have postulated at the beginning. In particular, the nasty parameter $\lambda$, which parametrizes the possible nonunique choices of measure change, has been nicely stuffed away into the parameters $\kappa$ and $\theta$ of the model, which have to be determined by calibration anyway, and this will take care of unique prices at the end.

Now the real work begins. By quite involved arguments, the paper [**36**] establishes

- for $-\kappa \leqslant \lambda < \infty$, $D := \mathcal{E}(G)$ is a martingale, and hence the measure $P^*$ defined by $dP^*/dP := D_T$ is a probability measure (*loc. cit.*, p. 8, THEOREM 3.5). In particular, $\widetilde{W}^Z$ is a BM and so the discounted stock price $\widetilde{S} = \mathcal{E}\left(\int \sqrt{V}\, d\widetilde{W}^Z\right)$ a local martingale;
- for $\rho\sigma - \kappa \leqslant \lambda < \infty$, the discounted stock price $\widetilde{S} = \mathcal{E}\left(\int \sqrt{V}\, d\widetilde{W}^Z\right)$ is a martingale and so $P^*$ a risk–neutral probability measure (*loc. cit.*, p. 8, THEOREM 3.6).

Thus our initial assumption of the existence of a risk–neutral measure for the HESTON model has been justified.

We next are going to see in which sense the HESTON model is affine. The affine process will not be $S_t$, but its logarithm $Z_t := \log S_t$. For this process, ITÔ's formula implies

$$d\log S_t = \frac{1}{S_t} dS_t - \frac{1}{2S_t^2} d\langle S\rangle_t = r dt + \sqrt{V_t} dW_t^Z - \frac{1}{2} V_t dt$$

and hence

$$\begin{aligned} dZ_t &= \left(r - \frac{1}{2}V_t\right) dt + \sqrt{V_t} dW_t^Z \\ &= \left(r - \frac{1}{2}V_t\right) dt + \sqrt{V_t}\rho dW_t^1 + \sqrt{V_t}\sqrt{1-\rho^2} dW_t^2 \,. \end{aligned} \tag{10.2}$$

Let us now check that the process $X := (V, Z)$ is an affine process.

First, note that $X$ is a stochastic process with canonical state space $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}$. The system (9.1) reads in this case

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t^1 \\ dZ_t &= \left(r - \frac{1}{2}V_t\right) dt + \sqrt{V_t}\rho dW_t^1 + \sqrt{V_t}\sqrt{1-\rho^2} dW_t^2 \,. \end{aligned}$$

With the notation from Section 9 we read off, with $x = (v, z) \in \mathcal{X}$ :

$$\begin{aligned} b(x) &= \begin{pmatrix} \kappa(\theta - v) \\ r - \frac{v}{2} \end{pmatrix} ; \\ \varrho(x) &= \begin{pmatrix} \sigma\sqrt{v} & 0 \\ \rho\sqrt{v} & \sqrt{1-\rho^2}\sqrt{v} \end{pmatrix} . \end{aligned}$$

Next we compute he matrix $a(x)$ as

$$a(x) := \varrho(x)\varrho(x)^\top = \begin{pmatrix} \sigma^2 v & \rho\sigma v \\ \rho\sigma v & v \end{pmatrix} .$$

Note that $a(x)$ and $b(x)$ are affine with parameters

$$a = \quad 0 \quad , \quad \alpha_1 = \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix} \quad , \quad \alpha_2 = 0 \, ;$$

$$b = \begin{pmatrix} \kappa\theta \\ r \end{pmatrix} \quad , \quad \beta_1 = \quad \begin{pmatrix} -\kappa \\ -\frac{1}{2} \end{pmatrix} \quad , \quad \beta_2 = 0 \,.$$

We now check that the parameters $a, \alpha_j, b, \beta_j$ are admissible. We have $I = \{1\}$, $J = \{2\}$, and

*(1) $a$, $\alpha_j$ are positiv semi–definite*: $a$ and $\alpha_2$ are trivially positiv semi–definite. For $\alpha_1$ either use the Main Minors Criterion (a matrix is positive semi–definite iff the determinant of ts main minors are non–negative). Or check it directly: one easily computes

$$\begin{aligned}(x_1, x_2) \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & v \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \sigma^2 x_1^2 + 2\sigma\rho x_1 x_2 + x_2^2 \\ &= (\sigma x_1 + \rho x_2)^2 + (1 - \rho^2) x_2^2 \geq 0 \,.\end{aligned}$$

*(2) $\alpha_j = 0$ for all $j \in J$*: clear.

*(3) $a_{II} = 0$ (and thus $a_{IJ} = a_{JI} = 0$)*: clear.

*(4) $\alpha_{i,kl} = \alpha_{i,lk} = 0$ for $k \in I \setminus \{i\}$ and $1 \leqslant i, l \leqslant d$*: $I \setminus \{1\} = \emptyset$, so no condition on $\alpha_1$. And $\alpha_2 = 0$, so these conditions are met.

*(5) $b \in \mathbb{R}_+^m \times \mathbb{R}^n$*: $\kappa \geq 0$, $\theta \geq 0$, hence $\kappa\theta \geq 0$.

*(6) $\mathcal{B}_{IJ} = 0$*: We have

$$\mathcal{B} = (\beta_1, \beta_2) = \begin{pmatrix} -\kappa & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} .$$

By inspection, $\mathcal{B}_{12} = 0$.

*(7) $\mathcal{B}_{II}$ has non–negative off–diagonal entries*: $\mathcal{B}_{II} = (-\kappa)$ has no off–diagonal entries at all.

So we can directly see from the parameters that $X$ is an affine process by Proposition 9.5. This conclusion is, however, not entirely satisfactory, since we did not prove this proposition. But I already announced, when I stated it, to prove it in the case of the HESTON model. It will then transpire that the same proof goes through in the general case virtually without a change but of notation which then will be much more elaborated and cumbersome, thus more hindering understanding than facilitating it. Hence, going through the proof in this special case hardly falls short in result of plodding through the general case, which, in contrast might be easy to understand after this special case has been digested. The general case, by the way, is done in [**10**], pp. 149-150.

So we are now going to check directly with Theorem 9.1 that $X$ is an affine process. The appropriate RICCATI equations read (where the prime denotes differentiation w.r.t. $t$) (see Theorem 9.1)

$$\begin{cases} \phi'(t, \mathrm{i}\, u) = \kappa\theta\psi_1(t, \mathrm{i}\, u) + r\psi_2(t, \mathrm{i}\, u) \,; \\ \phi(0, \mathrm{i}\, u) = 0 \,. \end{cases} \tag{10.3}$$

$$(10.4)\qquad \begin{cases} \psi_1'(t,\mathrm{i}\,u) &= \frac{1}{2}\psi(t,\mathrm{i}\,u)^\top \alpha_1 \psi(t,\mathrm{i}\,u) + {\beta_1}^\top \psi(t,\mathrm{i}\,u) \\ &= \frac{1}{2}\left(\sigma^2\psi_1^2 + 2\rho\sigma\psi_1\psi_2 + \psi_2^2\right) - \kappa\psi_1 - \frac{1}{2}\psi_2\,; \\ \psi_1(0,\mathrm{i}\,u) &= \mathrm{i}\,u_1\,. \end{cases}$$

$$(10.5)\qquad \begin{cases} \psi_2'(t,\mathrm{i}\,u) = 0\,; \\ \psi_2(0,\mathrm{i}\,u) = \mathrm{i}\,u_2\,. \end{cases}$$

We have to show that a solution $(\phi,\psi)$ exists and that $Re(\phi(t,\mathrm{i}\,u) + \psi(t,\mathrm{i}\,u)^\top x) \leqslant 0$ for all $t \in \mathbb{R}_+$, $u \in \mathbb{R}^d$, and $x \in \mathcal{X}$.

The solution of (10.5) obviously is given by

$$\psi_2(t,\mathrm{i}\,u) = \mathrm{i}\,u_2\,.$$

Now you could solve (10.4) explicitely. But I want to give instead an abstract general argument why a solution exists (it will be this argument which also applies to the general case).

**Lemma 10.1.** *For all $u \in \mathbb{R}^2$, the ODE* (10.4) *has a solution on $\mathbb{R}_+$. Moreover,* $\mathrm{Re}(\psi_1(t,\mathrm{i}\,u)) \leqslant 0$ *for all $t \in \mathbb{R}_+$, then.*

PROOF*. For $z = (z_1,z_2)^\top \in \mathbb{C}^2$ put

$$R(z) := \frac{1}{2}z^\top\alpha_1 z + {\beta_1}^\top z = \frac{1}{2}\left(\sigma^2 z_1^2 + 2\rho\sigma z_1 z_2 + z_2^2\right) - \kappa z_1 - \frac{1}{2}z_2\,.$$

The ODE (10.4) can be rewritten as

$$(\star)\qquad \psi_1'(t,\mathrm{i}\,u) = R(\psi_1(t,\mathrm{i}\,u),\mathrm{i}\,u_2) \quad , \quad \psi_1(0,\mathrm{i}\,u) = \mathrm{i}\,u_1\,.$$

$R$ is locally LIPSCHITZ continuous. Therefore, there exists $t_+ \in [0,\infty]$ such that $(\star)$ has a unique solution on $[0,t_+)$, and $t_+ = \infty$ or the solution explodes at $t = t_+$, i.e. $\lim_{t\uparrow t_+}|\psi_1(t,\mathrm{i}\,u)| = \infty$. What we want to show is $t_+ = \infty$, i.e. the modulus of the solution stays bounded on any finite interval.

But first we prove that $\mathrm{Re}(\psi_1(t,\mathrm{i}\,u)) \leqslant 0$ for all $t < t_+$. Note that $(\star)$ unfolds as (for clarity of notation we suppress the argument $(t,\mathrm{i}\,u)$)

$$\psi_1' = \frac{1}{2}\left(\sigma^2\psi_1^2 + 2\rho\sigma\,\mathrm{i}\,u\psi_1 - u_2^2\right) - \kappa\psi_1 - \frac{1}{2}\mathrm{i}\,u_2\,.$$

Let

$$f(t) := \mathrm{Re}(\psi_1(t,\mathrm{i}\,u))\,.$$

Then

$$f'(t) = \mathrm{Re}(\psi_1') = \frac{1}{2}\left(\mathrm{Re}(\sigma^2\psi_1^2) + 2\,\mathrm{Re}(\rho\sigma\,\mathrm{i}\,u\psi_1) - u_2^2\right) - \kappa f(t)$$

*This is FILIPOVIC's proof in [10] of Proposition 9.5, applied to this case. I use more or less his notation.

$$
\begin{aligned}
&= \frac{1}{2}\left(\sigma^2 f^2(t) - \sigma^2 \operatorname{Im}(\psi_1)^2 - 2\rho\sigma u_2 \operatorname{Im}(\psi_1) - u_2^2\right) - \kappa f(t)\\
&= \frac{1}{2}\sigma^2 f^2(t) - \frac{1}{2}\left(\sigma^2 \operatorname{Im}(\psi_1)^2 + 2\rho\sigma u_2 \operatorname{Im}(\psi_1) + u_2^2\right) - \kappa f(t)\\
&= \frac{1}{2}\sigma^2 f^2(t) - \kappa f(t) - \frac{1}{2}\left((\sigma \operatorname{Im}(\psi_1) + \rho u_2)^2 + (1-\rho^2)u_2^2\right) .
\end{aligned}
$$

For $f(t) = 0$ the RHS is $\leqslant 0$. Moreover, $f(0) = 0$. Hence $f$ must always stay below 0, i.e. we have indeed $\operatorname{Re}(\psi_1(t, \mathrm{i}\, u)) = f(t) \leqslant 0$ for all $t < t_+$.

We finally show that $t_+ = \infty$.

Note that $|\psi_1|^2 = \operatorname{Re}(\psi_1)^2 + \operatorname{Im}(\psi_1)^2$, hence

$$
\begin{aligned}
\frac{d}{dt}|\psi_1|^2 &= 2\operatorname{Re}(\psi_1)\frac{d}{dt}\operatorname{Re}(\psi_1) + 2\operatorname{Im}(\psi_1)\frac{d}{dt}\operatorname{Im}(\psi_1)\\
&= 2\operatorname{Re}(\psi_1)\operatorname{Re}\left(\frac{d}{dt}\psi_1\right) + 2\operatorname{Im}(\psi_1)\operatorname{Im}\left(\frac{d}{dt}\psi_1\right)\\
&= 2\operatorname{Re}(\psi_1)\operatorname{Re}(R(\psi_1, \mathrm{i}\, u_2)) + 2\operatorname{Im}(\psi_1)\operatorname{Im}(R(\psi_1, \mathrm{i}\, u_2)) \quad \text{(because of } (\star))\\
&= 2\operatorname{Re}[\overline{\psi}_1 R(\psi_1, \mathrm{i}\, u_2)] .
\end{aligned}
$$

This might look more complicated as it really is; we have just made use of the formula

$$\operatorname{Re}(z_1)\operatorname{Re}(z_2) + \operatorname{Im}(z_1)\operatorname{Im}(z_2) = \operatorname{Re}(\overline{z}_1 z_2)$$

which holds for any two complex numbers $z_1, z_2 \in \mathbb{C}$. To see this, either write $z_j = a_j + \mathrm{i}\, b_j$, $j = 1, 2$ and note that both sides equal $a_1a_2 + b_1b_2$. Or just observe that

$$\frac{z_1 + \overline{z}_1}{2} \cdot \frac{z_2 + \overline{z}_2}{2} + \frac{z_1 - \overline{z}_1}{2\,\mathrm{i}} \cdot \frac{z_2 - \overline{z}_2}{2\,\mathrm{i}} = \frac{\overline{z_1} z_2 + z_1 \overline{z}_2}{2} .$$

Now let us derive an estimate for this last expression. For $z = (z_1, z_2) \in \mathbb{C}^2$ we have

$$\operatorname{Re}[\overline{z}_1 R(z)] = \frac{1}{2}\overline{z}_1 z_1 \sigma^2 \operatorname{Re}(z_1) + \sigma\rho\overline{z}_1 z_1 \operatorname{Re}(z_2) + \frac{1}{2}\operatorname{Re}(\overline{z}_1 z_2^2) - \kappa\overline{z}_1 z_1 - \frac{1}{2}\operatorname{Re}(\overline{z}_1 z_2) .$$

Further, we have $\operatorname{Re}(z_1) \leqslant 1 + |z_1|^2$ and $\operatorname{Re}(z_2) \leqslant 1 + |z_2|^2$.

For this, we claim that for any $x \in \mathbb{C}$ we have $\operatorname{Re}(x) \leqslant 1 + |x|^2$: Let $x = a + \mathrm{i}\, b$, with $a, b \in \mathbb{R}$, then the claim is $a \leqslant 1 + a^2 + b^2$. This is equivalent to $a(1-a) \leqslant 1 + b^2$. If $a < 0$, the LHS is negative and things are OK. If $a \geq 0$, then either $a \geq 1$ and the LHS is $\leqslant 0$ whereas the RHS is $\geq 1$; or $a < 1$, in which case $a(1-a) < 1$, and still the RHS is $\geq 1$. So again things are OK.

Moreover, $\operatorname{Re}(\overline{z}_1 z_2^2) \leqslant |\overline{z}_1 z_2^2| = |z_1|\,|z_2|^2$. Checking each term in the formula for $\operatorname{Re}[\overline{z}_1 R(z)]$ above reveals that there exists $C \in \mathbb{R}_+$ such that for all $z \in \mathbb{C}^2$

$$\operatorname{Re}[\overline{z}_1 R(z)] \leqslant C(1 + \operatorname{Re}(z_1)^+ + |z_2|^2)(1 + |z_1|^2) .$$

Now let us apply this estimate to $\psi = (\psi_1, \psi_2)$:

$$\frac{d}{dt}|\psi_1|^2 \leqslant C(1 + \operatorname{Re}(\psi_1)^+ + |\mathrm{i}\, u_2|^2)(1 + |\psi_1|^2) .$$

From above we know that $\mathrm{Re}(\psi_1)^+ = 0$. Hence, with $g(t) := 1 + |\psi_1|^2$:

$$g'(t) \leqslant 2C(1+u_2^2)g(t)\,.$$

Now recall GRONWALL's Lemma:

*If* $g(t) \leqslant a + \int_0^t bg(s)\,ds$, *then* $g(t) \leqslant a + \int_0^t ab\,\mathrm{e}^{bt-s}\,ds = a\,\mathrm{e}^{bt}$, *where* $a, b \geq 0$.

Applying this here with $a := g(0)$ and $b := 2C(1+u_2^2))$ we see that $g$, hence $|\psi_1|$, remains bounded on any finite interval and so does not explode. Therefore, $t_+ = \infty$, and so, for each $u \in \mathbb{R}^2$, a solution $(\phi, \psi)$ exists globally on $\mathbb{R}_+$. QED

GRONWALL's Lemma exists in numerous variants. One formulation of which the above formulation is a special case goes as follows (see [**34**]):

**Lemma.** (GRONWALL's Lemma) *Let $I$ denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a < b$. Let $\alpha$, $\beta$ and $\gamma$ be real-valued functions defined on $I$. Assume that $\beta$ and $\gamma$ are continuous and that the negative part of $\alpha$ is integrable on every closed and bounded subinterval of $I$. If $\beta$ is non-negative and if $\gamma$ satisfies the integral inequality*

$$\forall\, t \in I\,:\ \gamma(t) \leqslant \alpha(t) + \int_a^t \beta(s)\gamma(s)\,ds$$

*then*

$$\forall\, t \in I\,:\ \gamma(t) \leqslant \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)\,dr\right)\,ds\,.$$

Whatever version one stumbles on, the gist is one has a function satisfying an integral or differential inequality involving itself on the RHS and is allowed then to conclude then this function satisfies the same inequality with itself on the RHS replaced by a solution of the corresponding integral or differential equation. So it gets eliminated on the RHS and one obtains an explicit inequality with no bootstraps. Correspondingly there exist versions of GRONWALL's Lemma in integral and differential formulations; the formulation above is an integral variant.

To conclude with, applying GRONWALL's Lemma in the situation above is cracking a peanut with a sledgehammer. Getting the wanted bound on $g(t)$ is straightforward: If

$$g'(t) \leqslant 2C(1+u_2^2)g(t)$$

we can, ince $g(t) > 0$ for all $t$, divide by $g(t)$ and obtain

$$\frac{g'(t)}{g(t)} = (\log g)'(t) \leqslant 2C(1+u_2^2) =: A\,.$$

Integrating this inequality over the intervall $[0,t]$ yields

$$\log g(t) - \log g(0) = \int_0^t (\log g)'(s)\,ds \leqslant At\,,$$

and applying the exponential function which s strictly monotonous:

$$g(t) \leqslant g(0)\,\mathrm{e}^{At}\,,$$

which definitely shows that $g$, and hence $|\psi_1|$, remains bounded on any finite interval and so does not explode.

Lemma 10.1 implies that the solution of (10.3) satisfies

$$\begin{aligned}\operatorname{Re}(\phi(t,\mathrm{i}\,u)) &= \int_0^t \left\{\kappa\theta\operatorname{Re}(\psi_1(s,\mathrm{i}\,u)) + r\,\underbrace{\operatorname{Re}(\psi_2(s,\mathrm{i}\,u))}_{=0}\right\} ds \\ &= \int_0^t \kappa\theta\,\underbrace{\operatorname{Re}(\psi_1(s,\mathrm{i}\,u))}_{\leqslant 0}\, ds \\ &\leqslant 0\,.\end{aligned}$$

Hence

$$\operatorname{Re}\left[\phi(t,\mathrm{i}\,u) + \psi(t,\mathrm{i}\,u)^\top x\right] \leqslant 0$$

for any $x \in \mathcal{X} = \mathbb{R}_+ \times \mathbb{R}$. This means we can apply Theorem 9.1, which says $X$ is affine. This completes our explicit check of affineness of the HESTON model.

The proof of the general case of admissible parameters is, modulo more cumbersome notation, the same; hence we consider Proposition 9.5 proved, too.

So we have seen the HESTON model is affine. In fact, you can explicitely solve the ODEs, but for the time being we have no need for these explicit solutions.

Our next aim is to derive the famous HESTON formula; it is an analytic expression for the price of an European call option (the formula for a put is similar).

The terminology often seen used in this context is “semi–analytic” in place of “analytic”; this refers to the fact that the underlying martingale measure will appear explicitely in the formula in shape of its characteristic function, in contrast to the BLACK–SCHOLES formula, which is called “analytic”, because it only contains numerical parameters of the model and the well–known simple analytic functions log and $\Phi$. The value of such a distinction may be considered questionable.

Recall the price paradigm. The arbitrage–free price of a call with maturity $T$ and strike $K$ is given by

$$\text{HESTON–call}(K,T) = \mathrm{e}^{-rT}\,\mathbb{E}^*\left[(S-K)^+\right] = \mathrm{e}^{-rT}\,\mathbb{E}^*\left[(\mathrm{e}^{Z_T} - K)^+\right]\,.$$

with $\mathbb{E}^*$ the expectation w.r.t the martingale measure $P^*$ which we regarded from the beginning as input to the model and which underlies the SDEs (10.1a) and (10.1b).

We split the expectation into two parts:

$$\begin{aligned}\mathbb{E}^*\left[(\mathrm{e}^{Z_T} - K)^+\right] &= \mathbb{E}^*\left[(\mathrm{e}^{Z_T} - K)\mathbb{1}_{\{Z_T \geq \log(K)\}}\right] \\ &= \mathbb{E}^*\left[(\mathrm{e}^{Z_T})\mathbb{1}_{\{Z_T \geq \log(K)\}}\right] - KP^*\left[Z_T \geq \log(K)\right]\end{aligned}$$

For $X$ a random variable there is a formula $P^*[X \geq x] = H(\varphi_X(x))$ in terms of the characteristic function of $X$ (see Corollary 10.3 below). The second part of the RHS has this form, and the strategy is now to convert the first part of the RHS also into this form, i.e. to convert the expectation into a probability of a random variable exceeding a given level $x$. The technique for doing so which presents itself is to make a measure change to a new probability measure $Q \sim P^*$ which involves $\exp Z_T$ as a density. Wanting $Q \sim P^*$ cries for Girsanov, and Girsanov cries for $\mathrm{e}^{-rt}\,\mathrm{e}^{Z_t} = \widetilde{S}_t$ being a martingale, which it is since we are working under a risk–neutral measure. We are thus in the position to define a new equivalent probability measure $Q$ via

$$\frac{dQ}{dP^*} := \mathrm{e}^{-rT}\,\frac{1}{S_0}\,\mathrm{e}^{Z_T}\ .$$

With this definition of $Q$ there comes

$$\mathbb{E}^*\left[(\mathrm{e}^{Z_T})\mathbb{1}_{\{Z_T \geq \log(K)\}}\right] = \mathrm{e}^{rT}\,S_0\mathbb{E}^Q\left[Z_T \geq \log(K)\right]\ .$$

Now note that $\mathrm{e}^{-rt}\,\mathrm{e}^{Z_t} = \mathrm{e}^{-rt}\,S_t$ is a stochastic exponential, namely $\mathrm{e}^{-rt}\,S_t = \mathcal{E}\left(\int_0^t \sqrt{V}_s\,dW_s^Z\right)$ (for a discussion of the stochastic exponential see page 82 ff). To see this, compute $d(\mathrm{e}^{-rt}\,S_t)$ by the product formula and use (10.1b) to show the SDE for the corresponding stochastic exponential is satisfied. Girsanov's theorem then implies that

$$\begin{aligned} dW_t^{1,Q} &= dW_t^1 - d\langle G, W^1\rangle = dW_t^1 - \rho\sqrt{V}_t dt \\ dW_t^{Z,Q} &= dW_t^Z - d\langle G, W^Z\rangle = dW_t^Z - \sqrt{V}_t dt \end{aligned}$$

are BMs. The SDEs for $V$ and $Z$ under the new measure $Q$ are

$$\begin{aligned} dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V}_t(dW_t^{1,Q} + \rho\sqrt{V}_t dt) \\ &= (\kappa\theta - (\kappa - \rho\sigma))V_t dt + \sigma\sqrt{V}_t dW_t^{1,Q}\ ; \\ dZ_t &= \left(r - \frac{V_t}{2}_t\right)dt + \sqrt{V}_t(dW_t^{Z,Q} + \sqrt{V}_t dt) \\ &= \left(r + \frac{V_t}{2}_t\right)dt + \sqrt{V}_t dW_t^{Z,Q}\ . \end{aligned}$$

Now let $\widetilde{X} := (U, Y)$ be the solution of the SDE system

$$\begin{aligned} dU_t &= (\kappa\theta - (\kappa - \rho\sigma))U_t dt + \sigma\sqrt{U}_t dW_t^1\ ; \\ dY_t &= \left(r + \frac{U_t}{2}_t\right)dt + \sqrt{U}_t dW_t^Z\ . \end{aligned}$$

From this we see that $\widetilde{X} = (U, Y)$ is also an affine process under $P^*$ with parameters

$$a = \quad 0 \quad , \quad \alpha_1 = \begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix} \quad , \quad \alpha_2 = 0\,;$$
$$b = \begin{pmatrix} \kappa\theta \\ r \end{pmatrix} \quad , \quad \beta_1 = \begin{pmatrix} -(\kappa - \varrho\sigma) \\ \frac{1}{2} \end{pmatrix} \quad , \quad \beta_2 = 0\,.$$

which will be important later for the complete formulation of the HESTON formula for a call.

Now $(\widetilde{X}, W) = ((U,Y),(W^1, W^2))$ and $(X, W^Q) = ((V,Z),(W^{1,Q}, W^{2,Q}))$ are two weak solutions of the SDE system for a process $(Q, R)$

$$dQ_t = (\kappa\theta - (\kappa - \rho\sigma))Q_t dt + \sigma\sqrt{Q}_t dB_t^1\,;$$
$$dR_t = \left(r + \frac{Q_t}{2}_t\right) dt + \sqrt{Q}_t dB_t^Z\,.$$

with $B = (B^1, B^2)$ some fixed reference BM and $B^Z := \rho B^1 + \sqrt{1-\rho^2}$. As described above, this SDE system has a unique strong solution. The more it has a unique weak solution, where uniqueness for weak solutions mean they are equivalent in law, i.e. have the same distributions. In particular, this entails

$$Q\left[Z_T \geq \log(K)\right] = P^*\left[Y_T \geq \log(K)\right]\,.$$

Then the pricing simplifies to the following preliminary version of the HESTON formula

(10.6) $\quad \text{HESTON–call}(K,T) = S_0 P^*\left[Y_T \geq \log(K)\right] - \mathrm{e}^{-rT} K P^*\left[Z_T \geq \log(K)\right]\,.$

This is not yet an analytical formula, but a probabilistic formula, since the probability measure $P^*, Q$ and the random variables $Y_T$, $Z_T$ enter explicitely. This implies that it is of not much practical use, since its numerical evaluation requires extensive computations, e.g. via Monte Carlo simulations. Therefore, providing an analytic formula is desirable, so now we set off for such a task.

**Remark**. It is interesting to note the structural similarity of this formula with the BS–formula; in both cases the option price is a difference of two terms, the first one involving the start value $S_0$ and the second one the discounted strike $K$. In fact, I claim that this formula is a generalization of the BS–formula. Namely, if the volatility $V_t$ is constant, $Y_t$ and $Z_t$ are are normally distributed, and (10.6) coincides with the BS–formula (see Proposition 7.10) for the call option price.

### EXCURSION: CHARACTERISTIC FUNCTION AND THE INVERSION THEOREM

First recall the definition of the characteristic function of a random variable. Let $X$ be an $\mathbb{R}$–valued random variable on some probability space $(\Omega, \mathcal{F}, P)$. The *characteristic function* $\varphi_X : \mathbb{R} \longrightarrow \mathbb{C}$ is defined by

$$\varphi_X(u) := \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\,uX}\right] .$$

Further recall that the *distribution* $\mu_X$ of $X$ is the probability measure on the measurable space $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by

$$\forall\, A \in \mathcal{B} \,:\, \mu_X\,[A] := P\,[X \in A]$$

(to enhance readability we write $P\,[X \in A]$ in place of $P\,[\{X \in A\}]$, leave alone $P\,[\{\,\omega \in \Omega \mid\; X(\omega) \in A\,\}]$). In particular, then,

$$\varphi_X(u) = \int_\Omega \mathrm{e}^{\mathrm{i}\,uX}\; dP = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,ux}\; \mu_X\,[dx]$$

and so we have expressed $\varphi_X$ in terms of $\mu_X$. The following theorem inverts this relation and expresses $\mu_X$ in terms of $\varphi_X$.

**Theorem 10.2.** (The Inversion Formula) *Let $X$ be an $\mathbb{R}$–valued integrable random variable on a probability space $(\Omega, \mathcal{F}, P)$ with distribution $\mu_X$ and characteristic function $\varphi_X$. Then for all $x \in \mathbb{R}$*

$$\mu_X\,[(-\infty, x)] + \frac{1}{2}\mu_X\,[\{x\}] = \frac{1}{2} + \lim_{c\to\infty} \frac{1}{2\pi} \int_0^c \frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi_X(-u) - \mathrm{e}^{-\mathrm{i}\,ux}\,\varphi_X(u)}{\mathrm{i}\,u}\,du\,.$$

One may wonder whether this formula indeed recovers the full probability measure $\mu_X$, in which case the characteristic function $\varphi_X$ and the distribution $\mu_X$ would determine each other. This is, in fact, the case, but it requires some work.

A somewhat roundabout abstract proof goes as follows. The distribution function $F_X : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$F_X(x) := P\,[X \leqslant x] = \mu_X\,[(-\infty, x]]\ .$$

It is monotonously increasing and thus is known to have at most countably many discontinuities. It is easy to see that the $\sigma$–algebra generated by the intervals $(-\infty, x]$, $x \in \mathbb{R}$, is the BOREL–algebra $\mathcal{B}(\mathbb{R})$ and so $\mu_X$ is determined by the values $\mu_X\,[(-\infty, x]]$, i.e. by $F_X$.

Now by the Inversion Theorem $\varphi_X$ determines the values $\mu_X\,[(-\infty, x)] = \mu_X\,[(-\infty, x]]$ for those $x$ such that $\{x\}$ s not an atom of $\mu_X$. These are the points of discontinuity of $F_X$, and so there are only at most countably many. If $A \subseteq \mathbb{R}$ is the set of those, $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$, and for any $x_0 \in A$ there is a monotone sequence $x_1 > x_2 > x_3 > \cdots$ such that $x_n \notin A$ and $\lim_{n\to\infty} = x_0$. Then $(-\infty, x_1] \supset (-\infty, x_2] \supset (-\infty, x_3] \supset \cdots \supset (-\infty, x_0]$ and $(-\infty, x_0] = \bigcap_n (-\infty, x_n]$. Then $\mu_X\,[(-\infty, x_0]] = \lim_{n\to\infty} \mu_X\,[(-\infty, x_n]]$, and so $\varphi_X$ determines all the $\mu_X\,[(-\infty, x]]$, $x \in \mathbb{R}$, which shows it determines $\mu_X$.

A proof more in the spirit of the calculations that follow is to show by the same methods as below

$$\mu_X\,[\{x\}] = \lim_{c\to\infty} \frac{1}{2c} \int_0^c \left\{\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi_X(u) - \mathrm{e}^{\mathrm{i}\,ux}\,\varphi_X(-u)\right\}\,du$$

(exercise). Then the formula of the Inversion Theorem and the last formula determine all the $\mu_X\left[(-\infty,x]\right]$ via

$$\mu_X\left[(-\infty,x]\right]=\left(\mu_X\left[(-\infty,x)\right]+\frac{1}{2}\mu_X\left[\{x\}\right]\right)+\frac{1}{2}\mu_X\left[\{x\}\right].$$

PROOF *(of Theorem 10.2).* Since $X$ stays fixed in the discussion, we write $\varphi:=\varphi_X$ and $\mu:=\mu_X$. We proceed in three steps.

*Step 1.* Note that

$$\begin{aligned}\left|\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\right| &= \left|\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(u)+\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\right|\\ &\leqslant \left|\mathrm{e}^{\mathrm{i}\,ux}\right|\left|\frac{\varphi(-u)-\varphi(u)}{\mathrm{i}\,u}\right|+\left|\varphi(u)\right|\left|\frac{\mathrm{e}^{\mathrm{i}\,ux}-\mathrm{e}^{-\mathrm{i}\,ux}}{\mathrm{i}\,u}\right|\\ &\leqslant \left|\frac{\varphi(-u)-\varphi(u)}{\mathrm{i}\,u}\right|+\left|\frac{\mathrm{e}^{\mathrm{i}\,ux}-\mathrm{e}^{-\mathrm{i}\,ux}}{\mathrm{i}\,u}\right|\\ &\leqslant \left|\frac{2\mathbb{E}\left[\sin uX\right]}{u}\right|+\left|\frac{2\sin ux}{u}\right|\end{aligned}$$

(since $\varphi(u)=\mathbb{E}\left[\cos uX\right]+\mathrm{i}\,\mathbb{E}\left[\sin uX\right]$ and so $\varphi(-u)=\mathbb{E}\left[\cos uX\right]-\mathrm{i}\,\mathbb{E}\left[\sin uX\right]$)

$$\begin{aligned}&= 2\mathbb{E}\left[\left|\frac{\sin uX}{u}\right|\right]+2\left|\frac{\sin ux}{u}\right|\\ &\leqslant 2\mathbb{E}\left[|X|\right]+2\left|x\right|\end{aligned}$$

(since $|\sin y|\leqslant|y|$ for all $y\in\mathbb{R}$). This shows that the integral in the theorem is defined.

*Step 2.* Let $\varepsilon>0$. Then

$$\int_{\varepsilon}^{c}\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\,du=-\int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)}\frac{\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\,du$$

(note that the integrand on the RHS is not LEBESGUE–integrable around 0)

$$\begin{aligned}&=-\int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)}\int\frac{\mathrm{e}^{-\mathrm{i}\,u(x-y)}}{\mathrm{i}\,u}\,\mu[dy]\,du\\ &=-\int\int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)}\frac{\mathrm{e}^{\mathrm{i}\,u(y-x)}}{\mathrm{i}\,u}\,du\,\mu[dy]\quad(\text{FUBINI}).\end{aligned}$$

We next perform the inner integral:

$$\begin{aligned}
\int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)} \frac{\mathrm{e}^{\mathrm{i}\,u(y-x)}}{\mathrm{i}\,u}\,du &= \int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)} \left\{\frac{\cos u(y-x)+\mathrm{i}\sin u(y-x)}{\mathrm{i}\,u}\right\} du \\
&= -\,\mathrm{i}\int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)} \frac{\cos u(y-x)}{u}\,du \;+ \int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)} \frac{\sin u(y-x)}{u}\,du \\
&= \int\limits_{[-c,c]\setminus(-\varepsilon,\varepsilon)} \frac{\sin u(y-x)}{u}\,du\,,
\end{aligned}$$

because the first integral vanishes as $\dfrac{\cos u(y-x)}{u}$ is odd in $u$. Note that the surviving integrand $\dfrac{\sin u(y-x)}{u}$ is Lebesgue-integrable around 0. There results

$$\int_0^c \frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\,du = -\int\int_{-c}^{c} \frac{\sin u(y-x)}{u}\,du\,\mu[dy]\,.$$

*Step 3.* One can show

$$\lim_{c\to\infty}\int_{-c}^{c} \frac{\sin u(y-x)}{u}\,du = \pi\,\mathrm{sign}(y-x)$$

where

$$\begin{aligned}
\mathrm{sign}(y-x) &= \begin{cases} 1 : & y>x\,; \\ 0 : & y=0\ : \\ -1 : & y<x\,. \end{cases} \\
&= \mathbb{1}_{(x,\infty)} - \mathbb{1}_{(-\infty,x)}\,.
\end{aligned}$$

This can be seen as follows. Let $a\in\mathbb{R}$. The function

$$f(u) := \frac{\sin ua}{u}$$

is continuous on $\mathbb{R}$ and so integrable over any bounded interval. Let $c>0$ be any positive real. Then, by making the substitution $x := ua$

$$\int_{-c}^{c} \frac{\sin ua}{u}\,du = \begin{cases} \displaystyle\int_{-ca}^{ca} \frac{\sin x}{x}\,dx & : a>0\,; \\ 0 & : a=0\,; \\ \displaystyle -\int_{ca}^{-ca} \frac{\sin x}{x}\,dx & : a<0\,. \end{cases}$$

With $c \to \infty$ there follows

$$\int_{-\infty}^{\infty} \frac{\sin ua}{u}\, du = \operatorname{sign} a \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

in the sense that if the right integral exists as a proper or improper integral so does the left and the values are equal. We will see below that the right integral does not exist in the proper sense. But it does exist as an improper, or conditional, integral; for this, write, for $N \in \mathbb{N}$,

$$\int_{0}^{N\pi} \frac{\sin x}{x}\, dx = \sum_{k=0}^{N-1} \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x}\, dx = \sum_{k=0}^{N-1} (-1)^k S_k$$

and observe that the sequence $(S_k)$ with

$$S_k := (-1)^k \int_{k\pi}^{(k+1)\pi} \frac{\sin x}{x}\, dx$$

is positive, monotonously decreasing, and satisfies $S_k \leqslant 1/(k\pi)$, hence $\lim_{k\to\infty} S_k = 0$. By the LEIBNIZ criterion for alternating series there follows (note that $\sin x/x$ is even)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \lim_{N\to\infty} \int_{-N\pi}^{N\pi} \frac{\sin x}{x}\, dx = 2\sum_{k=0}^{\infty} (-1)^k S_k < \infty\,.$$

The claim finally is

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi\,,$$

a very classical improper integral known as the DIRICHLET *integral* (one out of many).

The verification of this formula is quite subtle and often used on different occasions to demonstrate the various techniques to handle improper integrals:

1) *By Real Analysis*: As will be demonstrated below, the integrand is not integrable over $[0, \infty]$. The idea then is to force integrability by multiplying with a parametrized family of functions which decay fast enough at infinity to make the product integrable and then to remove the parameter in the end. The candidates of choice are, of course functions with an exponential decay, so we consider the family

$$f(\alpha, x) := \mathrm{e}^{-\alpha x}\, \frac{\sin x}{x} \quad , \quad \alpha \geq 0\,.$$

This function is continuous on $[0, \infty) \times \mathbb{R}$ and continuously differentiable on $(0, \infty) \times \mathbb{R}$. The usual theorems on integrals of functions depending on a parameter (see [**1**], p. 111, Theorem 3.18) tell us that the integral

$$F(\alpha) := \int_{0}^{\infty} \mathrm{e}^{-\alpha x}\, \frac{\sin x}{x}\, dx$$

is differentiable w.r.t. $\alpha$ on $(0, \infty)$ and the derivative can be obtained by "differentiating under the integral sign":

$$F'(\alpha) := \int_{0}^{\infty} \frac{\partial\, \mathrm{e}^{-\alpha x}}{\partial \alpha}\, \frac{\sin x}{x}\, dx = -\int_{0}^{\infty} \mathrm{e}^{-\alpha x}\, \sin x dx\,.$$

This is – oh lucky us – an integral we can handle; applying partial integration twice leaves us – doing the indefinite integrals first – with

$$\begin{aligned}\int \mathrm{e}^{-\alpha x}\sin x\,dx &= -\frac{1}{\alpha}\,\mathrm{e}^{-\alpha x}\sin x + \frac{1}{\alpha}\int \mathrm{e}^{-\alpha x}\cos x\,dx + C\\ &= -\frac{1}{\alpha}\,\mathrm{e}^{-\alpha x}\sin x - \frac{1}{\alpha^2}\,\mathrm{e}^{-\alpha x}\cos x - \frac{1}{\alpha^2}\int \mathrm{e}^{-\alpha x}\sin x\,dx + C\end{aligned}$$

and so

$$(*) \qquad \int \mathrm{e}^{-\alpha x}\sin x\,dx = -\frac{\mathrm{e}^{-\alpha x}}{1+\alpha^2}(\alpha\sin x+\cos x) + C\,,$$

where $C$ is an arbitrary integration constant. Hence

$$(**) \qquad \begin{aligned}\int_0^\infty \mathrm{e}^{-\alpha x}\sin x\,dx &= -\left.\frac{\mathrm{e}^{-\alpha x}}{1+\alpha^2}(\alpha\sin x+\cos x)\right|_{x=0}^{x=\infty}\\ &= \frac{1}{1+\alpha^2}\,.\end{aligned}$$

This immediately entails

$$F(\alpha) = -\int \frac{1}{1+\alpha^2}\,d\alpha = -\arctan\alpha + C\,.$$

The integration constant $C$ can be determined since

$$\forall\, x\in\mathbb{R}, x\neq 0\ :\ \left|\frac{\sin x}{x}\right| \leqslant 1$$

and so

$$\forall\, \alpha\in\mathbb{R}, \alpha>0\ :\ |F(\alpha)| \leqslant \left|\int_0^\infty \mathrm{e}^{-\alpha x}\ dx\right| = \frac{1}{\alpha}$$

whence $\lim_{\alpha\to\infty} F(\alpha) = 0$. We obtain

$$0 = \lim_{\alpha\to 0} F(\alpha) = -\lim_{\alpha\to\infty}\arctan\alpha + C = -\frac{\pi}{2} + C$$

hence

$$\forall\,\alpha>0\ :\ F(\alpha) = \frac{\pi}{2} - \arctan\alpha\,.$$

Thus we would get

$$\int_0^\infty \frac{\sin x}{x}\,dx = F(0) = \frac{\pi}{2}$$

if we knew *that $F$ be continuous in* 0. But – oh pitiful us – we do *not* know this yet, and the standard theorems on parameter integrals do not cover this particular case. So we have to find a workaround.

Writing

$$F(0) = \int_0^\infty F'(\alpha)\,d\alpha = -\int_0^\infty\int_0^\infty \mathrm{e}^{-\alpha x}\sin x\,dx\,d\alpha$$

if it were allowed (which it is not) would prove the claim, but it at least suggests to consider the double integral on the RHS. If it were allowed to apply FUBINI (which it is not), there would come

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, dx \, d\alpha = \int_0^\infty \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, d\alpha \, dx$$

The LHS would give, by equation $(**)$ above,

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, dx \, d\alpha = \int_0^\infty \frac{1}{1+\alpha^2} \, d\alpha = \frac{\pi}{2}$$

and the RHS weould give

$$\int_0^\infty \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, d\alpha \, dx = \int_0^\infty \left( -\frac{1}{x} \, \mathrm{e}^{-\alpha x} \right) \Big|_{\alpha=0}^{\alpha=\infty} \sin x \, dx = \int_0^\infty \frac{\sin x}{x} \, dx \, ,$$

which is our desideratum.

The situation changes on the spot, however, the moment one realizes that it *is* allowed to apply FUBINI over the region $[0, \infty) \times (0, c)$ where $c > 0$ is any positive real. So we fix $c > 0$ and consider the equality

$$\int_0^\infty \int_0^c \mathrm{e}^{-\alpha x} \sin x \, dx \, d\alpha = \int_0^c \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, d\alpha \, dx$$

which now *is* allowed by FUBINI. The LHS gives by equation $(*)$ above

$$\begin{aligned} \int_0^\infty \int_0^c \mathrm{e}^{-\alpha x} \sin x \, dx \, d\alpha &= \int_0^\infty \left( -\frac{\mathrm{e}^{-\alpha x}}{1+\alpha^2} (\alpha \sin x + \cos x) \right) \Big|_{x=0}^{x=c} d\alpha \\ &= -\int_0^\infty \frac{\mathrm{e}^{-\alpha c}}{1+\alpha^2} (\alpha \sin c + \cos c) \, d\alpha + \int_0^\infty \frac{1}{1+\alpha^2} \, d\alpha \\ &= \frac{\pi}{2} - \sin c \int_0^\infty \frac{\alpha \, \mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha - \cos c \int_0^\infty \frac{\mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha \, . \end{aligned}$$

The RHS gives

$$\int_0^c \int_0^\infty \mathrm{e}^{-\alpha x} \sin x \, d\alpha \, dx \int_0^c \left( -\frac{1}{x} \, \mathrm{e}^{-\alpha x} \right) \Big|_{\alpha=0}^{\alpha=\infty} \sin x \, dx = \int_0^c \frac{\sin x}{x} \, dx \, .$$

Collecting our findings we get

$$\int_0^c \frac{\sin x}{x} \, dx - \frac{\pi}{2} = -\sin c \int_0^\infty \frac{\alpha \, \mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha - \cos c \int_0^\infty \frac{\mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha$$

and so

$$\left| \int_0^c \frac{\sin x}{x} \, dx - \frac{\pi}{2} \right| \leqslant |\sin c| \int_0^\infty \frac{\alpha \, \mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha + |\cos c| \int_0^\infty \frac{\mathrm{e}^{-\alpha c}}{1+\alpha^2} \, d\alpha$$

$$
\begin{aligned}
&\leqslant \int_0^\infty \alpha\, \mathrm{e}^{-\alpha c}\, d\alpha + \int_0^\infty \mathrm{e}^{-\alpha c}\, d\alpha \\
&= -\frac{1}{c}\, \mathrm{e}^{-c\alpha} \left(\alpha + \frac{1}{c} + 1\right)\Big|_{\alpha=0}^{\alpha=\infty} \\
&= \frac{1}{c}\left(\frac{1}{c} + 1\right) \\
&\leqslant \frac{2}{c}
\end{aligned}
$$

for $c \geq 1$. There follows

$$\int_0^\infty \frac{\sin x}{x}\, dx = \lim_{c\to\infty} \int_0^c \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

and we are done.

For an enhanced FUBINI setup which applies in particular to the DIRICHLET integral here see [**25**].

b) *By Complex Analysis*: Improper integrals usually cry for treatment by the Residue Theorem. Let $f$ be the function $f(z) := \mathrm{e}^{iz}/z$. It has the form

$$f(z) = \frac{1}{z} + g(z) \quad , \quad g \text{ holomorphic with } g(0) \neq 0,$$

and so has a simple pole in 0.

Now fix two positive real numbers $0 < \varepsilon < R$. consider the points $P_\mp := \mp R + \mathrm{i}\,0$, $Q_\mp := \mp\varepsilon + \mathrm{i}\,0$ and finally the points $R_\mp := \mp R + \mathrm{i}\,R$ all in $\mathbb{R} + \mathrm{i}\,0 = \{\, x + \mathrm{i}\,0 \mid x \in \mathbb{R}\,\} \subseteq \mathbb{C}$.

We consider the following paths. Let $\gamma_{\mp\varepsilon}$ be the straight line segments from $P_-$ to $Q_-$ and $Q_+$ to $P_+$ respectively, $\sigma_\varepsilon$ the upper half–circle of radius $\varepsilon$ from $Q_-$ to $Q_+$ in the clockwise direction, and finally $\rho = \rho_+\rho_0\rho_-$ the linear path from $P_+$ over $R_+$ and $R_-$ to $P_-$ in the counter-clockwise direction along the boundary of the rectangle $P_+R_+R_-P_-$. Then $\gamma := \gamma_{-\varepsilon}\sigma_\varepsilon\gamma_{+\varepsilon}\rho$ is a closed circuit wich does not contain the only pole of $f$ in its interior and therefore by CAUCHY's Integral Theorem

$$
\begin{aligned}
0 = \int_\gamma f(z)\, dz &= \int_{\gamma_{-\varepsilon}} f(z)\, dz + \int_{\sigma_\varepsilon} f(z)\, dz + \int_{\gamma_{+\varepsilon}} f(z)\, dz + \int_\rho f(z)\, dz \\
&= \mathrm{i} \int_{-R}^{-\varepsilon} \frac{\sin x}{x}\, dx + \frac{1}{2} \int_{-\partial_\varepsilon} f(z)\, dz + \mathrm{i} \int_\varepsilon^R \frac{\sin x}{x}\, dx + \int_\rho f(z)\, dz \\
&= \mathrm{i} \int_{-R}^{-\varepsilon} \frac{\sin x}{x}\, dx + \mathrm{i} \int_\varepsilon^R \frac{\sin x}{x}\, dx + \frac{1}{2} \int_{-\partial_\varepsilon} f(z)\, dz + \int_\rho f(z)\, dz
\end{aligned}
$$

where $-\partial_\varepsilon$ is the boundary of the circle around the origin of radius $\varepsilon$ in the clockwise direction. Therefore

$$\mathrm{i} \int_{-R}^{-\varepsilon} \frac{\sin x}{x}\, dx + \mathrm{i} \int_\varepsilon^R \frac{\sin x}{x}\, dx = \frac{1}{2} \int_{\partial_\varepsilon} f(z)\, dz - \int_\rho f(z)\, dz\,.$$

where $\partial_\varepsilon$ is the boundary of the circle around the origin of radius $\varepsilon$ in the counter–clockwise (= mathematically positive direction. Now

$$\int\limits_{\partial_\varepsilon} f(z)\,dz = \int\limits_{\partial_\varepsilon} \frac{1}{z}\,dz + \int\limits_{\partial_\varepsilon} g(z)\,dz = 2\,\mathrm{i}\,\pi + \int\limits_{\partial_\varepsilon} g(z)\,dz$$

and so

$$\lim_{\varepsilon\to 0} \int\limits_{\partial_\varepsilon} f(z)\,dz = 2\,\mathrm{i}\,\pi + \lim_{\varepsilon\to 0} \int\limits_{\partial_\varepsilon} g(z)\,dz = 2\,\mathrm{i}\,\pi$$

since $\lim_{\varepsilon\to 0} \int\limits_{\partial_\varepsilon} g(z)\,dz = 0$. Thus

$$\mathrm{i}\int\limits_{-R}^{R} \frac{\sin x}{x}\,dx = \lim_{\varepsilon\to 0} \mathrm{i}\int\limits_{-R}^{-\varepsilon} \frac{\sin x}{x}\,dx + \lim_{\varepsilon\to 0} \mathrm{i}\int\limits_{\varepsilon}^{R} \frac{\sin x}{x}\,dx = \mathrm{i}\,\pi - \int\limits_{\rho} f(z)\,dz\,.$$

Now

$$\int\limits_{\rho} f(z)\,dz = \int\limits_{\rho_+} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz + \int\limits_{\rho_0} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz + \int\limits_{\rho_-} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz$$

and

$$\int\limits_{\rho_+} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz = \mathrm{e}^{\mathrm{i}\,R}\int\limits_{0}^{R} \frac{\mathrm{e}^{-y}}{R+\mathrm{i}\,y}\,dy \qquad \text{hence} \qquad \left|\int\limits_{\rho_+} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz\right| \leqslant \frac{1-\mathrm{e}^{-R}}{R}$$

$$\int\limits_{\rho_0} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz = \mathrm{e}^{-R}\int\limits_{R}^{-R} \frac{\mathrm{e}^{\mathrm{i}\,x}}{x+\mathrm{i}\,R}\,dx \qquad \text{hence} \qquad \left|\int\limits_{\rho_0} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz\right| \leqslant 2\,\mathrm{e}^{-R}$$

$$\int\limits_{\rho_-} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz = \mathrm{e}^{-\mathrm{i}\,R}\int\limits_{R}^{0} \frac{\mathrm{e}^{-y}}{-R+\mathrm{i}\,y}\,dy \quad \text{hence} \qquad \left|\int\limits_{\rho_-} \frac{\mathrm{e}^{\mathrm{i}\,z}}{z}\,dz\right| \leqslant \frac{1-\mathrm{e}^{-R}}{R}\,.$$

Therefore

$$\left|\int\limits_{\rho} f(z)\,dz\right| \leqslant \frac{4}{R}$$

and so

$$\mathrm{i}\int\limits_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{R\to\infty} \mathrm{i}\int\limits_{-R}^{R} \frac{\sin x}{x}\,dx = \mathrm{i}\,\pi - \lim_{R\to\infty} \int\limits_{\rho} f(z)\,dz = \mathrm{i}\,\pi$$

as was to be shown.

3) *By Trickery*: What makes Mathematics fascinating and amazing is the degree to which ideas, definitions and theorems of seemingly disparate disciplines are interrelated so that its universe appears to be subjected to the *small world paradigm* (i.e. to be modelled by a graph in which any two vertices are linked by a path of bounded length where the bound is small, see [**35**]). So there are always unexpected and astonishing shortcuts or elegant twists to obtain results in a way very different from the standard route. In particular, when a master opens his bag of tricks, this event may always be good for a surprise. The following road of attacking the DIRICHLET integral I did find in [**29**] , Kap. 14, §2, 3., pp. 324-5 (a book to be highly recommended); I somewhat modified the approach.

We begin by playing around with trigonometric identities and observe

$$\sin x = 2\sin(x/2)\cos(x/2) = (2\sin^2(x/2))'$$

and so by partial integration

$$\int \frac{\sin x}{x}\,dx = (2\sin^2(x/2))\frac{1}{x} + \int (2\sin^2(x/2))\frac{1}{x}\,dx = \frac{\sin^2(x/2)}{x/2} + \int \frac{\sin^2(x/2)}{x^2/4}\,d(x/2)$$

thus arriving at identity

$$\int_{-2c}^{2c} \frac{\sin x}{x}\,dx = 2\frac{\sin^2 c}{c} + \int_{-c}^{c} \frac{\sin^2 x}{x^2}\,dx$$

for any $c > 0$. Since $\left|\sin^2/x^2\right| \leqslant 1/x^2$ this immediately settles the question of existence of $\int_{-\infty}^{\infty} \sin^2/x^2\,dx$ and hence also of the existence of $\int_{-\infty}^{\infty} \sin/x\,dx$, and we have by letting $c \to \infty$:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx\,.$$

We now compute the integral on the RHS. Fix $N \in \mathbb{N}$; then

$$\begin{aligned}
\int_{-N\pi}^{(N+1)\pi} \frac{\sin^2 x}{x^2}\,dx &= \sum_{k=-N}^{k=N} \int_{k\pi}^{(k+1)\pi} \frac{\sin^2 x}{x^2}\,dx = \sum_{k=-N}^{k=N} \int_0^{\pi} \frac{\sin^2(x+k\pi)}{(x+k\pi)^2}\,dx \\
&= \sum_{k=-N}^{k=N} \int_0^{\pi} \frac{\sin^2 x}{(x+k\pi)^2}\,dx = \int_0^{\pi} \sin^2 x \sum_{k=-N}^{k=N} \frac{1}{(x+k\pi)^2}dx
\end{aligned}$$

Since the integral on the LHS and the series on the RHS are absolutely convergent, there is no problem with taking the limit $N \to \infty$ :

$$(\dagger) \qquad \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_0^{\pi} \sin^2 x \sum_{k=-\infty}^{k=\infty} \frac{1}{(x+k\pi)^2}dx\,.$$

One of the highlights of a first course in Complex Analysis is the partial fraction development of the classical functions, in particular that of the sine (see [**29**], Kap. 11, §2, 3. on p. 260):

$$\sum_{k=-\infty}^{\infty} \frac{1}{(z+k)^2} = \frac{\pi^2}{\sin^2 \pi z}$$

By putting $z = x/\pi$ we may rewrite this as

$$\sum_{k=-\infty}^{\infty} \frac{1}{(x+k\pi)^2} = \frac{1}{\sin^2 x}$$

thus arriving at

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \int_0^{\pi} dx = \pi\,.$$

Lovely.

**Remark**. 1) Of course one now is tempted to play the same game directly with $\int_{-\infty}^{\infty} \sin /x\, dx$; There is no need of a preparing partial integration then; directly proceeding to chopping the integral into pieces as above leads to

$$\int\limits_{-N\pi}^{(N+1)\pi} \frac{\sin x}{x}\, dx = \int\limits_{0}^{\pi} \sin x \sum_{k=-N}^{k=N} \frac{(-1)^k}{x+k\pi} dx\,.$$

One then would like to take limits:

$$\int\limits_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \int\limits_{0}^{\pi} \sin x \lim_{n\to\infty} \sum_{k=-N\pi}^{k=N} \frac{(-1)^k}{x+k\pi} dx\,,$$

but has to be careful here since the series on the RHS is only conditionally convergent and so the interchange of integration and summation needs further justification which makes this apparent direct approach more tedious. If one applies the convention to evaluate the sum by always pairing the summands corresponding to $\pm k$ (the so–call EISENSTEIN *summation*) one has indeed the partial development (see **[29]**, Kap. 11, §2, 3. on p. 261)

$$\begin{aligned}\sum_{k=-\infty}^{k=\infty} \frac{(-1)^k}{z+k} &:= \lim_{n\to\infty} \sum_{k=-N\pi}^{k=N} \frac{(-1)^k}{z+k} = \frac{1}{z} + \lim_{n\to\infty} \sum_{k=-N\pi}^{k=N} (-1)^k \frac{2z}{z^2-k^2} \\ &= \frac{1}{z} + \sum_{k=-\infty}^{k=\infty} (-1)^k \frac{2z}{z^2-k^2} = \frac{\pi}{\sin \pi z}\,,\end{aligned}$$

hence

$$\sum_{k=-\infty}^{k=\infty} \frac{(-1)^k}{x+k\pi} = \frac{1}{\sin x}\,,$$

and things go through as desired. The trick above of reducing $\int_{-\infty}^{\infty} \sin /x\, dx$ to $\int_{-\infty}^{\infty} \sin^2 /x^2\, dx$ circumvents these more cumbersome subtle convergence considerations.

2) In fact, all the integrals

$$I_{m,n} := \int\limits_{0}^{\infty} \frac{\sin^m x}{x^n}\, dx \quad , \quad m,n \in \mathbb{N}$$

are known. Of course, some of them are trivially properly or improperly divergent and so we exclude the cases $m = 0$ or $n = 0$, hence consider only $m, n \geq 1$. Likewise, we can exclude the cases $m < n$, since then $\sin^m x/x^n = (\sin x/x)^m (1/x^{n-m})$ and so we can find $a > 0$ such that $\sin^m x/x^n \geq 1/(2x^{n-m})$ on $[0, a]$ whence $I_{m,n}$ is properly divergent. Thus the interesting cases are $m \geq n \geq 1$. We then have (see **[27]**, the only source I found on these matters)

A) For $m \geq n \geq 1$ of equal parity the $I_{m,n}$ are rational multiples of $\pi$ and in fact

$$I_{m,n} = \frac{1}{2^m (n-1)!} \sum_{0 \leqslant k < m/2} (-1)^{\frac{m-n}{2}+k} \binom{m}{k} (m-2k)^{n-1} \pi\,.$$

B) For $m \geq n \geq 1$ of opposite parity the $I_{m,n}$ are

$$I_{m,n} = \begin{cases} \dfrac{1}{2^{m-1}(n-1)!} \displaystyle\sum_{0 \leqslant k < m/2} (-1)^{\frac{m-n-1}{2}+k} \binom{m}{k} (m-2k)^{n-1} \log(m-2k) & n \geq 2\,; \\ \infty & n = 1\,. \end{cases}$$

As an example for A), we know values of $I_n := I_{n,n}$ for $n = 1, 2$; the first new values are

$$I_3 = \frac{3}{8}\pi \quad , \quad I_4 = \frac{1}{3}\pi \quad , \quad I_5 = \frac{115}{384}\pi \quad , \quad I_6 = \frac{11}{40}\pi \,.$$

As an example for B), we have

$$I_{3,2} = \frac{3}{4}\log(3) \quad , \quad I_{4,3} = \log(2) \quad , \quad I_{5,2} = \frac{5}{16}\log\left(\frac{27}{5}\right) \quad , \quad I_{6,3} = \frac{3}{16}\log\left(\frac{256}{27}\right) .$$

The rest of this remark which follows is purely recreational; it is a reworking of the computations in [**27**]. For the proof we first consider the case $n \geq 2$. We use the formula

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty \mathrm{e}^{-xy}\, y^{n-1}\, dy$$

(easily proved by induction using partial integration) to write

$$(n-1)!\int_0^\infty \frac{\sin^m x}{x^n}\, dx = \int_0^\infty \sin^m x \int_0^\infty \mathrm{e}^{-xy}\, y^{n-1}\, dy\, dx \,.$$

Because of $n \geq 2$ there is no trouble to apply FUBINI, and so there comes

$$(n-1)!\int_0^\infty \frac{\sin^m x}{x^n}\, dx = \int_0^\infty y^{n-1}\int_0^\infty \sin^m x\, \mathrm{e}^{-xy}\; dx\, dy \,.$$

We now have

$$(*) \qquad \sin^m x = \left(\frac{\mathrm{e}^{\mathrm{i}\,x} - \mathrm{e}^{-\mathrm{i}\,x}}{2\,\mathrm{i}}\right)^m = \frac{1}{2^m\,\mathrm{i}^m}\sum_{k=0}^m (-1)^k \binom{m}{k} \mathrm{e}^{\mathrm{i}(m-2k)x}$$

by the bimomial theorem. Therefore

$$\begin{aligned} 2^m(n-1)!\int_0^\infty \sin^m x\, \mathrm{e}^{-xy}\; dx &= \frac{1}{\mathrm{i}^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\int_0^\infty \mathrm{e}^{(\mathrm{i}(m-2k)-y)x}\; dx \\ &= -\frac{1}{\mathrm{i}^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\frac{1}{\mathrm{i}(m-2k)-y} \end{aligned}$$

so that

$$\begin{aligned} 2^m(n-1)!\int_0^\infty \frac{\sin^m x}{x^n}\, dx &= \frac{1}{\mathrm{i}^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\int_0^\infty \frac{y^{n-1}}{y-\mathrm{i}(m-2k)}\, dy \\ &= \frac{1}{\mathrm{i}^m}\sum_{k=0}^m (-1)^k\binom{m}{k}\int_0^\infty \left[\frac{y^{n-1}-\mathrm{i}^{n-1}(m-2k)^{n-1}}{y-\mathrm{i}(m-2k)} + \frac{\mathrm{i}^{n-1}(m-2k)^{n-1}}{y-\mathrm{i}(m-2k)}\right] dy \end{aligned}$$

But

$$\frac{y^{n-1}-\mathrm{i}^{n-1}(m-2k)^{n-1}}{y-\mathrm{i}(m-2k)} = \sum_{p=0}^{n-1} y^{n-1-p}\,\mathrm{i}^p(m-2k)^p \,,$$

therefore

$$\sum_{k=0}^m (-1)^k\binom{m}{k}\int_0^\infty \frac{y^{n-1}-\mathrm{i}^{n-1}(m-2k)^{n-1}}{y-\mathrm{i}(m-2k)}\, dy = \sum_{p=0}^{n-1}\int_0^\infty y^{n-1-p}\,\mathrm{i}^p\sum_{k=0}^m (-1)^k\binom{m}{k}(m-2k)^p\, dy$$

Now observe that the TAYLOR development of the sine allows to write $\sin^m x = x^m \psi_m(x)$ with $\psi_m$ an analytic function, which implies that the $p$–fold derivative of $\sin^m$ vanishes at $x = 0$ for $0 \leqslant p < m$. Hence

$$(\sin^m)^{(p)}(x) = \frac{1}{2^m\,\mathrm{i}^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \mathrm{i}^p (m-2k)^p\, \mathrm{e}^{\mathrm{i}(m-2k)x}$$

and so

$$(**) \qquad \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-2k)^p = 0 \quad \text{for } 0 \leqslant p < m\,.$$

Therefore, the sum of integrals above vanishes since so does each summand, and we are left with

$$2^m(n-1)! \int_0^\infty \frac{\sin^m x}{x^n}\,dx = \frac{1}{\mathrm{i}^m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \int_0^\infty \frac{\mathrm{i}^{n-1}(m-2k)^{n-1}}{y - \mathrm{i}(m-2k)}\,dy\,.$$

For $m$ odd there is no summand with $k = m/2$ and for $m$ even the corresponding summand vanishes since $n \geq 2$ by assumption. For $k < m/2$ we group the summands corresponding to $k$ and $m-k$ together; these give

$$\begin{aligned}
&(-1)^k \binom{m}{k} \mathrm{i}^{n-1} \int_0^\infty \frac{(m-2k)^{n-1}}{y - \mathrm{i}(m-2k)}\,dy + (-1)^{m-k} \binom{m}{m-k} \mathrm{i}^{n-1} \int_0^\infty \frac{(m-2(m-k))^{n-1}}{y - \mathrm{i}(m-2(m-k))}\,dy \\
&= (-1)^k \binom{m}{k} \mathrm{i}^{n-1} \int_0^\infty \frac{(m-2k)^{n-1}}{y - \mathrm{i}(m-2k)}\,dy + (-1)^k \binom{m}{k} \mathrm{i}^{n-1} \int_0^\infty (-1)^m \frac{(-(m-2k))^{n-1}}{y + \mathrm{i}(m-2k)}\,dy \\
&= (-1)^k \binom{m}{k} \mathrm{i}^{n-1} \int_0^\infty \left[ \frac{(m-2k)^{n-1}}{y - \mathrm{i}(m-2k)} + (-1)^{m+n-1} \frac{(m-2k)^{n-1}}{y + \mathrm{i}(m-2k)} \right] dy\,.
\end{aligned}$$

We thus arrive at

$$\begin{aligned}
&2^m(n-1)! \int_0^\infty \frac{\sin^m x}{x^n}\,dx \\
&\qquad = \sum_{0 \leqslant k < m/2} (-1)^k \binom{m}{k} \frac{1}{\mathrm{i}^{m-n+1}} \int_0^\infty \left[ \frac{(m-2k)^{n-1}}{y - \mathrm{i}(m-2k)} + (-1)^{m+n-1} \frac{(m-2k)^{n-1}}{y + \mathrm{i}(m-2k)} \right] dy
\end{aligned}$$

A) Suppose $m$ and $n$ are of equal parity. Then $m+n-1$ is odd, and there comes

$$\begin{aligned}
2^m(n-1)! \int_0^\infty \frac{\sin^m x}{x^n}\,dx &= \sum_{0 \leqslant k < m/2} (-1)^k \binom{m}{k} \frac{1}{\mathrm{i}^{m-n+1}} 2\,\mathrm{i}(m-2k)^{n-1} \int_0^\infty \frac{m-2k}{y^2 + (m-2k)^2}\,dy \\
&= \sum_{0 \leqslant k < m/2} (-1)^k \binom{m}{k} \frac{1}{\mathrm{i}^{m-n}} 2(m-2k)^{n-1} \arctan \frac{y}{m-2k} \Big|_0^\infty \\
&= \sum_{0 \leqslant k < m/2} (-1)^k \binom{m}{k} (-1)^{\frac{m-n}{2}} 2(m-2k)^{n-1} \frac{\pi}{2}
\end{aligned}$$

and hence the claim A) above.

For $n = 1$ we directly get from $(*)$, since $m$ is odd,

$$\frac{\sin^m x}{x} = \frac{1}{2^{m-1}\,\mathrm{i}^{m-1}} \sum_{0\leqslant k<m/2} (-1)^k \binom{m}{k} \frac{\sin(m-2k)x}{x}$$

Integrating this over $(0,\infty)$ and using $\int_0^\infty \sin(m-2k)x/x\,dx = \int_0^\infty \sin x/x\,dx = \pi/2$ yields the formula for $I_{m,1}$ above.

B) Suppose $m$ and $n$ are of opposite parity. Then $m+n-1$ is even, and there comes

$$\begin{aligned}
2^m(n-1)! \int_0^\infty \frac{\sin^m x}{x^n}\,dx &= \sum_{0\leqslant k<m/2} (-1)^k \binom{m}{k} \frac{1}{\mathrm{i}^{m-n+1}} (m-2k)^{n-1} \int_0^\infty \frac{2y}{y^2+(m-2k)^2}\,dy \\
&= \lim_{s\to\infty} \sum_{0\leqslant k<m/2} (-1)^k \binom{m}{k} \frac{1}{\mathrm{i}^{m-n+1}} (m-2k)^{n-1} \int_0^s \frac{2y}{y^2+(m-2k)^2}\,dy \\
&= \lim_{s\to\infty} \sum_{0\leqslant k<m/2} (-1)^{\frac{m-n+1}{2}+k} \binom{m}{k} (m-2k)^{n-1} \log(y^2+(m-2k)^2)\Big|_0^s \\
&= \lim_{s\to\infty} \sum_{0\leqslant k<m/2} (-1)^{\frac{m-n+1}{2}+k} \binom{m}{k} (m-2k)^{n-1} \log\left(\frac{s^2}{(m-2k)^2}+1\right).
\end{aligned}$$

Now write

$$\frac{s^2}{(m-2k)^2}+1 = \frac{s^2}{(m-2k)^2}\left(1+\frac{(m-2k)^2}{s^2}\right),$$

whence

$$\log\left(\frac{s^2}{(m-2k)^2}+1\right) = 2\log(s) - 2\log(m-2k) + \log\left(1+\frac{(m-2k)^2}{s^2}\right).$$

In the resulting sum for $2^m(n-1)!\int_0^\infty \sin^m x/x^n\,dx$ the first term on the RHS sums up to 0 because of the relations $(**)$. The sum over the third term goes to 0 under $s\to\infty$. So after performing this limit we are left with

$$2^m(n-1)! \int_0^\infty \frac{\sin^m x}{x^n}\,dx = \sum_{0\leqslant k<m/2} (-1)^{\frac{m-n-1}{2}+k} \binom{m}{k} (m-2k)^{n-1} 2\log(m-2k),$$

and hence the claim B) above.

Finally, if $n=1$, $m$ must be even. As above, we chop the integral into pieces and obtain

$$\int_0^{N\pi} \frac{\sin^m x}{x}\,dx = \int_0^\pi \sin^m x \sum_{k=0}^{N-1} \frac{1}{x+k\pi}\,dx.$$

Since the harmonic series is properly divergent, so is the integral, which settles the last open case.

Since $\dfrac{\sin u(y-x)}{u}$ is locally LEBESGUE–integrable, the Dominated Convergence Theorem implies

$$\lim_{c\to\infty}\int_{\varepsilon}^{c}\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\,du=-\pi\int\operatorname{sign}(y-x)\,\mu[dy]\,.$$

Now

$$\begin{aligned}\operatorname{sign}(y-x)&=\mathbb{1}_{(x,\infty)}-\mathbb{1}_{(-\infty,x)}\\&=1-\mathbb{1}_{(-\infty,x]}-\mathbb{1}_{(-\infty,x)}\\&=1-\mathbb{1}_{\{x\}}-2\mathbb{1}_{(-\infty,x)}\end{aligned}$$

whence

$$\begin{aligned}\lim_{c\to\infty}\int_{\varepsilon}^{c}\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\,du&=-\pi\int\left\{1-\mathbb{1}_{\{x\}}-2\mathbb{1}_{(-\infty,x)}\right\}\,\mu[dy]\\&=-\pi\left\{1-\mu\left[\{x\}\right]-2\mu\left[(-\infty,x)\right]\right\}\\&=\pi\left\{-1+\mu\left[\{x\}\right]+2\mu\left[(-\infty,x)\right]\right\}\end{aligned}$$

and the theorem is proved. QED

**Remark**. Note that the integral in the theorem might *not* be defined in the LEBESGUE sense. Consider, for example, $X=0$, i.e. $\mu=\delta_0$, the DIRAC measure centered at 0. Then $\varphi(u)=\mathbb{E}\left[1\right]=1$ for all $u\in\mathbb{R}$ and

$$\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u)-\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}=\frac{\mathrm{e}^{\mathrm{i}\,ux}-\mathrm{e}^{-\mathrm{i}\,ux}}{\mathrm{i}\,u}=\frac{2\sin ux}{u}$$

which is not LEBESGUE–integrable on $[0,\infty)$.

By a simple variable rescaling it suffices to show that $f(u):=\sin u/u$ is not LEBESGUE–integrable on $[0,\infty)$. Suppose it were, then $|f|$ would also be LEBESGUE–integrable on $[0,\infty)$. Now for any $N\in\mathbb{N}$

$$\int_{[0,N\pi]}\left|\frac{\sin u}{u}\right|\,du=\sum_{k=0}^{N-1}\int_{[k\pi,(k+1)\pi)}\left|\frac{\sin u}{u}\right|\,du\,.$$

Since for all $k\in\mathbb{N}$

$$\left|\frac{\sin u}{u}\right|\geq\frac{|\sin u|}{(k+1)\pi}\quad\text{for}\quad u\in[k\pi,(k+1)\pi)$$

we get

$$\int_{[0,N\pi]}\left|\frac{\sin u}{u}\right|\,du\geq\sum_{k=0}^{N-1}\int_{[k\pi,(k+1)\pi)}\left|\frac{\sin u}{(k+1)\pi}\right|\,du=\frac{2}{\pi}\sum_{k=0}^{N-1}\frac{1}{k+1}\,.$$

But the harmonic series is unbounded, and we get a contradiction. QED

The slickest proof of the unboundedness of the harmonic series I know of runs as follows. Suppose it were bounded. It then would be abolutely convergent, and so any of its subseries would be so. Now put

$$\begin{aligned} H &:= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots, \\ O &:= 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots, \\ E &:= \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots. \end{aligned}$$

Then $H = O + E$. On the other hand,

$$E = \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = \frac{1}{2}H$$

and so $O = E$, which is plainly absurd.

**Corollary 10.3.** *Let $X$ be integrable and such that $P[X = x] = 0$ for all $x \in \mathbb{R}$. Then*

$$P[X \geq x] = \frac{1}{2} + \frac{1}{\pi} \lim_{c\to\infty} \int_0^c \operatorname{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi_X(u)}{\mathrm{i}\,u}\right] du\,.$$

PROOF. For $z \in \mathbb{C}$ we have

$$\frac{z - \overline{z}}{i} = 2\operatorname{Re}\frac{z}{\mathrm{i}} = -2\operatorname{Re}\frac{\overline{z}}{\mathrm{i}}\,.$$

Hence,

$$\frac{\mathrm{e}^{\mathrm{i}\,ux}\,\varphi(-u) - \mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u} = -2\operatorname{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi(u)}{\mathrm{i}\,u}\right],$$

and Theorem 10.2 implies

$$\begin{aligned} P[X \geq x] &= 1 - P[X < x] = 1 - \mu_X[(-\infty, x)] \\ &= \frac{1}{2} + \frac{1}{\pi} \lim_{c\to\infty} \int_0^c \operatorname{Re}\left[\frac{\mathrm{e}^{-\mathrm{i}\,ux}\,\varphi_X(u)}{\mathrm{i}\,u}\right] du\,. \end{aligned}$$

QED

END OF THE EXCURSION

Now back to the pricing formula of a call – here it is, the famous HESTON formula:

**Theorem 10.4.** (The HESTON pricing formula for European calls) *Let $\varphi_Y(u) := \mathbb{E}^*\left[\mathrm{e}^{iuY_T}\right]$ and $\varphi_Z(u) := \mathbb{E}^*\left[\mathrm{e}^{iuZ_T}\right]$, $u \in \mathbb{R}$. Then*

HESTON–call$(K, T)$

$$= S_0 \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[ \frac{\mathrm{e}^{-\mathrm{i}\,u \log(K)}\, \varphi_Y(u)}{\mathrm{i}\,u} \right] du \right\} -$$

$$- \mathrm{e}^{-rT} K \left\{ \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[ \frac{\mathrm{e}^{-\mathrm{i}\,u \log(K)}\, \varphi_Z(u)}{\mathrm{i}\,u} \right] du \right\} .$$

PROOF. Recall (see (10.6))

$$\text{HESTON–call}(K,T) = S_0 P^* \left[ Y_T \geq \log(K) \right] - \mathrm{e}^{-rT} K P^* \left[ Z_T \geq \log(K) \right] .$$

The result follows from Corollary 10.3. QED

The ultimate goal in this development of the HESTON formula finally is the explicit determination of the characteristic functions $\varphi_Y$ and $\varphi_Z$. Recall that the process $X = (V,Z)$ was shown to be affine. Hence, we can write by applying (9.2) with $t = 0$:

$$\mathbb{E}^* \left[ \mathrm{e}^{\mathrm{i}(u_1 V_T + u_2 Z_T)} \right] = \mathrm{e}^{\phi(T,\mathrm{i}\,u1,\mathrm{i}\,u_2) + \psi_1(T,\mathrm{i}\,u_1,\mathrm{i}\,u_2)V_0 + \psi_2(T,\mathrm{i}\,u_1,\mathrm{i}\,u_2)Z_0} .$$

where $\phi$, $\psi_1$, and $\psi_2$ satisfy the system of RICCATI ODEs (10.3) – (10.5). In particular, we know already that $\psi_2(t,\mathrm{i}\,u_1,\mathrm{i}\,u_2) = \mathrm{i}\,u_2$. If we put $u_1 := 0$, the LHS gives $\varphi_Z := \varphi_{Z_T}$, and so, writing $u$ in place of $u_2$ to simplify notation:

$$\varphi_Z(u) = \mathbb{E}^* \left[ \mathrm{e}^{\mathrm{i}\,u Z_T} \right] = \mathrm{e}^{\phi(T,0,\mathrm{i}\,u) + \psi_1(T,0,\mathrm{i}\,u)V_0 + \mathrm{i}\,u Z_0} ,$$

where $\phi$ and $\psi_1$ satisfy the equations (see equations (10.3) - (10.5)):

$$(10.7) \qquad \begin{cases} \phi' = \kappa\theta\psi_1 + r\,\mathrm{i}\,u\,; \\[2ex] \phi(0,0,\mathrm{i}\,u) = 0\,; \\[2ex] \psi_1' = \dfrac{1}{2}\sigma^2\psi_1^2 + (\rho\sigma\,\mathrm{i}\,u - \kappa)\psi_1 - \dfrac{u^2}{2} - \dfrac{\mathrm{i}\,u}{2}\,; \\[2ex] \psi_1(0,0,\mathrm{i}\,u) = 0\,. \end{cases}$$

Our aim now is to determine $\phi$ and $\psi_1$ explicitely; note that, once $\psi_1$ is knwn, $\phi$ can be found immediately by a direct integration.

**Lemma 10.5.** *Let $A; B, C \in \mathbb{C}$ be such that $B^2 - 4AC \neq 0$. Then the* RICCATI *equation with constant coefficients*

$$y'(t) = Ay^2(t) + By(t) + C \quad , \quad y(0) = 0$$

*has the unique local solution*

$$y(t) = \frac{2C\left(\mathrm{e}^{\vartheta t} - 1\right)}{\vartheta\left(\mathrm{e}^{\vartheta t} + 1\right) - B\left(\mathrm{e}^{\vartheta t} - 1\right)}$$

*where $\vartheta^2 := B^2 - 4AC$.*

Proof. Since locally the necessary Lipschitz condition is satisfied, the standard Existence and Uniqueness Theorem for ODEs shows that locally around 0 there is a unique solution. The proof then amounts to a direct verification that the explicitely given function is indeed a solution. QED

Proceeding this way is, of course, not entirely satisfactory since it leaves the origin of the explicitely given solution in complete darkness and loads one with the rather heavy burden of stupid verification. To get some enlightment and relief, let us try to solve the equation by some reasoning instead of violence.

First recall that it is common lore that the substitution $y = -z'/(Az)$ transforms a Riccati equation into a linear ODE of second order which then can be explicitely solved by traditional methods, e.g. by variation of constants in case of with constant coefficients. The solution process simplifies significantly, however, if one gets rid of the linear term beforehand:

First we treat the case $A = 0$. We then have the ODE

$$y' = By + C \quad , \quad y(0) = 0$$

which is readily solved; we have $B \neq 0$ and hence can write

$$y' = By + C = B\left(y + \frac{C}{B}\right)$$

and putting $y + C/B =: z$ we arrive at

$$z' = Bz \quad , \quad z(0) = C/B\,.$$

Therefore,

$$z = \frac{C}{B}\,\mathrm{e}^{Bt} \quad \text{hence} \quad y = \frac{C}{B}\left(\mathrm{e}^{Bt} - 1\right)$$

which is compliant with the above formula.

If $A \neq 0$, apply the thousands years old "completion of the square"

$$y' = A\left(y + \frac{B}{2A}\right)^2 + C - \frac{B^2}{4A} = A\left(y + \frac{B}{2A}\right)^2 - \frac{\vartheta^2}{4A}\,.$$

Substituting $y + B/(2A) =: u$ leaves us with a Riccati equation bare of the linear term:

$$u' = Au^2 - \frac{\vartheta^2}{4A} \quad , \quad u(0) = \frac{B}{2A}\,.$$

We now make the substitution $u = -z'/(Az)$. Then

$$u' = -\frac{Azz'' - A(z')^2}{A^2z^2}$$

and so the ODE for $u$ gets transformed into

$$\frac{-Azz'' + A(z')^2}{A^2z^2} = A\frac{(z'^2)}{A^2z^2} - \frac{\vartheta^2}{4A} \quad , \quad -\frac{z'(0)}{Az(0)} = \frac{B}{2A}$$

or

$$z'' = \frac{\vartheta^2}{4}z \quad , \quad \frac{z'(0)}{z(0)} = -\frac{B}{2}\,.$$

Now we have reached safe waters without having had to vary the constants. The solution to this equation is first–semester–stuff:

$$z = a\,\mathrm{e}^{\vartheta t/2} + b\,\mathrm{e}^{-\vartheta t/2} = \mathrm{e}^{-\vartheta t/2}\left(a\,\mathrm{e}^{\vartheta t} + b\right)$$

hence

$$z' = \frac{\vartheta}{2}\left(a\,\mathrm{e}^{\vartheta t/2} - b\,\mathrm{e}^{-\vartheta t/2}\right) = \frac{\vartheta}{2}\,\mathrm{e}^{-\vartheta t/2}\left(a\,\mathrm{e}^{\vartheta t} - b\right)$$

where we have to determine $a$ and $b$ so that the initial condition $z'(0)/z(0) = -B/2$ is satisfied. Since this is a single condition, but the ODE is of order 2, this does not uniquely fix the solution, but since $u = -z'/(Az)$ we can scale $z$, and hence $z'$, with a nonzero constant without changing the final solution $y$. So we only have to fix the ratio $a : b$, and for this the initial condition is sufficient; it yields

$$\frac{\vartheta}{2}\frac{a+b}{a-b} = -\frac{B}{2} \quad \text{or} \quad \frac{b+a}{b-a} = \frac{B}{\vartheta}$$

and so

$$a : b = (\vartheta - B) : (\vartheta + B)\,.$$

We thus arrive at

$$z = \mathrm{e}^{-\vartheta t/2}\left((\vartheta - B)\,\mathrm{e}^{\vartheta t} + (\vartheta + B)\right) = \mathrm{e}^{-\vartheta t/2}\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)$$

and

$$z' = \frac{\vartheta}{2}\,\mathrm{e}^{-\vartheta t/2}\left((\vartheta - B)\,\mathrm{e}^{\vartheta t} - (\vartheta + B)\right) = \frac{\vartheta}{2}\,\mathrm{e}^{-\vartheta t/2}\left(\vartheta(\mathrm{e}^{\vartheta t} - 1) - B(\mathrm{e}^{\vartheta t} + 1)\right)\,,$$

whence

$$u = -\frac{z'}{Az} = -\frac{\vartheta\left(\vartheta(\mathrm{e}^{\vartheta t} - 1) - B(\mathrm{e}^{\vartheta t} + 1)\right)}{2A\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)}$$

and so

$$\begin{aligned} y = u - \frac{B}{2A} &= \frac{\vartheta\left(-\vartheta(\mathrm{e}^{\vartheta t} - 1) + B(\mathrm{e}^{\vartheta t} + 1)\right) - B\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)}{2A\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)} \\ &= \frac{-\vartheta^2(\mathrm{e}^{\vartheta t} - 1) + \vartheta B(\mathrm{e}^{\vartheta t} + 1) - B\vartheta(\mathrm{e}^{\vartheta t} + 1) + B^2(\mathrm{e}^{\vartheta t} - 1)}{2A\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)} \\ &= \frac{(-\vartheta^2 + B^2)(\mathrm{e}^{\vartheta t} - 1)}{2A\left(\vartheta(\mathrm{e}^{\vartheta t} + 1) - B(\mathrm{e}^{\vartheta t} - 1)\right)}, \end{aligned}$$

which completes the proof. QED

We can then use this lemma to find the solution of (10.7); just put

$$A := \frac{1}{2}\sigma^2 \quad , \quad B := \rho\sigma\,\mathrm{i}\,u - \kappa \quad , \quad C := -\frac{u^2}{2} - \frac{\mathrm{i}\,u}{2}\,.$$

Finally, we observed on page 141 that the process $\widetilde{X} := (U, Y)$ is an affine process, too, and from the parameters given there we read off the RICCATI equations. Hence, there are functions $\widetilde{\phi}$ and $\widetilde{\psi}$ satisfying analogous RICCATI equations such that

$$\varphi_Y(u) = \mathrm{e}^{\widetilde{\phi}(T,0,\mathrm{i}\,u) + \widetilde{\psi}_1(T,0,\mathrm{i}\,u)V_0 + \mathrm{i}\,uY_0}$$

and one can again use Lemma 10.5 to obtain explicit expressions for $\widetilde{\phi}$ and $\widetilde{\psi}_1$. The corresponding system of RICCATI equations is very similar to the system

(10.7)

$$\begin{cases} \phi' = \kappa\theta\psi_1 + r\,\mathrm{i}\,u\,; \\ \phi(0,0,\mathrm{i}\,u) = 0\,; \\ \psi_1' = \dfrac{1}{2}\sigma^2\psi_1^2 + (\rho\sigma\,\mathrm{i}\,u - (\kappa-\rho\sigma))\psi_1 - \dfrac{u^2}{2} + \dfrac{\mathrm{i}\,u}{2}\,; \\ \psi_1(0,0,\mathrm{i}\,u) = 0\,. \end{cases}$$

so we see only the equation for $\psi'$ has slightly changed. We read off the parameters $A$, $B$, and $C$ as

$$A := \frac{1}{2}\sigma^2 \quad , \quad B := \rho\sigma\,\mathrm{i}\,u - (\kappa - \rho\sigma) \quad , \quad C := -\frac{u^2}{2} + \frac{\mathrm{i}\,u}{2}\,.$$

This concludes our derivation of a closed form for the HESTON formula.

Let us complete here the explicit derivation of HESTON's formula and present it in the form how it shows up in the literature (where virtually all authors appear to copy it from [**20**]). In general terms, Lemma 10.5 provides us with the solution for $\psi_1$, where we have to plug in the concrete values of $A, B$, and $C$. Before doing so, however, we complete the general solution to $\phi$ (recall that $\phi$, $\psi_1$, and $\psi_2$ together determine the characteristic function $\varphi_Z$). Recall that $\phi$ is given by 10.7 as

$$\phi'(t,\mathrm{i}\,u) = \kappa\theta\psi_1(t,\mathrm{i}\,u) + r\,\mathrm{i}\,u\,;$$
$$\phi(0,\mathrm{i}\,u) = 0\,,$$

and so by a direct integration.

**Lemma 10.6.** *Let $A; B, C \in \mathbb{C}$ be such that $B^2 - 4AC \neq 0$. Then the integral of the solution of Lemma 10.5 of the* RICCATI *equation with constant coefficients*

$$y'(t) = Ay^2(t) + By(t) + C \quad , \quad y(0) = 0$$

*is given by*

$$\int_0^t y(s)\,ds = \frac{1}{A}\log\left(\frac{2\vartheta}{\vartheta\left(\mathrm{e}^{\vartheta t}+1\right) - B\left(\mathrm{e}^{\vartheta t}-1\right)}\right) - \frac{2C}{\vartheta + B}t\,,$$

*where $\vartheta^2 := B^2 - 4AC$.*

PROOF. As in the case of proof of Lemma 10.5, a valid proof is provided by just checking, and this proof is equally unsatisfactory. So we give again a derivation of this formula.

We have for the indefinite integral

$$\int y(t)\,dt = \int \frac{2C\left(\mathrm{e}^{\vartheta t}-1\right)}{\vartheta\left(\mathrm{e}^{\vartheta t}+1\right) - B\left(\mathrm{e}^{\vartheta t}-1\right)}\,dt = \int \frac{2C\left(\mathrm{e}^{\vartheta t}-1\right)}{(\vartheta - B)\,\mathrm{e}^{\vartheta t} - (\vartheta + B)}\,dt$$

and so, to simplify notation, an integral of the type

$$\int \frac{\gamma\left(\mathrm{e}^{\vartheta t}-1\right)}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt \quad , \quad \alpha,\beta,\gamma\in\mathbb{C}\,.$$

Then

$$\int \frac{\gamma\left(\mathrm{e}^{\vartheta t}-1\right)}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt = \int \frac{\gamma\,\mathrm{e}^{\vartheta t}}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt - \int \frac{\gamma}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt$$

and we observe that the second integral is of the same type, since

$$\int \frac{\gamma}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt = \int \frac{\gamma\,\mathrm{e}^{-\vartheta t}}{\alpha+\beta\,\mathrm{e}^{-\vartheta t}}\,dt$$

and so

$$\int \frac{\gamma\left(\mathrm{e}^{\vartheta t}-1\right)}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt = \int \frac{\gamma\,\mathrm{e}^{\vartheta t}}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt - \int \frac{\gamma\,\mathrm{e}^{-\vartheta t}}{\beta\,\mathrm{e}^{-\vartheta t}+\alpha}\,dt\,.$$

The first integral on th RHS is easily calculated, once one has observed that the nominator is essentially the derivative of the denominator:

$$\int \frac{\gamma\,\mathrm{e}^{\vartheta t}}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt = \frac{\gamma}{\alpha\vartheta}\int \frac{\alpha\vartheta\,\mathrm{e}^{\vartheta t}}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt = \frac{\gamma}{\alpha\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right)$$

and so

$$\begin{aligned}\int \frac{\gamma\,\mathrm{e}^{-\vartheta t}}{\beta\,\mathrm{e}^{-\vartheta t}+\alpha}\,dt &= -\frac{\gamma}{\beta\vartheta}\log\left(\beta\,\mathrm{e}^{-\vartheta t}+\alpha\right) = -\frac{\gamma}{\beta\vartheta}\log\left(\left(\beta\,\mathrm{e}^{-\vartheta t}+\alpha\right)\mathrm{e}^{\vartheta t}\,\mathrm{e}^{-\vartheta t}\right)\\ &= -\frac{\gamma}{\beta\vartheta}\log\left(\left(\beta+\alpha\,\mathrm{e}^{\vartheta t}\right)\mathrm{e}^{-\vartheta t}\right) = -\frac{\gamma}{\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) - \frac{\gamma}{\beta\vartheta}\log\left(\mathrm{e}^{-\vartheta t}\right)\\ &= -\frac{\gamma}{\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) + \frac{\gamma}{\beta}t\,.\end{aligned}$$

There results

$$\begin{aligned}\int \frac{\gamma\left(\mathrm{e}^{\vartheta t}-1\right)}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\,dt &= \frac{\gamma}{\alpha\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) + \frac{\gamma}{\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) - \frac{\gamma}{\beta}t\\ &= \left(\frac{\gamma}{\alpha\vartheta}+\frac{\gamma}{\beta\vartheta}\right)\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) - \frac{\gamma}{\beta}t\\ &= \frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) - \frac{\gamma}{\beta}t\,.\end{aligned}$$

Therefore

$$\begin{aligned}\int_0^t \frac{\gamma\left(\mathrm{e}^{\vartheta s}-1\right)}{\alpha\,\mathrm{e}^{\vartheta s}+\beta}\,ds &= \left.\frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta s}+\beta\right) - \frac{\gamma}{\beta}s\right|_0^t\\ &= \frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log\left(\alpha\,\mathrm{e}^{\vartheta t}+\beta\right) - \frac{\gamma}{\beta}t - \frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log(\alpha+\beta)\\ &= \frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log\left(\frac{\alpha\,\mathrm{e}^{\vartheta t}+\beta}{\alpha+\beta}\right) - \frac{\gamma}{\beta}t\\ &= -\frac{(\alpha+\beta)\gamma}{\alpha\beta\vartheta}\log\left(\frac{\alpha+\beta}{\alpha\,\mathrm{e}^{\vartheta t}+\beta}\right) - \frac{\gamma}{\beta}t\,.\end{aligned}$$

Now put $\alpha := \vartheta - B$, $\beta := \vartheta + B$, $\gamma := 2C$, and use $\vartheta^2 = B^2 - 4AC$. QED

Lemma 10.5 and 10.6 together completely solve the problem of giving an analyical formula for the characteristic function $\varphi_Z(u)$:

$$\varphi_Z(u) = \mathrm{e}^{\phi_Z(T,\mathrm{i}\,u)+\psi_Z(T,\mathrm{i}\,u)V_0+\mathrm{i}\,Z_0}$$

with $\psi_Z(T,\mathrm{i}\,u) := \psi_1(T,0,\mathrm{i}\,u)$ and $\phi_Z(T,\mathrm{i}\,u) := \phi(T,0,\mathrm{i}\,u)$ given by Lemma 10.5 and Lemma 10.6, respectively, where

$$A := A_Z := \frac{1}{2}\sigma^2 \quad , \quad B := B_Z = \rho\sigma\,\mathrm{i}\,u - \kappa \quad , \quad C := C_Z := -\frac{u^2}{2} - \frac{\mathrm{i}\,u}{2}\,.$$

We finally rewrite the solutions in a way they appear in the literature since they show up there in an apparently very different form which may be difficult to match with our form given here. Rewrite the solution of Lemma 10.5 as

$$\begin{aligned}
\frac{2C\left(\mathrm{e}^{\vartheta t}-1\right)}{\vartheta\left(\mathrm{e}^{\vartheta t}+1\right)-B\left(\mathrm{e}^{\vartheta t}-1\right)} &= \frac{2C\left(\mathrm{e}^{\vartheta t}-1\right)}{(\vartheta-B)\,\mathrm{e}^{\vartheta t}-(\vartheta+B)}\\
&= \frac{2C\left(1-\mathrm{e}^{\vartheta t}\right)}{(\vartheta+B)-(\vartheta-B)\,\mathrm{e}^{\vartheta t}}\\
&= \frac{2C}{\vartheta+B}\left[\frac{1-\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}\right]\\
&= \frac{2C(\vartheta-B)}{(\vartheta+B)(\vartheta-B)}\left[\frac{1-\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}\right]\\
&= \frac{2C(\vartheta-B)}{\vartheta^2-B^2}\left[\frac{1-\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}\right]\\
&= \frac{2C(\vartheta-B)}{-4AC}\left[\frac{1-\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}\right]\\
&= \frac{2C(\vartheta-B)}{-4AC}\left[\frac{1-\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}\right].
\end{aligned}$$

Rewrite the solution of Lemma 10.6 as

$$\begin{aligned}
\frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\left(\mathrm{e}^{\vartheta t}+1\right)-B\left(\mathrm{e}^{\vartheta t}-1\right)}\right]-\frac{2C}{\vartheta+B}t &= \frac{1}{A}\log\left[\frac{2\vartheta}{(\vartheta-B)\,\mathrm{e}^{\vartheta t}-(\vartheta+B)}\right]-\frac{2C}{\vartheta+B}t\\
&= -\frac{1}{A}\log\left[\frac{(\vartheta-B)\,\mathrm{e}^{\vartheta t}-(\vartheta+B)}{2\vartheta}\right]-\frac{2C(\vartheta-B)}{(\vartheta+B)(\vartheta-B)}t\\
&= -\frac{1}{A}\log\left[\frac{(\vartheta+B)-(\vartheta-B)\,\mathrm{e}^{\vartheta t}}{-2\vartheta}\right]-\frac{2C(\vartheta-B)}{\vartheta^2-B^2}t\\
&= \frac{\vartheta-B}{2A}t-\frac{1}{A}\log\left[\frac{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}{\frac{-2\vartheta}{\vartheta+B}}\right]\\
&= \frac{1}{2A}\left\{(\vartheta-B)t-2\log\left[\frac{1-\frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta t}}{1-\frac{\vartheta-B}{\vartheta+B}}\right]\right\}.
\end{aligned}$$

The analytic expression for $\varphi := \varphi_Z$ in the Heston formula of Theorem 10.4 is then

$$\varphi(u) = \mathrm{e}^{\phi(T,\mathrm{i}\,u)+\psi(T,\mathrm{i}\,u)V_0+\mathrm{i}\,Z_0}$$

with

$$\phi(T, \mathrm{i}\,u) = r\,\mathrm{i}\,uT + \frac{\kappa\theta}{2A}\left\{(\vartheta - B)T - 2\log\left[\frac{1 - \frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta T}}{1 - \frac{\vartheta-B}{\vartheta+B}}\right]\right\}$$

and

$$\psi(T, \mathrm{i}\,u) = \frac{2C(\vartheta - B)}{-4AC}\left[\frac{1 - \mathrm{e}^{\vartheta T}}{1 - \frac{\vartheta-B}{\vartheta+B}\,\mathrm{e}^{\vartheta T}}\right],$$

where

$$\vartheta^2 := B^2 - 4AC$$

and

$$A := A_Z := \frac{1}{2}\sigma^2 \quad , \quad B := B_Z = \rho\sigma\,\mathrm{i}\,u - \kappa \quad , \quad C := C_Z := -\frac{u^2}{2} - \frac{\mathrm{i}\,u}{2}\,.$$

The analytic expression for $\varphi := \varphi_Y$ in the HESTON formula of Theorem 10.4 is given by just the same procedure of solving the appropiate RICCATI equations with only slightly different coefficients whose values we have seen above (on page 160) to be

$$A := A_Y := \frac{1}{2}\sigma^2 \quad , \quad B := B_Y = \rho\sigma\,\mathrm{i}\,u - (\kappa - \rho\sigma) \quad , \quad C := C_Y := -\frac{u^2}{2} + \frac{\mathrm{i}\,u}{2}\,.$$

If one now compares these formulas with the original ones of HESTON ([**20**], p. 331), one has to cope with two little bad surprises:

1) The formulas above for $\phi$ and $\psi$ do not match completely Heston's formulas for $C(\tau;\phi)$ and $D(\tau;\phi)$; one has to switch from $\vartheta$ to $-\vartheta$. This issue itself is not a surprise, since the solutions given in Lemma 10.5 and Lemma 10.6 have only $\vartheta^2$ as input, and so it may happen; the bad surprise is that it *does* happen. Of course, the solutions given above better be invariant under $\vartheta \to -\vartheta$ since they are unique, and it is a nice computational exercise to verify this. But it is conceptually more satisfying to see if they can be written in such a way that this symmetry becomes manifest, and this is indeed the case. One may write for the solution $y(t)$ of Lemma 10.5 by multiplying nominator and denominator by $\mathrm{e}^{-\vartheta t/2}$

$$y(t) = \frac{2C\left(\mathrm{e}^{\vartheta t} - 1\right)}{\vartheta\left(\mathrm{e}^{\vartheta t} + 1\right) - B\left(\mathrm{e}^{\vartheta t} - 1\right)} = \frac{2C\left(\sinh(\vartheta t/2)\right)}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\,.$$

Here the nominator and denominator are clearly odd and so $y(t)$ is even. To rewrite the solution of Lemma 10.6 is slightly more work. Again one starts with multiplying nominator and denominator by $\mathrm{e}^{-\vartheta t/2}$:

$$\begin{aligned}
\int y(t)dt &= \frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\left(\mathrm{e}^{\vartheta t} + 1\right) - B\left(\mathrm{e}^{\vartheta t} - 1\right)}\right] - \frac{2C}{\vartheta + B}t \\
&= \frac{1}{A}\log\left[\frac{2\vartheta\,\mathrm{e}^{-\vartheta t/2}}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \frac{2C}{\vartheta + B}t \\
&= \frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \left(\frac{\vartheta}{2A} + \frac{2C}{\vartheta + B}\right)t \\
&= \frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \left(\frac{\vartheta(\vartheta + B) + 4AC}{2A(\vartheta + B)}\right)t \\
&= \frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \left(\frac{\vartheta^2 + \vartheta B + 4AC}{2A(\vartheta + B)}\right)t \\
&= \frac{1}{A}\log\left[\frac{2\vartheta}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \left(\frac{B^2 - 4AC + \vartheta B + 4AC}{2A(\vartheta + B)}\right)t
\end{aligned}$$

$$= \frac{1}{A}\left\{\log\left[\frac{2\vartheta}{\vartheta\cosh(\vartheta t/2) - B\sinh(\vartheta t/2)}\right] - \frac{B}{2}t\right\},$$

which again is even. So if we now make the change $\vartheta \to -\vartheta$ and employ the relations

$$\frac{1-\mathrm{e}^{-x}}{1-h\,\mathrm{e}^{-x}} = h^{-1}\frac{1-\mathrm{e}^{x}}{1-h^{-1}\,\mathrm{e}^{x}} \quad , \quad \frac{1-h\,\mathrm{e}^{-x}}{1-h} = \mathrm{e}^{-x}\,\frac{1-h^{-1}\,\mathrm{e}^{x}}{1-h^{-1}}$$

HESTON's formulas should now match our formulas ...

2) ... if there were not little bad surprise #2: where we have a $\kappa$ in the parameter $B$, HESTON has a $\kappa + \lambda$ with an additional parameter $\lambda$. To see how this discrepancy arises, one has to realize that HESTON sets up his model under the real world measure $P$, whereas we set up our model under the assumption of a risk–neutral measure $P^*$. When trying to implement the GIRSANOV scheme on a measure change $P \to P^*$ making the discounted stock price $\mathrm{e}^{-rt}\,S_t$ into a martingale, one has to introduce an additional parameter $\lambda$ to parametrize the possible GIRSANOV transformations, because one now has two BMs. It thus turns out that the HESTON model is not complete and there are various EMMs parametrized by $\lambda$ (not all $\lambda$ can occur, see [**36**], Section 3, or p. 134 above). The choice made in [**20**] has the effect that, analogous to the situation in the BS model the equations maintain their form with the drift parameter $\mu$ replaced with the constant interest rate $r$ and the parameter $\lambda$ hidden in the parameters $\kappa$ and $\theta$ of the model under the risk–neutral measure (see pp. 131 – 134 for more details on how the measure change in the HESTON model works).

There seems to be a certain agreement in the literature that HESTON's choice is arbitrary and further inverstigation is needed by what methods to determine an optimal EMM to deal with the intrinsic risks existing in incomplete models which cannot be hedged away (see [**14**], [**19**] and [**11**]). At any rate, under a risk–neutral measure obtained this way and assumed in our approach from the beginning, HESTON's formulas in [**20**] take the form which has been derived here; the complete identification is accomplished by making use of the formula ($\star$) on p. 133. The parameter $\lambda$ is the only relic of the incompleteness of the HESTON model and has vanished completely from the formulas under a risk–neutral measure rendering them unique; it has been absorbed into the parameters $\kappa$ and $\theta$, which have to be determined by calibration (see the next topic) anyway. In this manner we end up with unambiguous price formulas. But keep in mind that the price for this was to choose the market price of volatility $\vartheta^1$ proportional to $\sqrt{V}$.

We now come to our last topic in this chapter.

### Calibration of the HESTON model

Given a parametric model, the first task to cope with before the model can be put to use is the determination of those numerical values of the parameters, which optimize the predictions of the model. This is a kind of inverse to the problems the model has been set up to solve. E.g. if the model is set up for predicting option prices, one first collects a sample of actual markt prices of liquidly traded options and sets out to find those values of the parameters which make the model produce the best approximations to the observed prices with respect to some chosen error distance. This process of determining optimal parameters is called *calibrating* to model. After this process of calibration the model can then be used to predict prices of other options, which then may also be exotic. So, while the standard task, which the model has been designed for, is to predict prices given the parameters, the calibration process predicts, given the prices, the optimal parameters, and so is inverse to the standard task of the model.

We describe this now in some more detail. First, introduce some notation. let $0 < T_1 < \cdots < T_M$ be a set of maturities, and $K_{ij}$, $i \in \{1, \ldots, M\}$ and $j \in \{1, \ldots, N(i)\}$ a set of strikes. Suppose that the call with maturity $T_i$ and strike $K_{ij}$ is liquidly traded at a market price $C_M(T_i, K_{ij})$ for all $1 \leqslant i \leqslant M$, $1 \leqslant j \leqslant N(i)$. Let $\Theta := (K, \theta, \sigma, \rho, V_0)$ be the HESTON model parameters. The *calibration of the model* consists in finding the $\Theta$ that minimizes

$$\sum_{i=1}^{M} \sum_{j=1}^{N(i)} \left[C_M(T_i, K_{ij}) - \text{HESTON–call}_\Theta(T_i, K_{ij})\right]^2 .$$

We write HESTON–call$_\Theta$ to remind of the fact that the HESTON formula for the price of a call is a function of the parameters. HESTON–call$_\Theta(T_i, K_{ij})$ and its derivatives (the Greeks) are known explicitely. Hence one can use one of the known efficient and fast optimization methods, e.g. the NEWTON method, for finding an optimal $\Theta$.

The HESTON model parameters can thus be approximated quickly. This is the strength of the HESTON model, and of, more generally, the affine models.

CHAPTER 11

# FOURIER Transform of Exponentially Weighted Plain Vanilla Options

From a numerical point of view, there is a problem with the HESTON formula: since it has $\mathrm{i}\,u$ in the denominator of the integrand, it has a singularity in $u = 0$ which makes the evaluation of the integral numerically unstable, so one has to be very careful about this point. In particular, this prevents the use of the FFT (Fast Fourier Transform) for evaluating the integral, which is highly regrettable in view of the speed advantages of this method.

Now numerical tractability of pricing formulae is quite an issue in daily financial practice; the methods of evaluation must be practicable, fast, and stable. One trick to circumvent the singularity at $u =$ is to use an exponential weight and is due to [**3**] (a very famous landmark paper, perhaps not quite in the league of [**2**] or [**20**] but surely among the Top Ten of the All–Time Greatest of Mathematical Finance ...) This makes the HESTON formula amenable to FFT calculations. We now explain the details.

Let $(\Omega, \mathcal{F}, P^*)$ be a probability space. We consider a model allowing analytical formulas for option prices (like the HESTON model, but there are others, see [**3**]). Let $S_t^0 = \mathrm{e}^{rt}$ be the price at time $t \in \mathbb{R}_+$ of a non–risky asset, and $S_t$ the price of a risky asset, the underlying. We assume that $\mathrm{e}^{-rt}\,S_t$ is a $P^*$–martingale; in other words, $P^*$ is supposed to be a risk–neutral measure.

We make a couple of assumptions. We assume that $X_T := \log(S_T)$ has a density w.r.t. the LEBESGUE measure on $\mathbb{R}$, denoted $q_T(x)$, $x \in \mathbb{R}$. Moreover, we suppose that the characteristic function $\varphi_T$ of $X_T$:

$$\varphi_T(u) := \int \mathrm{e}^{\mathrm{i}\,ux}\, q_T(x)\,dx$$

is “known” for $u \in \mathbb{C}$, i.e. given by some explicit analytic expression (which is for instance the case for affine processes).

Let $K$ be a strike price and $T > 0$ a maturity date. We define $k := \log(K)$. The price of a call with strike $K$ and maturity $T$ is given by

$$C_T(k) = \mathrm{e}^{-rT} \int_{-\infty}^{\infty} \left(\mathrm{e}^x - \mathrm{e}^k\right)^+ q_T(x)\,dx = \mathrm{e}^{-rT} \int_{k}^{\infty} \left(\mathrm{e}^x - \mathrm{e}^k\right) q_T(x)\,dx\,.$$

Now recall the FOURIER Inversion Theorem. Let $L^p(\mathbb{R})$ stand for the functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that $|f|^p$ is integrable. For $f \in L^2(\mathbb{R})$, define

$$\mathcal{F}[f](u) := \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\,ux} f(x)\, dx\,,$$

the FOURIER transform of $f$.

Here one has to be careful about the meaning of this formula. As a LEBESGUE integral, it defines the FOURIER transform on the subspace $L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subseteq L^2(\mathbb{R})$. In fact, this formula defines $\mathcal{F}$ on all of $L^1(\mathbb{R})$ with image contained in $\mathcal{C}^0_\infty(\mathbb{R})$, the space of continuous functions vanishing at infinity (RIEMANN–LEBESGUE *Lemma*). Unfortunately, $\mathcal{C}^0_\infty(\mathbb{R})$ is not contained in $L^1(\mathbb{R})$, so $\mathcal{F}$ does not map $L^1(\mathbb{R})$ to $L^1(\mathbb{R})$, which makes $L^1(\mathbb{R})$ a bad domain to study $\mathcal{F}$ and its inverse. However, $L^1(\mathbb{R})\cap L^2(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and can be shown to map into $\mathcal{C}^0_\infty(\mathbb{R})\cap L^2(\mathbb{R})$ under $\mathcal{F}$ ([**1**], Chap. X, Lemma 9.17). Hence $\mathcal{F}$ maps $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and extends to an isometry $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ ([**1**], Chap. X, Theorem 9.23). We thus have for any $f \in L^2(\mathbb{R})$:

$$\mathcal{F}[f] = \lim_{n\to\infty} \mathcal{F}[f_n]$$

for any sequence $(f_n)$ in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with $\lim_{n\to\infty} f_n = f$, the limits taken in the $L^2$–sense. In particular, one may take the sequence $f_n := f\mathbb{1}_{[-R_n,R_n]}$, where $(R_n)$ is any monotonously increasing unbounded sequence of real numbers. This shows

$$\mathcal{F}[f](u) = \lim_{R\uparrow\infty} \int_{-R}^{R} \mathrm{e}^{\mathrm{i}\,ux} f(x)\, dx = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\,ux} f(x)\, dx\,,$$

so that the integral above has to be read as an improper integral, which might not be LEBESGUE.

The FOURIER Inversion Theorem then is the statement

$$\frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\,ux} \mathcal{F}[f](u)\, du = f(x)\,.$$

Again, this equation has to be interpreted properly. As it stands, i.e. with the integral interpreted as a LEBESGUE integral, it holds for those $f \in L^1(\mathbb{R})$ such that $\mathcal{F}[f] \in L^1(\mathbb{R}) \cap \mathcal{C}^0_\infty(\mathbb{R})$. For $f \in L^2(\mathbb{R})$ it then holds again by density arguments, and the integral again has to be interpreted as an improper one.

Notice that one has $C_T(k) \to S_0$ as $k \to -\infty$; this prevents $C_T(k)$ from being in $L^1(\mathbb{R})$. This should be intuitively plausible, since $k \to -\infty$ corresponds to $K \to 0$; but the closer the strike $K$ approaches 0, the sooner the option becomes active and so, because the price process has continuous paths, the closer it stays to $S_0$.

For a formal proof, note that, because the discounted price process is a $P^*$–martingale,

$$S_0 = \mathrm{e}^{-rT}\, \mathbb{E}^*\left[S_T\right] = \mathbb{E}^*\left[\mathrm{e}^{-rT} S_T\right] = \mathbb{E}^*\left[\mathrm{e}^{X_T}\right] = \mathbb{E}^{X_T}\left[\mathrm{e}^{x}\right] = \int_{-\infty}^{\infty} \mathrm{e}^{x}\, q_T(x)\, dx$$

and so

$$\begin{aligned}
\mathrm{e}^{rT}(C_T(k) - S_0) &= \int_k^\infty \left(\mathrm{e}^x - \mathrm{e}^k\right) q_T(x)\,dx - \int_{-\infty}^\infty \mathrm{e}^x\, q_T(x)\,dx \\
&= -\int_{-\infty}^k \mathrm{e}^x\, q_T(x)\,dx - \int_k^\infty \mathrm{e}^k\, q_T(x)\,dx\,,
\end{aligned}$$

whence

$$\begin{aligned}
\mathrm{e}^{rT}\,|C_T(k) - S_0| &\leqslant \left|\int_{-\infty}^k \mathrm{e}^x\, q_T(x)\,dx\right| + \left|\int_k^\infty \mathrm{e}^k\, q_T(x)\,dx\right| \\
&\leqslant \mathrm{e}^k \left|\int_{-\infty}^k q_T(x)\,dx\right| + \mathrm{e}^k \left|\int_k^\infty q_T(x)\,dx\right| \\
&= \mathrm{e}^k \int_{-\infty}^k q_T(x)\,dx = \mathrm{e}^k
\end{aligned}$$

which proves the claim.

In order to be able to apply the FOURIER transform, the trick is to multiply $C_T(k)$ with an exponential weight to render it integrable. Suppose that there exists $\alpha > 0$ such that

$$D_T(k) := \mathrm{e}^{\alpha k}\, C_T(k)$$

is in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then the FOURIER transform

$$\psi_T(v) := \int \mathrm{e}^{\mathrm{i}\,vk}\, D_K(k)\,dk$$

is defined.

**Lemma 11.1.** *We have*

$$\psi_T(v) = \frac{\mathrm{e}^{-rT}\,\varphi_T(v - (\alpha+1)\,\mathrm{i})}{\alpha^2 + \alpha - v^2 + (2\alpha+1)\,\mathrm{i}\,v}\,.$$

PROOF. We compute

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}\,vk} \int_k^\infty \mathrm{e}^{\alpha k}\,\mathrm{e}^{-rT} \left(\mathrm{e}^x - \mathrm{e}^k\right) q_T(x)\,dx\,dk \\
&= \int_{-\infty}^\infty \mathrm{e}^{-rT}\, q_T(v) \int_{-\infty}^x \mathrm{e}^{\mathrm{i}\,vk} \left(\mathrm{e}^{x+\alpha k} - \mathrm{e}^{k+\alpha k}\right)\,dk\,dx
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \mathrm{e}^{-rT} q_T(v) \underbrace{\int_{-\infty}^{x} \left( \mathrm{e}^{x+\alpha k+\mathrm{i}\, vk} - \mathrm{e}^{k+\alpha k+\mathrm{i}\, vk} \right) dk}_{=:A} \; dx.$$

Then

$$\begin{aligned}
A &= \left. \frac{\mathrm{e}^{x+\alpha k+\mathrm{i}\, vk}}{\alpha + \mathrm{i}\, v} \right|_{-\infty}^{x} - \left. \frac{\mathrm{e}^{k+\alpha k+\mathrm{i}\, vk}}{1+\alpha + \mathrm{i}\, v} \right|_{-\infty}^{x} \\
&= \frac{\mathrm{e}^{x+\alpha x+\mathrm{i}\, vx}}{\alpha + \mathrm{i}\, v} - \frac{\mathrm{e}^{x+\alpha x+\mathrm{i}\, vx}}{1+\alpha + \mathrm{i}\, v} \\
&= \frac{(1+\alpha+\mathrm{i}\, v)\,\mathrm{e}^{(1+\alpha+\mathrm{i}\, v)x} - (\alpha+\mathrm{i}\, v)\,\mathrm{e}^{(1+\alpha+\mathrm{i}\, v)x}}{(\alpha+\mathrm{i}\, v)(1+\alpha+\mathrm{i}\, v)} \\
&= \frac{\mathrm{e}^{(1+\alpha+\mathrm{i}\, v)x}}{\alpha^2+\alpha-v^2+2\alpha\,\mathrm{i}\, v + \mathrm{i}\, v}\,,
\end{aligned}$$

which imples

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} \mathrm{e}^{-rT} q_T(v) \frac{\mathrm{e}^{(1+\alpha+\mathrm{i}\, v)x}}{\alpha^2+\alpha-v^2+(2\alpha+1)\,\mathrm{i}\, v} \\
&= \mathrm{e}^{-rT} \frac{1}{\alpha^2+\alpha-v^2+(2\alpha+1)\,\mathrm{i}\, v} \int_{-\infty}^{\infty} q_T(v)\,\mathrm{e}^{(1+\alpha+\mathrm{i}\, v)x}\, dx \\
&= \frac{\mathrm{e}^{-rT}}{\alpha^2+\alpha-v^2+(2\alpha+1)\,\mathrm{i}\, v} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(v-\mathrm{i}(1+\alpha))x}\, q_T(v) dx \\
&= \frac{\mathrm{e}^{-rT}}{\alpha^2+\alpha-v^2+(2\alpha+1)\,\mathrm{i}\, v} \varphi_T(v-\mathrm{i}(1+\alpha))\,.
\end{aligned}$$

QED

**Corollary 11.2.** *The call price satisfies*

$$C_T(k) = \mathrm{e}^{-\alpha k} \frac{1}{\pi} \mathrm{Re} \int_0^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv\,.$$

PROOF. The FOURIER Inversion Theorem implies

$$C_T(k) = \mathrm{e}^{-\alpha k}\, D_T(k) = \mathrm{e}^{-\alpha k}\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv\,.$$

Make the following observations:

$$\overline{\mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)} = \overline{\mathrm{e}^{-\mathrm{i}\, vk}}\; \overline{\psi_T(v)} = \mathrm{e}^{\mathrm{i}\, vk}\, \psi_T(-v)\,,$$

and

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv &= \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv \\
&= \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\, vk}\, \psi_T(-v)\, dv + \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv \\
&= \int_{0}^{\infty} \left( \overline{\mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)} + \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v) \right) dv \\
&= 2 \int_{0}^{\infty} \mathrm{Re}\left( \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v) \right) dv \\
&= 2\,\mathrm{Re} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}\, vk}\, \psi_T(v)\, dv\,.
\end{aligned}$$

QED

### Calibration using the Fast Fourier Transform (FFT)

The FFT is an efficient algorithm for computing sums of the following type:

$$\omega(k) = \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\, \frac{2\pi}{N}(j-1)(k-1)}\, x(j)$$

for $k = 1, \ldots, N \in \mathbb{N}$ (it brings the computational cost of $O(N^2)$ operations using the traditional implementation of the basic arithmetic operations down to $O(N \log N)$). This is of particular importance for the calibration process, where you have to compute not just one price, but a whole bunch of prices at one stroke. Applying it to our situation leads to the following scheme.

For simplicity, assume $S_0 = 1$. Let $\eta > 0$ and $v_j := n(j-1)$ for $j = 1, \dots, N$. Then

$$\frac{\mathrm{e}^{-\alpha k}}{\pi} \operatorname{Re} \sum_{j=1}^{N} \omega_j \, \mathrm{e}^{-\mathrm{i}\, v_j k} \, \psi_T(v_j) \eta$$

(you could interpret this as an approximating Riemann sum for the Fourier integral). The $\omega_j$ are weights depending on the numeric integration rule that you use. For the trapezoidal rule, for instance, we have $\omega_1 = \omega_N = 1/2$ and $\omega_j = 1$ for all other $j$.

Let $\lambda := (2\pi/N) \cdot (1/\eta)$ and define $b := N\lambda/2$. We set

$$k_n := -b + \lambda(n-1) \quad , \quad n = 1, \dots, N .$$

Then

$$C_T(k_n) \approx \frac{\mathrm{e}^{-\alpha k_n}}{\pi} \operatorname{Re} \sum_{j=1}^{N} \omega_j \, \mathrm{e}^{-\mathrm{i}\, v_j k_n} \, \psi_T(v_j) \eta .$$

Rewrite

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\, v_j k_n} &= \mathrm{e}^{-\mathrm{i}\, v_j (-b + \lambda(n-1))} = \mathrm{e}^{\mathrm{i}\, v_j b} \, \mathrm{e}^{-\mathrm{i}\, \lambda (n-1) v_j} \\ &= \mathrm{e}^{\mathrm{i}\, v_j b} \, \mathrm{e}^{-\mathrm{i}\, \lambda \eta (n-1)(j-1)} = \mathrm{e}^{\mathrm{i}\, v_j b} \, \mathrm{e}^{-\mathrm{i}\, \frac{2\pi}{N} (n-1)(j-1)} . \end{aligned}$$

To sum up, we get

$$C_T(k_n) \approx \frac{\mathrm{e}^{-\alpha k_n}}{\pi} \operatorname{Re} \sum_{j=1}^{N} \mathrm{e}^{-\mathrm{i}\, \frac{2\pi}{N} (n-1)(j-1)} \underbrace{\omega_j \, \mathrm{e}^{\mathrm{i}\, v_j b} \, \psi_T(v_j) \eta}_{=: x_j} .$$

Now one can apply the FFT algorithm to compute $C(k_n)$ quickly also for a whole range of strikes; this is what you need in applications, e.g., as mentioned above, in calibrations.

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