**Institute for Applied Mathematics**
**WS 2012/13**
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# „Mathematical Finance"

# Exercise sheet 1

**Please return your solution sheet in the lecture on November 7th**

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**1.** Recall the definition of the value-at-risk of a portfolio from the lecture. If the random variable $L$ on $(\Omega, \mathcal{F}, P)$ denotes the loss of the portfolio, then

$$\mathrm{VaR}_\alpha(L) = \inf\{c \in \mathbb{R} | P[L > c] \leq 1 - \alpha\}.$$

Let $(L_n)$ denote a sequence of integrable, independent and identically distributed random variables. Show that for any $\alpha \in (0, 1)$

$$\mathrm{VaR}_\alpha\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) \to E[L_1]$$

as $n \to \infty$. Choose the common distribution in such a way that convexity is violated for large $n$, i.e.

$$\mathrm{VaR}_\alpha\left(\frac{1}{n}\sum_{i=1}^{n} L_i\right) > \mathrm{VaR}_\alpha(L_1).$$

Thus, $\mathrm{VaR}_\alpha$ may penalize diversification instead of encouraging it.

**2.** Consider a market model with one bond and two risky assets. Let $\Omega = \{\omega_1, \omega_2\}$ and $P$ a probability measure with $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$. Let $r = 0.1$, and let the prices of the risky assets at time 0 be given by $\pi = (\pi^1, \pi^2) = (2, 2)$. Check whether the market is arbitrage free if the prices $S = (S^1, S^2)$ of the risky assets at time 1 satisfy

(a) $S(\omega_1) = (1, 3)$ and $S(\omega_2) = (3, 4)$,

(b) $S(\omega_1) = (3, 1)$ and $S(\omega_2) = (1, 3)$,

(c) $S(\omega_1) = (3, 1)$ and $S(\omega_2) = (1, 4)$.

Prove your answers!

**3.** Consider a market model with one bond and one risky asset with prices $S^0 = 1 + r$ and $S^1$ at time 1. Show that the following financial derivatives can be seen as linear combinations of $S^0$, $S^1$ and call options $(S^1 - K)^+$ on $S^1$ with strike $K$:

(a) put: $(K - S^1)^+$, with strike $K \geq 0$,

(b) $\min(S^1, K)$, with $K \geq 0$,

(c) convertible bond: $\max(S^1, S^0)$,

(d) butterfly spread: $f(S^1)$, where

$$f(x) = \begin{cases} x - a, & a \leq x \leq \frac{a+b}{2} \\ b - x, & \frac{a+b}{2} < x \leq b \\ 0, & \text{else} \end{cases}$$

Denote the arbitrage free prices of $S^0$, $S^1$ and $(S^1 - K)^+$ at time 0 by $\pi^0$, $\pi^1$ and $\pi(K)$, respectively. Calculate the arbitrage free prices of the derivatives $(a) - (d)$ in dependence of $\pi^0$, $\pi^1$ and $\pi(K)$.

**4.** Let $C_1$ and $C_2$ be two call options on the same asset, with strike prices $K_1 \leq K_2$. Suppose that both options are traded at prices $\pi(C_1)$ and $\pi(C_2)$ respectively. Show that for the market to be arbitrage free it is necessary that

(a) $\pi(C_1) \geq \pi(C_2)$,

(b) $\frac{K_2 - K_1}{1+r} \geq \pi(C_1) - \pi(C_2)$,

(c) if $C_3$ is an option with strike price $\lambda K_1 + (1-\lambda)K_2$, $\lambda \in [0,1]$, traded at $\pi(C_3)$, then $\lambda\pi(C_1) + (1-\lambda)\pi(C_2) \geq \pi(C_3)$.