Institute for Applied Mathematics
WS 2012/13
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„Mathematical Finance"

Exercise sheet 2

Please return your solution sheet in the lecture on November 21st

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1. Show that the supremum in Equation (2.1) from the lecture is indeed a maximum, i.e. the lower arbitrage bound of a contingent claim C is given by

$$\pi^{\downarrow}(C) = \max\{m \in [0,\infty) | \exists \xi \in \mathbb{R}^d \text{ with } m + \xi \cdot Y \leq \frac{C}{1+r} \quad P-\text{a.s.}\}$$

where $Y = \frac{S}{1+r} - \pi$ is the vector of discounted net gains.

2. Let $C^c = (S^1 - K)^+$ be a call option on a financial asset with price π^1 at time 0 and price S^1 at time 1, where $K > 0$ is the strike price.

(a) Show that any arbitrage free price π^c of C^c satisfies

$$\pi^c \geq (\pi^1 - K)^+, P - a.s.$$

(or in finance terms: that the call option price always exceeds its *intrinsic value.*)

(b) Let $C^p = (K - S^1)^+$ be the corresponding put option. Show that arbitrage free price π^p of C^p does *not* necessarily satisfy

$$\pi^p \geq (K - \pi^1)^+, P - a.s.$$

3. Consider an arbitrage free single period market model and let $C = (S^1 - K)^+$ be a call option on S^1 with strike price $K > 0$. Let $\pi^{\downarrow}(C)$ and $\pi^{\uparrow}(C)$ denote the arbitrage bounds of C. Show that:

(a) If $P(S^1 = (1+r)\pi^1) > 0$, then

$$\pi^{\downarrow}(C) = \left(\pi^1 - \frac{K}{1+r}\right)^+.$$

(b) If $\operatorname{ess\,sup} S^1 = \infty$ and $\operatorname{ess\,inf} S^1 = 0$, then $\pi^{\uparrow}(C) = \pi^1$.

4. Consider a multi period market model with time horizon T. Let Q denote a probability measure such that for all self-financing strategies the associated discounted value process $(D_t)_{0 \leq t \leq T}$ satisfies the condition:

$$\text{If } D_T \geq 0, \ Q\text{-a.s., then } E^Q[D_T] = D_0.$$

Show that Q is martingale measure.