Institute for Applied Mathematics
WS 2012/13
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„Mathematical Finance"

Exercise sheet 3

Please return your solution sheet in the lecture on December 5th

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1. Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_m)$ be vectors of random variables such that $\sigma(X_1, \dots, X_n)$ is independent of $\sigma(Y_1, \dots, Y_m)$. Let $\Phi : \mathbb{R}^{n+m} \to \mathbb{R}$ be such that $E|\Phi(X,Y)| < \infty$, and suppose that $\varphi(x) = E(\Phi(x,Y))$ is defined for all $x \in \mathbb{R}^n$. Show that

$$E(\Phi(X,Y)|\sigma(X_1, \dots, X_n)) = \varphi(X).$$

2. Consider a complete multi-period model. An owner of a *chooser option* has to decide, at some fixed time $t < T$, whether to turn the option into a call or into a put. The strike K and the expiration date T are fixed from the time of purchase. We suppose that the owner chooses the option type possessing a higher value at time t. Try to write the payoff of a chooser option as a random variable C.
Let $p(u,L)$ (resp. $c(u,L)$) denote the price at time 0 of a put (resp. call) with strike L and expiration date u. Show that

$$(1+r)^{-T} E^*(C) = c(T,K) + p(t, (1+r)^{t-T} K),$$

i.e. the arbitrage free price of the chooser option is equal to the sum of the price of a call with strike K and expiration date T, and the price of a put with strike $(1+r)^{t-T}K$ and expiration date t.
Hint: Use that $(K - S_T)^+ = (S_T - K)^+ - S_T + K$.

3. Consider the BOPM. Calculate the price of the up-and-out call

$$C = \begin{cases} 0 & \text{, if } \max_{0 \le t \le T} S_t \ge B \\ (S_T - K)^+ & \text{, else.} \end{cases}$$

Assume that $d = u^{-1}$ and $B > S_0 \wedge K$.

4. Let $C = (K - \min_{0 \le t \le T} S_t)^+$ be a lookback put with fixed strike K. Show that in the BOPM with $d = u^{-1}$ the arbitrage free price of C is given by

$$\frac{1}{(1+r)^T} \sum_{k=0}^{T} (K - S_0 u^{-k})^+ \sum_{l=l_k}^{T-k} (p^*)^l (1-p^*)^{T-l} \, \frac{2(k+l) - T + 1}{T+1} \binom{T+1}{T-l-k},$$

where $l_k = \min\{l : 2l - T \geq -k\}$.
You may assume, without proof, that for $k \in \mathbb{N}$ and $l \geq 0$,

$$P\left(\min_{0 \leq t \leq T} Z_t = -k, Z_T = -k + l\right) = \frac{2(k+l+1)}{T+1} P\left(Z_{T+1} = -k - l - 1\right).$$