Institute for Applied Mathematics
WS 2012/13
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„Mathematical Finance"

Exercise sheet 4

Please return your solution sheet in the lecture on December 12th

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1. Let $(C_t)_{t\in\{0,\dots,T\}}$ denote an American option in an arbitrage free market model. Denote by U the Snell envelope of the discounted process $\frac{C_t}{(1+r)^t}$ with respect to an EMM P^*. Moreover consider the corresponding European option C_T with discounted value process

$$D_t = E^*\left[\frac{C_T}{(1+r)^T}\middle|\mathcal{F}_t\right].$$

Show that

(a) $U_t \geq D_t$, P^*-a.s. for all $t = 0,\dots,T$.

(b) If $D_t \geq \frac{C_t}{(1+r)^t}$, P^*-a.s. for all t, then $U_t = D_t$, P^*-a.s. for all t.

(c) If $\frac{C_t}{(1+r)^t}$ is a submartingale, then $U_t = D_t$, P^*-a.s. for all t.

2. Consider an American call $C_t = (S_t - K)^+$, $t \in \{0,\dots,T\}$, in an arbitrage free market model. Show that it is always optimal to exercise an American call at the expiration date, i.e.

$$\sup_{\tau\in\mathcal{T}} E^*\left[\frac{C_\tau}{(1+r)^\tau}\right] = E^*\left[\frac{C_T}{(1+r)^T}\right].$$

Hint: Use Exercise 1(c).

3. In contrast to American calls, it is *not* always optimal to wait until the expiration date before exercising an *American put.*
In order to prove this, derive first the *put-call parity.* With V_t^c and V_t^p denoting the arbitrage free prices at time t of a European call respectively a European put with strike K and maturity date T, show that

$$V_t^p = V_t^c - S_t + (1+r)^{t-T}K.$$

Consider the BOPM and assume that $r > 0$. Then show that for large K, one has

$$(K - S_t)^+ > V_t^p.$$