Institute for Applied Mathematics
WS 2012/13
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„Mathematical Finance"

Exercise sheet 5

Please return your solution sheet in the lecture on January 16th

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1. Recall that a sequence of random variables $(X_n)_{n\geq 1}$ converges to X in distribution if and only if

$$\lim_{n\to\infty} Ef(X_n) = Ef(X) \quad \text{for all } f \in \mathcal{C}_b(\mathbb{R}). \tag{1}$$

Show that (1) is equivalent to the condition

$$\lim_{n\to\infty} Ef(X_n) = Ef(X) \quad \text{for all uniformly continuous } f \in \mathcal{C}_b(\mathbb{R}). \tag{2}$$

2. Let $M = (M_t)_{t\geq 0}$ be a non-negative local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ such that $M_0 \in \mathbb{R}_+$. Show that M is a supermartingale.

3. Consider the Black-Scholes model

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t,$$

where $B = (B_t)_{t\in[0,T]}$ is a Brownian motion under the EMM P^*. An Asian call option with strike $K > 0$ is a contingent claim with payoff

$$C = \left(\frac{1}{T}\int_0^T S_t\mathrm{d}t - K\right)^+ \tag{3}$$

at time T. Define the process $Y_t = \int_0^t S_r\mathrm{d}r$ and let V_t denote the arbitrage free price of C at time t. By Lemma 4.3 there exists a function $v : [0,T]\times[0,\infty)\times\mathbb{R} \to \mathbb{R}$ such that

$$V_t = e^{-r(T-t)}E^*[C|\mathcal{F}_t] = v(t, S_t, Y_t).$$

Show that v satisfies the partial differential equation

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2x^2v_xx(t,x,y) = rv(t,x,y)$$

for $(t,x,y) \in [0,T)\times(0,\infty)\times\mathbb{R}$ and the boundary conditions

$$\begin{array}{rcll} v(t,0,y) = & e^{-r(T-t)}\left(\frac{y}{T}-K\right)^+, & & 0 \leq t < T, y \in \mathbb{R} \\ \lim_{y\searrow-\infty} v(t,x,y) = & 0, & & 0 \leq t < T, x \geq 0 \\ v(T,x,y) = & \left(\frac{y}{T}-K\right)^+, & & x \geq 0, y \in \mathbb{R}. \end{array}$$

4. Consider the Black-Scholes model as in Exercise 3. Replacing the arithmetic average in (3) by a geometric average leads to an option with payoff

$$\hat{C} = \left(\exp\left(\frac{1}{T}\int_0^T \ln(S_t)\mathrm{d}t\right) - K\right)^+.$$

Denote by $\hat{V}_0$ the arbitrage free price of $\hat{C}$ at time 0.

(a) Show that $\hat{V}_0 \leq V_0$, where V_0 is the arbitrage free price of the Asian option (3).

(b) Show that

$$\hat{V}_0 = e^{-rT}E^*\left[\left(S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right)\frac{T}{2} + \sigma\sqrt{\frac{T}{3}}N\right) - K\right)^+\right],$$

where N is a standard normal variable. Hint: Show that for a Brownian motion B the random variable $\int_0^T B_t \mathrm{d}t$ is normal with zero mean and a variance equal to $T^3/3$.

(c) Prove the inequality

$$V_0 - \hat{V}_0 \leq S_0 e^{-rT}\left(\frac{e^{rT}-1}{rT} - e^{r\frac{T}{2} - \sigma^2\frac{T}{12}}\right).$$