Institute for Applied Mathematics
WS 2012/13
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„Mathematical Finance"

Exercise sheet 6

Please return your solution sheet in the lecture on January 30th

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1. Let $A \in \mathbb{R}^{d,d}$ be a positive semi-definite matrix. Prove the following claims.

(a) If $a_{ii} = 0$ for all $i \in \{1, \dots, d\}$, then $A = 0$.

(b) If there exist $B \in \mathbb{R}^{n,m}$ and $C \in \mathbb{R}^{m,m}$ with $m + n = d$ and

$$A = \begin{pmatrix} 0 & B \\ B^T & C \end{pmatrix},$$

then $B = 0$.

2. Show that

$$\lim_{c\to\infty} \int_0^c \frac{\sin(x)}{x} dx = \frac{\pi}{2}.$$

Hint: Fix $c > 0$ and show that $(x, y) \mapsto e^{-xy} \sin(x)$ is Lebesgue integrable on $(0, c)\times(0, \infty)$. Apply Fubini's theorem to obtain

$$\int_0^c \frac{\sin(x)}{x} dx = \frac{\pi}{2} - \cos(c) \int_0^\infty \frac{e^{-cy}}{1 + y^2} dy - \sin(c) \int_0^\infty \frac{ye^{-cy}}{1 + y^2} dy.$$

Use $1 + y^2 \geq 1$ to conclude that $| \int_0^c \frac{\sin(x)}{x} dx - \frac{\pi}{2}| \to 0$ as $c \to \infty$.

3. Let B be a Brownian motion and define the $\mathbb{R}^2_+$-valued process X by $X^i_t = (\sqrt{x^i} + B_t)^2$ for $i = 1, 2$ and $x \in \mathbb{R}^2_+$.

(a) Show that X satisfies

$$dX^i_t = dt + 2\sqrt{X^i_t} dB_t$$

for $i = 1, 2$ and $X_0 = x$.

(b) Is X an affine process? Prove your answer by computing the characteristic function of X_t. You can use the following representation of the characteristic function of a noncentral chi-squared distributed random variable Z^2

$$E\left[e^{iuZ^2}\right] = \frac{1}{\sqrt{1 - 2i\sigma^2 u}} e^{\frac{iu\mu^2}{1-2i\sigma^2 u}},$$

where $Z \sim N(\mu, \sigma^2)$.

4. In the Vasicek model the short-rate X evolves according to an Ornstein-Uhlenbeck process

$$dX_t = (b + \beta X_t)dt + \sigma dB_t,$$

where $b \in \mathbb{R}$, $\beta < 0$, $\sigma > 0$ and B is a Brownian motion. Show that X is an affine process by using Theorem 9.1 and solving the associated Riccati equations

$$\begin{aligned} \Psi_t(t, iu) &= \beta \Psi(t, iu), \\ \Psi(0, iu) &= iu, \\ \Phi(t, iu) &= \frac{1}{2}\sigma^2 \int_0^t \Psi^2(s, iu)ds + b \int_0^t \Psi(s, iu)ds. \end{aligned}$$

Conclude that X_t is normally distributed with mean $X_0 e^{\beta t} + \frac{b}{\beta}(e^{\beta t} - 1)$ and variance $\sigma^2 \frac{e^{2\beta t} - 1}{2\beta}$.