# Optimal Bankruptcy Time and Consumption/Investment Policies on an Infinite Horizon with a Continuous Debt Repayment until Bankruptcy

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**Abstract.** In this paper we consider the optimization problem of an agent who wants to maximize the total expected discounted utility from consumption over an infinite horizon. The agent is under obligation to pay a debt at a fixed rate until he/she declares bankruptcy. At that point, after paying a fixed cost, the agent will be able to keep a certain fraction of the present wealth, and the debt will be forgiven. The selection of the bankruptcy time is taken to be at the discretion of the agent. The novelty of this paper is that at the time of bankruptcy the wealth process has a discontinuity, and that the agent continues to invest and consume after bankruptcy. We show that the solution of a free boundary problem satisfying some additional conditions is the value function of the above optimization problem. Particular examples such as the logarithmic and the power utility functions will be provided, and in these cases explicit forms will be given for the optimal bankruptcy time, investment and consumption processes.

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## 1. Introduction

We consider a financial market that consists of a risky asset (stock) with price $S_t$ and a fixed interest rate $r$. We assume that the stock follows a geometric Brownian Motion with a constant average rate of return $\mu$ and dispersion coefficient $\sigma$. There is an agent in this market who is under the obligation to pay a debt at a fixed continuous rate $d$ until the time he/she declares bankruptcy. At that point the debt is erased, the agent will pay a fixed cost $F$ and keep a given fraction $\alpha$ of the remaining wealth. The decision variables are the consumption and investment processes $\{(c_t, \pi_t),\ 0 \leq t < \infty\}$ and a stopping time $\tau \leq \infty$ representing the time of bankruptcy. We allow $\tau = \infty$ which corresponds to the event that bankruptcy is never declared. We will maximize the quantity $E\left[\int_0^\infty e^{-\gamma t} U(c_t) dt\right]$ where $U(\cdot)$ is a utility function, and $\gamma$ is a given discount factor. The novelty in this optimization problem is that the agent keeps investing and consuming after the possible bankruptcy, and utility is derived from consumption after time $\tau$ as well. Examples for this situation include holders of housing loans, student loans, or small business loans where the business proprietor is also personally liable for repayment of the loan.

Another mixed optimal stopping/control problem has been studied by Karatzas & Wang (2001). In that paper the "duality method" have been used. In order to derive explicit solutions, in particular for the optimal wealth level at bankruptcy, we are going to use the dynamic programming method. The initial impetus for this work came from the above mentioned paper where in Appendix B it is suggested that an optimization problem over a consumption stream that extends beyond a discretionarily selected stopping time $\tau$ is an interesting open problem. It is an important feature of our model that the wealth process is discontinuous at time $\tau$.

We shall prove that the solution of a particular free boundary problem satisfying some additional conditions is actually the value function of our optimization problem, and that it is optimal to declare bankruptcy if and only if the wealth is less or equal than the free boundary. It will be shown also that under the optimal policy bankruptcy will be declared in finite time with probability one . Explicit solution for the free boundary problem will be presented for the logarithmic and the power utility functions.

## 2. The optimization problem.

We consider a financial market that consists of a risky asset whose price $S_t$ evolves according to the dynamics

$$dS_t = \mu S_t dt + \sigma S_t dw_t \ ; \quad t \in [0, \infty)$$

and of a fixed interest rate $r$. The time horizon is $[0, \infty)$, the process $\{w_t, t < \infty\}$ is a standard Brownian Motion on the filtered probability space $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t,\ t < \infty\}$, where the filtration is the augmentation of the natural filtration of the Brownian Motion $\{w_t,\ t < \infty\}$. Let $\pi_t$ be the amount of money invested in the risky asset, and $c_t$ be the consumption rate at time $t$. The agent is under obligation to repay a debt at a fixed rate $d > 0$ until he/she declares bankruptcy (which may never occur). Let $X_t$ be the wealth at time $t$. At bankruptcy time $\tau$ the agent has to pay $F \geq 0$ as a fixed cost, and will be able to keep the amount $\alpha(X_\tau - F)$ where $0 < \alpha < 1$ is a given constant. In order to guarantee that the agent is able to cover the fixed cost of the bankruptcy and possibly still has some minimum amount of wealth after that time, it will be required that the wealth before bankruptcy is not below the level $F + \eta$, where $\eta$ is a "small" non-negative constant. After bankruptcy the debt is forgiven, and our agent continues to consume and invest. Let $x \geq F + \eta$ be the initial endowment. We postulate that the wealth process satisfies the following equation:

$$X_t = x + \int_0^t \big[(\mu - r)\pi_s + rX_s - d - c_s\big]ds + \int_0^t \sigma\pi_s dw_s \ , \quad t \le \tau, \tag{2.1}$$

$$X_{\tau+} = \alpha(X_\tau - F) \ ,$$

$$X_t = X_{\tau+} + \int_\tau^t \big[(\mu - r)\pi_s + rX_s - c_s\big]ds + \int_\tau^t \sigma\pi_s dw_s \ , \quad t > \tau.$$

**2.1 Definition:** A strictly increasing, strictly concave function $U : (0,\infty) \mapsto \Re$ is called a utility function if it is three times continuously differentiable on $(0,\infty)$, and $\lim_{c\to\infty} U'(c) = 0$ .

We extend $U(\cdot)$ to $[0,\infty)$ by $U(0) \triangleq \lim_{c\to 0+} U(c) \ge -\infty$. The quantity $\lim_{c\to 0+} U'(c)$ may be both finite or plus infinity. Notice that $U'(\cdot)$ is strictly decreasing on $(0,\infty)$.We define the pseudo-inverse of $U'(\cdot)$ as

$$I(y) = \inf\{c \ge 0;\ U'(c) \le y\}$$

which becomes the inverse of $U'(\cdot)$ if $\lim_{c\to 0+} U'(c) = \infty$.

**2.2 Definition:** We call a policy $(\tau, \{(\pi_t, c_t),\ t < \infty\})$ *admissible* if

(i) $\tau \le \infty$ is a stopping time (not necessarily finite);

(ii) $\{c_t;\ t \ge 0\}$ is a measurable, adapted non-negative process such that $\int_0^t c_s ds < \infty$, a.s., for all $t < \infty$;

(iii) $\{\pi_t;\ t \ge 0\}$ is a measurable adapted process such that $\int_0^t \pi_s^2 ds < \infty$, a.s., for all $t < \infty$;

(iv) $X_t \ge 0$ , a.s. $\quad t < \infty$;

(v) $X_{t\wedge\tau} \ge F + \eta$ , a.s., $\quad t < \infty$;

(vi) $E\Big[\int_0^\infty e^{-\gamma t} U^-(c_t)dt\Big] < \infty$ .

We denote by $\mathcal{A}(x)$ the class of admissible policies.

Let $\gamma > 0$ be a fixed discount factor. Our optimization problem is to maximize

$$E\Big[\int_0^\infty e^{-\gamma t} U(c_t)dt\Big] \tag{2.2}$$

over all admissible policies $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)$. We define the value function of this problem as

$$V(x) = \sup\Big\{E\Big[\int_0^\infty e^{-\gamma t} U(c_t)dt\Big];\ (\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)\Big\} \ .$$

We shall call the above optimization problem with $d = 0$ *Merton's problem.* In that case obviously the agent will never declare bankruptcy since there is nothing to gain from it (there is no debt). We denote by $\tilde{V}(x)$ the value function for Merton's problem and by $\tilde{\pi}_t = \tilde{\Pi}(X_t), \tilde{c}_t = \tilde{C}(X_t)$ the feedback form for the optimal investment/consumption policies. The functions $\tilde{V}(\cdot), \tilde{\Pi}(\cdot), \tilde{C}(\cdot)$ are computed in Karatzas et al. (1986) for different choices of the utility function, and in the present paper we regard these functions as known. Throughout this paper we shall assume that

$$\tilde{V}(x) < \infty, \quad x \in (0, \infty). \tag{2.3}$$

A sufficient condition for (2.3) is available in Karatzas et al. (1986). Obviously $V(x) \leq \tilde{V}(x)$ for all $x \in [F + \eta, \infty)$ thus also

$$V(x) < \infty\ , \quad x \in [F + \eta, \infty). \tag{2.4}$$

An intuitively obvious fact is that after bankruptcy an optimally behaving investor will follow the optimal investment/consumption policies $\tilde{\Pi}(X_t), \tilde{C}(X_t)$ for Merton's problem, thus we are only interested in the optimal bankruptcy time $\tau^*$ and the optimal investment/consumption policies $(\pi^*, c^*)$ (for our original optimization problem) *up to* $\tau^*$.

**2.3 Definition:** We denote by $\mathcal{A}_1(x) \subset \mathcal{A}(x)$ the class of admissible controls such that

$$E\left[\int_\tau^\infty e^{-\gamma t} U(c_t) dt\right] = E\left[e^{-\gamma\tau} \tilde{V}\big(\alpha(X_\tau - F)\big) 1_{\{\tau < \infty\}}\right]. \tag{2.5}$$

We have the following (rather obvious) proposition:

**2.4 Proposition:** If $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)$ is an admissible policy and $\{(\pi_t, c_t),\ \tau < t < \infty\} \equiv \{(\tilde{\Pi}(X_t), \tilde{C}(X_t)),\ \tau < t < \infty\}$, then $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}_1(x)$. Additionally, for every $x \in [F + \eta, \infty)$

$$V(x) = \sup\left\{E\left[\int_0^\infty e^{-\gamma t} U(c_t) dt\right]; (\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}_1(x)\right\}. \tag{2.6}$$

We defer the proof of this proposition to Appendix A4.

By the above proposition it is sufficient to maximize (2.2) over the class $\mathcal{A}_1(x)$, and we are going to do exactly that in the following sections.

## 3. Analytic characterization of the value function and the optimal policy

For every $p \in \Re$ and $c \geq 0$ we define the second order differential operator $\mathcal{D}^{(p,c)}$ in the following way. For every open set $G \subseteq \Re$ and $\phi \in C^2(G)$ the function $\mathcal{D}^{(p,c)}\phi : G \mapsto \Re$ will be defined as

$$\mathcal{D}^{(p,c)}\phi(x) = \phi'(x)\Big[p(\mu - r) + rx - d - c\Big] + \frac{1}{2}\phi''(x)\sigma^2 p^2 + U(c) - \gamma\phi(x), \quad x \in G. \tag{3.1}$$

We also define $\mathcal{D}\phi : G \mapsto \Re \cup \{\infty\}$ for every open set $G \subseteq \Re$ and $\phi \in C^2(G)$ as

$$\mathcal{D}\phi(x) = \sup_{p \in \Re, c \geq 0} \mathcal{D}^{(p,c)}\phi(x) \ , \quad x \in G.$$

The following theorem will characterize the value function and the optimal policy in terms of a solution of a free-boundary problem.

**3.1 Theorem:** Assume that there exist a continuous, strictly increasing function $\phi : [F+\eta, \infty) \mapsto \Re$ and a constant $b > F + \eta$ satisfying the following conditions:

(i) $\phi \in C^1\Big([F+\eta,\infty)\Big) \cap C^2\Big((F+\eta,b)\cup(b,\infty)\Big)$, and the limits $\lim_{x\to b+}\phi''(x)$ and $\lim_{x\to b-}\phi''(x)$ exist and are finite;

(ii) $\phi(x) \geq \tilde{V}(\alpha(x-F)), \quad x \in [b,\infty)$;

(iii) $\phi(x) = \tilde{V}(\alpha(x-F)), \quad x \in [F+\eta, b]$;

(iv) $\mathcal{D}\phi(x) = 0, \quad x \in (b,\infty)$;
$\mathcal{D}\phi(x) \leq 0, \quad x \in (F+\eta, b)$;

(v) There exist positive constants $K$, $K_1$, and $K_2$ such that

$$-\frac{\phi'(x)}{\phi''(x)} \leq Kx + K_1, \quad x \in (b,\infty), \tag{3.2}$$

and the Lipschitz-condition $|a(x)-a(y)| \leq K_2|x-y|$ for all $x, y \in (b,\infty)$ holds with both $a(\cdot) = \Big(\phi'/\phi''\Big)(\cdot)$ and $a(\cdot) = I(\phi'(\cdot))$. Furthermore, the constants $K$ and $K_1$ can be selected so that

$$\gamma > 3\max\{\theta(K_1+K) + \frac{r}{2} \ , \ 2\theta K^2\} \tag{3.3}$$

where $\theta = \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2$;

(vi) In the case when $\mu \neq r$, the scale-function $s(\cdot)$ corresponding to the diffusion on $(b,\infty)$

$$dY_t = \left[-2\theta\frac{\phi'}{\phi''}(Y_t) + rY_t - d - I(\phi'(Y_t))\right]dt - \frac{\mu-r}{\sigma}\frac{\phi'}{\phi''}(Y_t)dw_t \tag{3.4}$$

satisfies

$$s(\infty) = \infty. \tag{3.5}$$

If $\mu = r$ then we assume

$$rx - I(\phi'(x)) < d, \quad x \geq b \ .$$

Under the above conditions the value function of our optimization problem is

$$V(x) = \phi(x), \quad x \in [F+\eta,\infty),$$

and an optimal policy is given by

$$\tau^* = \inf\{t \geq 0: \ X_t \leq b\}, \tag{3.6}$$

$$\pi_t^* = \begin{cases} \frac{r-\mu}{\sigma^2}\frac{\phi'}{\phi''}(X_t), & \text{if } t \leq \tau^*; \\ \tilde{\Pi}(X_t), & \text{if } t > \tau^*, \end{cases} \tag{3.7}$$

$$c_t^* = \begin{cases} I(\phi'(X_t)), & \text{if } t \leq \tau^*; \\ \tilde{C}(X_t), & \text{if } t > \tau^*. \end{cases} \tag{3.8}$$

Furthermore,

$$P(\tau^* < \infty) = 1 \ . \tag{3.9}$$

**3.2 Remark:** Condition (i) and(iv) imply that $\phi(\cdot)$ is strictly concave on $[F+\eta, \infty)$. Also, (i) and (iii) imply

$$\phi'(b) = \frac{\partial}{\partial x}\tilde{V}(\alpha(x-F))\bigg|_{x=b}. \tag{3.10}$$

Notice also the implicit assumption in (i) that $\phi(F+\eta) > -\infty$.

The proof of this theorem will be deferred to Appendix A1.

In Sections 4 and 5 we are going to solve explicitly the free-boundary problem represented by (iv), (iii), and (3.10) for the power and logarithmic utility functions, and then show that the solution satisfies all assumptions of the above theorem. For an arbitrary utility function the first relation of assumption (iv) is

$$\gamma\phi(x) = \max_{p\in\Re, c\geq 0}\Big\{\phi'(x)\Big[p(\mu-r)+rx-d-c\Big]+\frac{1}{2}\phi''(x)\sigma^2p^2+U(c)\Big\}; \quad x > b. \tag{3.11}$$

Under $\phi''(x) < 0$ the maximum on the right-hand side is achieved in $p^* = \frac{r-\mu}{\sigma^2}\frac{\phi'}{\phi''}(x)$ and $c^* = I(\phi'(x))$, and (3.11) becomes

$$\gamma\phi = -\theta\frac{(\phi')^2}{\phi''} + \Big(rx - d - I(\phi')\Big)\phi' + U(I(\phi')), \quad x > b \ . \tag{3.12}$$

This nonlinear equation can be linearized. We borrow the idea from Karatzas et al. (1986) and apply a change of variable $c = I(\phi'(x))$. Let as assume (it will be justified in the coming sections) that the function

$$C(x) \stackrel{\triangle}{=} I(\phi'(x)); \quad x \in (b, \infty) \tag{3.13}$$

is strictly increasing and has an inverse function $X : (C(b), \infty) \mapsto (b, \infty)$. After this change of variable (3.12) becomes

$$\gamma\phi(X(c)) = -\theta\frac{(U'(c))^2X'(c)}{U''(c)} + \Big(rX(c) - d - c\Big)U'(c) + U(c). \tag{3.14}$$

After differentiation with respect to $c$ and using the identity

$$\phi'(X(c)) = U'(c) \tag{3.15}$$

we arrive to the following second order *linear* ordinary differential equation:

$$\theta X''(c) = X'(c)\Big[(-\gamma - 2\theta + r)\frac{U''(c)}{U'(c)} + \theta\frac{U'''(c)}{U''(c)}\Big] - (c+d)\left(\frac{U''(c)}{U'(c)}\right)^2 + r\left(\frac{U''(c)}{U'(c)}\right)^2 X(c). \tag{3.16}$$

## 4. Explicit solution for the power utility function

We are going to consider the power utility function

$$U(c) = \frac{1}{\delta}c^\delta; \quad c \geq 0 \tag{4.1}$$

with some constant $\delta < 1$, $\delta \neq 0$. It is known from Karatzas et al. (1986) that in the case of $\delta \in (0,1)$ the condition

$$\gamma > r\delta + \frac{\theta\delta}{1-\delta} \tag{4.2}$$

guarantees $\tilde{V}(x) < \infty$, so we are going to assume in this section that (4.2) holds (if $\delta$ is negative then (4.2) is obviously satisfied since $\gamma$ is positive). Under this assumption

$$\tilde{V}(x) = \frac{1}{\delta}D^{1-\delta}x^\delta; \quad x \geq 0, \tag{4.3}$$

where $D$ is the positive constant

$$D = \frac{1-\delta}{\gamma - \frac{\delta}{1-\delta}\theta - r\delta} \tag{4.4}$$

(see Karatzas et al. (1986), (14.6)). The linear ODE (3.16) with the power utility function becomes

$$\theta X'' = X'\Big[(r-\gamma)(\delta-1) - \theta\delta\Big]\frac{1}{c} + r(\delta-1)^2\frac{1}{c^2}X - (c+d)(\delta-1)^2\frac{1}{c^2}, \tag{4.5}$$

and (3.13) becomes

$$\phi'(x) = (C(x))^{\delta-1}, \quad x \geq b\ . \tag{4.6}$$

A particular solution to the inhomogeneous equation (4.5) is $X_1(c) = Dc + \frac{d}{r}$, where the constant $D$ is given in (4.4). The general solution of the corresponding homogeneous equation is

$$X(c; B, \lambda, B_1, \lambda_1) = Bc^{\lambda(\delta-1)} + B_1c^{\lambda_1(\delta-1)}, \tag{4.7}$$

where $\lambda > 0$, $\lambda_1 < 0$ are the two solutions of the quadratic equation

$$\theta\lambda^2 - (r - \gamma - \theta)\lambda - r = 0.$$

At this point we are only searching for a solution, and the correctness of the result will be justified later. In this spirit we simply postulate that $B_1 = 0$ and search for a solution of the free boundary problem in the form of

$$X(c) = Dc + \frac{d}{r} + Bc^{\lambda(\delta-1)}. \tag{4.8}$$

We anticipate that $B$ will turn out to be negative, thus $X(\cdot)$ is increasing. For any $B < 0$ we have $X : (0, \infty) \overset{\text{onto}}{\longmapsto} (-\infty, \infty)$, thus its inverse $C : (-\infty, \infty) \overset{\text{onto}}{\longmapsto} (0, \infty)$ is well-defined and strictly increasing. Condition (iii) with $x = b$ becomes

$$\phi(b) = \frac{1}{\delta} D^{1-\delta}\alpha^\delta(b - F)^\delta, \tag{4.9}$$

thus by (4.6) our candidate solution must have the form

$$\phi(x) = \int_b^x (C(y))^{\delta-1} dy + \frac{1}{\delta} D^{1-\delta}\alpha^\delta(b - F)^\delta, \quad x \geq b. \tag{4.10}$$

For every $B < 0$ and $b > 0$ this determines a candidate solution for $\phi(\cdot)$. We need to write two equations for the unknown parameters $B, b$. The "smooth fit" condition (3.10) becomes $\phi'(b) = D^{1-\delta}\alpha^\delta(b - F)^{\delta-1}$, and using (4.6) this becomes

$$C(b) = \frac{1}{D}\alpha^{\frac{\delta}{\delta-1}}(b - F), \tag{4.11}$$

i.e.,

$$b = X\Big(\frac{1}{D}\alpha^{\frac{\delta}{\delta-1}}(b - F)\Big)\ . \tag{4.12}$$

Substituting (4.8) into this gives the following equation:

$$b = \alpha^{\frac{\delta}{\delta-1}}(b - F) + \frac{d}{r} + BD^{\lambda(1-\delta)}\alpha^{\delta\lambda}(b - F)^{\lambda(\delta-1)}. \tag{4.13}$$

We still need another equation. We arrived from (3.14) to (3.16) by taking a derivative. If we want our solution $X(\cdot)$ to satisfy (3.14) as well as (3.16), then (3.14) must be satisfied in a particular point $c = c_1$. We substitute $c = C(b)$ into (3.14), use (4.9), (4.11), (4.8), and derive the following equation:

$$\gamma D^{1-\delta}\alpha^\delta(b - F)^\delta = -\theta\frac{\delta}{\delta - 1}D^{-\delta}\alpha^{\frac{\delta^2}{\delta-1}}(b - F)^\delta\Big[D + B\lambda(\delta - 1)D^{-\lambda\delta+\lambda+1}\alpha^{\frac{\delta}{\delta-1}(\lambda\delta-\lambda-1)}$$

$$(b-F)^{\lambda\delta-\lambda-1}\Big] + \Big[rb - d - \frac{1}{D}\alpha^{\frac{\delta}{\delta-1}}(b - F)\Big]\delta D^{1-\delta}\alpha^\delta(b - F)^{\delta-1} + D^{-\delta}\alpha^{\frac{\delta^2}{\delta-1}}(b - F)^\delta. \tag{4.14}$$

We have now two equations, (4.13) and (4.14). One can express $B$ from (4.13), substitute it into (4.14) and after some elementary but rather long algebra an explicit solution to $b$ arises, that is

$$b = \Big(1 - b_0 \frac{r}{d}\Big) F + b_0, \tag{4.15}$$

where

$$b_0 = \frac{\delta(r - \theta\lambda)}{r\Big(\alpha^{\frac{\delta}{\delta-1}} - 1\Big)\Big(\gamma - \delta(r - \theta\lambda)\Big)} d. \tag{4.16}$$

One can easily see that

$$r > \theta\lambda \tag{4.17}$$

thus $b_0 > 0$ (in the case of $\delta \in (0,1)$ recall our assumption (4.2)). An other restriction on $\eta$ (see the second inequality of (4.19) below) will guarantee $b > F + \eta$, as required. From (4.13) we get now

$$B = \Big(b - \alpha^{\frac{\delta}{\delta-1}}(b - F) - \frac{d}{r}\Big) D^{-\lambda+\lambda\delta} \alpha^{-\lambda\delta} (b - F)^{-\lambda\delta+\lambda}, \tag{4.18}$$

which is negative provided that $b > F$ and $F < d/r$. Indeed, by (4.15)

$$b - \alpha^{\frac{\delta}{\delta-1}}(b - F) - \frac{d}{r} = F\bigg(1 - \frac{r}{d} b_0 \Big(1 - \alpha^{\frac{\delta}{\delta-1}}\Big)\bigg) + b_0(1 - \alpha^{\frac{\delta}{\delta-1}}) - \frac{d}{r}.$$

The coefficient of $F$ in the last expression is obviously positive if $\delta \in (0,1)$. It can be seen easily that it is also positive if $\delta$ is negative. Now $F < d/r$ indeed guarantees that the above expression is negative.

Now we are ready to state and prove the following theorem:

**4.1 Theorem:** Suppose that the utility function is the power utility of (4.1) with $\delta < 1$, $\delta \neq 0$. Assume (4.2) (which is active only if $\delta \in (0,1)$), and also

$$\gamma > \max\Big\{3\Big(\frac{\theta}{1-\delta} + d + \frac{r}{2}\Big), \frac{6\theta}{(1-\delta)^2}, \theta + r, -\frac{\delta}{1-\delta}\Big(\frac{r}{\lambda} - \theta\Big)\Big\} \quad \text{and} \quad 0 \le \eta < b_0\Big(1 - F\frac{r}{d}\Big). \tag{4.19}$$

Additionally, assume that $\eta$ is strictly positive whenever $\delta < 0$. Let the constants $D, b, B$ be as in (4.4), (4.15), (4.16) and (4.18), the increasing function $X : (0,\infty) \overset{\text{onto}}{\longmapsto} (-\infty,\infty)$ given by (4.8), and let $C : \Re \overset{\text{onto}}{\longmapsto} (0,\infty)$ be the inverse function of $X(\cdot)$. Then the function $\phi : [F + \eta, \infty) \mapsto \Re$ given by (4.10) on $[b,\infty)$ and $\phi(x) = \frac{1}{\delta} D^{1-\delta} \alpha^{\delta} (x - F)^{\delta}$ on $[F + \eta, b]$ satisfies all conditions (i)-(vi) of Theorem 3.1.

The proof of this theorem is deferred to Appendix A2.

**4.2 Corollary:** Suppose that the utility function is the power utility (4.1), all assumptions of Theorem 4.1 hold, and $C(\cdot)$, $\phi(\cdot)$ and $b$ are as in that theorem. Then the value function of our optimization problem is

$$V(x) = \phi(x), \quad x \in [F + \eta, \infty).$$

Furthermore, the optimal policy is given by (3.6) and

$$c_t^* = C(X_t)1_{\{[0,\tau^*]\}}(t) + \frac{1}{1-\delta}\Big(\gamma - r\delta - \frac{\theta\delta}{1-\delta}\Big)X_t 1_{\{(\tau*,\infty)\}}(t)$$

$$\pi_t^* = \frac{\mu - r}{\sigma^2(1-\delta)}\Big(\frac{C(X_t)}{C'(X_t)}1_{\{[0,\tau^*]\}}(t) + X_t 1_{\{(\tau*,\infty)\}}(t)\Big)\ .$$

Furthermore, $\tau^*$ is almost surely finite.

**Proof:** This is a straightforward consequence of Theorems 3.1, 4.1, identities (3.13) and (4.10), Theorem 14.1 in Karatzas et al. (1986), and the identities above (14.6) and below (14.8) in the same reference.

### 4.3 Static analysis of the solution

It is interesting to examine what kind of monotonicity and limit properties does the solution posses as a function of the different model parameters. We start with the exercise boundary $b$. Straightforward differentiation reveals that $b = b(\theta)$ is a decreasing function of $\theta$. This means that if the interest rate $r$ is fixed then $b$ is a decreasing function of $|\mu - r|$, and the function $\sigma^2 \mapsto b$ is increasing. All these make perfect intuitive sense; the stock is getting more attractive if $|\mu - r|$ is increasing or if $\sigma^2$ is decreasing. The function $F \mapsto b$ is an affine function, may be either increasing or decreasing. The function $d \mapsto b$ is affine, increasing. For the limit properties of $b$ we shall consider first $d \to 0$. In that case we must assume $F \to 0$ as well in order to satisfy $F < \frac{d}{r}$. Clearly then we have $\lim_{d,F\to 0} b = 0$. On the other hand, if $\alpha \to 0$ then in the case of $\delta > 0$ we must have $\eta \to 0$ in order to satisfy the second inequality of (4.19). We thus get that if $\delta > 0$ then $\lim_{\alpha,\eta\to 0} b = F$. If $\delta < 0$ then $\lim_{\alpha\to 0} b > F$. Finally, for all possible $\delta$ we have $\lim_{\alpha\to 1} b = \infty$.

Next we look at the behavior of the consumption function $C(x)$. One can easily see that $C(x;\alpha) = C(x)$ is an increasing function of $\alpha$. Indeed, straightforward calculus shows that $\alpha \mapsto B$ is decreasing, hence $\alpha \mapsto X(c)$ is also decreasing, thus $\alpha \mapsto C(x;\alpha)$ is increasing. We can also verify that

$$\lim_{\alpha\to 1} C(x;\alpha) = \infty. \tag{4.20}$$

Indeed, $\lim_{\alpha\to 1} B = -\infty$, hence $\lim_{\alpha\to 1} X(c;\alpha) = -\infty$, and (4.20) now follows. We must interpret (4.20) together with the fact that $\lim_{\alpha\to 1} b = \infty$. If $\alpha$ is very close to 1 then one should claim bankruptcy unless the wealth level is very large, and in that case consumption is very large until bankruptcy.

Considering the fixed cost $F$ as a variable, the function $F \mapsto C(x;F)$ is decreasing. Indeed, $F \mapsto B$ is increasing, thus $F \mapsto X(c;F)$ is also increasing, hence $C(x;F)$ is decreasing as the function of $F$.

The limit of the value function as $\alpha \to 1$ is given by $\lim_{\alpha\to 1} V(x;\alpha) = \tilde{V}(x - F) = \frac{1}{\delta}D^{1-\delta}(x-F)^\delta$. This follows immediately from $\lim_{\alpha\to 1} b = \infty$.

Finally, we anticipate that the solution of our problem converges to that of Merton's problem if $d \to 0$. We must assume $F \to 0$ as well in order to satisfy the constraint $F < \frac{d}{r}$. Now we have

$$\lim_{d,F\to 0} C(x;d,F) = \frac{1}{D}x \tag{4.21}$$

$$\lim_{d,F\to 0} \Pi(x;d,F) = \frac{\mu - r}{\sigma^2(1-\delta)}x \tag{4.22}$$

$$\lim_{d,F\to 0} V(x;d,F) = \frac{1}{\delta}D^{1-\delta}x^{\delta}, \tag{4.23}$$

where $\Pi(x) = \frac{\mu-r}{\sigma^2(1-\delta)}\frac{C(x)}{C'(x)}$, i.e., the feedback form of the optimal investment policy up to bankruptcy. The right-hand sides of the above three relations characterize Merton's solution for the $d = 0$ case. Relations (4.21) and (4.22) follow immediately from $\lim_{d,F\to 0} B = 0$. Relation (4.23) can be verified in the following way. A change of variable $c = C(y)$ in (4.10) gives for $x \geq b$

$$\phi(x) = \frac{D}{\delta}\left[C^{\delta}(x) - C^{\delta}(b)\right] + \frac{B\lambda}{\lambda+1}\left[C^{(\lambda+1)(\delta-1)}(x) - C^{(\lambda+1)(\delta-1)}(b)\right] + \frac{1}{\delta}D^{1-\delta}\alpha^{\delta}(b-F)^{\delta}.$$

From this last identity, (4.21), (4.11) and $B \to 0$ follows that

$$\lim_{d,F\to 0}\phi(x) = \frac{1}{\delta}D^{1-\delta}x^{\delta} +$$

$$D^{1-\delta}\alpha^{\delta}\lim_{d,F\to 0}\left\{(b-F)^{\delta-1}\left[\left(1-\alpha^{\frac{\delta}{\delta-1}}\right)\frac{1}{\delta}(b-F) - \frac{B\lambda}{\lambda+1}D^{\lambda(1-\delta)}\alpha^{\lambda\delta}(b-F)^{\lambda(\delta-1)}\right]\right\}.$$

One can see that the expression in the square bracket of the right-hand side of the last identity is zero; one needs to substitute $F = 0$, use (4.15), (4.16), (4.18), and the definition of $\lambda$.

## 5. The logarithmic utility function

In this section we specialize the results of Section 3 to the logarithmic utility

$$U(c) = \log c\ . \tag{5.1}$$

Several of the computations here are similar to those in the previous section, one only needs to substitute $\delta = 0$. We shall include in this section only those details that are substantially different from the corresponding ones in Section 4.

We know from Karatzas et al. (1986) that with the logarithmic utility

$$\tilde{V}(x) = \frac{1}{\gamma}\log(\gamma x) + \frac{r-\gamma+\theta}{\gamma^2}\ .$$

Equation (3.16) becomes

$$\theta X'' = (\gamma - r)\frac{1}{c}X' + \frac{r}{c^2}X - \frac{c+d}{c^2} \tag{5.2}$$

and the "free boundary conditions" are

$$\phi(b) = \frac{1}{\gamma}\log\{\gamma\alpha(b-F)\} + \frac{r-\gamma+\theta}{\gamma^2} \tag{5.3}$$

$$\phi'(b) = \frac{1}{\gamma(b-F)} \ . \tag{5.4}$$

A particular solution to (5.2) is $X_1(c) = \frac{1}{\gamma}c + \frac{d}{r}$, and the homogeneous equation has the general solution (4.7) with the $\delta = 0$ substitution. We again set $B_1 = 0$ and search for a solution to (5.2)-(5.4) in the form

$$X(c) = \frac{1}{\gamma}c + \frac{d}{r} + Bc^{-\lambda} \ . \tag{5.5}$$

We hope that $B$ will turn out to be negative, and so $X(\cdot)$ is increasing and has an inverse function $C(\cdot)$. By (3.15)

$$\phi'(x) = \frac{1}{C(x)} \tag{5.6}$$

thus by (5.3) we get

$$\phi(x) = \int_b^x \frac{1}{C(y)}dy + \frac{1}{\gamma}\log\{\gamma\alpha(b-F)\} + \frac{r-\gamma+\theta}{\gamma^2}, \quad x \in [b, \infty) \ . \tag{5.7}$$

Formulae (5.6) and (5.4) give

$$b = X\Big(\gamma(b-F)\Big), \quad C(b) = \gamma(b-F). \tag{5.8}$$

This last identity and (5.5) yield

$$F - \frac{d}{r} = B\gamma^{-\lambda}(b-F)^{-\lambda} \ . \tag{5.9}$$

Next we substitute $c_1 = C(b)$ into (3.14) which after some algebra becomes

$$(b-F)\Big(\log\alpha + \frac{r}{\gamma}\Big) = -\lambda B\theta\gamma^{-\lambda-1}(b-F)^{-\lambda} + \frac{rb-d}{\gamma} \ . \tag{5.10}$$

We solve (5.9)-(5.10) for the unknown constants $B, b$ and get

$$b = F + \frac{r-\lambda\theta}{\gamma\log\frac{1}{\alpha}}\Big(\frac{d}{r} - F\Big) \tag{5.11}$$

$$B = -\Big(\frac{d}{r} - F\Big)^{\lambda+1}\Bigg(\frac{r-\lambda\theta}{\log\frac{1}{\alpha}}\Bigg)^{\lambda}. \tag{5.12}$$

**5.1 Theorem:** Suppose that the utility function is the logarithmic utility, assume

$$\frac{r-\lambda\theta}{\gamma\log\frac{1}{\alpha}}\Big(\frac{d}{r} - F\Big) > \eta > 0 \tag{5.13}$$

and

$$\gamma > 3\max\Big\{\theta(K_1+1) + \frac{r}{2}\ ,\ 2\theta\Big\} \tag{5.14}$$

where

$$K_1 = \Bigg(\lambda\Big(\frac{d}{r} - F\Big) - F\Bigg)^{+}. \tag{5.15}$$

Let the constants $B$ and $b$ be given by (5.11)-(5.12), the increasing function $X : (0,\infty) \overset{\text{onto}}{\longmapsto} \Re$ be given by (5.5), $C : \Re \mapsto (0,\infty)$ be the inverse of $X(\cdot)$, and the function $\phi : [F+\eta,\infty) \mapsto \Re$ be given by (5.7) on $[b,\infty)$ and by

$$\phi(x) = \frac{1}{\gamma}\log\{\gamma\alpha(x-F)\} + \frac{r-\gamma+\theta}{\gamma^2}, \quad x \in [F+\eta, b).$$

Then $b$ and $\phi(\cdot)$ satisfy all conditions of Theorem 3.1.

The proof of this theorem will be deferred to Appendix A3.

**5.2 Corollary:** Suppose that the utility function is the logarithmic utility (5.1), all assumptions of Theorem 5.1 hold, and $C(\cdot)$, $\phi(\cdot)$ and $b$ are as in that theorem. Then the value function of our optimization problem is

$$V(x) = \phi(x), \quad x \in [F+\eta,\infty).$$

Furthermore, the optimal policy is given by (3.6) and

$$c_t^* = C(X_t)1_{\{[0,\tau^*]\}}(t) + \gamma X_t 1_{\{(\tau*,\infty)\}}(t)$$

$$\pi_t^* = \frac{\mu-r}{\sigma^2}\Big(\frac{C(X_t)}{C'(X_t)}1_{\{[0,\tau^*]\}}(t) + X_t 1_{\{(\tau*,\infty)\}}(t)\Big).$$

Additionally, $\tau^*$ is almost surely finite.

**Proof:** This is a straightforward consequence of Theorems 3.1, 5.1, Theorem 14.1 in Karatzas et al. (1986), and the identities above (14.6) and below (14.8) in the same reference.

**5.3 Static analysis of the solution:** All statements in section 4.3 remain true in the case of the logarithmic utility function. We note that $\lim_{\alpha,\eta\to 0} b = F$ (this relation does not follow from the corresponding relation in the power utility case since there we separated the $\delta > 0$ and $\delta < 0$ cases). It is worth spelling out the limits as $d, F \to 0$ in the logarithmic utility case. We have

$$\lim_{d,F\to 0} C(x; d, F) = \gamma x \tag{5.16}$$

$$\lim_{d,F\to 0} \Pi(x; d, F) = \frac{\mu - r}{\sigma^2} x$$

$$\lim_{d,F\to 0} V(x; d, F) = \frac{1}{\gamma} \log \gamma x + \frac{r - \gamma + \theta}{\gamma^2}. \tag{5.17}$$

Here $\Pi(x) = \frac{\mu - r}{\sigma^2} \frac{C(x)}{C'(x)}$ is the feedback form of the optimal trading strategy. The right-hand sides of these three relations describe Merton's solution to the $d = 0$ case. In order to verify (5.17) we note that a change of variable $c = C(y)$ in (5.7) gives for $x \geq b$
$\phi(x) =$

$$\frac{1}{\gamma} [\log C(x) - \log C(b)] + \frac{B\lambda}{\lambda + 1} \left[C^{-\lambda-1}(x) - C^{-\lambda-1}(b)\right] + \frac{r - \gamma + \theta}{\gamma^2} + \frac{1}{\gamma} \log (\gamma\alpha(b - F)).$$

We use now relations (5.16),(5.8), $B \to 0$, substitute $F = 0$ and see that

$$\lim_{d,F\to 0} \phi(x) = \frac{1}{\gamma} \log \gamma x + \frac{r - \gamma + \theta}{\gamma^2} + \lim_{d\to 0} \left\{ -B \frac{\lambda}{\lambda + 1} (\gamma b)^{-\lambda-1} + \frac{1}{\gamma} \log \alpha \right\}.$$

After a substitution of (5.12), (5.11), and $F = 0$, using the definition of $\lambda$ one can verify that the expression in the curly brackets is zero.

**5.4 Remark:** Using elementary calculus one can see that the optimal bankruptcy level for the power utility case (given by (4.15)-(4.16)) converges to the corresponding optimal bankruptcy level for the logarithmic utility (given by (5.11)) as $\delta \to 0$. Also, one can formally derive the optimal consumption and investment policies from the corresponding optimal policies for the power utility by substituting $\delta = 0$. Indeed, if we substitute $\delta$ with zero in the $X(\cdot)$ function for the power utility (given by (4.8)) we get exactly the corresponding function for the logarithmic utility. Hence a formal substitution of $\delta = 0$ in the optimal policies for the power utility case (see Corollary 4.2) will lead to the corresponding optimal policies for the logarithmic case (given in Corollary 5.2). However, the value function for the power utility given by (4.10) converges to infinity as $\delta \to 0$, not to the corresponding value function for the logarithmic case. This is quite intuitive, since the power utility itself converges to infinity as $\delta \to \infty$.

## 6. Previous literature and application of the results

Utility maximization problems up to a possible bankruptcy have been addresses by several authors in the literature. In their 1986 paper, Karatzas, Lehoczky, Sethi, and Shreve maximize the expected value of the discounted total utility from consumption up to a possible bankruptcy plus the expected discounted fixed payment at bankruptcy. Bankruptcy is defined as the first time the wealth of the consumer reaches zero. An explicit solution has been presented for a general concave utility function and for various values of the payment at bankruptcy. Also, the probability of bankruptcy has been computed under the optimal policy.

Presman and Sethi (1991) examined the effect of the various parameters and the wealth level on the absolute and relative risk aversion coefficients of the value function. These authors also show that for a HARA type utility function the value function is not necessarily of HARA type. Papers by Sethi, Taksar, & Presman (1992) and Presman & Sethi (1997) present an analysis of the same maximization problem under a constraint that consumption can never fall below a minimum (subsistence) rate. Cadenillas & Sethi (1997) extended the model to include random coefficients in the equation for the security prices and random interest rates. Lehoczky, Sethi, & Shreve (1983 and 1985) introduced a short-selling constraint in addition to the subsistence constraint and solved the optimization problem explicitly. Closest in spirit to our paper is the one by Sethi & Taksar (1992) since in that model bankruptcy is non-terminal.

Our paper differs from all the above papers in that the selection of bankruptcy time is taken to be at the discretion of the investor, hence it becomes one of the control variables. The optimization problem examined here is a major concern for investors who have an outstanding liability that is to be repayed at a fixed rate. Examples for such loans include housing loans, student loans, and small business loans, as already pointed out in the introduction. Since we have explicit formulas for the optimal bankruptcy time as well as for the optimal portfolio and consumption processes, the basic remaining question for practical applicability of these results is the estimation of the directly unobservable parameters $\mu$ and $\sigma$ and the selection of the utility function. Assuming that the present time is $t$ and that we are observing the continuous path $\{S_u,\ u \le t\}$, the volatility $\sigma$ causes no problem since it can be read out exactly from the continuous path of the price process. Indeed, the quadratic variation of the price can be written as $< S >_t = \int_0^t \sigma^2 S_u^2 du$ and it is the limit of the sequence $\sum_{i=1}^{m(n)} \big[S(t_i^{(n)}) - S(t_{i-1}^{(n)})\big]^2$ as $n \to \infty$. In this formula for every positive integer $n$, $\mathcal{D}_n = \big(0 = t_0^{(n)}, t_1^{(n)}, \ldots, t_{m(n)}^{(n)} = t\big)$ is a partition of the time-interval $\big[0,t\big]$ into $m(n)$ subintervals such that the length of the largest subinterval in partition $\mathcal{D}_n$ converges to zero as $n \to \infty$.

A more interesting question concerns the estimation of the mean rate of return $\mu$ based on the continuous path up to time $t$. We suggest the maximum likelihood estimator (MLE) $\hat{\mu}$ which can be explicitly computed for this model in the following way. We denote by $P_\mu$ the measure generated by the price process $\{S_u,\ u \le t\}$ on the Borel-sets of $C([0,t])$, and by $P_0$ the measure generated by the same process for the parameter choice $\mu = 0$. It is well-known (see, for example, Feigin (1976)) that the log-likelihood function is

$$\log\left(\frac{dP_\mu}{dP_0}\right) = \frac{\mu}{\sigma^2}\int_0^t \frac{1}{S_u} dS_u - \frac{\mu^2}{2\sigma^2} t \ ,$$

hence the MLE for the instantaneous rate of return is

$$\hat{\mu} = \frac{1}{t} \int \frac{1}{S_u} dS_u \ .$$

The choice of the utility function is, of course, subjective. A more concave utility function will lead to less consumption at large wealth levels thus smaller $\delta$ values in the power-utility case will lead to less greedy optimal strategies. Naturally the logarithmic utility fits into this scheme, if regarded as a power utility function with $\delta = 0$.

## 7. Conclusion

Theorem 3.1 provides a general framework for solving our optimization problem with an optional bankruptcy. The method is the following: solve the second order ordinary differential equation (3.16) with free boundary conditions $\phi(b) = \tilde{V}(\alpha(b - F))$ and (3.10), then prove that under appropriate constraints on the parameters the other conditions of Theorem 3.1 are also satisfied. We completed this analysis for the power and the logarithmic utility functions. In both cases we presented explicit formulas for the free boundary $b$ and for the optimal consumption/investment processes. The conditions for the validity of our analysis require only that the discount factor $\gamma$ is sufficiently large, and that the fixed cost of bankruptcy $F$ and the parameter $\eta$ are sufficiently small (see (4.2), (4.19), (5.13), and (5.14)).

## A1 Appendix, Proof of Theorem 3.1

Let $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}_1(x)$ arbitrary. We are going to show first that

$$E\left[\int_0^\infty e^{-\gamma t} U(c_t) dt\right] \le \phi(x). \tag{$A1.1$}$$

For every positive integer $n$ we define the stopping times

$$\tau_n = \inf\left\{ t \ge 0 : \ \int_0^t \left(e^{-\gamma s} \phi'(X_s) \pi_s\right)^2 ds \ge n \right\},$$

$$T_n = \tau_n \wedge \tau \wedge n,$$

thus

$$\lim_{n \to \infty} T_n = \tau, \quad \text{a.s.} \tag{$A1.2$}$$

We note that $\tau_n$ and $\tau$ may take the value $\infty$ with positive probability. By (2.5) we have

$$E\left[\int_0^\infty e^{-\gamma t} U(c_t) dt\right] = E\left[\int_0^{T_n} e^{-\gamma t} U(c_t) dt\right] + E\left[\int_{T_n}^{\tau} e^{-\gamma t} U(c_t) dt\right] +$$

$$+E\Big[e^{-\gamma\tau}\tilde{V}(\alpha(X_\tau-F))1_{\{\tau<\infty\}}\Big]. \qquad (A1.3)$$

Relation (2.4) and item (vi) of Definition 2.2 imply that

$$E\Big[\int_0^\infty e^{-\gamma t}|U(c_t)|dt\Big]<\infty. \qquad (A1.4)$$

This implies that all the expectations on the right-hand side of (A1.3) are finite. By assumption (i) we can apply the generalized version of Ito's rule (Karatzas & Shreve (1988), Problem 3.7.3), and using assumption (iv) we have

$$e^{-\gamma T_n}\phi(X_{T_n})-\phi(x)=$$

$$=\int_0^{T_n}e^{-\gamma t}\Big\{-\gamma\phi(X_t)+\phi'(X_t)\Big[\pi_t(\mu-r)+rX_t-d-c_t\Big]+\frac{1}{2}\phi''(X_t)\sigma^2\pi_t^2\Big\}dt+$$

$$+\int_0^{T_n}e^{-\gamma t}\phi'(X_t)\pi_t\sigma dw_t\le-\int_0^{T_n}e^{-\gamma t}U(c_t)dt+\int_0^{T_n}e^{-\gamma t}\phi'(X_t)\pi_t\sigma dw_t\ .$$

The Optional Sampling Theorem and $T_n\le\tau_n$ imply that the last (stochastic) integral has zero expectation, thus

$$E\Big[\int_0^{T_n}e^{-\gamma t}U(c_t)dt\Big]\le\phi(x)-E\Big[e^{-\gamma T_n}\phi(X_{T_n})\Big]\ . \qquad (A1.5)$$

The expectation on the right-hand side is finite because of (A1.4) and $\phi(F+\eta)>-\infty$. We use (A1.3), (A1.5) and see that

$$E\Big[\int_0^\infty e^{-\gamma t}U(c_t)dt\Big]\le\phi(x)-E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau<\infty\}}\Big]-E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau=\infty\}}\Big]+$$

$$+E\Big[\int_{T_n}^\tau e^{-\gamma t}U(c_t)dt\Big]+E\Big[e^{-\gamma\tau}\tilde{V}(\alpha(X_\tau-F))1_{\{\tau<\infty\}}\Big],$$

thus by the elementary $\liminf(a_n-b_n-c_n)\le\liminf a_n-\liminf b_n-\liminf c_n$ we also have

$$E\Big[\int_0^\infty e^{-\gamma t}U(c_t)dt\Big]\le\phi(x)-\liminf_{n\to\infty}E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau<\infty\}}\Big]-$$

$$-\liminf_{n\to\infty}E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau=\infty\}}\Big]+\liminf_{n\to\infty}E\Big[\int_{T_n}^\tau e^{-\gamma t}U(c_t)dt\Big]+$$

$$+E\Big[e^{-\gamma\tau}\tilde{V}(\alpha(X_\tau-F))1_{\{\tau<\infty\}}\Big]. \qquad (A1.6)$$

The function $\phi(\cdot)$ is bounded below on $[F+\eta,\infty)$ by the finite $\phi(F+\eta)$, thus an application of Fatou's Lemma yields

$$\liminf_{n\to\infty}E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau<\infty\}}\Big]\ge E\Big[e^{-\gamma\tau}\phi(X_\tau)1_{\{\tau<\infty\}}\Big]. \qquad (A1.7)$$

Another application of Fatou's Lemma gives the bound

$$\liminf_{n\to\infty} E\Big[e^{-\gamma T_n}\phi(X_{T_n})1_{\{\tau=\infty\}}\Big] \geq \phi(F+\eta)E\Big[\lim_{n\to\infty} e^{-\gamma T_n}1_{\{\tau=\infty\}}\Big] = 0. \qquad (A1.8)$$

Relation (A1.4) implies

$$\int_0^\infty e^{-\gamma t}|U(c_t)|dt < \infty, \quad \text{a.s.} \qquad (A1.9)$$

Now by (A1.2) and (A1.9) we have

$$\lim_{n\to\infty}\int_{T_n}^{\tau} e^{-\gamma t}U(c_t)dt = 0, \quad \text{a.s.} \qquad (A1.10)$$

Since

$$\left|\int_{T_n}^{\tau} e^{-\gamma t}U(c_t)dt\right| \leq \int_0^\infty e^{-\gamma t}|U(c_t)|dt, \qquad (A1.11)$$

thus by (A1.10), (A1.11), (A1.4) and the Dominated Convergence Theorem

$$\lim_{n\to\infty} E\Big[\int_{T_n}^{\tau} e^{-\gamma t}U(c_t)dt\Big] = 0. \qquad (A1.12)$$

We pull (A1.6), (A1.7), (A1.8) and (A1.12) together thus we have

$$E\Big[\int_0^\infty e^{-\gamma t}U(c_t)dt\Big] \leq \phi(x) - E\Big[e^{-\gamma\tau}\phi(X_\tau)1_{\{\tau<\infty\}}\Big] + E\Big[e^{-\gamma\tau}\tilde{V}(\alpha(X_\tau - F))1_{\{\tau<\infty\}}\Big]$$

$$\leq \phi(x),$$

where the last inequality is a consequence of (ii) and (iii). Now we finished proving (A1.1) It remains to show that the policy given by (3.6)-(3.8) is indeed optimal and (3.9) holds. Proposition 2.4 guarantees that $(\tau^*, \{(\pi_t^*, c_t^*),\ t<\infty\}) \in \mathcal{A}_1(x)$. We start with showing (3.9). If $\mu \neq r$ then up to $\tau^*$ the wealth process corresponding to our candidate optimal policy is evolving according to the diffusion (3.4) (see (2.1), (3.7), (3.8)). Proposition 5.5.22 of Karatzas & Shreve (1988) and relation (3.5) imply that with probability one each path of the diffusion $Y$ reaches the level $b$ in finite time. If $\mu = r$, i.e., $\theta = 0$ then the wealth process is deterministic and satisfies the ordinary differential equation

$$dX_t = \Big[rX_t - d - I(\phi'(X_t))\Big]dt,$$

and now the second part of condition (vi) guarantees that $X$ reaches the level $b$ in finite (deterministic) time.

Now we proceed similarly to the first part of the proof and see that by (3.9), (2.5) and condition (iii)

$$E\Big[\int_0^\infty e^{-\gamma t}U(c_t^*)\Big]dt = E\Big[\int_0^{\tau^*} e^{-\gamma t}U(c_t^*)dt\Big] + E\Big[e^{-\gamma\tau^*}\phi(b)\Big]. \qquad (A1.13)$$

By Ito's rule

$$e^{-\gamma\tau^*}\phi(X_{\tau^*}) - \phi(x)$$

$$= \int_0^{\tau^*} e^{-\gamma t}\Big\{-\gamma\phi(X_t) + \phi'(X_t)\Big[\pi_t^*(\mu - r) + rX_t - d - c_t^*\Big] + \frac{1}{2}\phi''(X_t)\sigma^2(\pi_t^*)^2\Big\}dt$$

$$+ \int_0^{\tau^*} e^{-\gamma t}\phi'(X_t)\pi_t^*\sigma dw_t. \tag{A1.14}$$

The selection of $\pi_t^*, c_t^*$ and condition (iv) imply that on $\{t \le \tau^*\}$

$$0 = \mathcal{D}\phi(X_t) = \mathcal{D}^{(\pi_t^*, c_t^*)}\phi(X_t), \tag{A1.15}$$

thus by (A1.14), (A1.15) and (3.1) we have

$$e^{-\gamma\tau^*}\phi(X_{\tau^*}) - \phi(x) = -\int_0^{\tau^*} e^{-\gamma t}U(c_t^*)dt + \int_0^{\tau^*} e^{-\gamma t}\phi'(X_t)\pi_t^*\sigma dw_t, \tag{A1.16}$$

and (A1.13),(A1.16) now imply

$$E\Big[\int_0^{\infty} e^{-\gamma t}U(c_t^*)dt\Big] = \phi(x) + E\Big[\int_0^{\tau^*} e^{-\gamma t}\phi'(X_t)\pi_t^*\sigma dw_t\Big].$$

In order to complete the proof we need to show that

$$E\Big[\int_0^{\tau^*} e^{-\gamma t}\phi'(X_t)\pi_t^*\sigma dw_t\Big] = 0,$$

for which it is sufficient to show that

$$E\Big[\int_0^{\tau^*}\Big(e^{-\gamma t}\phi'(X_t)\pi_t^*\Big)^2 dt\Big] < \infty.$$

We define the functions $f, g : [b, \infty) \mapsto \Re$ by

$$f(x) = -2\theta\frac{\phi'}{\phi''}(x) + rx - d - I(\phi'(x)), \quad x \ge b \tag{A1.17}$$

and

$$g(x) = \frac{r-\mu}{\sigma}\frac{\phi'}{\phi''}(x), \quad x \ge b, \tag{A1.18}$$

and extend both functions to $\Re$ by $f(x) = e^{x-b}f(b)$ and $g(x) = g(b)$ for $x \le b$. From condition (v) follows that both $f(\cdot)$ and $g(\cdot)$ are Lipschitz-continuous on $\Re$. We want to apply Lemma A4.1 of Appendix A4, so we need to bound $xf(x)$ and $g^2(x)$. Assumption (3.2) guarantees that for every $x \ge b$

$$xf(x) \le -2\theta x\frac{\phi'}{\phi''}(x) + rx^2 \le 2\theta x(Kx + K_1) + rx^2 \le x^2\Big(2\theta(K_1 + K) + r\Big) + 2\theta K_1.$$

The function $x \mapsto xf(x)$ is bounded on $(-\infty, b]$, thus for some constant $A_1$ we have

$$xf(x) \le x^2\Big(2\theta(K_1+K)+r\Big)+A_1; \quad x \in \Re.$$

Assumption (3.2) implies that for every $x \ge b$

$$g^2(x) \le 2\theta(Kx+K_1)^2 \le 4\theta K^2x^2 + 4\theta K_1^2.$$

The function $g^2(\cdot)$ is constant on $(-\infty, b]$ thus for some constant $A_2$

$$g^2(x) \le 4\theta K^2 x^2 + A_2; \quad x \in \Re.$$

It follows that with

$$K_3^2 = \max\Big\{2\theta(K_1+K)+r \ , \ 4\theta K^2\Big\}$$

and $A = A_1 + A_2$ we have

$$xf(x) \le K_3^2 x^2 + A; \quad x \in \Re \tag{A1.19}$$

and

$$g^2(x) \le K_3^2 x^2 + A; \quad x \in \Re. \tag{A1.20}$$

Now we consider the diffusion on $\Re$

$$dZ_t = f(Z_t)dt + g(Z_t)dw_t; \quad Z_0 = x \tag{A1.21}$$

(it differs from the diffusion $Y$ defined in (3.4) only insofar as the coefficients of $Y$ have been extended to $\Re$, so $Y$ "explodes" at the hitting time of the boundary $b$ whereas $Z$ happily diffuses further). Relations (A1.19), (A1.20), and the Lipschitz-continuity of $f(\cdot)$ and $g(\cdot)$ guarantee that (A1.21) has a unique strong solution. Relation (A1.19) is unusual (it is more standard to require that $f$ satisfies a linear growth condition similar to (A1.20)); see, however, Gihman & Skorohod (1972), Chapter 2, Remark 3. The paths of the processes $X$ and $Z$ coincide up to $\tau^*$. Lemma A4.1 guarantees that

$$EZ_t^2 \le e^{3K_3^2 t}\left(\frac{A}{K_3^2} + x^2\right); \quad t < \infty \ . \tag{A1.22}$$

By (3.7)

$$E\left[\int_0^{\tau^*}\left(e^{-\gamma t}\phi'(X_t)\pi_t^*\right)^2 dt\right] = \text{const} \times E\left[\int_0^{\tau^*} e^{-2\gamma t}\frac{(\phi'(Z_t))^4}{(\phi''(Z_t))^2}dt\right] \le$$

$$\le \text{const} \times (\phi'(b))^2 E\left[\int_0^{\tau^*} e^{-2\gamma t}\left(\frac{\phi'(Z_t)}{\phi''(Z_t)}\right)^2 dt\right].$$

Now (A1.22) and our assumptions (3.2), (3.3) guarantee that the above quantity is finite.

## A2 Appendix, Proof of Theorem 4.1

Conditions (i), (iii), and the first identity of (iv) follow obviously from our construction. We are going to verify the second relation of (iv). Notice that $\phi(F+\eta)$ is finite, because we assumed $\eta > 0$ whenever $\delta < 0$. One can see that

$$\mathcal{D}\phi(x) = -\theta\frac{(\phi')^2}{\phi''}(x) + \Big(rx - d - I(\phi'(x))\Big)\phi'(x) + U(I(\phi'(x)) - \gamma\phi(x).$$

Using the definition of $\phi(\cdot)$ on $(F+\eta, b)$ this becomes

$$\mathcal{D}\phi(x) = D^{-\delta}\alpha^{\delta}(x-F)^{\delta-1}(1-\delta)(d-rF)\left[\Big(\alpha^{\frac{\delta}{\delta-1}} - 1\Big)\frac{1}{\delta}(x-F)\frac{1}{d-rF} - \frac{D}{1-\delta}\right],$$

$$x \in (F+\eta, b)\ .$$

By the second inequality of (4.19) we have $d > rF$ hence we need to show only that the quantity in the brackets is non-positive. It is bounded above by

$$\Big(\alpha^{\frac{\delta}{\delta-1}} - 1\Big)\frac{1}{\delta}(b-F)\frac{1}{d-rF} - \frac{D}{1-\delta},$$

and using (4.15), (4.16) and (4.4) this can be written as

$$\frac{r-\theta\lambda}{r(\gamma-\delta r+\delta\theta\lambda)} - \frac{1}{\gamma - \frac{\delta}{1-\delta}\theta - r\delta}\ .$$

A bit of algebra shows that this will be non-positive if

$$\gamma > -\frac{\delta}{1-\delta}\Big(\frac{r}{\lambda} - \theta\Big),$$

which has been assumed in (4.19) (this assumption is active only when $\delta$ is negative).
In order to show (ii), by (iii) it is sufficient to show that

$$\phi'(x) \geq \frac{\partial}{\partial x}\tilde{V}(\alpha(x-F)); \quad x \geq b, \tag{A2.1}$$

Using (4.6) and (4.3), inequality (A2.1) becomes

$$C(x) \leq \frac{1}{D}\alpha^{\frac{\delta}{\delta-1}}(x-F); \quad x \geq b, \tag{A2.2}$$

and we note that equality holds in (A2.2) in $x = b$ (see (4.11)). Substituting $C(x) = c$ this becomes

$$c \leq \frac{1}{D}\alpha^{\frac{\delta}{\delta-1}}(X(c)-F); \quad c \geq C(b).$$

Notice that equality holds in $c = C(b)$ thus it suffices to show that

$$1 \leq \frac{1}{D}\alpha^{\frac{\delta}{\delta-1}} X'(c),$$

but substituting (4.8) one can immediately see that this is true.
Next we are going to show that

$$-\frac{\phi'(x)}{\phi''(x)} \leq \frac{1}{1-\delta}x + \frac{d}{\theta}; \quad x > b.$$

By (4.10)

$$-\frac{\phi'(x)}{\phi''(x)} = \frac{1}{1-\delta}\frac{C(x)}{C'(x)}, \tag{A2.3}$$

thus we need to show that

$$\frac{1}{1-\delta}\frac{C(x)}{C'(x)} \leq \frac{1}{1-\delta}x + \frac{d}{\theta}; \quad x > b.$$

Substituting $c = C(x)$ this becomes

$$\frac{1}{1-\delta}cX'(c) \leq \frac{1}{1-\delta}X(c) + \frac{d}{\theta}; \quad c > C(b).$$

Using (4.8) this becomes

$$0 \leq Bc^{\lambda(\delta-1)}\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta}; \quad c > C(b). \tag{A2.4}$$

The right-hand side is an increasing function of $c$ thus it is sufficient to show that (A2.4) holds in $c = C(b)$. By (4.11), (4.13) and (4.15)

$$\begin{aligned}
&BC(b)^{\lambda(\delta-1)}\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta} \\
&= BD^{\lambda(1-\delta)}\alpha^{\lambda\delta}(b-F)^{\lambda(\delta-1)}\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta} \\
&= \Big(b - \alpha^{\frac{\delta}{\delta-1}}(b-F) - \frac{d}{r}\Big)\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta} \\
&= \Big(F - b_0\frac{r}{d}F(1-\alpha^{\frac{\delta}{\delta-1}}) + b_0(1-\alpha^{\frac{\delta}{\delta-1}}) - \frac{d}{r}\Big)\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta} \\
&\geq \Big(b_0(1-\alpha^{\frac{\delta}{\delta-1}}) - \frac{d}{r}\Big)\Big(\frac{1}{1-\delta} + \lambda\Big) + \frac{d}{r(1-\delta)} + \frac{d}{\theta}
\end{aligned} \tag{A2.5}$$

We separate two cases now. If $\delta < 0$ then this is larger or equal than

$$-\frac{d}{r}\Big(\frac{1}{1-\delta}+\lambda\Big)+\frac{d}{r(1-\delta)}+\frac{d}{\theta}=d\Big(\frac{1}{\theta}-\frac{\lambda}{r}\Big)$$

which is positive by (4.17). If $\delta \in (0,1)$ then by (4.2) and (4.16) we have

$$b_0 \leq \frac{d(r-\theta\lambda)}{r\Big(\alpha^{\frac{\delta}{\delta-1}}-1\Big)\theta\Big(\frac{1}{1-\delta}+\lambda\Big)}.$$

It follows that the last expression of (A2.5) is bounded below by

$$-\left(\frac{d(r-\theta\lambda)}{r\theta\Big(\frac{1}{1-\delta}+\lambda\Big)}+\frac{d}{r}\right)\Big(\frac{1}{1-\delta}+\lambda\Big)+\frac{d}{r(1-\delta)}+\frac{d}{\theta},$$

and this expression is algebraically equal to zero. We have now proven that (3.2) holds with $K=\frac{1}{1-\delta}$ and $K_1=\frac{d}{\theta}$. Our assumption (4.19) implies that (3.3) is also satisfied. Next we show that $-\Big(\phi'/\phi''\Big)(\cdot)$ is Lipschitz-continuous on $(b,\infty)$. By (A2.3) it suffices to show that $\Big(C/C'\Big)(\cdot)$ has bounded derivatives on $(b,\infty)$, i.e., that

$$1-\frac{C(x)C''(x)}{(C'(x))^2}$$

is bounded on $(b,\infty)$. After substitution of $c=C(x)$ this boils down to showing that the function

$$c \mapsto \frac{cC''(X(c))}{\Big(C'(X(c))\Big)^2} \tag{A2.6}$$

is bounded on $(C(b),\infty)$ (the function $C(\cdot)$ is increasing and convex). We note that

$$C'(X(c))=\frac{1}{X'(c)}$$

and

$$C''(X(c))=-\frac{X''(c)}{(X'(c))^3},$$

thus (A2.6) becomes $-c\frac{X''(c)}{X'(c)}$, and a substitution of (4.8) immediately shows that this expression is bounded (on the entire $(0,\infty)$). In order to show that $I(\phi'(\cdot))=C(\cdot)$ is Lipschitz-continuous on $(b,\infty)$ it is sufficient to show that $C'(x)$ is bounded on $(b,\infty)$, but this follows from the fact that $1/X'(c)$ is bounded on the entire $(0,\infty)$ by $1/D$.

Finally we show that (vi) is also satisfied. We start with the case $\mu \neq r$. The drift and diffusion coefficients in (3.4) are (see also (A1.17), (A1.18), (A2.3))

$$f(x) = -2\theta \frac{\phi'}{\phi''}(x) + rx - d - I(\phi'(x)) = \frac{2\theta}{1-\delta} \frac{C(x)}{C'(x)} + rx - d - C(x); \quad x > b$$

and

$$g(x) = \frac{r-\mu}{\sigma} \frac{\phi'}{\phi''}(x) = \frac{\mu - r}{\sigma(1-\delta)} \frac{C(x)}{C'(x)}; \quad x > b.$$

The scale-function corresponding to the diffusion given by (3.4) is

$$s(x) = \int_a^\infty \exp\Big\{-2\int_a^y \frac{f(z)}{g^2(z)} dz\Big\} dy, \quad x > b,$$

where $a > b$ is an arbitrary constant. It follows that in order to prove (3.8) it suffices to show that

$$\frac{f(x)}{g^2(x)} \leq \frac{1}{2x}$$

for sufficiently large values of $x$. Using the definitions of $f, g$ and substituting $c = C(x)$ this becomes

$$\frac{\frac{2\theta}{1-\delta} cX'(c) + rX(c) - d - c}{\frac{2\theta}{(1-\delta)^2} c^2 (X'(c))^2} \leq \frac{1}{2X(c)},$$

which should hold for sufficiently large values of $c$. This inequality becomes

$$X(c)\Big[\frac{2\theta}{1-\delta} cX'(c) + rX(c) - d - c\Big] \leq \frac{\theta}{(1-\delta)^2} c^2 (X'(c))^2.$$

It is sufficient to show that the coefficient of $c^2$ on the left-hand side is larger than that on the right-hand side, i.e.,

$$\frac{2\theta}{1-\delta} D^2 + rD^2 - D < \frac{\theta}{(1-\delta)^2} D^2.$$

However, substituting (4.4) this becomes $\theta + r < \gamma$, and this follows from assumption (4.19).
If $\mu = r$ then condition (vi) becomes $rx - C(x) < d$ for all $b \leq x$, which amounts to $rX(c) - c < d$ for all $c \geq C(b)$, and this follows from our assumption $\gamma > r$ (see (4.19), (4.8), and recall that now $\theta = 0$).

### A3 Appendix , Proof of Theorem 5.1

**Proof:** Most parts of this proof can be transplanted in a straightforward manner from the corresponding computations in Appendix A2. Formula (4.17) and our assumption (5.13) imply $d/r > F$. The only essentially different derivation in the logarithmic case is the proof of conditions (3.2) and (3.3), so we include this derivation here. We are going to show that

$$-\frac{\phi'(x)}{\phi''(x)} \leq x + K_1 \ , \quad x \in (b, \infty)$$

where $K_1$ is given in (5.15). By (5.6) this amounts to showing that

$$\frac{C(x)}{C'(x)} \leq x + K_1 \ , \quad x \in (b, \infty)$$

i.e.,

$$cX'(c) \leq X(c) + K_1 \ , \quad c \geq C(b).$$

By (5.5) this becomes

$$0 \leq c^{-\lambda} B(1+\lambda) + \frac{d}{r} + K_1 \ , \quad c \geq C(b) \ .$$

The right-hand side of this inequality is increasing in $c$ so it suffices to show that the inequality holds in $c = C(b)$. By (5.8) and (5.9)

$$\Big(C(b)\Big)^{-\lambda} B(1+\lambda) + \frac{d}{r} + K_1 = \gamma^{-\lambda}(b-F)^{-\lambda} B(1+\lambda) + \frac{d}{r} + K_1 = -\lambda\Big(\frac{d}{r} - F\Big) + F + K_1$$

which is indeed non-negative by the definition of $K_1$. Now (3.3) follows from our assumption (5.14).

## A4 Appendix

We shall consider the diffusion

$$dZ_t = f(Z_t)dt + g(Z_t)dw_t \ , \quad Z_0 = z \tag{A4.1}$$

where the measurable functions $f(\cdot)$ and $g(\cdot)$ satisfy the following growth and Lipschitz conditions:

$$xf(x) \leq K_3 x^2 + A \ , \quad x \in \Re \tag{A4.2}$$

$$g^2(x) \leq K_3^2 x^2 + A, \quad x \in \Re \tag{A4.3}$$

and

$$|f(x) - f(y)| + |g(x) - g(y)| \leq K_2 |x - y| \ , \quad x, y \in \Re \tag{A4.4}$$

for some positive constants $K_2$, $K_3$. It is known that under these conditions (A4.1) has a unique strong solution and the second moment $EZ_t^2$ is bounded by $(1+z^2)\exp\{\text{constant}\times t\}$ (see Gihman & Skorohod (1972), Chapter 2, Theorems 3,4 and Remark 3). In this paper we need to identify the constant in the exponent.

**A4.1 Lemma:** Let $\{Z_t,\ t \geq 0\}$ be the unique strong solution of (A4.1) where the functions $f(\cdot)$ and $g(\cdot)$ satisfy conditions (A4.2)-(A4.4). Then

$$EZ_t^2 \leq \exp\{3K_3^2 t\}\Big(\frac{A}{K_3^2} + z^2\Big) \ . \qquad (A4.5)$$

**Proof:** We are going to imitate the proof of Theorem 4 in Gihman & Skorohod (1972), Chapter 2, paying extra attention to the constants. For every $n \geq 1$ we set $z_n = z$ if $|z| \leq n$ and $z_n = n$ sign $z$ if $|z| > n$. We define $f_n(x) = f(x)$ for $|x| \leq n$ and $f_n(x) = f(n \text{ sign } x)$ for $|x| > n$. Similarly, we set $g_n(x) = g(x)$ for $|x| \leq n$ and $g_n(x) = g(n \text{ sign } x)$ for $|x| > n$. Let $\{Z_n(t),\ t \geq 0\}$ be the unique strong solution of

$$dZ_n(t) = f_n(Z_n(t))dt + g_n(Z_n(t))dw_t \ , \quad Z_n(0) = z_n \ .$$

By Ito's Lemma

$$Z_n^2(t) = z_n^2 + \int_0^t 2Z_n(s)f_n(Z_n(s))ds + \int_0^t g_n^2(Z_n(s))ds + \int_0^t 2Z_n(s)g_n(Z_n(s))dw_s \quad (A4.6)$$

Taking expectations in (A4.6) we get

$$EZ_n^2(t) = z_n^2 + \int_0^t E\Big[2f_n(Z_n(s))Z_n(s) + g_n^2(Z_n(s))\Big]ds \leq z_n^2 + 3K_3^2\int_0^t E\left[Z_n^2(s)\right]ds + 3At \ .$$

By the Gronwall inequality (Gihman & Skorohod (1972), Ch. 2, Lemma 1)

$$EZ_n^2(t) \leq z_n^2 + 3At + 3K_3^2\int_0^t e^{3K_3^2(t-s)}(3As + z_n^2)ds = e^{3K_3^2 t}\Big(\frac{A}{K_3^2} + z_n^2\Big) - \frac{A}{K_3^2}$$

$$\leq e^{3K_3^2 t}\Big(\frac{A}{K_3^2} + z_n^2\Big) \ .$$

The fact that $Z_n(t) \to Z_t$, almost surely as $n \to \infty$ (see the proof of Theorem 3 in Gihman & Skorohod (1972), Chapter 2) and Fatou's Lemma imply (A4.5).

We proceed to prove Proposition 2.4.

**Proposition 2.4, proof:** Let $(\tau, \{(\pi_t, c_t),\ t < \infty\}) \in \mathcal{A}(x)$ such that $\{(\pi_t, c_t),\ \tau < t < \infty\} = \{(\tilde{\Pi}(X_t), \tilde{C}(X_t)),\ \tau < t < \infty\}$. Karatzas et al. (1986) guarantees that for every $(t, y) \in [0, \infty) \times [F + \eta, \infty)$

$$E\Big[\int_t^\infty e^{-\gamma s}U(c_s)ds \mid \tau = t,\ X_\tau = y\Big] = e^{-\gamma t}\tilde{V}(\alpha(y - F)) \ .$$

Integrating this inequality on $[0, \infty) \times [F + \eta, \infty)$ with respect to the law of $(\tau, X_\tau)$ gives (2.5). Next let $(\tau, \{(\pi_1(t), c_1(t)),\ t < \infty\}) \in \mathcal{A}(x)$ arbitrary. We define

$$\pi_2(t) = \pi_1(t)1_{\{t \leq \tau\}} + \tilde{\Pi}(X_t^{(2)})1_{\{t > \tau\}}$$

and

$$c_2(t) = c_1(t) 1_{\{t \le \tau\}} + \tilde{C}(X_t^{(2)}) 1_{\{t > \tau\}} ,$$

where the superscript (2) above the $X$ indicates that it is the wealth process corresponding to $(\tau, \{(\pi_2(t), c_2(t)),\ t < \infty\})$. It develops that $(\tau, \{(\pi_2(t), c_2(t)),\ t < \infty\}) \in \mathcal{A}_1(x)$, and for every $(t, y) \in [0, \infty) \times [F + \eta, \infty)$

$$E\Big[\int_t^\infty e^{-\gamma s} U(c_2(s)) ds \mid \tau = t,\ X_\tau = y\Big] \ge E\Big[\int_t^\infty e^{-\gamma s} U(c_1(s)) ds \mid \tau = t,\ X_\tau = y\Big] .$$

In this inequality we dropped the superscript (2) from $X_\tau$ since the wealth processes corresponding to the two policies coincide up to $\tau$. Integrating this inequality on $[0, \infty) \times [F + \eta, \infty)$ with respect to the law of $(\tau, X_\tau)$ gives

$$E\Big[\int_\tau^\infty e^{-\gamma s} U(c_2(s)) ds\Big] \ge E\Big[\int_\tau^\infty e^{-\gamma s} U(c_1(s)) ds\Big] ,$$

which implies (2.6).

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