

# Optimal Investment and Consumption Decisions when Time-Horizon is Uncertain

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May 11, 2003

## Abstract

Many investors do not know with certainty when their portfolio will be liquidated. Should their portfolio selection be influenced by the uncertainty of exit time? In order to answer that question, we consider a suitable extension of the familiar optimal investment problem of Merton (1971), where we allow the conditional distribution function of an agent's time-horizon to be stochastic and correlated to returns on risky securities. In contrast to existing literature, which has focused on an independent time-horizon, we show that the portfolio decision is affected. For CRRA preferences, we also show that a solution formally similar to the one obtained in the case of a constant time-horizon can be recovered at the cost of a suitable adjustment to the drift process of the risky assets.

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## Abstract

Many investors do not know with certainty when their portfolio will be liquidated. Should their portfolio selection be influenced by the uncertainty of exit time? In order to answer that question, we consider a suitable extension of the familiar optimal investment problem of Merton (1971), where we allow the conditional distribution function of an agent's time-horizon to be stochastic and correlated to returns on risky securities. In contrast to existing literature, which has focused on an independent time-horizon, we show that the portfolio decision is affected. For CRRA preferences, we also show that a solution formally similar to the one obtained in the case of a constant time-horizon can be recovered at the cost of a suitable adjustment to the drift process of the risky assets.

There is an interesting discrepancy between finance theory and practice. On the one hand, most of standard financial economics is based on the assumption that, at the moment of making an investment decision, an investor knows with certainty the time of eventual exit. Such an assumption can be traced back to the origins of modern financial economics, and, in particular, to the development of portfolio selection theory by Markowitz (1952). On the other hand, most investors would acknowledge the fact that, upon entering the market, they never know with certainty the time of exiting the market. Factors which can potentially affect the time of exit are, for example, securities markets behavior, changes in the opportunity set, uncertainty of order execution time, changes in an investor's endowment, or time of an exogenous shock to an investor's consumption process (e.g., purchasing or selling of a house), for example.

Since it is obvious that an investment horizon is hardly ever known with certainty at the date when the initial investment decisions are made, it is both of practical and theoretical interest to develop a comprehensive theory of optimal investment and consumption under uncertain time horizon. One would expect uncertainty over time-of-exit to be a serious complication, because, as underlined in the afore-mentioned examples, it can be in general dependent on risky securities returns. Existing literature, however, has only focused on an independent time-horizon, and, as a result, has led to the misleading conclusion that the optimal portfolio selection is essentially not affected by uncertainty over exit time. Research on the subject actually started as early as Yaari (1965), who addresses the problem of optimal consumption for an individual with uncertain date of death, in a simple setup with a pure deterministic investment environment. Hakansson (1969, 1971) extends this work to a discrete-time setting under uncertainty including risky assets. Merton (1971), as a special case, also addresses a dynamic optimal portfolio selection problem for an investor retiring at an uncertain date, defined as the date of the first jump of an independent Poisson process with constant intensity. This was done in a continuous-time setting with no bequest motive. In a subsequent work, Richard (1975) generalizes these results to the presence of life insurance. More recently, Martellini and Urošević (1999) extend Markowitz analysis to the situations involving an uncertain independent time of exit. In all these papers, the random time-horizon is assumed to be independent of all other sources of uncertainty. To the best of our knowledge, the only exception is Karatzas and Wang (2001), who solve the optimal dynamic investment problem in the case of complete markets and when the uncertain time horizon is a stopping time of asset price filtration. This is also an extremely stylized assumption, which in essence states that randomness of time-horizon, fully dependent upon asset prices, induces no new uncertainty in the economy.<sup>1</sup>

In this paper, we attempt to cover some of the ground between these two extreme assump-

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<sup>1</sup>Somewhat related also are recent papers by Collin-Dufresne and Hugonnier (2001) and El Karoui and Martellini (2002) who study the problem of valuation and hedging of cash-flows affected by some event occurring at a random time.

tions, an independent time-horizon on the one hand, and a time-horizon that is a stopping time of the asset price filtration on the other hand. Our results can be summarized as follows. We first provide an elegant sufficient condition for optimality in the presence of an uncertain time-horizon. We then apply that result to solve the optimal investment problem in a set up with CRRA preferences, constant expected return and drift parameters, and a deterministic distribution function of time-horizon. In that framework, we confirm and extend a result obtained by Merton (1971) and Richard (1975), as we show that the optimal portfolio selection is not affected by the presence of an uncertain time-horizon, even though the value function is not identical to the one corresponding to the standard fixed-horizon case. One of our contribution is to obtain explicit solutions for the optimal strategy and the wealth process as a function of the distribution of the uncertain-time horizon in the case of CRRA utility functions. We also consider the case of an economy with an infinite time span. We then strongly depart from existing literature by relaxing the assumption of an independent time-horizon. In a model where the conditional distribution function of the random time-horizon is assumed to follow an Itô process correlated to stock returns, we show that a serious complication occurs as the portfolio decision is affected. For CRRA preferences, we show, however, that a solution formally similar to the one obtained in the case of a constant time-horizon can be recovered at the cost of a suitable adjustment to the drift process of the risky assets. We find that, if the probability of exiting the market increases (respectively, decreases) with the return on the risky asset, then the fraction invested in the risky asset is lower than (respectively, greater than) in the case of a certain time-horizon. Our results have natural interpretations and important potential implications for optimal investment decisions when exit time can be impacted by returns on risky securities.

The rest of the paper is organized as follows. In section 1, we introduce the model of an economy with an uncertain time-horizon. Section 2 is devoted to the problem of optimal dynamic investment decision in the presence of an uncertain time-horizon when the conditional probability of exiting is deterministic. In section 3, we present explicit solutions in the case of CRRA utility functions. In section 4, we discuss the case of a non bounded time-horizon, and we introduce consumption in section 5. In section 6, we extend the setup to a stochastically time-varying conditional distribution of the time-horizon. Our conclusions are presented in section 7.

## 1 The Economy

In this section, we introduce a general model for the economy in the presence of an uncertain time-horizon. Let  $[0, T]$ , with  $T \in \mathbb{R}_*^+$ , denote the (finite) time span of the economy. Uncertainty in the economy is described through a probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  on which is defined a

$n$ -dimensional standard Brownian motion  $W$ .

## 1.1 Asset Prices

We consider  $n$  risky assets, the prices of which,  $S_t^i$ ,  $i = 1, \dots, n$ , are given by

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j), \quad i = 1, \dots, n \quad (1)$$

We shall sometimes use the shorthand notation  $\mu = (\mu^i)_{i=1, \dots, n}$  and  $\sigma = (\sigma^{i,j})_{i,j=1, \dots, n}$ . A risk-free asset is also traded in the economy. The return on that asset, typically a default free bond, is given by  $\frac{dB_t}{B_t} = r_t dt$ , where  $r$  is the risk-free rate in the economy. Agents' basic information set is captured by the filtration  $\mathbb{F} = \{\sigma(S_s, s \leq t); t \geq 0\}$ , with  $\mathcal{F}_\infty \subset \mathcal{A}$  and  $\mathcal{F}_0$  is trivial. We also assume that:

- (i) the coefficients  $\mu$ ,  $r$  are bounded and deterministic and  $r_t \geq 0$ ,
- (ii) the coefficient  $\sigma$  is bounded, invertible, deterministic and the inverse  $\sigma^{-1}$  is also a bounded function,
- (iii)  $W = (W^i)_{i=1, \dots, n}$  is an  $\mathbb{F}$ -Brownian motion.

Under these assumptions, the market is arbitrage-free (see for example Karatzas (1996)). We denote by  $\mathbb{P}^0$  the equivalent martingale measure which the Radon-Nikodym density with respect to  $\mathbb{P}$ ,  $Z$ , is the solution to  $dZ_t = -Z_t \theta_t dW_t$ , where  $\theta_t = \sigma_t^{-1}(\mu_t - \mathbf{1}r_t)$ , where  $\mathbf{1}$  is a  $n$ -dimensional vector of ones.

A  $n$ -dimensional process  $\pi$  is said to be a *weak admissible* strategy if  $\pi$  is an  $\mathbb{F}$ -predictable almost surely square integrable process, i.e., if  $\sum_{i=1}^n \int_0^T (\pi_s^i)^2 ds < +\infty$ , almost surely. Consider now an investor who uses a weak admissible strategy  $\pi$ , i.e., invests the amount  $\pi_s^i$  in each of the risky securities at date  $s$ . If the strategy  $\pi$  is used in a self-financing way, i.e., if the wealth invested in the riskless asset is  $X_s - \sum_{i=1}^n \pi_s^i$ , then the wealth process  $(X_s^{t,\pi,x}, t \leq s \leq T)$ , with  $X_t^{t,\pi,x} = x$ , evolves according to the following stochastic differential equation

$$dX_s^{t,\pi,x} = X_s^{t,\pi,x} r_s ds + \pi_s [(\mu_s - \mathbf{1}r_s) ds + \sigma_s dW_s]. \quad (2)$$

**Definition 1** A portfolio  $(\pi_t, 0 \leq t \leq T)$  is said to be *admissible* if it is a weak admissible strategy and if the associated wealth process is non negative.

Let  $P(t, x)$  denote the set of admissible portfolios starting from a wealth level  $x$  at date  $t$

$$P(t, x) = \{\pi_u, t \leq u \leq T, X_t^\pi = x, \pi \text{ is weak admissible and } X_u^\pi \geq 0\}$$

where  $X_u^\pi := X_u^{t,\pi,x}$ .

## 1.2 Timing Uncertainty

In this paper, we assume that an agent’s time-horizon  $\tau$ , i.e., “the maximum length of time for which the investor gives any weight in his utility function” (Merton (1975)), is a positive random variable measurable with respect to the sigma-algebra  $\mathcal{A}$ .

Importantly, we do *not* assume that  $\tau$  is a stopping time of the filtration  $\mathbb{F}$  generated by asset prices. In other words, we do not assume that observing asset prices up to date  $t$  implies full knowledge about whether  $\tau$  has occurred or not by time  $t$ . Formally, it means that there are some dates  $t \geq 0$  such that the event  $\{t < \tau\}$  does not belong to  $\mathcal{F}_t$ . When  $\tau$  is an  $\mathbb{F}$ -stopping time, e.g., the first hitting time of a deterministic barrier by asset prices, it is possible, although sometimes difficult, to apply the standard tools of dynamic valuation and optimization problems (see Karatzas and Wang (2001)). In this paper, we are instead interested in situations such that the presence of an uncertain time-horizon induces some *new* uncertainty in the economy.<sup>2</sup>

There are two sources of uncertainty related to optimal investment in the presence of an uncertain time-horizon, one stemming from the randomness of prices (market risk), the other stemming from the randomness of the timing of exit  $\tau$  (timing risk). A serious complication, which we explicitly address in Section 4, is that, in general, these two sources of uncertainty are not independent. Separating out these two sources of uncertainty is a useful operation that may be achieved as follows. Conditioning upon  $\mathcal{F}_t$  allows one to isolate a pure timing uncertainty component. Since  $\mathcal{F}_t$  contains information about risky asset prices up to time  $t$ ,  $\mathbb{P}[\tau > t | \mathcal{F}_t]$ , for example, is the probability that the agent has not reached his time-horizon at date  $t$ , given all possible information about asset prices. We denote by  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ , the conditional distribution function of timing uncertainty.

We further make the following assumption.

**[G]** :  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$  is an increasing absolutely continuous process with respect to Lebesgue measure, with a density denoted by  $f$ , e.g.,  $F_t = \int_0^t f_s ds$ .

**Remark 1** *A sufficient condition for (G) to hold is  $\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \mathbb{P}(\tau \leq t | \mathcal{F}_\infty)$ , i.e., when  $\tau$  is modelled as a Cox process. On the other hand, one can find examples such that (G) holds and  $\mathbb{P}(\tau \leq t | \mathcal{F}_t) \neq \mathbb{P}(\tau \leq t | \mathcal{F}_\infty)$  (cf. Yor (2002)).*

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<sup>2</sup>This does not imply that a random time under consideration in this paper may not be dependent upon asset prices behaviors. On the contrary, it is, in general, dependent upon asset prices, yet not depend *only* on asset prices as in the case of a stopping time of the asset price filtration.

### 1.3 Preferences

We first consider a case with no intermediate consumption. Interestingly enough, this case happens to be more challenging than the case with intermediate consumption.<sup>3</sup> The agent's preferences are captured by an utility function  $U$ , which is a continuous, strictly increasing, strictly concave and continuously differentiable function defined on  $(0, \infty) \rightarrow \mathbb{R}$ , satisfying the following two conditions:  $\lim_{x \rightarrow +\infty} U'(x) = 0$  and  $\lim_{x \rightarrow 0} U'(x) = +\infty$ . Under these assumptions, the function  $U'$  is invertible; the inverse function is denoted as  $I$  defined on  $\mathbb{R}_*^+$ . The agent's portfolio problem choice problem is to find an admissible strategy  $\pi$  which maximizes the expected utility of terminal wealth

$$V(x) = \sup_{\pi \in P(0,x)} \mathbb{E}[U(X_{\tau \wedge T}^{\pi,x})] \quad (3)$$

where  $\tau$  is an agent uncertain time-horizon, e.g., the date of death of the agent.

Using  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t)$ , and a result from Dellacherie (1972), this problem can be re-written as

$$V(x) = \sup_{\pi \in P(0,x)} \mathbb{E}\left[\int_0^\infty U(X_{t \wedge T}^{\pi,x}) dF_t\right] = \sup_{\pi \in P(0,x)} \mathbb{E}\left[\int_0^T U(X_u^{\pi,x}) dF_u + U(X_T^{\pi,x})(1 - F_T)\right]$$

## 2 Deterministic Probability of Exiting

In this section, we consider the case where the process  $(F_t, t \geq 0)$  is a deterministic function, hence equals the cumulative function of  $\tau$ , with a derivative  $f$ . Note that a sufficient condition for the conditional distribution  $F_t$  to be a deterministic function of time is to have  $\tau$  independent of  $\mathbb{F}$ , the case on which previous literature has focused on. As in the case of fixed time-horizon, the problem can be solved either using dynamic programming or the martingale/duality approach to utility maximization (Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987)).

### 2.1 Dynamic Programming Approach

Using the density  $f$  of the law of  $\tau$ , we may re-write the value function at the initial date as

$$V(0, x) = \sup_{\pi \in P(0,x)} \mathbb{E}\left[\int_0^\infty f(t)U(X_{t \wedge T}^{\pi,x})dt\right]$$

Let us introduce the value function  $V$  at any time  $t$

$$V(t, x) = \max \mathbb{E}\left\{\int_t^T ds f(s)U(X_s^{t,\pi,x}) + (1 - F(T))U(X_T^{t,\pi,x})\right\} \quad (4)$$

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<sup>3</sup>In the latter case, one can actually show that the problem with uncertain time-horizon can be mapped into a standard problem with infinite time-horizon (see Merton (1971) and section 5 of this paper).

**Lemma 1** For all  $t$  in  $[0, T]$ , the function  $V(t, \cdot)$  is increasing and strictly concave.

**Proof.** If  $x \leq y$ , then  $P(t, x) \subset P(t, y)$  and the result is based on the properties of increasing and concavity of the function  $U$  and the linearity property of  $X$  in  $x$ . ■

We recall some facts on dynamic programming (cf. Fleming-Soner p. 163 theorem 3.1).

**Theorem 1** *i) If  $Y$  is the  $C^{1,2}$  solution to the following Hamilton-Jacobi-Bellman equation*

$$0 = f(t)U(x) + \left( \frac{\partial Y(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^d} A(t, x, \pi) \right) \quad (5)$$

where  $A(t, x, \pi) = [xr_t + \pi(\mu_t - r_t)]Y_x(t, x) + \frac{1}{2}\pi^2\sigma_t^2Y_{xx}(t, x)$  and  $Y_x$  (resp.  $Y_{xx}$ ) denotes the first (resp. second) derivative with respect to the space variable, the boundary condition being

$$Y(T, x) = U(x)(1 - F(T)),$$

then  $V = Y$ .

*ii) If the conditions of (i) hold, the optimal portfolio strategy is given by*

$$\pi_t = -\sigma_t^{-1}\theta_t \frac{V_x(t, x)}{V_{xx}(t, x)}$$

These conditions are the usual optimality conditions. One still needs to solve equation (5) subject to the appropriate boundary condition to obtain an explicit solution to the problem of optimal investment when time-horizon is uncertain, which is not easy in general. The martingale approach to optimal investment problem allows us to provide an explicit characterization of the optimal wealth process.

## 2.2 Martingale Approach - Sufficient Condition

The following theorem, which is the main result of this section, provides a sufficient condition for optimality.

**Theorem 2** Define  $I$  as the inverse of the first derivative of the utility function, i.e.,  $I(x) = (U')^{-1}(x)$ . If  $I$  is  $C^2$  and if there exists a deterministic function  $\nu$  satisfying  $I(\nu(0)) = x$  such that the process  $(H_t I(\nu(t)H_t), t \geq 0)$  is a martingale, then the wealth process  $X^*$  defined by  $X_t^* = I(\nu(t)H_t)$  is optimal and the portfolio strategy  $\pi^*$  is defined by

$$\pi_t^* = -I'(\nu(t)H_t)\nu(t)H_t\sigma_t^{-1}\theta_t$$

where  $H_t = \exp(-\int_0^t r_s ds)Z_t$ .

**Proof.** Let  $X$  be a non negative wealth process with initial value  $x$ . The concavity of  $U$  implies that

$$\mathbb{E}[U(X_{\tau \wedge T}) - U(X_{\tau \wedge T}^*)] \leq \mathbb{E}[(X_{\tau \wedge T} - X_{\tau \wedge T}^*)U'(X_{\tau \wedge T}^*)]$$

Now,

$$\begin{aligned} & \mathbb{E}[(X_{\tau \wedge T} - X_{\tau \wedge T}^*)U'(X_{\tau \wedge T}^*)] \\ &= \mathbb{E}\left[\int_0^T f(t)(X_t - X_t^*)\nu(t)H_t dt + (1 - F(T))(X_T - X_T^*)\nu(T)H_T\right] \\ &= \int_0^\infty dt f(t)\nu(t)E[(X_t - X_t^*)H_t] + (1 - F(T))\nu(T)E[(X_T - X_T^*)H_T] \\ &\leq 0 \end{aligned}$$

since  $HX^*$  is a martingale and  $HX$  a supermartingale with same initial value  $x$  (and using the fact that  $f$  and  $\nu$  are two deterministic functions of time). The optimal portfolio  $\pi^*$  is obtained by applying Itô's lemma to the process  $I(\nu(t)H_t)$  and identifying the  $dW$  terms. ■

That result provides an explicit characterization of the optimal strategy. One needs, however, to first check the conditions for existence of such a function  $\nu$ . It is in general difficult to give a closed-form solution for  $\nu$ . One can actually show that for constant coefficients only CRRA utility functions are consistent with the existence of  $\nu$ .

**Proposition 1** *If  $r \neq 0$  and if the coefficients  $\mu, r, \sigma$  are constant, the existence of a differentiable function  $\nu$  satisfying  $I(\nu(0)) = x$  such that the process  $H_t I(\nu(t)H_t)$  is a martingale can only be obtained for logarithmic and power utility functions  $U$ .*

**Proof.** To see this, first apply Itô's formula to the process  $H_t I(\nu(t)H_t)$  for a deterministic function  $\nu$  to obtain (we denote by  $\xi_t = \nu(t)H_t$ )

$$\begin{aligned} dH_t I(\xi_t) &= H_t \theta [-\xi_t I'(\xi_t) - I(\xi_t)] dW_t \\ &\quad + \{-r H_t I(\xi_t) + I'(\xi_t) \xi_t H_t (\theta^t \theta - r) + H_t^2 I'(\xi_t) \nu'(t) + \frac{1}{2} \xi_t^2 H_t \theta^t \theta I''(\xi_t)\} dt \end{aligned}$$

The martingale condition implies that (after dividing by  $H$ )

$$-r I(\xi_t) + I'(\xi_t) \xi_t (\theta^t \theta - r) + \frac{1}{2} \xi_t^2 \theta^t \theta I''(\xi_t) + I'(\xi_t) \xi_t \frac{\nu'(t)}{\nu(t)} = 0 \quad (6)$$

For a fixed  $t$ ,  $\{\xi_t = \nu(t)H_t(\omega), \omega \in \cdot\}$  describes the set  $\mathbb{R}^+$ . So, if the coefficients  $r, \mu, \sigma$  are constant, then the equation (6) has a deterministic solution  $\nu$  if and only if the function  $I$  satisfies the following equation

$$-r I(z) + \frac{1}{2} z^2 \theta^t \theta I''(z) = C z I'(z), \forall z \geq 0 \text{ and } \frac{\nu'}{\nu} = \text{constant}$$

where  $C$  is a constant. If  $r \neq 0$ , that equation is only satisfied for those functions  $I$  which are power functions, which corresponds to the logarithmic and power utility functions  $U$ . If  $r = 0$ , that equation is only satisfied for those functions  $I'$  which are power functions, which corresponds the power, logarithmic and the exponential utility functions  $U$ . ■

A generalisation of this result is done in El Karoui, Huang and Jeanblanc (2002). We give here this result.

**Proposition 2** a) *If the problem (3) admits a solution  $X^*$ , then  $X_t^* = I(\nu_t H_t)$  where  $\nu$  is an adapted process which satisfies*

i) *The process  $H_t I(\nu_t H_t)$  is a local martingale,*

(ii) *The random variable  $\int_0^T \nu_s f(s) ds + (1 - F(T)) H_T^{-1} U'(X_T^*)$  is a constant.*

b) *If there exists an adapted process  $\nu$  such that (i) and (ii) hold, then  $X_t = I(\nu_t H_t)$  is an optimal solution of problem (3).*

**Remark 2** *Before giving the proof, notice that the condition (ii) can be written as*

$$\int_0^T H_s^{-1} U'(X_s^*) f(s) ds + (1 - F(T)) H_T^{-1} U'(X_T^*)$$

*which is exactly the first order Lagrange equation. It can be directly obtained using duality principle.*

**Proof.** a) Setting  $\nu_t = H_t^{-1} U'(H_t X_t^*)$ , we obtain  $X_t^* = I(\nu_t H_t)$ , and since  $X_t^*$  is a wealth process,  $(H_t X_t^* = H_t I(\nu_t H_t); t \geq 0)$  is a local-martingale. Let  $X$  be another wealth process, i.e., a process such that  $(X_t H_t, t \geq 0)$  is a local-martingale with initial value  $x$ . The function  $\Phi$  defined as

$$\Phi(\epsilon) = E \left( \int_0^T U(\epsilon X_s^* + (1 - \epsilon) X_s) f(s) ds + (1 - F(T)) U'(\epsilon X_T^* + (1 - \epsilon) X_T) \right)$$

admits a maximum for  $\epsilon = 1$ . From

$$\begin{aligned} \Phi'(1) &= E \left( \int_0^T f(s) U'(X_s^*) (X_s^* - X_s) ds + (1 - F(T)) U'(X_T^*) (X_T^* - X_T) \right) \\ &= E \left( \int_0^T f(s) U'(X_s^*) H_s^{-1} (H_s X_s^* - H_s X_s) ds + (1 - F(T)) U'(X_T^*) H_T^{-1} (H_T X_T^* - H_T X_T) \right) \end{aligned}$$

and using the fact that  $(H_s X_s^* - H_s X_s, s \geq 0)$  is a martingale, we get

$$\Phi'(1) = E \left( H_T (X_T^* - X_T) \left[ \int_0^T U'(X_s^*) H_s^{-1} f(s) ds + (1 - F(T)) U'(X_T^*) H_T^{-1} \right] \right) = 0.$$

This equality holds for any  $X_T \in \mathcal{F}_T$  such that  $E(X_T H_T) = x$ . Setting

$$Z = \int_0^T U'(X_s^*) H_s^{-1} f(s) ds + (1 - F(T)) U'(X_T^*) H_T^{-1}$$

leads to  $E(H_T Y Z) = 0$  for any  $Y \in \mathcal{F}_T$  such that  $E(Y H_T) = 0$ , or  $E(\tilde{Y} Z) = 0$  for any  $\tilde{Y} \in \mathcal{F}_T$  such that  $E(\tilde{Y}) = 0$ . For  $\tilde{Y} = Z - E(Z)$ , we get  $E(Z^2) = E^2(Z)$ , therefore  $Z$  is a constant.

It remains to note that the definition of  $\nu$  leads to

$$Z = \int_0^T U'(X_s^*) H_s^{-1} f(s) ds + (1 - F(T)) U'(X_T^*) H_T^{-1} = \int_0^T \nu_s f(s) ds + (1 - F(T)) U'(I(\nu_T H_T)) H_T^{-1}.$$

b) Let  $X$  be a wealth process, with the initial wealth  $x$  and  $X_t^* = I(\nu_t H_t)$ . Then

$$\begin{aligned} & E \left( \int_0^T dt f(t) [U(X_t) - U(X_t^*)] + (1 - F(T))(U(X_T) - U(X_T^*)) \right) \\ & \leq E \left( \int_0^T dt f(t) U'(X_t^*) (X_t - X_t^*) + (X_T - X_T^*) (1 - F(T)) U'(X_T^*) \right) \\ & = E \left( \int_0^T dt f(t) \nu_t H_t (X_t - X_t^*) + (X_T - X_T^*) (1 - F(T)) U'(X_T^*) \right) \\ & = E \left( H_T (X_T - X_T^*) \left[ \int_0^T dt f(t) \nu_t + \frac{(1 - F(T)) U'(X_T^*)}{H_T} \right] \right) = 0. \end{aligned}$$

■

In what follows, we provide an explicit solution to the optimal dynamic investment problem in the presence of an uncertain time-horizon.

### 3 Examples

For simplicity, we focus on the case of a one-dimensional Brownian motion and constant coefficients  $\mu$ ,  $r$  and  $\sigma$ .<sup>4</sup>

**Proposition 3** *Let us assume that  $U(x) = \ln x$ . The value function and the optimal investment policy are respectively given by*

$$V(t, x) = p(t) + q(t) \ln x$$

where  $q(t) = 1 - F(t)$ ,  $p(t) = a[(T - t) - \int_t^T F(s) ds]$ ,  $a = r + \frac{1}{2}\theta^2$ , and

$$\begin{aligned} X_t^* &= \frac{x}{H_t} \\ \pi_t^* &= \frac{\mu - r}{\sigma^2} X_t^* \end{aligned}$$

**Proof.** Using theorem 2 and proposition 1, obviously  $\nu(t) = x$  is a solution to (6), hence the optimal wealth is  $X_t^* = \frac{x}{H_t}$  and the optimal portfolio is  $\frac{\mu - r}{\sigma^2} X_t^*$ .

<sup>4</sup>A generalization to the case of general coefficients for CRRA utility functions is done in El Karoui, Huang and Jeanblanc (2002).

We are looking for a function  $V(t, x)$  that can be written as  $V(t, x) = p(t) + q(t) \ln x$ . Introducing the expression of the optimal portfolio given in ii) in equation (5), we obtain the following equation for  $p$  and  $q$

$$0 = f(t) \ln x + p'(t) + q'(t) \ln x + q(t)[r + \frac{1}{2}\theta^2]$$

A sufficient and necessary condition for this equation to be satisfied is

$$\begin{aligned} q'(t) &= -f(t) \\ p'(t) &= -q(t)[r + \frac{1}{2}\theta^2] \end{aligned}$$

which is equivalent to

$$\begin{aligned} q(t) &= \lambda_1 - F(t) \\ p(t) &= \lambda_2 - a \int_0^t du \{\lambda_1 - F(u)\} \end{aligned}$$

where  $\lambda_1$  and  $\lambda_2$  are two real numbers and  $a = r + \frac{1}{2}\theta^2$ . Using the boundary condition  $V(T, x) = \ln(x)(1 - F(T))$ , we get  $p(T) = 0$  and  $q(T) = 1 - F(T)$  from which we obtain  $\lambda_1 = \int_0^{+\infty} f(s)ds = 1$  and  $\lambda_2 = a \int_0^T du \{1 - F(u)\} = a(T - \int_0^T F(u)du)$ . ■

We also obtain an explicit solution in the case of power utility.

**Proposition 4** *Let us assume that  $U(x) = \frac{x^\alpha}{\alpha}$  with  $0 < \alpha < 1$ . Then the value function is*

$$V(t, x) = q(t) \frac{x^\alpha}{\alpha}$$

with  $q(t) = e^{-at}[e^{aT}(1 - F(T)) + \int_t^T e^{as} f(s)ds]$ ,  $a = r - \frac{\theta^2}{2(\alpha-1)}$ .

The optimal wealth and portfolio are

$$\begin{aligned} X_t^* &= (\nu(t)H_t)^\beta \\ \pi_t^* &= -\frac{\mu - r}{(\alpha - 1)\sigma^2} X_t^* \end{aligned}$$

where  $\nu(t) = x \exp[-\frac{\beta+1}{\beta}(-r + \frac{\beta}{2}\theta^2)t]$  and  $\beta = \frac{1}{\alpha-1}$ .

**Proof.** The explicit form of  $\nu(t)$  follows from (6). The optimal wealth is  $X_t^* = (\nu(t)H_t)^\beta$  and the optimal portfolio is  $-\frac{\mu-r}{(\alpha-1)\sigma^2} X_t^*$ .

We are looking for a function  $V(t, x)$  that can be written as  $V(t, x) = q(t)U(x)$ . We thus obtain the differential equation which is satisfied by  $q$

$$0 = f(t) + q'(t) + q(t)[r - \frac{\theta^2}{2(\alpha - 1)}]$$

Hence  $q(t) = e^{-at}[\gamma - \int_0^t e^{as} f(s)ds]$ . From the boundary condition  $q(T) = 1 - F(T)$ , we obtain  $q(t) = e^{-at}[e^{aT}(1 - F(T)) + \int_t^T e^{as} f(s)ds]$ . ■

We find that the optimal strategies coincide to these obtained when the time-horizon is fixed, whatever the distribution  $F$  of the time-horizon. In the simple case of deterministic distribution of the time-horizon and coefficients, it is a striking result that the portfolio strategy expressed in term of optimal wealth is independent of the exit time distribution and identical to the one obtained in the case of a fixed time-horizon. This result is a confirmation and an extension of Merton (1973) and Richard (1975), who had focused on an exponentially distributed time-horizon.

This result intuitively comes from the fact that optimal portfolio strategies in these simple cases (CRRA utility functions and deterministic coefficients) do not depend on the time-horizon (“myopic strategies”). In other words, if the proportion of wealth optimally invested in the risky versus riskfree assets with a fixed time-horizon is independent of the maturity, then these proportions will be optimal in the case of a random time-horizon. On the other hand, the value function is affected by the presence of a random time of exit. This is consistent with a result obtained in Martellini and Urošević (1999) who show, in a static framework, that minimum variance portfolio for an uncertain exit time has a strictly lower (scaled) expected return than the fixed-exit-time one. Hence, uncertainty over time-horizon leads mean-variance investors to select a less risky portfolio than what they would have selected had they known in advance their exact date of exit.

We are able to obtain explicit solutions for the optimal strategy and the wealth process as a function of the distribution of the uncertain-time horizon. We also notice that, if the coefficients are not constant but deterministic functions of time, the solutions obtained just above are always solutions to the equation (6).

We now discuss how these results transport to the case of an economy with infinite time span.

## 4 The Case of an Economy with Infinite Time Span

One problem in the case of an infinite time span is that stronger restrictions than the one in definition 1 have to be imposed on admissible strategies so as to avoid arbitrage. As in Huang and Pagès (1992), a strategy is now said to be admissible if it satisfies the following definition.

**Definition 2** *A portfolio  $(\pi_t, 0 \leq t)$  is said to be admissible if it is a weak admissible strategy, if the wealth process is positive and if*

$$\int_0^{T_n} \left| \pi_s \sigma_s \exp\left(-\int_0^s r_u du\right) \right|^2 ds < \infty \quad \mathbb{P} - a.s.$$

*for a sequence of stopping times  $T_n \uparrow \infty$   $\mathbb{P}$ -a.s.*

**Remark 3** As  $\sigma$  is deterministic and bounded, if the process  $\pi$  is continuous, then the sequence  $T_n = n$  satisfies the previous condition.

As previously, let  $P(t, x)$  denote the set of admissible portfolios starting from a wealth level  $x$  at date  $t$

$$P(t, x) = \{\pi_u, t \leq u, X_t^{t, \pi, x} = x, \pi \text{ is admissible}\}$$

We define

$$V(t, x) = \sup_{\pi \in P(t, x)} \mathbb{E} \left[ \int_t^\infty f(u) U(X_u^{t, \pi, x}) du \right] \quad (7)$$

and obtain that  $V$  is a solution to the Hamilton-Jacobi-Bellman equation (5), except for the fact that the boundary condition has now changed (cf. Fleming-Soner p. 172, theorem 5.1).

**Theorem 3** If  $Y$  is the  $C^{1,2}$  solution to the following Hamilton-Jacobi-Bellman equation

$$0 = f(t)U(x) + \frac{\partial Y(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^d} A(t, x, \pi) \quad (8)$$

where  $A(t, x, \pi) = [xr_t + \pi(\mu_t - r_t)]Y_x(t, x) + \frac{1}{2}\pi^2\sigma_t^2Y_{xx}(t, x)$  and  $Y_x$  (resp.  $Y_{xx}$ ) denotes the first (resp. second) derivative with respect to the space variable, the boundary conditions being  $\lim_{t \rightarrow +\infty} Y(t, x) = 0$ , then  $V = Y$ .

As previously, we can determine the solution for some utility functions. These solutions are the same as before, except that we need to account for the stricter conditions for portfolios to be admissible.

**Proposition 5** *i) Let us assume that  $U(x) = \ln x$ . The value function and the optimal investment policy are respectively given by*

$$\begin{aligned} V(t, x) &= p(t) + q(t) \ln x \\ \pi_t^{\log, *} &= \frac{\mu - r}{\sigma^2} X_t^{\log, *} \end{aligned}$$

where  $q(t) = 1 - \int_0^t f(s) ds$ ,  $p(t) = \lambda_2 - at + \int_0^t ads \int_0^s f(u) du$ ,  $a = r + \frac{1}{2}\theta^2$ ,  $\lambda_2 = a\mathbb{E}(\tau)$ .

*ii) Let us assume that  $U(x) = \frac{x^\alpha}{\alpha}$  with  $0 < \alpha < 1$ ,  $(r - \frac{\theta^2}{2(\alpha-1)}) > 0$ , and  $\mathbb{E}[e^{(r - \frac{\theta^2}{2(\alpha-1)})\tau}] < \infty$ .*

*Then the value function is*

$$V(t, x) = q(t) \frac{x^\alpha}{\alpha}$$

with  $q(t) = e^{-at} [\int_t^{+\infty} e^{as} f(s) ds]$ ,  $a = r - \frac{\theta^2}{2(\alpha-1)}$ .

*The optimal portfolio strategy is*

$$\pi_t^{(\alpha), *} = -\frac{\mu - r}{(\alpha - 1)\sigma^2} X_t^{(\alpha), *}$$

**Proof.** It is exactly the same that in propositions 2 et 3, just changing the boundary condition. And using the remark (3) we notice that the portfolios are admissible. ■

## 5 Introducing Consumption

We now consider a setup with intertemporal utility emanating from lifetime consumption.<sup>5</sup> We use the same setup as in the previous section in terms of assumptions about asset prices. A process  $(\pi, c)$  is now said to be a weak *admissible* strategy if  $\pi$  and  $c$  are two  $\mathbb{F}$ -predictable satisfying  $\sum_{i=1}^n \int_0^\infty (\pi_s^i)^2 ds < +\infty$ ,  $\int_0^\infty c_s ds < +\infty$ . A portfolio/consumption strategy  $(c_t, \pi_t, t \geq 0)$  is said to be admissible if it is a weak admissible strategy, if the wealth process is non negative and if

$$\int_0^{T_n} \left| \pi_s \sigma_s \exp\left(-\int_0^s r_u du\right) \right|^2 ds < \infty \quad \mathbb{P} - \text{a.s.}$$

for a sequence of stopping times  $T_n \uparrow \infty$   $\mathbb{P}$ -a.s. We let  $P(t, x)$  denote the set of admissible portfolios starting from a wealth level  $x$  at date  $t$

$$P(t, x) = \{c_u, \pi_u, t \leq u, X_t^\pi = x, (\pi, c) \text{ is admissible}\}$$

If the strategy  $(\pi, c)$  is used in a self-financing way, then the wealth process  $(X_t^{\pi, x}, 0 \leq t)$ , with  $X_0^{\pi, x} = x$ , now evolves according to the following stochastic differential equation

$$dX_t^{\pi, x} = X_t^{\pi, x} r_t dt + \pi_t [(\mu_t - \mathbf{1}r_t)dt + \sigma_t dW_t] - c_t dt$$

The agent's consumption and portfolio problem is to find a strategy  $(c, \pi)$  which maximizes the following time-additive functional representing that agent's preferences over intertemporal consumption up until the random date  $\tau$

$$\sup_{\pi, c} \mathbb{E} \int_0^\tau U(s, c_s) ds$$

where the utility function  $U$  is assumed to be a continuous, strictly increasing, strictly concave and continuously differentiable function in the space variable defined on  $(0, \infty) \times (0, \infty) \rightarrow \mathbb{R}$  and satisfying the following two conditions:  $\lim_{x \rightarrow +\infty} U'_x(t, x) = 0$  and  $\lim_{x \rightarrow 0} U'_x(t, x) = +\infty$  for all  $t$ , where  $U'_x$  is the first derivative of  $U$  with respect to the space variable  $x$ . We now define the value function for the problem as

$$V(x) = \sup_{\pi, c \in P(0, x)} \mathbb{E} \int_0^\tau U(s, c_s) ds$$

It turns out that the problem involving utility from intertemporal consumption is, at least for deterministic  $f$ , easier to solve than the one involving utility from terminal wealth, because it

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<sup>5</sup>While we only consider, for simplicity, the case with utility over consumption but no bequest motive, one can use the results from the previous sections to handle the general case of utility over consumption and terminal wealth.

can easily be mapped into an equivalent fixed time-horizon problem. To see this, we transform the objective as follows

$$\mathbb{E} \int_0^\tau U(s, c_s) ds = \mathbb{E} \int_0^\infty dF_t \int_0^t U(s, c_s) ds = \mathbb{E} \int_0^\infty U(s, c_s) (1 - F_s) ds$$

Therefore, the problem is formally equivalent to solving

$$\sup_{\pi, c} \mathbb{E} \int_0^\tau \tilde{U}(s, c_s, \cdot) ds$$

with  $\tilde{U}(t, c, \omega) = U(t, c)(1 - F_t(\omega))$ . We now get more specific and assume that  $U(t, c) = e^{-\delta t} u(c)$ , where  $\delta$  describes the agent subjective rate of intertemporal preferences and  $u$  his preference over consumption. Hereafter we denote by  $I$  the inverse of  $u'$ .

We also specialize our setup by assuming an exponential distribution for the uncertain time-horizon

$$1 - F_t = \exp \left[ - \int_0^t \lambda_s ds \right]$$

where  $\lambda$  that can be interpreted as the conditional rate of arrival of the event at time  $t \leq \tau$ , given all information available up to that time (to relate to previous notation, we essentially have that  $\lambda = \frac{f}{1-F}$  and  $\lambda$  is called the *intensity* of  $\tau$ ).<sup>6</sup> It is actually very convenient to study random times through their intensity process. This is because the probability of default may be expressed in terms of the intensity process through a formula which exhibits a striking resemblance to the familiar discount factor, as can be seen from the following transformation

$$\mathbb{E} \int_0^\tau U(c_s) e^{-\delta s} ds = \mathbb{E} \int_0^\infty \tilde{U}(s, c_s) ds = \mathbb{E} \left[ \int_0^\infty \exp \left[ - \int_0^t (\lambda_s + \delta) ds \right] u(c_t) dt \right] \quad (9)$$

From equation (9), we see that a problem with random time-horizon may be conveniently turned into a problem with infinite time-horizon and adjusted subjective discount rate, where the adjusted discount rate is  $\hat{\delta}_t = \delta + \bar{\lambda}_{0,t}$ , where  $\bar{\lambda}_{0,t} = \frac{1}{t} \int_0^t \lambda_u du$  is the average intensity over the period  $[0, t]$ . Hence, everything goes as if the investor uses the original rate of intertemporal preferences augmented by the average force of mortality, a result already noted by Yaari (1965) and Merton (1971). In summary, we therefore obtain the following result.

**Theorem 4** *The optimal consumption and the optimal wealth are given by*

$$\begin{aligned} c_t^* &= I \left( \nu H_t \exp \left( \int_0^t (\lambda_u + \delta) du \right) \right) \\ X_t^* &= \frac{1}{H_t} E \left[ \int_t^{+\infty} H_s c_s^* ds / \mathcal{F}_t \right] \end{aligned}$$

where  $\nu$  is the Lagrange coefficient which satisfied the constraint

$$E \left[ \int_0^{+\infty} H_s I \left( \nu H_s \exp \left( \int_0^s (\lambda_u + \delta) du \right) \right) ds \right] = x.$$

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<sup>6</sup>See for example Brémaud (1981).

**Proof.** Using equation (9), the problem is equivalent to optimize

$$\mathbb{E} \left[ \int_0^\infty \exp \left[ - \int_0^t (\lambda_s + \delta) ds \right] u(c_t) dt \right]$$

under the constraint

$$E \left[ \int_0^{+\infty} H_s c_s ds \right] = x.$$

Using standard arguments, we obtain the result. ■

**Proposition 6** *When the intensity  $\lambda$  is a deterministic function of time, the optimal fraction of wealth dedicated to consumption for an investor endowed with CRRA preferences is equal to that of an investor using the original rate of intertemporal preferences augmented by the average force of mortality. The portfolio selection decision, however, is not affected by uncertainty over an investor's time-horizon.*

**Proof.** The same calculus as in the case of fixed horizon give solutions which are similar to the well-known expressions, and can be written as  $\pi_t^* = -\frac{\mu-r}{(\alpha-1)\sigma^2}$ , and

$$\frac{c_t^*}{X_t^*} = \frac{\widehat{\delta}_t}{\alpha - 1} - \left( \frac{\alpha}{\alpha - 1} \right) \left( r + \frac{\beta^2}{2(\alpha - 1)} \right)$$

where  $\beta = \frac{\mu-r}{\sigma}$ . Note that  $\widehat{\delta}_t = \delta + \bar{\lambda}_{0,t} \geq \delta$  since  $\bar{\lambda}_{0,t} \geq 0$ . Noting that  $\widehat{\delta}_t$  is an increasing function of time when  $\bar{\lambda}_{0,t}$ , that is when  $\lambda_t$  is, concludes the proof. Note simply that, in the case of a random time with unbounded support, conditions are needed for the total expected utility to remain finite. This is similar to a case with infinite time-horizon;  $\delta$  needs to be sufficiently large to satisfy the *transversality condition*. An extension of the standard condition (Merton (1969), equation 39) to our context is  $\widehat{\delta}_t > \alpha \left( r + \frac{\beta^2}{2(\alpha-1)} \right)$  or  $\delta > \alpha \left( r + \frac{\beta^2}{2(\alpha-1)} \right) - \bar{\lambda}_{0,t}$  for all  $t$ . ■

Note that the fraction of wealth invested in stock is not affected by uncertainty over time-horizon. It is a standard result (see Samuelson (1969) for the discrete-time case and Merton (1971) for the continuous-time case) that the portfolio selection decision is independent of the consumption decision for the class of utility functions displaying constant relative risk aversion. This is because constant relative risk aversion implies that the agent's attitude toward financial risk is independent from her financial wealth, and the assumption of a geometric Brownian motion (independent increments and constant coefficients) makes the problem stationary.

The fraction of wealth dedicated to consumption, on the other hand, has been affected. In the case of an uncertain time-horizon with increasing intensity, we obtain that the optimal fraction of wealth devoted to consumption increases with time, because  $\bar{\lambda}_{0,t}$ , and therefore  $\widehat{\delta}_t$ , is an increasing function of time when  $\lambda_t$  is. This is consistent with the life cycle hypothesis:

as her mortality rate rises with age, the probability increases that the agent will die with substantial wealth. The response is to increase consumption so that wealth declines with age.

From previous literature and the above discussions, it seems that the optimal portfolio selection is not affected by the presence of an uncertain time-horizon. We now argue that this is not true in general.

## 6 Stochastic Probability of Exiting

We extend the analysis to a framework with a stochastically time-varying density process. This generalization shall prove useful because the assumption of a deterministic density is neither realistic nor rich enough, as it implies independence between time-horizon and stock returns. In particular, one would like to allow for possible correlation between the instantaneous probability of exit and risky asset returns, a feature of high potential practical relevance.

In this section, we enlarge the probability space and we introduce  $W^f$ , assumed to be a one-dimensional Brownian motion.  $W^f$  and  $W^i$ ,  $i = 1, \dots, n$ , are correlated Brownian motions, with correlation  $\sigma_{S,f} = (\sigma_{if})_{1 \leq i \leq n}$ . In what follows, we assume that  $F_t = \mathbb{P}(\tau \leq t | \mathcal{F}_t) = \int_0^t f_s ds$  with  $f$  a solution to the following stochastic differential equation<sup>7</sup>

$$df_s = f_s(a(s)ds + b(s)dW_s^f), \quad f_0 = y, \quad 0 \leq s \leq T \quad (10)$$

where we assume that the coefficients  $a, b$  are deterministic and  $\int_0^\infty du \exp(\int_0^u a(s)ds) < \infty$ .

In an explicit form

$$f_s = y \exp\left(\int_0^s a(u)du\right) \xi_s$$

where  $\xi_s = \exp(\int_0^s b(u)dW_u^f - \frac{1}{2} \int_0^s b^2(u)du)$ .

The value function  $V$  is now depending on  $t, x, y$ . At initial time 0, one has

$$\begin{aligned} V(0, x, y) &= \sup_{\pi \in P(0, x)} \mathbb{E}[\int_0^\infty f_u U(X_{u \wedge T}^x) du] \\ &= \sup_{\pi \in P(0, x)} \mathbb{E}_{\mathbb{Q}}[y \int_0^\infty \exp(A(u)) U(X_{u \wedge T}^x) du] \end{aligned}$$

where  $\mathbb{Q}$  the probability defined on  $\mathcal{F}_T$  by

$$d\mathbb{Q} = \xi_T d\mathbb{P},$$

and  $A$  is a primitive of  $a$  vanishing at 0.

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<sup>7</sup>Note that nothing prevents  $F$  to take on values greater than 1 under such a general specification. This is similar to the Vasicek (1977) term structure model that allows negative values with positive probabilities. The rule of thumb is that the problem may be neglected in practice as long as the probability of getting forbidden values remains sufficiently small for reasonable values of the parameters.

We obtain that

$$V(0, x, y) = y \left[ \int_0^\infty du \exp(A(u)) \sup_{\pi \in P(0, x)} \mathbb{E}_{\mathbb{Q}} \left[ \int_0^\infty \varphi(u) U(X_{u \wedge T}^x) du \right] \right]$$

where  $\varphi(u)$  is the deterministic density  $\varphi(u) = \frac{1}{\int_0^\infty \exp(A(s)) ds} \exp(A(u))$

$$V(0, x, y) = y \left[ \int_0^\infty du \exp(A(u)) V^1(0, x) \right]$$

where  $V^1$  is the solution associated with the density  $\varphi$  and a wealth process

$$dX_s^{t, \pi, x} = X_s^{t, \pi, x} r_s ds + \pi_s [(\mu_s + \sigma_s \sigma_{S, f} b(s) - r_s) ds + \sigma_s dW_s^{\mathbb{Q}}]. \quad (11)$$

where  $W_t^{\mathbb{Q}} = W_t + \int_0^t \sigma_{S, f} b(s) ds$ .

Under the assumption of constant coefficients  $r, \mu, \sigma, a, b$  and CRRA preferences, one may obtain explicit solutions.

**Proposition 7** *i) If  $U(x) = \log x$  and  $a < 0$ , then the value function  $V$  is given by*

$$V(t, x, y) = -e^{at} \frac{y}{a} \left( \frac{C}{a} (e^{a(T-t)} - 1) + \log(x) \right)$$

where  $C = r + \frac{1}{2} \frac{(\mu - r + \sigma b \sigma_{S, f})^2}{\sigma^2}$ . The optimal portfolio strategy is

$$\pi_t^* = \left[ \frac{\mu - r + \sigma b \sigma_{S, f}}{\sigma^2} \right] X_t^* \quad (12)$$

*ii) If  $U(x) = \frac{x^\alpha}{\alpha}$  and  $a < 0$ , the value function and optimal strategy are*

$$V(t, x, y) = -\frac{y}{a} \frac{x^\alpha}{\alpha} e^{-Ct} \left[ \left( \frac{C}{C+a} \right) e^{(a+C)T} + \frac{a}{C+a} e^{(a+C)t} \right]$$

and

$$\pi_t^* = - \left[ \frac{\mu - r + \sigma b \sigma_{S, f}}{\sigma^2} \right] \frac{X_t^*}{\alpha - 1}$$

where  $C = \left( r - \frac{1}{2} \frac{(\mu - r + \sigma b \sigma_{S, f})^2}{\sigma^2(\alpha - 1)} \right)$ .

**Proof.** Using the result of proposition 2 and the equation 11. ■

**Remark 4** *In the case of deterministic coefficients, using the dynamic programming, we can obtain that  $V$  is solution of the following HJB equation*

$$0 = yU(x) + \frac{\partial V(t, x)}{\partial t} + \sup_{\pi \in \mathbb{R}^d} A(t, x, y, \pi) \quad (13)$$

with

$$\begin{aligned} A(t, x, y, \pi) = & [xr(t) + \pi(\mu(t) - r(t) + \sigma(t)b(t)\sigma_{S, f})]V_x(t, x, y) + \frac{1}{2}\pi^2\sigma^2(t)V_{xx}(t, x, y) \\ & + V_y(t, x, y)ya(t, y) + V_{xy}(t, x, y)yb(t, y)\pi\sigma(t)\sigma_{S, f} + \frac{1}{2}V_{yy}(t, x, y)y^2b^2(t, y) \end{aligned}$$

and

$$V(T, x, y) = U(x)E(1 - F_T / f_T = y) = yU(x) \int_T^{+\infty} \exp \int_T^u a(v) dv du \quad (14)$$

By taking  $b = 0$ , we recover that a random time-horizon with deterministic conditional density leads to the same portfolio strategy as in the case of a fixed time-horizon. In general, however, agents invest more or less in the risky assets than when time horizon is fixed depending on the sign of the correlation. Let us consider for simplicity of exposure the case of a single risky asset. The result is as follows: if  $\sigma_{Sf} < 0$  (resp.  $\sigma_{Sf} = 0$ , resp.  $\sigma_{Sf} > 0$ ), i.e., if the probability of exiting the market is negatively correlated (resp. is not correlated, resp. positively correlated) with the return on the risky asset, then the fraction invested in the risky asset is lower than (resp. equal to, resp. greater than) what it is in the case of a certain time-horizon:  $\frac{\mu-r+\sigma b\sigma_{Sf}}{\sigma^2} < \frac{\mu-r}{\sigma^2}$  (resp.  $\frac{\mu-r+\sigma b\sigma_{Sf}}{\sigma^2} = \frac{\mu-r}{\sigma^2}$ , resp.  $\frac{\mu-r+\sigma b\sigma_{Sf}}{\sigma^2} > \frac{\mu-r}{\sigma^2}$ ).

One natural question is whether we have reasons to believe that  $\sigma_{Sf}$  should be different from zero, and if so, whether it should be positive or negative. This, of course, depends on the situation. There is for example abundant anecdotal evidence that agents may postpone their retirement decision in relation to stock market downturns that negatively affect the level of their expected pension benefits. This paper shows that such complex features can be accounted for in a tractable framework.

## 7 Conclusion

Uncertainty over exit time is an important practical issue facing most, if not all, investors. In order to address that question, we consider a suitable extension of the familiar optimal investment problem of Merton (1971), where we allow the conditional distribution function of an agent's time-horizon to be stochastic and correlated to returns on risky securities. In contrast to existing literature, which has focused on an independent time-horizon, we show that the portfolio decision is affected.

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