MODELLING AND HEDGING OF DEFAULT RISK

Monique Jeanblanc
Equipe d'Analyse et Probabilités, Université d'Evry Val d'Essonne
Boulevard François Miterrand, 91025 Evry Cedex, France
jeanbl@maths.univ-evry.fr

Marek Rutkowski
Faculty of Mathematics and Information Science
Warsaw University of Technology
00-661 Warszawa, Poland
markrut@mini.pw.edu.pl

June 18, 2003

Contents

1 Introduction

2 Art of Modelling

2.1 Model Building

- Specification of risk-sensitive contracts under considerations.
- Identification of all essential risk factors.
- Choice of a family of tradable (liquid) instruments.
- Choice of the most convenient and adequate models.
- Construction of self-financing replicating strategies.
- Valuation of standard derivative products.
- Calibration of the model to market prices of standard derivatives.
- Valuation of exotic products.
- Calculation of sensitivities.

2.2 Hedging of Credit Derivatives

As soon as a model for the underlying is chosen, one may apply one of the following mathematical methods developed to value and hedge non-attainable contingent claims:

- Static hedging: Greenfield (2000),
- Dynamic replication: Wong (1998), Bélanger, Shreve and Wong (2001), Blanchet-Scalliet and Jeanblanc (2001), Vaillant (2001),
- Superhedging: Collin-Dufresne and Hugonnier (1999),
- Quantile hedging: Lotz (1998),
- Utility-based hedging: Collin-Dufresne and Hugonnier (1999),
- Risk-return approach: Bielecki and Jeanblanc (2002).

Credit risk models are covered by the general theory, through an introduction of specific discontinuous martingales. Credit-risk specific issues encompass:

- Correct specification of the credit/default risk,
- Proper choice of tradable instruments,
- Valuation of non-tradable claims,
- Determination of particular trading strategies.

3 Structural Approach to Default Event

3.1 Merton's Model

One of the simplifying assumptions in the Merton (1974) model is that the short-term interest rate is constant and equals r. Therefore, the price at time t of the unit default-free zero-coupon bond with maturity T is easily seen to be $B(t,T) = e^{-r(T-t)}$. The latter formula can be extended to the case of a deterministic continuously compounded interest rate $r : \mathbb{R}_+ \to \mathbb{R}$. We denote by $E(V_t)$ ($D(V_t)$, resp.) the value of the firm's equity (debt, resp.) at time t; hence, the total value of firm's assets satisfies $V_t = E(V_t) + D(V_t)$. We postulate that the firm's value process V follows a geometric Brownian motion under the spot martingale measure $\mathbf{P}^*$, specifically,

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma\,dW_t^*\big), \tag{1}$$

where σ is the constant volatility coefficient of the value process V and the constant κ represents the payout ratio, provided that it is non-negative. Otherwise, κ reflects an inflow of capital to the firm. The process W^* is the one-dimensional standard Brownian motion under $\mathbf{P}^*$, with respect to some reference filtration $\mathbb{F}$ (it is common to take $\mathbb{F} = \mathbb{F}^{W^*}$; this is not essential, though). Notice that dynamics (1) is justified only under the assumption that the total value of the firm's assets represents a traded security.

We postulate that the default event may only occur at the debt's maturity date T. Specifically, if at the maturity T the total value V_T of the firm's assets is less than the notional value L of the firm's debt, the firm defaults and the bondholders receive the amount V_T.

Otherwise, the firm does not default, and its liability is repaid in full. We are thus dealing here with a rather elementary example of a defaultable claim with recovery at maturity. We have

$$D(T,T) = L\mathbb{1}_{\{\tau>T\}} + V_T\mathbb{1}_{\{\tau\le T\}} = L\mathbb{1}_{\{V_T\ge L\}} + V_T\mathbb{1}_{\{V_T<L\}}$$

or, equivalently,

$$D(T,T) = \min(V_T,L)\mathbb{1}_{\{V_T\ge L\}} + \min(V_T,L)\mathbb{1}_{\{V_T<L\}} = \min(V_T,L).$$

The fixed amount L may be interpreted as the face value (or par value) of a corporate zero-coupon bond maturing at time T. Since

$$D(T,T) = \min(V_T,L) = L - (L-V_T)^+,$$

where $x^+ = \max(x,0)$ for every $x \in \mathbb{R}$, the price process $D(t,T)$ of a defaultable zero-coupon bond is manifestly equal to the difference of the value of a default-free zero-coupon bond with the face value L and the value of a European put option written on the firm's assets, with the strike price L and the exercise date T. This put option, with the terminal payoff $(L-V_T)^+$, is commonly referred in the present context as the *put-to-default*. Formally, the value of the firm's debt at time t thus equals

$$D(V_t) = D(t,T) = LB(t,T) - P_t, \tag{2}$$

where P_t is the price of the put-to-default, and where, for the sake of notational convenience, we write $D(t,T)$ to denote the price of a defaultable bond:

$$D(t,T) = B_t\,\mathbf{E}_{\mathbf{P}^*}(B_T^{-1}D(T,T)\,|\,\mathcal{F}_t).$$

It is apparent from (2) that the value at time t of the firm's equity satisfies

$$E(V_t) = V_t - D(V_t) = V_t - LB(t,T) + P_t = C_t, \tag{3}$$

where C_t denotes the price at time t of a call option written on the firm's assets, with the strike price L and the exercise date T. To justify the last equality in (3), we may observe that at time T we have

$$E(V_T) = V_T - D(V_T) = V_T - \min(V_T,L) = (V_T-L)^+,$$

and thus the firm's equity can be seen as a call option on the firm's assets. Alternatively, we may directly use the so-called *put-call parity* relationship for European-style options:

$$C_t - P_t = V_t - LB(t,T).$$

Combining (2) with the Black-Scholes formula for the arbitrage price of a European put option, Merton (1974) derived a closed-form expression for the arbitrage price of a corporate bond. In what follows, N denotes the standard Gaussian cumulative distribution function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du, \quad \forall\, x \in \mathbb{R}.$$

Proposition 3.1 *For every* $0 \le t < T$ *we have*

$$D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big) + LB(t,T) N\big(d_2(V_t, T-t)\big), \tag{4}$$

where

$$d_{1,2}(V_t, T-t) = \frac{\ln(V_t/L) + \big(r - \kappa \pm \frac{1}{2}\sigma^2\big)(T-t)}{\sigma\sqrt{T-t}}\,. \tag{5}$$

The bond price can be re-expressed as follows:

$$D(t,T) = LB(t,T) - DP \times EDL$$

where $DP = N(-d_2) = \mathbf{P}^*(V_T \le L)$ is the (risk-neutral) default probability and EDL is the expected default loss

$$EDL = B(t,T)\left(L - \frac{N(-d_1)}{N(-d_2)} V_t e^{(r-\kappa)(T-t)}\right).$$

3.2 Hedging of a Corporate Bond

Merton's formula can be seen as a variant of the Black-Scholes valuation result. Therefore, the form of the replicating (self-financing) trading strategy for a defaultable bond can be easily deduced from the well-known expressions for the Black-Scholes hedging strategy for a European put option. For the sake of completeness, we state the following corollary to Proposition 3.1, in which we write $D(t,T) = u(V_t, t)$.

Corollary 3.1 *The unique replicating strategy for a defaultable bond involves holding at any time* $0 \le t < T$ *the* $\phi_t^1 V_t$ *units of cash invested in the firm's value and* $\phi_t^2 B(t,T)$ *units of cash invested in default-free bonds, where*

$$\phi_t^1 = u_V(V_t, t) = e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big)$$

and

$$\phi_t^2 = \frac{D(t,T) - \phi_t^1 V_t}{B(t,T)} = LN\big(d_2(V_t, T-t)\big).$$

3.3 Credit Spreads

An important characteristic of a defaultable bond is the difference between its yield and the yield of an equivalent default-free bond, i.e., the *credit spread.* Recall that the credit spread $S(t,T)$ is defined through the formula $S(t,T) = Y^d(t,T) - Y(t,T)$, where $Y^d(t,T)$ and $Y(t,T)$ are given by:

$$Y(t,T) = -\frac{\ln B(t,T)}{T-t}, \quad Y^d(t,T) = -\frac{\ln D(t,T)}{T-t}\,,$$

In Merton's model the yield on a default-free bond is equal to the short-term interest rate; i.e., $Y(t,T)=r$.

Using the formula for $D(t,T)$ with $L=1$, we arrive at the following representation for the credit spread in Merton's model

$$S(t,T)=-\frac{\ln\Big(V_t e^{-\kappa(T-t)}N\big(-d_1(V_t,T-t)\big)/(LB(t,T))+N\big(d_2(V_t,T-t)\big)\Big)}{T-t}.$$

Let us now analyse the behaviour of the credit spread when time converges to the debt's maturity. For this purpose, observe that:

$$\lim_{t\to T}N\big(-d_1(V_t,T-t)\big) \;=\; \begin{cases}1, & \text{on } \{V_T<L\},\\ 0, & \text{on } \{V_T>L\},\end{cases}$$

and

$$\lim_{t\to T}N\big(d_2(V_t,T-t)\big) \;=\; \begin{cases}0, & \text{on } \{V_T<L\},\\ 1, & \text{on } \{V_T>L\}.\end{cases}$$

The reader can readily verify that

$$\lim_{t\to T}S(t,T)=\begin{cases}+\infty, & \text{on } \{V_T<L\},\\ 0, & \text{on } \{V_T>L\}.\end{cases} \tag{6}$$

An essential feature of Merton's model is that the default time τ appears to be a predictable stopping time with respect to the filtration $\mathbb{F}^V$ generated by the value process V, as it is announced, for instance, by the following sequence of $\mathbb{F}^V$-stopping times:

$$\tau_n=\inf\{\,t\geq T-\tfrac{1}{n}:V_t<L\} \tag{7}$$

with the usual convention that $\inf\emptyset=\infty$.

3.4 Black and Cox Model

Black and Cox (1976) have made an attempt to relax the most unrealistic features of Merton's (1974) model. Their approach makes account for such specific features of debt contracts as safety covenants and debt subordination. Since they assume that the firm's stockholders (or bondholders) receive a continuous dividend payment proportional to the current value of the firm, the risk-neutral dynamics of the firm's value are

$$dV_t=V_t\big((r-\kappa)\,dt+\sigma\,dW_t^*\big), \tag{8}$$

where the constants $\kappa\geq 0$ and $\sigma>0$ represent the payout ratio and the volatility coefficient, respectively. As in Merton's model, the short-term interest rate r is assumed to be constant, so that the interest rate risk is neglected.

3.4.1 Safety Covenants

Let us first focus on the safety covenants in the firm's indenture provisions. Generally speaking, safety covenants provide the firm's bondholders with the right to force the firm to bankruptcy or reorganization if the firm is doing poorly according to a set standard. The standard for a poor performance is set in Black and Cox (1976) in terms of a time-dependent deterministic barrier $\bar{v}(t)=Ke^{-\gamma(T-t)}$, $t\in[0,T)$, for some constant $K>0$. They postulate that as soon as the value of firm's assets crosses this lower threshold, the bondholders take over the firm. Otherwise, default takes place at debt's maturity or not depending on whether $V_T < L$ or not. Let us set:

$$v_t = \begin{cases} \bar{v}(t), & \text{for } t < T, \\ L, & \text{for } t = T. \end{cases} \tag{9}$$

The default event occurs at the first time $t \in [0,T]$ at which the firm's value V_t falls below the level v_t, or the default event does not occur at all. The default time τ thus equals (as usual, $\inf \emptyset = +\infty$):

$$\tau = \inf\{\, t \in [0,T] : V_t < v_t\}.$$

If default happens at maturity date T, the recovery payoff is proportional to the value process at this date, specifically, it equals $\beta_1 V_T$ for some constant $\beta_1 \in [0,1]$. The recovery payoff in case of premature default (i.e., on the set $\{\tau < T\}$) equals $\beta_2 V_\tau$ where $\beta_2 \in [0,1]$ is constant. In the original paper by Black and Cox (1976), we have $\beta_1 = \beta_2 = 1$.

Let us notice that the default time equals $\tau = \bar{\tau} \wedge \hat{\tau}$ where the *early default time* $\bar{\tau}$ equals

$$\bar{\tau} = \inf\{\, t \in [0,T) : V_t < \bar{v}(t)\},$$

and $\hat{\tau}$ represents Merton's default time: $\hat{\tau} = T\mathbb{1}_{\{V_T < L\}} + \infty \mathbb{1}_{\{V_T \geq L\}}$. It is natural to postulate that $\bar{v}(t) \leq LB(t,T)$ or, more explicitly,

$$Ke^{-\gamma(T-t)} \leq Le^{-r(T-t)}, \quad \forall\, t \in [0,T], \tag{10}$$

so that, in particular, $K \leq L$. Condition (10) ensures that the payoff to the bondholder at the default time τ never exceeds the face value of debt, discounted at a risk-free rate.

Since the interest rate r is assumed to be constant, the pricing function $u = u(V,t)$ of a defaultable bond solves the following PDE:

$$u_t(V,t) + (r-\kappa)Vu_V(V,t) + \tfrac{1}{2}\sigma^2 V^2 u_{VV}(V,t) - ru(V,t) = 0$$

with the boundary condition $u(Ke^{-\gamma(T-t)}, t) = \beta_2 Ke^{-\gamma(T-t)}$ and the terminal condition $u(V,T) = \min(\beta_1 V, L)$. The boundary problem above is rather difficult to solve, however.

An alternative approach relies on the following probabilistic representation of the price $D(t,T)$ prior to default

$$\begin{aligned} D(t,T) \;=\; & \mathbf{E}_{\mathbf{P}^*}\Big(Le^{-r(T-t)}\mathbb{1}_{\{\bar{\tau} \geq T,\, V_T \geq L\}}\,\Big|\,\mathcal{F}_t\Big) \\ & + \mathbf{E}_{\mathbf{P}^*}\Big(\beta_1 V_T e^{-r(T-t)}\mathbb{1}_{\{\bar{\tau} \geq T,\, V_T < L\}}\,\Big|\,\mathcal{F}_t\Big) \\ & + \mathbf{E}_{\mathbf{P}^*}\Big(K\beta_2 e^{-\gamma(T-\bar{\tau})}e^{-r(\bar{\tau}-t)}\mathbb{1}_{\{t < \bar{\tau} < T\}}\,\Big|\,\mathcal{F}_t\Big). \end{aligned}$$

Using the well known results concerning the law of the first hitting time, Black and Cox (1976) were able to derive an explicit formula for the bond's price. We denote:

$$\nu = r - \kappa - \tfrac{1}{2}\sigma^2, \quad \tilde{\nu} = \nu - \gamma = r - \kappa - \gamma - \tfrac{1}{2}\sigma^2,$$

and $\tilde{a} = \tilde{\nu}\sigma^{-2}$.

Proposition 3.2 *Assume that $\tilde{\nu}^2 + 2\sigma^2(r-\gamma) > 0$. Then the price process $D(t,T) = u(V_t,t)$ of a defaultable bond equals, on the set $\{\tau > t\}$,*

$$\begin{aligned} D(t,T) = {} & LB(t,T)\big(N\big(h_1(V_t,T-t)\big) - R_t^{2\tilde{a}}N\big(h_2(V_t,T-t)\big)\big) \\ & + \beta_1 V_t e^{-\kappa(T-t)}\big(N\big(h_3(V_t,T-t)\big) - N\big(h_4(V_t,T-t)\big)\big) \\ & + \beta_1 V_t e^{-\kappa(T-t)} R_t^{2\tilde{a}+2}\big(N\big(h_5(V_t,T-t)\big) - N\big(h_6(V_t,T-t)\big)\big) \\ & + \beta_2 V_t\big(R_t^{\theta+\zeta}N\big(h_7(V_t,T-t)\big) + R_t^{\theta-\zeta}N\big(h_8(V_t,T-t)\big)\big), \end{aligned}$$

where $R_t = \bar{v}(t)/V_t$,

$$\theta = \tilde{a} + 1, \quad \zeta = \sigma^{-2}\sqrt{\tilde{\nu}^2 + 2\sigma^2(r-\gamma)}$$

and

$$\begin{aligned}
h_1(V_t, T-t) &= \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}},\\
h_2(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}},\\
h_3(V_t, T-t) &= \frac{\ln(L/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}},\\
h_4(V_t, T-t) &= \frac{\ln(K/V_t) - (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}},\\
h_5(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(LV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}},\\
h_6(V_t, T-t) &= \frac{\ln \bar{v}^2(t) - \ln(KV_t) + (\nu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}},\\
h_7(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) + \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}},\\
h_8(V_t, T-t) &= \frac{\ln(\bar{v}(t)/V_t) - \zeta\sigma^2(T-t)}{\sigma\sqrt{T-t}}.
\end{aligned}$$

3.4.2 Special cases

Let us now analyse some special cases of the Black-Cox valuation formula. We shall assume that $\beta_1 = \beta_2 = 1$, and the barrier function $\bar{v}$ is chosen in such a way that $K = L$. Then necessarily $\gamma \geq r$ (otherwise, condition (10) would be violated). Obviously, if $K = L$, $D(t,T) = D_1(t,T) + D_2(t,T)$, where:

$$D_1(t,T) = LB(t,T)\big(N\big(h_1(V_t, T-t)\big) - R_t^{2\tilde{a}} N\big(h_2(V_t, T-t)\big)\big) \tag{11}$$

and

$$D_2(t,T) = V_t\big(R_t^{\theta+\zeta} N\big(h_7(V_t, T-t)\big) + R_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big)\big). \tag{12}$$

Case $\gamma = r$. If we also assume that $\gamma = r$, then $\zeta = -\sigma^{-2}\tilde{\nu}$, and thus

$$V_t R_t^{\theta+\zeta} = LB(t,T), \quad V_t R_t^{\theta-\zeta} = V_t R_t^{2\tilde{a}+1} = LB(t,T) R_t^{2\tilde{a}}.$$

Moreover, it is also easy to see that in this case

$$h_1(V_t, T-t) = \frac{\ln(V_t/L) + \nu(T-t)}{\sigma\sqrt{T-t}} = -h_7(V_t, T-t),$$

while

$$h_2(V_t, T-t) = \frac{\ln \bar{v}^2(t) - \ln(LV_t) + \nu(T-t)}{\sigma\sqrt{T-t}} = h_8(V_t, T-t).$$

We conclude that if $\bar{v}(t) = Le^{-r(T-t)} = LB(t,T)$, then $D(t,T) = LB(t,T)$. This result is quite intuitive; a defaultable bond with a safety covenant represented by the barrier function, which equals the discounted value of the bond's face value, is obviously equivalent to a default-free bond with the same face value and maturity. Notice also that when $\gamma = r$ but $K < L$, then we have: $D_2(t,T) = KB(t,T)\mathbf{P}^*\{\tau < T \,|\, \mathcal{F}_t\}$.

Case $\gamma > r$. If $K = L$ but $\gamma > r$ then one would expect that $D(t,T)$ would be smaller than $LB(t,T)$. We shall show that when γ tends to infinity (all other parameters being fixed), then the Black and Cox price converges to Merton's price, that is,

$$\lim_{\gamma\to\infty} D(t,T) = V_t e^{-\kappa(T-t)} N\big(-d_1(V_t, T-t)\big) + LB(t,T)\big(d_2(V_t, T-t)\big).$$

First, it is clear that $h_1(V_t, T-t) = d_2(V_t, T-t)$. Furthermore, straightforward calculations show that

$$\lim_{\gamma\to\infty} R_t^{2\tilde{a}} N\big(h_2(V_t, T-t)\big) = \lim_{\gamma\to\infty} R_t^{\theta-\zeta} N\big(h_8(V_t, T-t)\big) = 0$$

and thus the second term on the right-hand side of (11), as well as the second term on the right-hand side of (12), vanish. Finally,

$$\lim_{\gamma\to\infty} R_t^{\theta+\zeta} N\big(h_8(V_t, T-t)\big) = e^{-\kappa(T-t)}\, N\big(-d_1(V_t, T-t)\big),$$

since $\lim_{\gamma\to\infty} R_t^{\theta+\zeta} = e^{-\kappa(T-t)}$ and $\lim_{\gamma\to\infty} h_7(V_t, T-t) = -d_1(V_t, T-t)$.

3.5 Market Completeness

In most models developed within the structural approach, the presence of default does not induce a new kind of incompleteness. If the default-free market is complete, as in Merton's and Black and Cox models, the defaultable market is complete too, since the default time is a stopping time in the asset's filtration. Note that the definition of completeness depends strongly on the choice of the filtration (not to mention the choice of tradable assets).

In the two previous models, the default time was also a predictable stopping time. This property is no longer valid in the case of Zhou's (1996) model, where the dynamics of the firm's value is driven by a jump-diffusion process. In that case, not only the defaultable market, but also the default-free market, are incomplete.

3.6 Valuation and Hedging of Credit Derivatives

As soon as the default-free market is complete, one can hedge any contingent claim in the terminal filtration of the prices, hence any defaultable contingent claim. In particular, hedging of credit derivatives is similar to that of standard or exotic options. Since this area is fairly well known, we do not go into details here.

3.7 Partial Observations

Until now, we have implicitly assumed that all the agent have a perfect knowledge of the current state of the firm's value. Following Duffie and Lando (2001), we shall now examine the case of a market model in which the agents have only partial information about the firm's assets. Specifically, the agents get the full observation of the prices only at discrete (fixed) times $t_1, t_2, \dots, t_i$. We postulate that

$$dV_t = V_t\big((r-\kappa)\,dt + \sigma\, dW_t^*\big).$$

Let $\tilde{\mathbb{F}} \subset \mathbb{F}^V$ be the filtration generated by the information flow available to an agent:

$$\begin{aligned}
\tilde{\mathcal{F}}_t &= \{\emptyset, \Omega\} \quad \text{for } t < t_1 \\
\tilde{\mathcal{F}}_t &= \mathcal{F}^V_{t_1} = \sigma(V_s,\, s \le t_1) \quad \text{for } t_1 \le t < t_2 \\
\tilde{\mathcal{F}}_t &= \mathcal{F}^V_{t_n} = \sigma(V_s,\, s \le t_n) \quad \text{for } t_n \le t < t_{n+1}
\end{aligned}$$

The default time is modelled as

$$\tau = \inf\{t \,:\, V_t \le \alpha\}$$

for some constant $\alpha > 0$. In the case of deterministic interest rate r, the t-time value of a defaultable zero-coupon bond with zero recovery is $D^0(t,T) = B(t,T)\mathbf{P}^*(\tau > T \,|\, \tilde{\mathcal{F}}_t)$. We write $V_t = V_0 e^{\sigma X_t}$

where $X_t = \nu t + W^*_t$, $\nu = (r/\sigma) - \sigma/2$ and we denote $a = (1/\sigma)\ln(V_0/\alpha)$. Using standard computations based on the Markov property, we obtain that on the set $t_i \le t < t_{i+1}$

$$\begin{aligned} D^0(t,T) &= B(t,T)\,\mathbf{P}^*(\tau > T \mid \tilde{\mathcal{F}}_t) = B(t,T)\,\mathbf{P}^*(\tau > T \mid \mathcal{F}^V_{t_i}) \\ &= B(t,T)\,\mathbf{P}^*\Big(\inf_{s<T} X_s > a \mid \mathcal{F}^V_{t_i}\Big) = 1\!\!1_{\{\tau > t_i\}}\, B(t,T)\,\mathbf{P}^*\Big(\inf_{t_i \le s < T} X_s > a \mid \mathcal{F}^V_{t_i}\Big) \\ &= 1\!\!1_{\{\tau > t_i\}}\, B(t,T)\,\Phi(T - t_i, a - X_{t_i}) \end{aligned}$$

where

$$\begin{aligned} \Phi(t,z) &= \mathbf{P}^*\Big(\inf_{u\le t}(\nu u + W^*_u) > z\Big) \\ &= N\left(\frac{\nu t - z}{\sqrt{t}}\right) - e^{2\nu z} N\left(\frac{z + \nu t}{\sqrt{t}}\right) \quad \text{for } z < 0 \text{ and } t > 0, \\ &= 0 \quad \text{for } z \ge 0 \text{ and } t \ge 0, \\ \Phi(0,z) &= 1 \quad \text{for } z < 0. \end{aligned}$$

The process $D^0(t,T)/B(t,T)$ is increasing on the interval $[t_i, t_{i+1}[$, its jump at time t_i on the set $\{\tau \le t_i\}$ is

$$\Delta^0(D(t_i,T)/B(t,T)) = 1\!\!1_{\{\tau > t_{i-1}\}}\Phi(t_i - t_{i-1}, a - X_{t_{i-1}}),$$

while on the set $\{\tau > t_i\}$ the jump equals

$$\Delta^0(D(t_i,T)/B(t,T)) = \Phi(t_i - t_{i-1}, a - X_{t_{i-1}}) - 1.$$

4 Intensity-Based Approach to Default Event

One of the major drawbacks of the structural approach is that the default time is a predictable stopping time (at least, this holds in Merton's and Black and Cox's models). In order to capture the idea of a default that occurs by a total surprise, one can use a *totally inaccessible* stopping time (i.e., a stopping times that can not coincide with a predictable one on a set of a strictly positive probability). The most typical example of these times is the jump time of a Poisson process.

Let N be a Poisson process with the intensity $\lambda > 0$, defined on a probability space $(\Omega, \mathcal{G}, \mathbf{Q}^*)$. We denote by τ the random time of the first jump of N, that is,

$$\tau := \inf\{t \ge 0 \,:\, N_t \ne 0\}.$$

This time can not be announced, however, its law is well known: this is an exponential law with parameter λ. The *intensity coefficient* λ is characterized by the fact that the *compensated process* $N_t - \lambda t$ is a martingale with respect to the filtration generated by N (hence the stopped process $M_t = N_{t\wedge\tau} - \lambda(t \wedge \tau)$ is a martingale as well).

This can be generalized to the case of an inhomogeneous Poisson process, i.e., a process N such that $N_t - \int_0^t \lambda(u)\,du$ is a martingale with respect to the filtration generated by N. If λ is a given non-negative function (or even a non-negative stochastic process) it is easy to construct such a process. Indeed, let $\tilde{N}$ be a Poisson process with the constant intensity equal to 1, and let us set $N_t = \tilde{N}_{\Lambda(t)}$, where $\Lambda(t) = \int_0^t \lambda(u)\,du$. Then, the moment of the first jump time of $\tilde{N}$ has the law

$$\mathbf{Q}^*(\tau \le t) = 1 - \exp\left(-\int_0^t \lambda(u)\,du\right).$$

In this case, the process

$$N_{t\wedge\tau} - \int_0^{t\wedge\tau} \lambda(u)\,du = H_t - \int_0^{t\wedge\tau} \lambda(u)\,du$$

is a martingale with respect to the filtration $\mathbb{H}$ generated by the process H. However, we do not need the whole structure of the Poisson process, since only the first jump is usually taken into account. In the case of stochastic intensity process λ, one starts with a given process λ, adapted to a given filtration $\mathbb{F}$, and one enlarge the filtration in order to construct the Poisson process with a stochastic intensity λ.

4.1 Intensity Process

We shall now formally introduce the notion of a stochastic intensity of default. Let τ be a non-negative random variable on a probability space $(\Omega, \mathcal{G}, \mathbf{Q}^*)$, referred to as the *default time.* We introduce the jump process $H_t = \mathbb{1}_{\{\tau \geq t\}}$ and we denote by $\mathbb{H}$ the filtration generated by this process.

4.1.1 The Hazard Process

We now assume that some reference filtration $\mathbb{F}$ is given. We set $\mathbb{G} := \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every t. $\mathbb{G}$ is referred to as to the *full filtration*: it includes the observations of default event. Of course, τ is an $\mathbb{H}$-stopping time, as well as a $\mathbb{G}$-stopping time (but not necessarily an $\mathbb{F}$-stopping time).

The concept of the *hazard process* of a random time τ is closely related to the conditional distribution function F_t of τ. We define

$$F_t = \mathbf{Q}^*(\tau \leq t \,|\, \mathcal{F}_t)$$

and $G_t = 1 - F_t = \mathbf{Q}^*(\tau > t \,|\, \mathcal{F}_t)$. We postulate that $G_t > 0$, hence, we exclude the case where τ is an $\mathbb{F}$-stopping time. The process $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ given by the formula

$$\Gamma_t = -\ln(1 - F_t) = -\ln G_t$$

is called the *hazard process* of a random time τ with respect to the reference filtration $\mathbb{F}$.

4.1.2 Deterministic Intensity

The study of a simple case when the reference filtration $\mathbb{F}$ trivial, and thus the hazard process is deterministic, may be instructive. Assume that τ is a given random time. Assume that $\mathbf{Q}^*(\tau > t) > 0$ for every $t \in \mathbb{R}_+$ and that the cumulative distribution function $F(t) = \mathbf{Q}^*(\tau \leq t)$ is an absolutely continuous function:

$$F(t) = \int_0^t f(u)\, du$$

for some function $f : \mathbb{R}_+ \to \mathbb{R}_+$. Then we have

$$F(t) = 1 - e^{-\Gamma(t)} = 1 - e^{-\int_0^t \gamma(u)\, du}$$

where

$$\gamma(t) = \frac{f(t)}{1 - F(t)}\,.$$

It is clear that the function $\gamma : \mathbb{R}_+ \to \mathbb{R}$ is non-negative and satisfies $\int_0^\infty \gamma(u)\, du = \infty$. The function γ is called the *intensity function* or the *hazard rate* of τ. It can be checked by direct calculations that the process $H_t - \int_0^{t \wedge \tau} \gamma(u)\, du$ is an $\mathbb{H}$-martingale.

Note that the case of partial information where the agent has no information on the prices can be considered as the case $\mathbb{F}$ is the trivial information.

4.1.3 Stochastic Intensity

In terms of the stochastic intensity, the conditional probability of the default event $\{t < \tau \le T\}$, given the full information $\mathcal{G}_t$ available at time t, equals

$$\mathbf{Q}^*(t < \tau \le T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbf{E}_{\mathbf{Q}^*}\Big(1 - e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big).$$

Thus

$$\mathbf{Q}^*(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}}\, \mathbf{E}_{\mathbf{Q}^*}\Big(e^{-\int_t^T \gamma_u\, du} \,\Big|\, \mathcal{F}_t\Big)$$

and consequently,

$$\mathbf{Q}^*(t < \tau \le t + dt \,|\, \mathcal{G}_t) \approx \mathbb{1}_{\{\tau > t\}} \gamma_t\, dt.$$

The latter formula shows that $\gamma_t\, dt$ gives the conditional probability of the default occurrence in a short time after t, given that it has not yet occurred and that we observe all events from $\mathcal{F}_t$. Also, it can be shown that the process

$$\hat{M}_t := H_t - \int_0^{\tau \wedge t} \gamma_u\, du = H_t - \Gamma_{\tau \wedge t} = \int_0^t (1 - H_u)\gamma_u\, du.$$

is a (purely discontinuous) $\mathbb{G}$-martingale

4.2 Explicit Construction of a Default Time

We shall now briefly describe the most commonly used construction of a default time associated with a given hazard process Γ. It should be stressed that the random time obtained through this particular method – which will be called the *canonical construction* in what follows – has certain specific features that are not necessarily shared by all random times with a given $\mathbb{F}$-hazard process Γ. We start by assuming that we are given an $\mathbb{F}$-adapted, right-continuous, increasing process Γ defined on a filtered probability space $(\tilde{\Omega}, \mathbb{F}, \mathbf{P}^*)$. As usual, we assume that $\Gamma_0 = 0$ and $\Gamma_\infty = +\infty$. In many instances, Γ is given by the equality

$$\Gamma_t = \int_0^t \gamma_u\, du,$$

for some non-negative, $\mathbb{F}$-progressively measurable intensity process γ.

To construct a random time τ such that Γ is the $\mathbb{F}$-hazard process of τ, we need to enlarge the underlying probability space $\tilde{\Omega}$. This also means that Γ is not the $\mathbb{F}$-hazard process of τ under $\mathbf{P}^*$, but rather the $\mathbb{F}$-hazard process of τ under a suitable extension $\mathbf{Q}^*$ of the probability measure $\mathbf{P}^*$. Let ξ be a random variable defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{Q}})$, uniformly distributed on the interval $[0, 1]$ under $\hat{\mathbf{Q}}$. We consider the product space $\Omega = \tilde{\Omega} \times \hat{\Omega}$, endowed with the product σ-field $\mathcal{G} = \mathcal{F}_\infty \otimes \hat{\mathcal{F}}$ and the product probability measure $\mathbf{Q}^* = \mathbf{P}^* \otimes \hat{\mathbf{Q}}$. The latter equality means that for arbitrary events $A \in \mathcal{F}_\infty$ and $B \in \hat{\mathcal{F}}$ we have $\mathbf{Q}^*\{A \times B\} = \mathbf{P}^*\{A\}\hat{\mathbf{Q}}\{B\}$.

An alternative way of achieving basically the same goal relies on postulating that the underlying probability space $(\tilde{\Omega}, \mathbb{F}, \mathbf{P}^*)$ is sufficiently rich to support a random variable ξ, uniformly distributed on the interval $[0, 1]$, and independent of the filtration $\mathbb{F}$ under $\mathbf{P}^*$. In this version of the construction, Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbf{P}^*$.

We define the random time $\tau : \Omega \to \mathbb{R}_+$ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ : e^{-\Gamma_t} \le \xi \,\} = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t \ge \eta \,\}, \tag{13}$$

where the random variable $\eta = -\ln \xi$ has a unit exponential law under $\mathbf{Q}^*$. It is not difficult to find the process $F_t = \mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}$. Indeed, since clearly $\{\tau > t\} = \{\xi < e^{-\Gamma_t}\}$ and the random variable Γ_t is $\mathcal{F}_\infty$-measurable, we obtain

$$\mathbf{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbf{Q}^*\{\xi < e^{-\Gamma_t} \,|\, \mathcal{F}_\infty\} = \hat{\mathbf{Q}}\{\xi < e^x\}_{x = \Gamma_t} = e^{-\Gamma_t}. \tag{14}$$

Consequently, we have

$$1 - F_t = \mathbf{Q}^*\{\tau > t \,|\, \mathcal{F}_t\} = \mathbf{E}_{\mathbf{Q}^*}\big(\mathbf{Q}^*\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Gamma_t}, \tag{15}$$

and so F is an $\mathbb{F}$-adapted, right-continuous, increasing process. It is also clear that the process Γ represents the $\mathbb{F}$-hazard process of τ under $\mathbf{Q}^*$.

4.2.1 Technical Comment

As an immediate consequence of (14) and (15), we obtain the following interesting property of the canonical construction of the default time:

$$\mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\}. \tag{16}$$

Let us now analyse some important consequences of (16). First, we obtain

$$\mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_u\} = \mathbf{Q}^*\{\tau \le t \,|\, \mathcal{F}_t\} = e^{-\Gamma_t} \tag{17}$$

for arbitrary two dates $0 \le t \le u$. Notice that only the last equality in (17) is necessarily satisfied by the $\mathbb{F}$-hazard process Γ of τ; the first two equalities are additional features of the canonical construction of τ, meaning that they are not necessarily valid in a general set-up. Equality (17) entails the conditional independence under $\mathbf{Q}^*$ of the σ-fields $\mathcal{H}_t$ and $\mathcal{F}_t$, given the σ-field $\mathcal{F}_\infty$. Such a property of the two filtrations $\mathbb{H}$ and $\mathbb{F}$ is termed Hypothesis (H). It can be shown that Hypothesis (H) is equivalent to the following condition: an arbitrary $\mathbb{F}$-martingale also follows a $\mathbb{G}$-martingale under $\mathbf{Q}^*$. The latter condition was previously studied by, among others, Brémaud and Yor (1978), Kusuoka (1999) and Elliott et al. (2000).

5 Pricing and Hedging of Credit Derivatives

5.1 Overview

In this section, we analyse the possibility of an exact replication of defaultable claims within various models of defaultable markets. We argue that the choice of a particular model (in particular, of default-free and defaultable tradable assets) is an essential step in model building. From the practical perspective, the valuation and hedging of credit derivatives should be done with respect to liquid credit-risk sensitive instruments of a similar nature; that is, of a similar exposure with respect to the relevant risk factors. The latter point is illustrated by means of a credit default swap.

5.1.1 Default-Free Market

Consider an economy in continuous time, with time parameter $t \in \mathbb{R}_+$. We are given a filtered probability space $(\Omega, \mathbb{F}, \mathbf{P}^*)$ endowed with a one-dimensional standard Brownian motion W^*. It is convenient to assume that $\mathbb{F}$ is the $\mathbf{P}^*$-augmented and right-continuous version of the natural filtration generated by W^*. As we shall see in what follows, the filtration $\mathbb{F}$ will also play the role of the *reference filtration* for the default intensity. It is important to notice that all martingales with respect to a Brownian filtration $\mathbb{F}$ are continuous processes; this property will be of frequent use in what follows.

In the first step, we shall introduce an arbitrage-free market model for default-free securities. Notice that all price processes introduced in this subsection are $\mathbb{F}$-adapted and continuous. In the *default-free market* we have the following primary tradable assets:

- a *money market account* B satisfying

$$dB_t = r_t B_t \, dt, \quad B_0 = 1,$$

or, equivalently,

$$B_t = \exp\left(\int_0^t r_u\,du\right) \tag{18}$$

where r is an $\mathbb{F}$-adapted stochastic process,

- a *default-free zero-coupon bond* with the price process

  $$B(t,T) = B_t\,\mathbf{E}_{\mathbf{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t), \quad \forall\, t \le T, \tag{19}$$

  where T is the bond's maturity date,

- a *risky asset* whose price dynamics under $\mathbf{P}^*$ are

  $$dS_t = S_t\left(r_t\,dt + \sigma_t\,dW_t^*\right), \quad S_0 > 0, \tag{20}$$

  for some $\mathbb{F}$-progressively measurable volatility process σ.

We make an important assumption that our model of default-free market is complete. The probability $\mathbf{P}^*$ is thus the unique martingale measure for the default-free market model. Let us finally notice that we may equally well assume that the Wiener process W^* is d-dimensional and $S = (S^1, \dots, S^d)$ is the vector of cash prices of d risky assets. The completeness of such a market is essentially equivalent to non-degeneracy of the volatility matrix σ_t.

5.1.2 Defaultable Market

Most approaches have a certain common features, which are listed below:

- Completeness of a default-free market model is assumed.
- Introduction of default event is done through a suitable enlargement of the reference filtration.
- Specification of the default time is done by postulating the form of the default intensity either under an objective or under a martingale probability.
- The enlarged model, with possibility of default, is thus incomplete, unless tradable defaultable securities are introduced.
- Pricing of default-free and defaultable contingent claims is done through the risk-neutral valuation formula.

From the practical viewpoint, we are mainly interested in:

- Hedging of default (jump) risk with the use defaultable securities with similar features (jump and recovery rate) as the considered defaultable claim.
- Hedging of residual risk (credit spread risk) using either defaultable or default-free securities.
- Specific issue: the hedging of a non-standard credit default swap using liquid (standard) credit default swaps of different maturities.

5.1.3 Simplified Approach

In the financial industry, it is common to use a simplified practical approach to the hedging of credit risk. It is based on the following presumptions:

- A pure credit risk instrument (e.g., a basic credit default swap) is considered.
- One considers a one-sided counterparty risk with a fixed recovery rate (the recovery rate is the same for a derivative product and for a corporate bond).
- Nonnegativity of the marked-to-market value of the contract to a non-defaultable counterparty is postulated (defaultable loans and vulnerable options are thus covered, but defaultable swaps are beyond the scope).
- Independence of market and credit risks is assumed.
- Existence of a non-defaultable version of the contract and of a liquid market in corporate bonds is taken for granted.

5.2 Basic Valuation Formulae

5.2.1 Default Time

Let us assume that we are given the $\mathbb{F}$-adapted, right-continuous, increasing process Γ on $(\Omega, \mathbb{F}, \mathbf{P}^*)$. The default time τ and the probability measure $\mathbf{Q}^*$ are assumed to be constructed as in Section 4.2. The probability $\mathbf{Q}^*$ will play the role of the *martingale probability* for the defaultable market. It is essential to observe that

- the Wiener process W^* is also a Wiener process with respect to $\mathbb{G}$ under the probability measure $\mathbf{Q}^*$,
- we have $\mathbf{Q}^*_{|\mathcal{F}_t} = \mathbf{P}^*_{|\mathcal{F}_t}$ for every $t \in [0, T^*]$.

5.2.2 Defaultable Term Structure

For a defaultable discount bond with zero recovery[1] it is natural to adopt the following definition of the price

$$D^0(t,T) = B_t \, \mathbf{E}_{\mathbf{Q}^*}(B_T^{-1} \mathbb{1}_{\{\tau > T\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau > t\}} \tilde{D}^0(t,T)$$

where $\tilde{D}^0(t,T)$ stands the pre-default value of the bond, given by the equality

$$\tilde{D}^0(t,T) = \hat{B}_t \, \mathbf{E}_{\mathbf{Q}^*}(\hat{B}_T^{-1} \,|\, \mathcal{F}_t),$$

and where the *risk-adjusted savings account* $\hat{B}_t$ equals

$$\hat{B}_t = \exp\left(\int_0^t (r_u + \gamma_u)\, du \right).$$

Since $\mathbb{F}$ is assumed to be the Brownian filtration, the process $\tilde{D}^0(t,T)/\hat{B}_t$ is a continuous, strictly positive, $\mathbb{F}$-martingale. Therefore, the pre-default bond price $\tilde{D}^0(t,T)$ is a continuous, strictly positive $\mathbb{F}$-semimartingale. In the very special case when r and γ are constants, we get a simple representation

$$\tilde{D}^0(t,T) = e^{-(r+\gamma)(T-t)} = e^{-\gamma(T-t)} B(t,T).$$

Remarks. The probability measure $\mathbf{Q}^*$ introduced above is an essential input in the specification of defaultable term structure. We deal here with the modelling of bond prices $D^0(t,T)$, rather than with the arbitrage valuation of contingent claims. In this sense, the probability $\mathbf{Q}^*$ is "given by the market", rather than derived using some formal mathematical arguments.

---

[1] The superscript 0 refers to the postulated zero recovery scheme.

5.2.3 Value of a Defaultable Claim

Let Y be an $\mathcal{F}_T$-measurable random variable representing the *promised payoff* at maturity date T. We consider a defaultable claim of the form $X = 1\!\!1_{\{\tau>T\}}Y$, that is, a defaultable claim with zero recovery in case of default. A common approach to that valuation of defaultable claims is based on the following *risk-neutral valuation formula*

$$B_t\,\mathbf{E}_{\mathbf{Q}^*}(B_T^{-1}X \,|\, \mathcal{G}_t) = B_t\,\mathbf{E}_{\mathbf{Q}^*}(B_T^{-1}Y 1\!\!1_{\{\tau>T\}} \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}}V_t(X) \tag{21}$$

where

$$V_t(X) = \hat{B}_t\,\mathbf{E}_{\mathbf{Q}^*}(\hat{B}_T^{-1}Y \,|\, \mathcal{F}_t). \tag{22}$$

The process $V_t(X)$ represents the *pre-default value* of X at time t. Notice that the process $V_t(X)/\hat{B}_t$ is a continuous $\mathbb{F}$-martingale (thus, $V_t(X)$ is a continuous $\mathbb{F}$-semimartingale). The valuation formula (21), as well as the concept of pre-default value, should be supported by replication arguments. To this end, we need first to construct a suitable model of a defaultable market. In fact, if we wish to use formula (21), we need to know the joint law of all random variables involved, and this appears to be a non-trivial issue.

5.3 Trading Strategies: Default-Free Case

Our goal in this section is to present basic results related to the concept of a self-financing trading strategy for a market model involving default-free and defaultable securities. For the sake if the reader's convenience, we shall first discuss briefly the classic concepts of self-financing cash and futures strategies in the context of default-free market model. It appears that in case of defaultable securities only minor adjustments in definitions and results are needed (see, Blanchet-Scalliet and Jeanblanc (2000) or Vaillant (2001)),

5.3.1 Cash Strategies

Let Z_t^0 and Z_t^1 be cash prices at time t of two tradable assets, where Z^0 and Z^1 are continuous semimartingales. We assume, in addition, that the process Z^0 is strictly positive. We denote by V_t the wealth of the cash strategy (ϕ^0, ϕ^1) at time t, so that

$$V_t = \phi_t^0 Z_t^0 + \phi_t^1 Z_t^1.$$

We say that the cash strategy (ϕ^0, ϕ^1) is self-financing if

$$dV_t = \phi_t^0\,dZ_t^0 + \phi_t^1\,dZ_t^1.$$

This yields

$$dV_t = (V_t - \phi_t^1 Z_t^1)(Z_t^0)^{-1}\,dZ_t^0 + \phi_t^1\,dZ_t^1.$$

Let us introduce discounted (relative) values: $\tilde{V}_t = V_t/Z_t^0$ and $\tilde{Z}_t^1 = Z_t^1/Z_t^0$. Using Itô's lemma, we get

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \phi_u^1\,d\tilde{Z}_u^1.$$

A similar result holds for any finite number of cash assets. Let $Z_t^0, Z_t^1, \dots, Z_t^k$ be cash values at time t of $k+1$ primary assets (as before, Z^0, $Z^1, \dots, Z^k$ are continuous semimartingales). Then the wealth process equals

$$V_t = \phi_t^0 Z_t^0 + \phi_t^1 Z_t^1 + \ldots + \phi_t^k Z_t^k$$

and the strategy $(\phi^0, \phi^1, \dots, \phi^k)$ is said to be self-financing if

$$dV_t = \phi_t^0\,dZ_t^0 + \phi_t^1\,dZ_t^1 + \ldots + \phi_t^k\,dZ_t^k.$$

By combining the last two formulae, we obtain

$$dV_t = \Big(V_t - \sum_{i=1}^{k} \phi^i_t Z^i_t\Big)(Z^0_t)^{-1}\, dZ^0_t + \sum_{i=1}^{k} \phi^i_t\, dZ^i_t.$$

Choosing Z^0 as a numeraire and denoting $\tilde{V}_t = V_t/Z^0_t$, $\tilde{Z}^i_t = Z^i_t/Z^0_t$, we get the standard result

$$\tilde{V}_t = \tilde{V}_0 + \sum_{i=1}^{k} \int_0^t \phi^i_u\, d\tilde{Z}^i_u.$$

5.3.2 Futures Strategies

Let Z^0_t and Z^1_t represent the cash and futures prices at time t, respectively, where Z^0 and Z^1 are continuous semimartingales and Z^0_t is assumed to be a strictly positive process. In view of specific features of futures contracts, it is natural to postulate that the wealth V satisfies[2]

$$V_t = \phi^0_t Z^0_t + \phi^1_t 0 = \phi^0_t Z^0_t.$$

The futures strategy (ϕ^0, ϕ^1) is self-financing if

$$dV_t = \phi^0_t\, dZ^0_t + \phi^1_t\, dZ^1_t. \tag{23}$$

We thus have

$$dV_t = V_t (Z^0_t)^{-1}\, dZ^0_t + \phi^1_t\, dZ^1_t.$$

Proposition 5.1 *The process* $\tilde{V}_t = V_t/Z^0_t$ *satisfies*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \hat{\phi}^1_u\, d\hat{Z}^1_u$$

where $\hat{\phi}^1_t = \phi^1_t e^{\alpha_t}/Z^0_t$, $\hat{Z}^1_t = Z^1_t e^{-\alpha_t}$ *and*

$$\alpha_t = \langle \ln Z^0, \ln Z^1 \rangle_t = \int_0^t (Z^0_u)^{-1} (Z^1_u)^{-1}\, d\langle Z^0, Z^1\rangle_u$$

Proof. Itô's formula combined with (23) yield

$$\begin{aligned} d\tilde{V}_t &= (Z^0_t)^{-1} dV_t + V_t\, d(Z^0_t)^{-1} + d\langle (Z^0)^{-1}, V\rangle_t \\ &= \phi^0_t (Z^0_t)^{-1}\, dZ^0_t + \phi^1_t (Z^0_t)^{-1}\, dZ^1_t + \phi^0_t Z^0_t\, d(Z^0_t)^{-1} - \phi^0_t (Z^0_t)^{-2}\, d\langle Z^0, Z^0\rangle_t - \phi^1_t (Z^0_t)^{-2}\, d\langle Z^0, Z^1\rangle_t \\ &= \phi^1_t (Z^0_t)^{-1} dZ^1_t - \phi^1_t (Z^0_t)^{-2}\, d\langle Z^0, Z^1\rangle_t = \phi^1_t e^{\alpha_t} (Z^0_t)^{-1} \big(e^{-\alpha_t} dZ^1_t - Z^1_t e^{-\alpha_t} d\alpha_t\big) \end{aligned}$$

and the result follows. □

5.3.3 Special Cash Strategies

Assume now that three assets are traded on the market and let (ϕ^0, ϕ^1, ϕ^2) be a self-financing strategy such that at any time there is zero net investment in Z^1 and Z^2 so that

$$\phi^1_t Z^1_t + \phi^2_t Z^2_t = 0$$

---

[2] Let us recall that the futures price Z^1_t, that is, the current quotation of a futures contract, has different practical features than the cash price of an asset. We make here the standard assumption that it costs nothing to enter a futures contract.

or, equivalently, $\phi^2_t = -\phi^1_t Z^1_t / Z^2_t$. Then, from

$$V_t = \phi^0_t Z^0_t,\ dV_t = \phi^0_t dZ^0_t + \phi^1_t dZ^1_t + \phi^2_t dZ^2_t$$

we get

$$dV_t = V_t (Z^0_t)^{-1}\, dZ^0_t + \phi^1_t \left(dZ^1_t - Z^1_t (Z^2_t)^{-1}\, dZ^2_t \right). \tag{24}$$

Let us denote $\bar{Z}^0_t = Z^0_t / Z^2_t$, $\bar{Z}^1_t = Z^1_t / Z^2_t$. The following result extends Proposition 5.1.

Proposition 5.2 *The process* $\tilde{V}_t = V_t / Z^0_t$ *satisfies*

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \hat{\phi}^1_u \, d\hat{Z}^1_u$$

where

$$\hat{\phi}^1_t = \phi^1_t e^{\bar{\alpha}_t} / Z^0_t,\ \hat{Z}^1_t = \bar{Z}^1_t e^{-\bar{\alpha}_t},\ \bar{\alpha}_t = \langle \ln \bar{Z}^0, \ln \bar{Z}^1 \rangle_t.$$

Proof. It suffices to consider the discounted values of all considered processes, with the price Z^2 being chosen as a numeraire. Then equation (24) becomes

$$d\bar{V}_t = \bar{V}_t (\bar{Z}^0_t)^{-1}\, d\bar{Z}^0_t + \phi^1_t d\bar{Z}^1_t$$

where $\bar{V}_t = V_t / Z^2_t$. To conclude, it suffices to apply Proposition 5.1, and to note that $\tilde{V}_t = V_t / Z^0_t = \bar{V}_t / \bar{Z}^0_t$. □

Remarks. Suppose that $\bar{\sigma}^0$ and $\bar{\sigma}^1$ are the volatilities of $\bar{Z}^0$ and $\bar{Z}^1$ respectively, so that

$$d\bar{Z}^i_t = \bar{Z}^i_t \big(\bar{\mu}^i_t \, dt + \bar{\sigma}^i_t \, dW^*_t \big)$$

for $i = 0, 1$. Then clearly

$$\bar{\alpha}_t = \int_0^t \bar{\sigma}^0_u \cdot \bar{\sigma}^1_u \, du$$

where $\cdot$ denotes the inner product. In a typical case, $\bar{\sigma}^0$ and $\bar{\sigma}^1$ (and thus also) $\bar{\alpha}_t$ will be deterministic functions.

5.4 Self-financing Strategies with Defaultable Assets

We shall first examine basic properties of general financial models involving default-free and defaultable securities. At this stage, our goal will be to derive fundamental relationships. Subsequently, we shall be more specific about the nature of these securities, and we shall furnish closed-form solutions for specific defaultable claims.

5.4.1 Case A. Single defaultable tradable asset and two default-free assets

For the sake of simplicity, we assume zero recovery for the defaultable contingent claim X, as well as for the defaultable tradable asset with the price process Z^0. Thus, at time τ the wealth process of any strategy that replicates X should necessarily jump to zero. The process Z^0 vanishes at time of default (and thus also after this date). Nevertheless, it can be used as a numeraire prior to τ. Indeed, we have

$$Z^0_t = \mathbb{1}_{\{\tau > t\}} \tilde{Z}^0_t$$

for some $\mathbb{F}$-adapted process $\tilde{Z}^0$. We assume that $\tilde{Z}^0$ is a strictly positive continuous $\mathbb{F}$-semimartingale (clearly $\tilde{Z}^0 = V(X)$ for some defaultable claim X that settles at T, cf. (21)-(22)). On the other hand, it is obvious that the price process Z^0 jumps from $\tilde{Z}^0_{\tau-}$ to 0 at default time τ.

Continuous $\mathbb{F}$-semimartingales Z^1 and Z^2 are assumed to model cash prices of tradable default-free securities. We postulate that Z^2 is a strictly positive process. It is convenient to assume that the processes Z^1 and Z^2 are stopped at τ. Since we are going to deal with defaultable claims that are subject to the zero recovery scheme, it will be sufficient to examine replicating strategies on the random interval $[\![0, \tau \wedge T]\!]$. For this reason, we shall postulate throughout that processes ϕ^0, ϕ^1 and ϕ^2 are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable. In fact, it can be formally shown that for any $\mathbb{G}$-predictable process ϕ there exists a unique $\mathbb{F}$-predictable process ψ such that $\mathbb{1}_{\{\tau \geq t\}} \phi_t = \mathbb{1}_{\{\tau \geq t\}} \psi_t$ for every $t \in \mathbb{R}_+$.

We consider a self-financing cash strategy (ϕ^0, ϕ^1, ϕ^2) such that at any time there is zero net investment in Z^1 and Z^2

$$\phi^1_t Z^1_t + \phi^2_t Z^2_t = 0, \tag{25}$$

that is, $\phi^2_t = -\phi^1_t Z^1_t / Z^2_t$. We thus have $V_t = \phi^0_t Z^0_t$ and

$$dV_t = V_{t-} (\tilde{Z}^0_t)^{-1} \, dZ^0_t + \phi^1_t \left(dZ^1_t - Z^1_t (Z^2_t)^{-1} \, dZ^2_t \right).$$

Let us denote $\bar{Z}^0_t = \tilde{Z}^0_t / Z^2_t$, $\bar{Z}^1_t = Z^1_t / Z^2_t$. The next result is a direct counterpart of Proposition 5.2.

Proposition 5.3 *The process V satisfies for $t \in [0, T]$*

$$V_t = Z^0_t \Big(\tilde{V}_0 + \int_0^t \hat{\phi}^1_u \, d\hat{Z}^1_u \Big)$$

where $\tilde{V}_0 = V_0 / \tilde{Z}^0_0 = V_0 / Z^0_0$ and

$$\hat{\phi}^1_t = \phi^1_t e^{\bar{\alpha}_t} / \bar{Z}^0_t, \ \hat{Z}^1_t = \bar{Z}^1_t e^{-\bar{\alpha}_t}, \ \bar{\alpha}_t = \langle \ln \bar{Z}^0, \ln \bar{Z}^1 \rangle_t.$$

5.4.2 Case B. Two defaultable tradable assets

Assume that Z^0 and Z^1 are defaultable tradable assets with zero recovery, and a common default time τ. Then $Z^0_t = \mathbb{1}_{\{\tau > t\}} \tilde{Z}^0_t$, $Z^1_t = \mathbb{1}_{\{\tau > t\}} \tilde{Z}^1_t$ for some processes $\tilde{Z}^0, \tilde{Z}^1$, that are assumed to be strictly positive, continuous, $\mathbb{F}$-semimartingales. In this case, for any trading strategy (ϕ^0, ϕ^1) we have

$$V_t = \phi^0_t Z^0_t + \phi^1_t Z^1_t = 0$$

on the set $\{\tau \leq t\}$, that is, after default. In principle, we may thus directly apply such a strategy to the replication of a defaultable claim with zero recovery. We say that (ϕ^0, ϕ^1) is self-financing provided that

$$dV_t = \phi^0_{t-} \, dZ^0_t + \phi^1_{t-} \, dZ^1_t.$$

Simple considerations show that the wealth process V satisfies

$$dV_t = (V_{t-} - \phi^1_{t-} Z^1_{t-})(Z^0_{t-})^{-1} \, dZ^0_t + \phi^1_{t-} \, dZ^1_t.$$

or, equivalently,

$$dV_t = (V_{t-} - \phi^1_{t-} \tilde{Z}^1_t)(\tilde{Z}^0_t)^{-1} \, dZ^0_t + \phi^1_{t-} \, dZ^1_t.$$

Proposition 5.4 *The wealth process V satisfies for $t \in [0, T]$*

$$V_t = Z^0_t \Big(\tilde{V}_0 + \int_0^t \phi^1_u \, dZ^{1*}_u \Big)$$

where $\tilde{V}_0 = V_0 / \tilde{Z}^0_0 = V_0 / Z^0_0$ and $Z^{1*}_t = \tilde{Z}^1_t / \tilde{Z}^0_t$.

Proof. It suffices to note that, setting $\tilde{V}_t = \phi^0_t \tilde{Z}^0_t + \phi^1_t \tilde{Z}^1_t$, we have $d\tilde{V}_t = \phi^0_t \, d\tilde{Z}^0_t + \phi^1_t \, d\tilde{Z}^1_t$. □

5.4.3 Case C. Single defaultable tradable asset and a single default-free asset

Let us finally consider the case of two tradable assets, with prices $Z^0_t = \mathbb{1}_{\{\tau>t\}}\tilde{Z}^0_t$ and Z^1_t, where $\tilde{Z}^0, Z^1$ are strictly positive, continuous, $\mathbb{F}$-semimartingales. We now have

$$V_t = \phi^0_t Z^0_t + \phi^1_t Z^1_t = \phi^0_t \mathbb{1}_{\{\tau>t\}}\tilde{Z}^0_t + \phi^1_t Z^1_t$$

and

$$dV_t = (V_{t-} - \phi^1_t Z^1_t)(\tilde{Z}^0_t)^{-1} dZ^0_t + \phi^1_t \, dZ^1_t.$$

It is clear that equality $V_t = 0$ on $\{\tau \le t\}$ implies that $\phi^1_t = 0$ for every $t \in [0,T]$. Therefore, $dV_t = V_{t-}(\tilde{Z}^0_t)^{-1} dZ^0_t$ and the possibility of replication of a defaultable claim with zero-recovery is unlikely within this setup (except for some trivial cases).

5.5 Replication and Valuation of Defaultable Claims

We shall first examine the possibility of an exact replication of a generic defaultable contingent claim with zero recovery. Formally, we consider a claim of the form $X = \mathbb{1}_{\{\tau>T\}}Y$, where Y is some $\mathcal{F}_T$-measurable random variable, representing the *promised payoff.*

5.5.1 Case A. Single defaultable tradable asset and two default-free assets

In view of Proposition 5.3, it is clear that a trading strategy (ϕ^0, ϕ^1, ϕ^2) replicates X whenever the following equality holds:

$$\tilde{Z}^0_T \Big(\tilde{V}_0 + \int_0^T \hat{\phi}^1_t \, d\hat{Z}^1_t \Big) = Y.$$

As usual, we say that a defaultable claim is attainable if it admits at least one replicating strategy.

Corollary 5.1 *Suppose that a defaultable claim X is attainable. Let $\hat{\mathbf{Q}}$ be a probability measure such that $\hat{Z}^1$ is an $\mathbb{F}$-martingale under $\hat{\mathbf{Q}}$. Then the value at time 0 of X equals*

$$V_0(X) = Z^0_0 \, \mathbf{E}_{\hat{\mathbf{Q}}}(Y/\tilde{Z}^0_T).$$

It is useful to notice that within the present setup the defaultable market is complete, provided that the underlying default-free market is complete (this assumption was done in Section 5.1.1). Generally speaking, the attainability of a defaultable claim X is equivalent to attainability of the promised payoff Y in the default-free market.

5.5.2 Case B. Two defaultable tradable assets

We shall now make use of Proposition 5.4. Let us consider two defaultable tradable assets Z^0 and Z^1, and an associated trading strategy (ϕ^0, ϕ^1). Clearly, it replicates a defaultable claim X whenever

$$\tilde{Z}^0_T \Big(\tilde{V}_0 + \int_0^T \phi^1_t \, dZ^{1*}_t \Big) = Y.$$

Corollary 5.2 *Suppose that a defaultable claim X is attainable. Let $\tilde{\mathbf{Q}}$ be a probability measure such that Z^{1*} is an $\mathbb{F}$-martingale under $\tilde{\mathbf{Q}}$. Then the value at time 0 of X equals*

$$V_0(X) = Z^0_0 \, \mathbf{E}_{\tilde{\mathbf{Q}}}(Y/\tilde{Z}^0_T).$$

From the viewpoint of market completeness, the situation in different than in the previous case. Indeed, a defaultable claim X is attainable if and only if the associated promised payoff Y can be replicated with the use of pre-default value processes $\widetilde{Z}^0$ and $\widetilde{Z}^1$. In addition, even if a default-free asset in introduced, a replicating strategy for an arbitrary defaultable claim will always involve a null position in this asset. Therefore, the introduction of a tradable default-free asset is not relevant if we restrict our attention to defaultable claims.

5.6 Equity Derivatives

The dynamics of the stock price S are given by (20). As the first example, we shall show how to value and hedge a vulnerable European call option with the terminal payoff

$$\hat{C}_T = \mathbb{1}_{\{\tau>T\}}(S_T-K)^+.$$

Formally, we have $\hat{C}_T = \tilde{C}_T$ where we denote

$$\tilde{C}_T = \mathbb{1}_{\{\tau>T\}}(S_T\mathbb{1}_{\{\tau>T\}}-K)^+ = (S_T\mathbb{1}_{\{\tau>T\}}-K)^+,$$

and thus the contract can also be seen as either a vulnerable or non-vulnerable option on a defaultable stock. We argue that the financial interpretation of a particular real-life derivative contract is of great importance here. To show this, we shall consider several possible models, with different choices of tradable assets that are used for hedging purposes, and we shall show that both the claim's price and its hedging strategy depends on the model's choice.

5.6.1 Case A. Single defaultable tradable asset and two default-free assets

We first consider the case of a vulnerable option written on a non-defaultable stock. Specifically, the stock price process is assumed to be tradable and default-free. In addition, we postulate that the default-free and defaultable zero-coupon bonds, maturing at time T, are also tradable.

Valuation. To value a vulnerable call option, it suffices to apply Corollary 5.1. Let us denote $F^S_t = S_t/B(t,T)$ and $\Gamma(t,T) = \tilde{D}^0(t,T)/B(t,T)$. It is important to observe that the process $\Gamma(t,T)$ is a continuous $\mathbb{F}$-submartingale, and it is an increasing process (in fact, an increasing function) if and only if the intensity γ is deterministic (either a deterministic or a random character of interest rates is not relevant here). Indeed, in this case $\Gamma(t,T) = \exp\big(-\int_t^T \gamma(u)\,du\big)$.

Corollary 5.1 yields

$$\hat{C}_0 = D^0(0,T)\,\mathbf{E}_{\hat{\mathbf{Q}}}(Y) = \Gamma(0,T)B(0,T)\,\mathbf{E}_{\hat{\mathbf{Q}}}(Y)$$

where $\hat{\mathbf{Q}}$ is the martingale measure for the process $\hat{S}_t = F^S_t e^{-\bar{\alpha}_t}$ where $\bar{\alpha}_t = \langle \ln\Gamma(t,T), \ln F^S\rangle_t$. If $\Gamma(t,T)$ is an increasing process, we have $\hat{S}_t = F^S_t$ is the forward price of the stock. If interest rate r is deterministic then (cf. (20))

$$d\hat{S}_t = \hat{S}_t\sigma\,dW^*_t, \quad \hat{S}_0 = S_0/B(0,T). \tag{26}$$

The price $\hat{C}_0$ thus equals $\Gamma(0,T)C_0$, where C_0 denotes the Black-Scholes price of a (non-vulnerable) European call. This result can be easily generalized to the case of random interest rates (e.g., within the Gaussian HJM framework)

Hedging. Let us now examine hedging of a vulnerable option. In general, the replicating strategy for X satisfies on the set $\{\tau>t\}$

$$\phi^0_t\tilde{D}^0(t,T) + \phi^1_t S_t + \phi^2_t B(t,T) = V_t(X)$$

and

$$\phi^0_t\,d\tilde{D}^0(t,T) + \phi^1_t\,dS_t + \phi^2_t\,dB(t,T) = dV_t(X).$$

To hedge perfectly the jump risk we need to take

$$\phi_t^0 = V_t(X)/\tilde{D}^0(t,T).$$

Therefore necessarily $\phi_t^1 S_t + \phi_t^2 B(t,T) = 0$. The *spread risk* (or, *volatility risk*) is hedged by matching the diffusion terms (recall that completeness of the default-free market was postulated). Notice that the trading strategy introduced above replicates the claim after the default time as well. The component ϕ^0 hedges the *jump risk* and the components ϕ^1, ϕ^2 the *spread risk.*

Formally, we consider a self-financing cash strategy (ϕ^0, ϕ^1, ϕ^2) such that at any time t there is zero net investment in stock and default-free bond, so that

$$\phi_t^1 S_t + \phi_t^2 B(t,T) = 0.$$

The following result is a straightforward consequence of Proposition 5.3.

Corollary 5.3 *Assume that the default intensity γ is deterministic. Then the process V satisfies for $t \in [0,T]$*

$$V_t = D^0(t,T)\Big(\frac{V_0}{D^0(0,T)} + \int_0^t \hat{\phi}_u^1 \, d\hat{S}_u\Big)$$

where $\hat{\phi}_t^1 = \phi_t^1/\Gamma(t,T)$ and $\hat{S}_t = F_t^S$.

Proof. Since $\Gamma(t,T)$ is of finite variation, we have $\bar{\alpha}_t = 0$, $\hat{\phi}_t^1 = \phi_t^1/\Gamma(t,T)$ and $\hat{S}_t = F_t^S$. □

Consider the defaultable claim $\hat{C}_T = \mathbb{1}_{\{\tau > T\}}(S_T - K)^+$. In view of Corollary 5.3, we need to find a constant c and a process $\hat{\phi}^1$ such that

$$c + \int_0^T \hat{\phi}_t^1 \, d\hat{S}_t = c + \int_0^T \hat{\phi}_t^1 \, dF_t^S = (S_T - K)^+.$$

It is clear that $\hat{\phi}^1$ coincides with the Black-Scholes replicating strategy and $V_0/D^0(0,T) = C_0/B(0,T)$, where C_0 is the Black-Scholes price of a European call option. Thus the price at time 0 of $\hat{C}_T$ equals $\Gamma(0,T)C_0$.

Corollary 5.4 *We have $\hat{\phi}^1 = \psi$ where ψ is the Black-Scholes hedge ratio for the call option. The pre-default price at time t of $\hat{C}_T$ satisfies $V_t(X) = \Gamma(t,T)C_t$.*

The component ϕ^0 of the replicating strategy for the vulnerable call option satisfies on the set $\{\tau > t\}$

$$\phi_t^0 = V_t(X)/\tilde{D}(t,T) = C_t/B(t,T).$$

Moreover $\phi_t^1 = \psi_t \Gamma(t,T)$ and $\phi_t^2 = -\psi_t \Gamma(t,T) F_t^S$.

Special case. Assume that r and γ are constant. Then

$$\phi_t^0 = C_t e^{r(T-t)}, \ \phi_t^1 = \psi_t e^{-\gamma(T-t)}, \ \phi_t^2 = -\psi_t e^{(r-\gamma)(T-t)} S_t$$

where $\psi_t = N(d_1(S_t, T-t))$ is the classic Black-Scholes delta of a European call option.

5.6.2 Case B. Two defaultable tradable assets

We shall now consider the payoff

$$\tilde{C}_T = \mathbb{1}_{\{\tau > T\}}(\tilde{S}_T \mathbb{1}_{\{\tau > T\}} - K)^+ = (\tilde{S}_T \mathbb{1}_{\{\tau > T\}} - K)^+$$

representing a (vulnerable or non-vulnerable) option written on a defaultable stock. To replicate this claim, we postulate that the stock price process is assumed to be a tradable, but defaultable, asset.

Thus the price process S of the stock admits the following generic representation $S_t = \mathbb{1}_{\{\tau > t\}} \tilde{S}_t$ where, by assumption, the pre-default price is governed by

$$d\tilde{S}_t = \tilde{S}_t \big(\mu_t \, dt + \sigma \, dW^*_t \big).$$

In addition, we postulate that that the defaultable zero-coupon bond with maturity T is tradable, with the price $D^0(t,T)$ and the pre-default price $\tilde{D}^0(t,T)$.

Valuation. The valuation procedure is based on Corollary 5.2. In the case of deterministic default intensity $\gamma(t)$, the martingale property of the process $S^*_t = \tilde{S}_t / \tilde{D}^0(t,T)$ under $\tilde{\mathbf{Q}}$ is equivalent to

$$d\tilde{S}_t = \tilde{S}_t \big((\gamma(t) + r_t) \, dt + \sigma \, dW^*_t \big).$$

The price at time 0 of the contract equals

$$\tilde{C}_0 = D^0(0,T) \, \mathbf{E}_{\tilde{\mathbf{Q}}}(Y) = \Gamma(0,T) B(0,T) \, \mathbf{E}_{\tilde{\mathbf{Q}}}(Y)$$

where $\tilde{\mathbf{Q}}$ is the martingale measure for the process S^*, or, more explicitly,

$$\tilde{C}_0 = \Gamma(0,T) B(0,T) \, \mathbf{E}_{\tilde{\mathbf{Q}}} (\tilde{S}_T - K)^+.$$

In general, we have

$$d\tilde{S}_t = \tilde{S}_t \big((\gamma_t + r_t + \sigma\beta_t - \beta^2_t) \, dt + \sigma \, dW^*_t \big)$$

where $\sigma\beta_t = \langle \ln \tilde{D}^0(\cdot,T), \ln \tilde{S} \rangle_t$ and $\beta^2_t = \langle \ln \tilde{D}^0(\cdot,T), \ln \tilde{D}^0(\cdot,T) \rangle_t$.

Special case. In the case of constant r and γ, the result is exactly the same as the Black-Scholes price of a European call, under the assumption that the interest rate equals $\hat{r} = r + \gamma$.

Hedging. Using Proposition 5.4, we arrive at the following equality for the wealth process V of a self-financing strategy:

$$V_t = D^0(t,T) \Big(\tilde{V}_0 + \int_0^t \phi^1_u \, dS^*_u \Big)$$

where $\tilde{V}_0 = V_0 / \tilde{S}_0 = V_0 / S_0$. Replication of the claim $\tilde{C}_T$ is thus equivalent to the following equality

$$c + \int_0^T \phi^1_t \, dS^*_t = (\tilde{S}_T - K)^+.$$

It is rather clear that in the case of constant r and γ, the hedging strategy will be exactly the same as in the Black-Scholes model, but with the default-free interest rate substituted with the risk-adjusted interest rate $\hat{r} = r + \gamma$.

5.7 Credit Derivatives

The most popular credit derivatives are: credit default swap, total rate of return swap and credit linked note. Furthermore, typical credit derivatives are linked to the default (credit) risk of several reference entities.

5.7.1 Credit Default Swap

A generic *credit default swap* is a derivative contract which allows to directly transfer the credit risk of the reference entity from one party (the risk seller) to another party (the risk buyer). The contingent payment is triggered by the default event, provided that it occurs before the maturity of the contract. In one version of a credit default swap, the contract is settled at time τ and the recovery payoff equals

$$Z_\tau = \big(1 - \delta B(\tau,T) \big).$$

The following two market conventions are common:

(1) The buyer pays a lump sum at inception (*default option*),

(2) The buyer pays annuities κ at the dates $0 < t_1, \dots, t_{n-1} < t_n = T$ prior to τ (*default swap*).

In case (1), the value V_0 of the default option equals

$$V_0 = \mathbf{E}_{\mathbf{Q}^*}\Big(B_\tau^{-1}\left(1 - \delta B(\tau, T)\right)\mathbb{1}_{\{\tau \le T\}}\Big).$$

In case (2), the value of the contract at time 0 should be equal to zero. The right level of the annuity κ can be found from the equality

$$V_0 = \kappa\, \mathbf{E}_{\mathbf{Q}^*}\Big(\sum_{i=1}^{n} B_{t_i}^{-1}\, \mathbb{1}_{\{t_i < \tau\}}\Big).$$

5.7.2 Digital Credit Default Swap

The fixed leg of a CDS can be represented as the sequence of payoffs:

$$c_i = \kappa \mathbb{1}_{\{\tau > t_i\}}$$

at the dates t_i for $i = 1, \dots, n$. The fixed leg of a CDS is thus simply a portfolio of defaultable zero-coupon bonds.

We consider a digital CDS, specifically, we postulate that the floating leg is represented by the following sequence of payoffs:

$$d_i = \delta \mathbb{1}_{\{t_i < \tau \le t_{i+1}\}} = \delta \mathbb{1}_{\{\tau \le t_{i+1}\}} - \delta \mathbb{1}_{\{\tau \le t_i\}}$$

at the dates t_{i+1} for $i = 1, \dots, n-1$. Clearly

$$d_i = \delta(1 - \mathbb{1}_{\{\tau > t_{i+1}\}}) - \delta(1 - \mathbb{1}_{\{\tau > t_i\}})$$

It thus suffices to value and replicate the payoff of the form $\mathbb{1}_{\{\tau > t_i\}}$ which occurs at time t_{i+1}.

5.7.3 Delayed Defaultable Bond

Let us consider the payoff of the form $\mathbb{1}_{\{\tau > T\}}$ which occurs at time $U > T$. To replicate this claim, it suffices to assume that default-free bonds with maturities T and U, as well as the defaultable bond with maturity T are tradable.

The payoff $\mathbb{1}_{\{\tau > T\}}$ at time U is equivalent to $B(T, U)\mathbb{1}_{\{\tau > T\}}$ at time T. We are in the position to apply Proposition 5.3 and Corollary 5.1. Thus we have $\phi_t^1 B(t, U) + \phi_t^2 B(t, T) = 0$ and the total pre-default value is invested in defaultable bonds. Moreover,

$$\bar{\alpha}_t = \int_0^t d(s, T) \cdot b(s, T)\, ds$$

where $d(s, T)$ and $b(s, T)$ are the volatilities of $\tilde{D}^0(t, T)/B(t, T)$ and $B(t, U)/B(t, T)$. These volatilities depend on the model of default-free term structure and on the default intensity.

5.7.4 Basket Credit Derivatives

We shall now examine very briefly the valuation and hedging of *basket* credit derivatives. The main issue arising in this context is the modelling of dependent ("correlated") defaults. We shall focus on the intensity-based approach to this problem (see, for instance, Kusuoka (1999), Jarrow and Yu (2000) or Bielecki and Rutkowski (2001) for more details).

Basic assumption: the reference filtration is trivial (no "volatility risk" is present). It can be shown that the jump risk can be hedged using the underlying defaultable zero-coupon bonds with default times $\tau_1, \dots, \tau_n$.

Recovery schemes and the values of (deterministic) recovery rates should be specified. This does not make the problem more complicated if the recovery schemes for the derivative contract and for the hedging instruments are the same (with possibly different values of recovery rates).

Let us describe briefly Kusuoka's (1999) construction of default times (for details, see Chapter 7 in Bielecki and Rutkowski (2002)) Under the original probability $\mathbf{Q}$ the random times τ_i, $i = 1, 2$ are independent random variables with exponential laws with parameters λ_1 and λ_2, resp. For a fixed $T > 0$, we define

$$\frac{d\mathbf{Q}^*}{d\mathbf{Q}} = \eta_T, \quad \mathbf{Q}\text{-a.s.}$$

where η_t, $t \in [0, T]$, satisfies

$$\eta_t = 1 + \sum_{i=1}^{2} \int_{]0,t]} \eta_{u-} \kappa_u^i \, d\hat{M}_u^i$$

where $\hat{M}_t^i = \mathbb{1}_{\{\tau_i \le t\}} - \int_0^{t \wedge \tau_i} \lambda_i \, du$ and

$$\kappa_t^1 = \mathbb{1}_{\{\tau_2 < t\}} \left(\frac{\alpha_1}{\lambda_1} - 1 \right), \quad \kappa_t^2 = \mathbb{1}_{\{\tau_1 < t\}} \left(\frac{\alpha_2}{\lambda_2} - 1 \right).$$

It appears that the 'martingale intensities' under $\mathbf{Q}^*$ are

$$\lambda_t^1 = \lambda_1 \mathbb{1}_{\{\tau_2 > t\}} + \alpha_1 \mathbb{1}_{\{\tau_2 \le t\}}, \ \lambda_t^2 = \lambda_2 \mathbb{1}_{\{\tau_1 > t\}} + \alpha_2 \mathbb{1}_{\{\tau_1 \le t\}}.$$

The following result shows that the intensities λ^1 and λ^2 can be interpreted as the *local intensities* of default with respect to the information available at time t, and thus the model can be reformulated as a two-dimensional Markov chain (cf. Lando (1998)).

Proposition 5.5 *For $i = 1, 2$ and every $t \in [0, T]$ we have*

$$\lambda_i = \lim_{h \downarrow 0} h^{-1} \, \mathbf{Q}^* \{ t < \tau_i \le t + h \,|\, \tau_1 > t, \, \tau_2 > t \}.$$

Moreover:

$$\alpha_1 = \lim_{h \downarrow 0} h^{-1} \, \mathbf{Q}^* \{ t < \tau_1 \le t + h \,|\, \tau_1 > t, \, \tau_2 \le t \}$$

and

$$\alpha_2 = \lim_{h \downarrow 0} h^{-1} \, \mathbf{Q}^* \{ t < \tau_2 \le t + h \,|\, \tau_2 > t, \, \tau_1 \le t \}.$$

The next step is the valuation of defaultable bonds. As before, we assume that the defaultable bond is subject to zero recovery rule. Then the price of the bond issued by the i^{th} firm is given by

$$D_1^0(t, T) = \mathbf{Q}^* (\tau_1 > T \,|\, \mathcal{H}_t^1 \vee \mathcal{H}_t^2)$$

provided that the interest rate $r = 0$. The case of non-zero (even random) interest rates can be dealt with in an analogous way.

Proposition 5.6 *Assume that $\lambda_1 + \lambda_2 - \alpha_1 \neq 0$. Then on the set $\{\tau_1 > t\}$ the pre-default bond price $\tilde{D}_1^0(t, T)$ equals*

$$\tilde{D}_1^0(t, T) = \mathbb{1}_{\{\tau_2 > t\}} \frac{1}{\lambda - \alpha_1} \left(\lambda_2 e^{-\alpha_1 (T-t)} + (\lambda_1 - \alpha_1) e^{-\lambda (T-t)} \right) + \mathbb{1}_{\{\tau_2 \le t\}} \, e^{-\alpha_1 (T-t)}$$

where $\lambda = \lambda_1 + \lambda_2$.

Of course, an analogous formula holds for the price $D_2^0(t,T)$ of the bond issued by the second firm. As an example of a basket credit derivative, we consider the first-to-default zero-coupon bond under zero recovery rule. Thus we deal with the claim $X = \mathbb{1}_{\{\tau > T\}}$ where $\tau = \min(\tau_1, \tau_2)$. As hedging instruments we shall take defaultable zero-coupons with default times τ_1 and τ_2.

Replicating strategy: at time t we hold ϕ_t^i units of the bond defaulting at τ_i where $(j \neq i)$

$$\phi_t^i = \frac{\lambda_1 \lambda_2 \alpha_j}{(\alpha_1 - \lambda_1)(\alpha_2 - \lambda_2) - \lambda_1 \lambda_2}\,.$$

On the set $\{\tau > t\}$ we have

$$V_t(X) = \phi_t^1 \tilde{D}_1^0(t,T) - \phi_t^2 \tilde{D}_2^0(t,T)$$

where $\tilde{D}_i^0(t,T)$ are pre-default values of defaultable bonds. On the set $\{\tau \leq t\}$ the wealth of the replicating strategy as well as the value $V_t(X)$ vanish.

5.8 Conclusions

5.8.1 Complete Case: Replication

- Specification of essential contractual features of a credit-risk-sensitive contract under study.
- Identification of risks: in most cases both the market and the credit risks are involved.
- Choice of the most convenient and adequate model: structural, reduced-form or hybrid.
- Arbitrage-free valuation with respect to a martingale measure.
- Identification of tradable (liquid) instruments (e.g., corporate bonds) that can be used for hedging.
- Construction of a self-financing strategy, which replicates the value of a contract up to default time.
- Calibration of the model to market prices of liquid instruments.

Drawback: since the method relies on the exact specification of the model for the underlying default-free market and the default time, the valuation and (in-model) hedging of credit derivatives is merely quasi-dynamic.

5.8.2 Incomplete Case: Mean-Variance Hedging

If the set of tradable assets is not a defaultable market is incomplete, and thus exact replication of defaultable claims is not feasible, in general. Therefore, the issue of imperfect hedging of contingent claims arises in a natural way in this context. For more details on the mean-variance hedging of credit derivatives, the interested reader is referred to Bielecki and Jeanblanc (2002).

References

[1] A. Bélanger, S. Shreve, D. Wong (2001) A unified model for credit derivatives. To appear in *Mathematical Finance.*

[2] T. Bielecki, M. Jeanblanc (2002) Mean-variance hedging of credit risk: a case study. Working paper.

[3] T. Bielecki, M. Rutkowski (2001) Dependent defaults and credit migrations. To appear in *Applicationes Mathematicae.*

[4] T. Bielecki, M. Rutkowski: *Credit Risk: Modelling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York, 2002.

[5] F. Black, J.C. Cox (1976) Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance* 31, 351–367.

[6] F. Black, M. Scholes (1973) The pricing of options and corporate liabilities. *J. Political Econ.* 81, 637–654.

[7] C. Blanchet-Scalliet, M. Jeanblanc (2000) Hazard rate for credit risk and hedging defaultable contingent claims. To appear in *Finance and Stochastics.*

[8] P. Collin-Dufresne and J.-N. Hugonnier (1999) On the pricing and hedging of contingent claims in the presence of extraneous risks. Working paper, Carnegie Mellon University.

[9] D. Duffie, D. Lando (2001) The term structure of credit spreads with incomplete accounting information. *Econometrica* 69, 633–664.

[10] R.J. Elliott, M. Jeanblanc, M. Yor (2000) On models of default risk. *Mathematical Finance* 10, 179–195.

[11] Y. Greenfield (2000) Hedging of credit risk embedded in derivative transactions. Thesis, Carnegie Mellon University.

[12] F. Jamshidian (2002) Valuation of credit default swap and swaptions. Working paper.

[13] M. Jeanblanc, M. Rutkowski (2002) Default risk and hazard processes. In: *Mathematical Finance – Bachelier Congress 2000.* H. Geman, D. Madan, S.R. Pliska and T. Vorst, eds., Springer-Verlag, Berlin Heidelberg New York, pp. 281–312.

[14] S. Kusuoka (1999) A remark on default risk models. *Adv. Math. Econom.* 1, 69-82.

[15] D. Lando (1998) On Cox processes and credit-risky securities. *Rev. Derivatives Res.* 2, 99–120.

[16] C. Lotz (1998) Locally minimizing the credit risk. Working paper, University of Bonn.

[17] S. Lukas (2001) On pricing and hedging defaultable contingent claims. Thesis, Humboldt University.

[18] R. Merton (1974) On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* 29, 449–470.

[19] P.J. Schönbucher (1998) Pricing credit risk derivatives. Working paper, University of Bonn.

[20] N. Vaillant (2001) A beginner's guide to credit derivatives. Working paper, Nomura International.

[21] D. Wong (1998) A unifying credit model. Working paper, Research Advisory Services, Capital Markes Group, Scotia Capital Markets.

[22] C. Zhou (1996) A jump-diffusion approach to modeling credit risk and valuing defaultable securities. Working paper, Federal Reserve Board, Washington.