

# MEAN-VARIANCE HEDGING OF DEFAULTABLE CLAIMS

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## 1 Introduction

In this note, we formulate a new paradigm for pricing and hedging financial risks in incomplete markets, rooted in the classical Markowitz mean-variance portfolio selection principle. We consider an underlying market of liquid financial instruments that are available to an investor (or an agent) for investment. We assume that the *underlying market* is arbitrage-free and complete. We also consider an investor who is interested in dynamic selection of her portfolio, so that the expected value of her wealth at the end of the pre-selected planning horizon is no less than some floor value, and so that the associated risk, as measured by the variance of the wealth at the end of the planning horizon, is minimized.

When a new investment opportunity becomes available for the agent, in a form of some contingent claim, she needs to decide how much she is willing to pay for acquiring the opportunity. More specifically, she has to decide what portion of her current endowment she is willing to invest in a new opportunity. It is assumed that the new claim, if acquired, is held until the horizon date, and the remaining part of the endowment is dynamically invested in primary (liquid) assets.

If the cash-flows generated by the new opportunity can be perfectly replicated by the existing liquid market instruments already available for trading, then the price of the opportunity will be uniquely determined by the wealth of the replicating trading strategy. However, if the perfect replication is not possible, then the determination of a purchase (or bid) price that the investor is willing to pay for the opportunity, will become subject to the investor's overall attitude towards trading. In case of our investor, the bid price and the corresponding hedging strategy will be determined in accordance with the mean-variance paradigm. Analogous remarks apply to an investor who engages in creation of an investment opportunity and needs to decide about its selling (or ask) price.

As explained above, it suffices to focus on a situation when the newly available investment opportunity can not be perfectly replicated by the instruments existing in the underlying market. Thus, the emerging investment opportunity is not *attainable*, and consequently the market model (that is the underlying market and new investment opportunities) is *incomplete*. It is well known (see, e.g., El Karoui and Quenez (1995) or Kramkov (1996)) that when a market is incomplete then for any non-attainable contingent claim  $X$  there exists a non-empty interval of arbitrage prices, determined by the maximum bid price  $\pi^u(X)$  (the upper price) and the minimum ask price  $\pi^l(X)$  (the lower price). The maximum bid price represents the price of the most expensive portfolio that can be used to perfectly hedge the long position in the contingent claim. The minimum ask price represents the price of the cheapest portfolio that can be used to perfectly hedge the short position in the contingent claim. From another perspective, the maximum bid price is the maximum amount that the agent purchasing the contingent claim can afford to pay for the claim, and still be sure to find an admissible portfolio that would fully manage her debt and repay it with cash flows generated by the strategy and the contingent claim, and end up with a non-negative wealth at the maturity date of the claim. Likewise, the minimum ask price is the minimum amount that the agent selling the claim can afford to accept to charge for the claim, and still be sure to find an admissible portfolio that would generate enough cash flow to make good on her commitment to buyer of the claim, and end up with a non-negative wealth at the maturity date of the claim.

As is well known, the arbitrage opportunities are precluded if and only if the actual price of the contingent claim belongs to the arbitrage interval. But this means, of course, that perfect hedging will not be accomplished by neither the short party, nor by the long party. Thus, any price that precludes arbitrage, enforces possibility of a financial loss for either party at the maturity date. This observation gave rise to quite abundant literature regarding selection a specific price within the arbitrage interval by means of minimizing some functional that assesses the risk associated with potential loss. We shall not be discussing this extensive literature here. We shall only observe that much work within this line of research has been done with regard to so called *mean-variance hedging*; we refer to the recent paper by Schweizer (2001) for an exhaustive survey of relevant results. The

interpretation of the term “mean-variance hedging”, as defined in these works, is entirely different from what we mean here by a mean-variance hedging.

## 2 Mean-Variance Pricing and Hedging

We consider an economy in continuous time,  $t \in [0, T^*]$ , and the underlying probability space  $(\Omega, \mathcal{G}, \mathbb{P})$  endowed with a one-dimensional standard Brownian motion  $W$  (with respect to its natural filtration) The probability  $\mathbb{P}$  plays the role of the statistical probability. We denote by  $\mathbb{F}$  the  $\mathbb{P}$ -augmentation of the filtration generated by  $W$ . In our economy, we consider an agent who initially has two liquid assets available to invest in:

- a risky asset whose price dynamics under  $\mathbb{P}$  are

$$dZ_t^1 = Z_t^1(\nu dt + \sigma dW_t), \quad Z_0^1 > 0,$$

for some constants  $\nu$  and  $\sigma > 0$ ,

- a money market account whose price dynamics under  $\mathbb{P}$  are

$$dZ_t^2 = rZ_t^2 dt, \quad Z_0^2 = 1,$$

where  $r$  is a constant interest rate.

Suppose for the moment that  $\mathcal{G} = \mathcal{F}_{T^*}$ . It is well known that in this case the *underlying market*, consisting of the two above assets, is complete. Thus the fair value of any claim contingent  $X$  which settles at time  $T \leq T^*$ , and thus is formally defined as an  $\mathcal{F}_T$ -measurable random variable, is the (unique) no-arbitrage price of  $X$ , denoted as  $\pi_0(X)$  in what follows.

Now let  $\mathbb{H}$  be another filtration in  $(\Omega, \mathcal{G}, \mathbb{P})$ , which satisfies the usual conditions. We consider the enlarged filtration  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$  and we postulate that  $\mathcal{G} = \mathcal{G}_{T^*}$ . We shall refer to  $\mathbb{G}$  as to the *full filtration*; the Brownian filtration  $\mathbb{F}$  will be called the *reference filtration*. We make an important assumption that  $W$  is a standard Brownian motion with respect to  $\mathbb{G}$  under  $\mathbb{P}$ .

Let  $\phi_t^i$  represent the number of shares of asset  $i$  held in the agent’s portfolio at time  $t$ . We consider trading strategies  $\phi = (\phi^1, \phi^2)$ , where  $\phi^1$  and  $\phi^2$  are  $\mathbb{G}$ -predictable processes. A strategy  $\phi$  is *self-financing* if

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^1 dZ_u^1 + \int_0^t \phi_u^2 dZ_u^2, \quad \forall t \in [0, T^*],$$

where  $V_t(\phi) = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2$  is the *wealth* of  $\phi$  at time  $t$ . Thus, we postulate the absence of outside endowments and/or consumption.

**Definition 2.1** We say that a self-financing strategy  $\phi$  is *admissible* on the interval  $[0, T]$  if and only if for any  $t \in [0, T]$  the wealth  $V_t(\phi)$  is a  $\mathbb{P}$ -square-integrable random variable.

The condition

$$\mathbb{E}_{\mathbb{P}} \left( \int_0^T (\phi_u^i Z_u^i)^2 du \right) < \infty, \quad i = 0, 1,$$

is manifestly sufficient for the admissibility of  $\phi$  on  $[0, T]$ . Let us fix  $T$  and let us denote by  $\Phi(\mathbb{G})$  the linear space of all admissible trading strategies on  $[0, T]$ .

Suppose that the agent has at time  $t = 0$  a positive amount  $v > 0$  available for investment. It is easily seen that for any  $\phi \in \Phi(\mathbb{G})$  the wealth process satisfies the following SDE

$$dV_t^v(\phi) = rV_t^v(\phi) dt + \phi_t^1 (dZ_t^1 - rZ_t^1 dt), \quad V_0^v(\phi) = v.$$

This shows that the wealth at time  $t$  depends exclusively on the initial wealth  $v$  and the component  $\phi^1$  of a self-financing strategy  $\phi$ .

Now, imagine that a new investment opportunity becomes available for the agent. Namely, the agent may purchase at time  $t = 0$  a contingent claim  $X$ , whose corresponding cash-flow of  $X$  units of cash occurs at time  $T$ . We assume that  $X$  is not an  $\mathcal{F}_T$ -measurable random variable. Notice that this requirement alone may not suffice for the non-attainability of  $X$ . Indeed, in the present setup, we have the following definition of attainability.

**Definition 2.2** A contingent claim  $X$  is *attainable* if there exists a strategy  $\phi \in \Phi(\mathbb{G})$  such that  $X = V_T(\phi)$  or, equivalently,

$$X = V_0(\phi) + \int_0^T \phi_u^1 dZ_u^1 + \int_0^T \phi_u^2 dZ_u^2.$$

If a claim  $X$  can be replicated by means of a trading strategy  $\phi \in \Phi(\mathbb{F})$ , we shall say that  $X$  is  $\mathbb{F}$ -attainable. According to the definition of admissibility, the square-integrability of  $X$  under  $\mathbb{P}$  is a necessary condition for attainability. Notice, however, that it may happen that  $X$  is not an  $\mathcal{F}_T$ -measurable random variable, but it represents an attainable contingent claim according to the definition above.

Suppose now that a considered claim  $X$  is not attainable. The main question that we want to study is: how much would the agent be willing to pay at time  $t = 0$  for  $X$ , and how the agent should hedge her investment? A symmetric study can be conducted for an agent creating such an investment opportunity by selling the claim. In what follows, we shall first present our results in a general framework of a generic  $\mathcal{G}_T$ -measurable claim; then we shall examine a particular case of defaultable claims.

## 2.1 Mean-Variance Portfolio Selection

We postulate that the agent's objective for investment is based on the classical *mean-variance portfolio selection*. Specifically, for any fixed date  $T$ , any initial wealth  $v > 0$ , and any given  $d \in \mathbb{R}$ , the agent is interested in solving the following problem:

**Problem MV( $d, v$ ):** Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$  over all strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) \geq d$ .

We shall show that, given the parameters  $d$  and  $v$  satisfy certain additional conditions, the above problem admits a solution, so that there exists an optimal trading strategy, say  $\phi^*(d, v)$ . Let  $V^*(d, v) = V(\phi^*(d, v))$  stand for the optimal wealth process, and let us denote by  $w^*(d, v)$  the value of the variance  $\mathbb{V}_{\mathbb{P}}(V_T^*(d, v))$ .

For simplicity of presentation, we did not postulate above that agent's wealth should be non-negative at any time. Problem MV( $d, v$ ) with this additional restriction has been recently studied in Bielecki et al. (2004b).

**Remarks.** It is apparent that the problem MV( $d, v$ ) is non-trivial only if  $d > ve^{rT}$ . Otherwise, investing in the money market alone generates the wealth process  $V_t^v(\phi) = ve^{rt}$ , that obviously satisfies the terminal condition  $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) = ve^{rT} \geq d$ , and for which the variance of the terminal wealth  $V_T^v(\phi)$  is zero. Thus, when considering the problem MV( $d, v$ ) we shall always assume that  $d > ve^{rT}$ . Put another way, we shall only consider trading strategies  $\phi$  for which the expected return satisfies  $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi)/v) \geq e^{rT}$ , that is, it is strictly higher than the return on the money market account.

Assume that a claim  $X$  is available for purchase at time  $t = 0$ . We postulate that the random variable  $X$  is  $\mathcal{G}_T$ -measurable and square-integrable under  $\mathbb{P}$ . The agent shall decide whether to

purchase  $X$ , and what is the maximal price she could offer for  $X$ . According to the mean-variance paradigm, her decision will be based on the following reasoning. First, for any  $p \in [0, v]$  the agent needs to solve the related mean-variance problem.

**Problem MV**( $d, v, p, X$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$  over all trading strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$ .

We shall show that if  $d, v, p$  and  $X$  satisfy certain sufficient conditions, then there exists an optimal strategy, say  $\phi^*(d, v, p, X)$ , for this problem. We denote by  $V_T^*(d, v, p, X)$  the value of  $V_T^{v-p}(\phi^*(d, v, p, X))$  and we set  $w^*(d, v, p, X) = \mathbb{V}_{\mathbb{P}}(V_T^*(d, v, p, X) + X)$ .

It is reasonable to expect that the agent will be willing to pay for the claim  $X$  the price that is no more than (by convention,  $\sup \emptyset = -\infty$ )

$$p^{d,v}(X) := \sup \{p \in [0, v] : \text{MV}(d, v, p, X) \text{ admits a solution and } w^*(d, v, p, X) \leq w^*(d, v)\}.$$

This leads to the following definition of mean-variance price and hedging strategy.

**Definition 2.3** The number  $p^{d,v}(X)$  is called the (*buying*) *agent's mean-variance price* of a claim  $X$ . The corresponding optimal strategy  $\phi^*(d, v, p^{d,v}(X), X)$  is called the agent's *mean-variance hedging strategy* for  $X$ .

Of course, in order to make the last definition operational, we need to be able to solve explicitly problems  $\text{MV}(d, v)$  and  $\text{MV}(d, v, p, X)$ , at least in some special cases of a common interest. These issues will be examined in some detail in the remaining part of this note, first for the special case of  $\mathbb{F}$ -adapted trading strategies (see Section 3), and subsequently, in the general case of  $\mathbb{G}$ -adapted strategies (see Section 4).

**Remarks.** Let us denote  $\mu_X = \mathbb{E}_{\mathbb{P}}X$ . Inequality  $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$  is equivalent to  $\mathbb{E}_{\mathbb{P}}V_T^{v-p}(\phi) \geq d - \mu_X$ . Observe that, unlike as in the case of the problem  $\text{MV}(d, v)$ , the problem  $\text{MV}(d, v, p, X)$  may be non-trivial even if  $d - \mu_X \leq e^{rT}(v - p)$ . Although investing in a money market alone will produce in this case a wealth process for which the condition  $\mathbb{E}_{\mathbb{P}}V_T^{v-p}(\phi) \geq d - \mu_X$  is manifestly satisfied, the corresponding variance  $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) = \mathbb{V}_{\mathbb{P}}(X)$  is not necessarily minimal.

## 2.2 Comments

Let us denote by  $\mathcal{N}(X)$  the no-arbitrage interval for the claim  $X$ , that is,  $\mathcal{N}(X) = [\pi^l(X), \pi^u(X)]$ . It may well happen that the mean-variance price  $p^{d,v}(X)$  is outside this interval. Since this possibility may appear as an unwanted feature of the approach to pricing and hedging presented in this note, we shall comment briefly on this issue. When we consider the valuation of a claim  $X$  from the perspective of the entire market, then we naturally apply the no-arbitrage paradigm.

According to the no-arbitrage paradigm, the financial market as a whole will accept only those prices of a financial asset, which fall into the no-arbitrage interval. Prices from outside this interval can't be sustained in a longer term due to market forces, which will tend to eliminate any arbitrage opportunity.

Now, let us consider the same issue from the perspective of an individual. Suppose that an individual investor is interested in putting some of her initial endowment  $v > 0$  into an investment opportunity provided by some claim  $X$ . Thus, the investor needs to decide whether to acquire the investment opportunity, and if so then how much to pay for it, based on her overall attitude towards risk and reward.

The number  $p^{d,v}(X)$  is the price that investor is willing to pay for the investment opportunity  $X$ , given her initial capital  $v$ , given her attitude towards risk and reward, and given the primary market. The investor "submits" her price to the market. Now, suppose that the market recognized

no-arbitrage interval for  $X$  is  $\mathcal{N}(X)$ . If it happens that  $p \in \mathcal{N}(X)$  then the investor's bid price for  $X$  can be accepted by the market. In the opposite case, the investor's bid price may not be accepted by the market, and the investor may not enter into the investment opportunity.

### 3 Strategies Adapted to the Reference Filtration

In this section, we shall first solve the problem  $MV(d, v)$  under the restriction that trading strategies are  $\mathbb{F}$ -adapted. In other words, we postulate that  $\phi$  belongs to the class  $\Phi(\mathbb{F})$  of all admissible and  $\mathbb{F}$ -predictable strategies  $\phi$ . In this case, we shall say that a strategy  $\phi$  is  $\mathbb{F}$ -admissible. The assumption that  $\phi$  is  $\mathbb{F}$ -admissible implies, of course, that the corresponding terminal wealth  $V_T^v(\phi)$  is an  $\mathcal{F}_T$ -measurable random variable.

#### 3.1 Solution to $MV(d, v)$ in the Class $\Phi(\mathbb{F})$

A general version of the problem  $MV(d, v)$  has been studied in Bielecki et al. (2004b). Because our problem is a very special version of the general one, we give below a complete solution tailored to present set-up.

##### 3.1.1 Reduction to Zero Interest Rate Case

Recall our standing assumption that  $d > ve^{rT}$ . Problem  $MV(d, v)$  is clearly equivalent to: minimize the variance  $\mathbb{V}_{\mathbb{P}}(e^{-rT}V_T(\phi))$  under the constraint

$$\mathbb{E}_{\mathbb{P}}(e^{-rT}V_T(\phi)) \geq e^{-rT}d.$$

For the sake of notational simplicity, we shall write  $V_t$  instead of  $V_t^v(\phi)$ . We set  $\tilde{V}_t = V_t(Z_t^2)^{-1} = e^{-rt}V_t$ , so that

$$d\tilde{V}_t = \phi_t^1 d\tilde{Z}_t^1 = \phi_t^1 \tilde{Z}_t^1 (\hat{\nu}dt + \sigma dW_t) \quad (1)$$

where we denote  $\hat{\nu} = \nu - r$ . So we can and do restrict our attention to the case  $r = 0$ . Thus, in what follows, we shall have  $Z_t^2 = 1$  for every  $t \in \mathbb{R}_+$ . In the rest of this note, unless explicitly stated otherwise, we assume that  $d > v$ .

##### 3.1.2 Decomposition of the Problem $MV(d, v)$

Let  $\mathbb{Q}$  be a (unique) equivalent martingale measure on  $(\Omega, \mathcal{F}_{T^*})$  for the underlying market. It is easily seen that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \eta_t, \quad \forall t \in [0, T^*],$$

where we denote by  $\eta$  the Radon-Nikodým density process. Specifically, we have

$$d\eta_t = -\theta\eta_t dW_t, \quad \eta_0 = 1, \quad (2)$$

or, equivalently,

$$\eta_t = \exp\left(-\theta W_t - \frac{1}{2}\theta^2 t\right),$$

where  $\theta = \nu/\sigma$  (recall that we have formally reduced the problem to the case  $r = 0$ ). The process  $\eta$  is a  $\mathbb{F}$ -martingale under  $\mathbb{P}$ . Moreover,

$$\mathbb{E}_{\mathbb{P}}(\eta_T^2 | \mathcal{F}_t) = \eta_t^2 \exp(\theta^2(T-t)), \quad (3)$$

and thus  $\mathbb{E}_{\mathbb{P}}(\eta_t^2) = \exp(\theta^2 t)$  for  $t \in [0, T^*]$ . It is easily seen that the price  $Z^1$  is an  $\mathbb{F}$ -martingale under  $\mathbb{Q}$ , since

$$dZ_t^1 = \sigma Z_t^1 d(W_t + \theta t) = \sigma Z_t^1 d\tilde{W}_t \quad (4)$$

for the  $\mathbb{Q}$ -Brownian motion  $\widetilde{W}_t = W_t + \theta t$ . The measure  $\mathbb{Q}$  is thus the equivalent martingale measure for our primary market.

From (1), we have that

$$V_t = v + \int_0^t \phi_u^1 dZ_u^1 = v + \int_0^t \phi_u^1 \sigma Z_u^1 d\widetilde{W}_u. \quad (5)$$

Recall that if  $\phi$  is an  $\mathbb{F}$ -admissible strategy, that is,  $\phi \in \Phi(\mathbb{F})$ , then  $V_T$  is an  $\mathcal{F}_T$ -measurable random variable, which is square-integrable under  $\mathbb{P}$ .

Let  $X$  be a square-integrable under  $\mathbb{P}$  and  $\mathcal{F}_T$ -measurable random variable. It is easily seen that  $X$  is integrable with respect to  $\mathbb{Q}$  (since  $\eta_T$  is square-integrable with respect to  $\mathbb{P}$ ). The existence of a self-financing trading strategy that replicates  $X$  can be justified by the predictable representation theorem combined with the Bayes formula. We thus have the following result.

**Lemma 3.1** *Let  $X$  be a square-integrable under  $\mathbb{P}$  and  $\mathcal{F}_T$ -measurable random variable. Then  $X$  is an  $\mathbb{F}$ -attainable contingent claim, i.e., there exists a strategy  $\phi^X$  in  $\Phi(\mathbb{F})$  such that  $V_T(\phi) = X$ .*

We shall argue that problem  $MV(d, v)$  can be split into two problems (see also Pliska (2001) and Bielecki et al. (2004b) in this regard). We first focus on the optimal terminal wealth  $V_T^*(d, v)$ . Let  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  denote the collection of  $\mathbb{P}$ -square-integrable random variables that are  $\mathcal{F}_T$ -measurable. Thus the first problem we need to solve is:

**Problem MV1:** Minimize  $\mathbb{V}_{\mathbb{P}}(\xi)$  over all  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , subject to  $\mathbb{E}_{\mathbb{P}}\xi \geq d$  and  $\mathbb{E}_{\mathbb{Q}}\xi = v$ .

**Lemma 3.2** *Suppose that  $\phi^* = \phi^*(d, v)$  solves the problem  $MV(d, v)$ , and let  $V^*(d, v) = V(\phi^*)$ . Then the random variable  $\xi^* = V_T^*(d, v)$  solves the problem MV1.*

*Proof.* We argue by contradiction. Suppose that there exists a random variable  $\widehat{\xi} \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  such that  $\mathbb{E}_{\mathbb{P}}\widehat{\xi} \geq d$ ,  $\mathbb{E}_{\mathbb{Q}}\widehat{\xi} = v$  and  $\mathbb{V}_{\mathbb{P}}(\widehat{\xi}) < \mathbb{V}_{\mathbb{P}}(\xi^*)$ . Since  $\widehat{\xi}$  is square-integrable under  $\mathbb{P}$  and  $\mathcal{F}_T$ -measurable, it represents an attainable contingent claim, so that there exists an  $\mathbb{F}$ -admissible strategy  $\widehat{\phi}$  such that  $\widehat{\xi} = V_T(\widehat{\phi})$ . Of course, this contradicts the assumption that  $\phi^*$  solves  $MV(d, v)$ .  $\square$

Denoting by  $\xi^*$  the optimal solution to problem MV1, the second problem is:

**Problem MV2:** Determine an  $\mathbb{F}$ -admissible strategy  $\phi^*$  such that  $V_T(\phi^*) = \xi^*$ .

The next result is analogous to Theorem 2.1 in Bielecki et al. (2004b), and thus its proof is omitted. It demonstrates that solving problem  $MV(d, v)$  is indeed equivalent to successive solving problems MV1 and MV2.

**Proposition 3.1** *Suppose that the problem MV1 has a solution  $\xi^*$ . Then, the following BSDE*

$$dv_t = -\theta z_t dt + z_t dW_t, \quad v_T = \xi^*, \quad t \in [0, T], \quad (6)$$

*has a unique,  $\mathbb{P}$ -square-integrable solution, denoted as  $(v^*, z^*)$ , which is adapted to  $\mathbb{F}$ . Moreover, if we define a process  $\phi^{1*}$  by*

$$\phi_t^{1*} = z_t^*(\sigma Z_t^1)^{-1}, \quad \forall t \in [0, T],$$

*then the  $\mathbb{F}$ -admissible strategy  $\phi^* = (\phi^{1*}, \phi^{2*})$  with the wealth process  $V(\phi^*) = v^*$  solves  $MV(d, v)$ .*

For the last statement, recall that if the first component of a self-financing strategy  $\phi$  and its wealth process  $V(\phi)$  is known, then the component  $\phi^2$  is uniquely determined through the equality  $V_t(\phi) = \phi_t^1 Z_t^1 + \phi_t^2 Z_t^2$ .

**Remarks.** In what follows, we shall derive closed-form expressions for  $\phi^*$  and  $V(\phi^*)$ . It will be easily seen that the process  $V(\phi^*)$  is not only square-integrable with respect to  $\mathbb{P}$ , but also square-integrable with respect to  $\mathbb{Q}$ . It should be stressed that Proposition 3.1 will not be used in the derivation of a solution to problem  $MV(d, v)$ . In fact, we shall find a solution to  $MV(d, v)$  through explicit calculations.

### 3.1.3 Solution of the Problem MV1

In order to make the problem MV1 non-trivial, we need to make an additional assumption that  $\theta \neq 0$ . Indeed, if  $\theta = 0$  then we have  $\mathbb{P} = \mathbb{Q}$ , and thus the problem MV1 becomes: minimize  $\mathbb{V}_{\mathbb{P}}(\xi)$  over all  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , subject to  $\mathbb{E}_{\mathbb{P}}\xi \geq d$  and  $\mathbb{E}_{\mathbb{P}}\xi = v$ . It is easily seen that this problem admits a solution for  $d = v$  only, and the optimal solution is trivial, in the sense that the optimal variance is null. Consequently, for  $\theta = 0$ , the solution to  $MV(d, v)$  exists if and only if  $d = v$ , and it is trivial:  $\phi^* = (0, 1)$ . Let us reiterate that we postulate that  $d > v$  in order to avoid trivial solutions to  $MV(d, v)$ .

From now on, we assume that  $\theta \neq 0$ . We begin with the following auxiliary problem:

**Problem MV1A:** Minimize  $\mathbb{V}_{\mathbb{P}}(\xi)$  over all  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , subject to  $\mathbb{E}_{\mathbb{P}}\xi = d$  and  $\mathbb{E}_{\mathbb{Q}}\xi = v$ .

The last problem is manifestly equivalent to:

**Problem MV1B:** Minimize  $\mathbb{E}_{\mathbb{P}}\xi^2$  over all  $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , subject to  $\mathbb{E}_{\mathbb{P}}\xi = d$  and  $\mathbb{E}_{\mathbb{Q}}\xi = v$ .

Since  $\mathbb{E}_{\mathbb{Q}}\xi = \mathbb{E}_{\mathbb{P}}(\eta_T\xi)$ , the corresponding Lagrangian is

$$\mathbb{E}_{\mathbb{P}}(\xi^2 - \lambda_1\xi - \lambda_2\eta_T\xi) - d^2 + \lambda_1d + \lambda_2v.$$

The optimal random variable is given by  $2\xi^* = \lambda_1 + \lambda_2\eta_T$ , where the Lagrange multipliers satisfy

$$2d = \lambda_1 + \lambda_2, \quad 2v = \lambda_1 + \lambda_2 \exp(\theta^2 T).$$

Hence, we have

$$\xi^* = (de^{\theta^2 T} - v + (v - d)\eta_T)(e^{\theta^2 T} - 1)^{-1}, \quad (7)$$

and the corresponding minimal variance is

$$\mathbb{V}_{\mathbb{P}}(\xi^*) = \mathbb{E}_{\mathbb{P}}(\xi^*)^2 - d^2 = (d - v)^2(e^{\theta^2 T} - 1)^{-1}. \quad (8)$$

Since we assumed that  $d \geq v$ , the minimal variance is an increasing function of the parameter  $d$  for any fixed value of the initial wealth  $v$ , we conclude that we have solved not only the problem MV1A, but the problem MV1 as well. We thus have the following result.

**Proposition 3.2** *The solution  $\xi^*$  to problem MV1 is given by (7) and the minimal variance  $\mathbb{V}_{\mathbb{P}}(\xi^*)$  is given by (8).*

### 3.1.4 Solution of the Problem MV2

We maintain the assumption that  $\theta \neq 0$ . Thus, the optimal wealth for the terminal time  $T$  is given by (7), that is,  $V_T(\phi^*) = \xi^*$ . Our goal is to determine an  $\mathbb{F}$ -admissible strategy  $\phi^*$  for which the last equality is indeed satisfied. In view of (5), it suffices to find  $\phi^{1*}$  such that the process  $V_t^*$  given by

$$V_t^* = v + \int_0^t \phi_u^{1*} dZ_u^1 \quad (9)$$

satisfies  $V_T = \xi^*$ , and the strategy  $\phi^* = (\phi^{1*}, \phi^{2*})$ , where  $\phi^{2*}$  is derived from  $V_t = \phi_t^{1*} Z_t^1 + \phi_t^{2*} Z_t^2$ , is  $\mathbb{F}$ -admissible.

To this end, let us introduce an  $\mathbb{F}$ -martingale  $V$  under  $\mathbb{Q}$  by setting  $V_t = \mathbb{E}_{\mathbb{Q}}(\xi^* | \mathcal{F}_t)$  (the integrability of  $\xi^*$  under  $\mathbb{Q}$  is rather obvious). It is easy to see that  $V_T^* = \xi^*$  and  $V_0^* = v$ . It thus remains to find the process  $\phi^{1*}$ . Using (3), we obtain

$$V_t^* = (de^{\theta^2 T} - v + (v - d)\eta_t e^{\theta^2(T-t)})(e^{\theta^2 T} - 1)^{-1}.$$

Consequently, in view of (2) and (4), we have

$$\begin{aligned} dV_t^* &= \frac{v - d}{e^{\theta^2 T} - 1} (e^{\theta^2(T-t)} d\eta_t - \eta_t e^{\theta^2(T-t)} \theta^2 dt) \\ &= e^{\theta^2(T-t)} \frac{\theta \eta_t (v - d)}{e^{\theta^2 T} - 1} (dW_t - \theta dt) \\ &= e^{\theta^2(T-t)} \frac{d - v}{e^{\theta^2 T} - 1} \frac{\nu \eta_t}{\sigma^2} \frac{dZ_t^1}{Z_t^1}. \end{aligned}$$

This shows that we may choose

$$\phi_t^{1*} = e^{\theta^2(T-t)} \frac{d - v}{e^{\theta^2 T} - 1} \frac{\nu}{\sigma^2} \frac{\eta_t}{Z_t^1}. \quad (10)$$

It is clear that  $\phi^*$  is  $\mathbb{F}$ -admissible, since it is  $\mathbb{F}$ -adapted, self-financing, and  $V_t(\phi^*)$  is square-integrable under  $\mathbb{P}$  for every  $t \in [0, T]$ .

### 3.1.5 Solution of the Problem $MV(d, v)$

By virtue of Lemma 3.2, we conclude that  $\phi^*$  solves  $MV(d, v)$ . In view of (8), the variance under  $\mathbb{P}$  of the terminal wealth of the optimal strategy is

$$w^*(d, v) = \mathbb{E}_{\mathbb{P}}(V_T^*)^2 - d^2 = \frac{(d - v)^2}{e^{\theta^2 T} - 1}.$$

Let us stress that since we did not impose any no-bankruptcy condition, that is we do not require that the agent's wealth is non-negative, we see that  $d$  can be any number greater than  $v$ .

We are in the position to state the following result, which summarizes the analysis above. For a fixed  $T > 0$ , we denote  $\rho(\theta) = e^{\theta^2 T} (e^{\theta^2 T} - 1)^{-1}$  and  $\eta_t(\theta) = \eta_t e^{\theta^2 t}$ , so that  $\eta_0(\theta) = 1$ .

**Proposition 3.3** *Assume that  $\theta \neq 0$  and let  $d > v$ . Then a solution  $\phi^*(d, v) = (\phi^{*1}(d, v), \phi^{*2}(d, v))$  to  $MV(d, v)$  is given by*

$$\phi_t^{1*}(d, v) = (d - v) \rho(\theta) \frac{\nu \eta_t(\theta)}{\sigma^2 Z_t^1} \quad (11)$$

and  $V_t^*(d, v) = V_t(\phi^*(d, v)) = \phi_t^{*1}(d, v) Z_t^1 + \phi_t^{*2}(d, v)$ , where the optimal wealth process equals

$$V_t^*(d, v) = v + (d - v) \rho(\theta) (1 - \eta_t(\theta)). \quad (12)$$

The minimal variance  $w^*(d, v)$  is given by

$$w^*(d, v) = \mathbb{E}_{\mathbb{P}}(V_T^*(d, v))^2 - d^2 = \frac{(d - v)^2}{e^{\theta^2 T} - 1}. \quad (13)$$

Notice that the optimal trading strategy  $\phi^*(d, v)$ , the minimal variance  $w^*(d, v)$  and the optimal gains process  $G_t^*(d, v) = V_t^*(d, v) - v$  depend exclusively on the difference  $d - v > 0$ , rather than on parameters  $d$  and  $v$  themselves.

### 3.1.6 Efficient Portfolio

As it was observed above the function  $f(d) := w^*(d, v)$  is (strictly) increasing for  $d \geq v$ . Consider the following problem (as usual, for  $d \geq v$ ):

**Problem ME**( $d, v$ ): Maximize  $\mathbb{E}_{\mathbb{P}} V_T^v(\phi)$  over all  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{V}_{\mathbb{P}} V_T^v(\phi) = w^*(d, v)$ .

Denote the maximal expectation in the above problem by  $\mu^*(d, v)$ . In view of the strict monotonicity of the function  $f(d)$  for  $d \geq v$ , it is clear that  $\mu^*(d, v) = d$ . Consequently, the minimum variance portfolio  $\phi^*$  is in fact an *efficient portfolio*.

## 3.2 Solution to MV( $d, v, p, X$ ) in the class $\Phi(\mathbb{F})$

First, we consider the special case of an attainable claim, which is  $\mathcal{F}_T$ -measurable. Subsequently, we shall show that in general it suffices to decompose a general claim  $X$  into an attainable component  $\tilde{X} = \mathbb{E}_{\mathbb{P}}(X | \mathcal{F}_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ , and a component  $X - \tilde{X}$  which is orthogonal in  $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$  to the subspace  $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$  of admissible terminal wealths.

### 3.2.1 Case of an Attainable Claim

We shall verify that the mean-variance price coincides with the (unique) arbitrage price for any contingent claim that is attainable. Of course, this feature is a standard requirement for any reasonable valuation mechanism for contingent claims. Since in this section we consider only  $\mathbb{F}$ -adapted strategies, we postulate here that a claim  $X$  is  $\mathcal{F}_T$ -measurable; the general case of a  $\mathcal{G}_T$ -measurable claim is considered in Section 4.1.

Let  $\phi^X \in \Phi(\mathbb{F})$  be a replicating strategy for  $X$ , so that  $X$  is  $\mathbb{F}$ -attainable, and let  $\pi_0(X) = \mathbb{E}_{\mathbb{Q}} X$  be the arbitrage price of  $X$ . Since  $\Phi(\mathbb{F})$  is a linear space, it is easily seen that  $\Phi(\mathbb{F}) = \Phi(\mathbb{F}) + \phi^X = \Phi(\mathbb{F}) - \phi^X$ . The following lemma is thus easy to prove.

**Lemma 3.3** *Let  $X$  be an  $\mathbb{F}$ -attainable claim. Then the problem  $MV(d, v, p, X)$  is equivalent to the problem  $MV(d, \hat{v})$  with  $\hat{v} = v - p + \pi_0(X)$ .*

Equivalence of problems  $MV(d, v, p, X)$  and  $MV(d, \hat{v})$  is understood in the following way: first, the minimal variance for both problems is identical. Second, if a strategy  $\psi^*$  is a solution to  $MV(d, v - p + \pi_0(X))$ , then a strategy  $\phi^* = \psi^* - \phi^X$  is a solution to the original problem  $MV(d, v, p, X)$ .

**Corollary 3.1** *Suppose that an  $\mathcal{F}_T$ -measurable random variable  $X$  represents an  $\mathbb{F}$ -attainable claim.*

- (i) *If the arbitrage price  $\pi_0(X)$  satisfies  $\pi_0(X) \in [0, v]$  then  $p^{d,v}(X) = \pi_0(X)$ .*
- (ii) *If the arbitrage price  $\pi_0(X)$  is strictly greater than  $v$  then  $p^{d,v}(X) = v$ .*

*Proof.* By definition, the mean-variance price of  $X$  is the maximal value of  $p \in [0, v]$  for which  $w^*(d, v, p, X) = w^*(d, \hat{v}) \leq w^*(d, v)$ . Recall that we assume that  $d > v$  so that, in view of (17),

$$w^*(d, v) = \frac{(d - v)^2}{e^{\theta^2 T} - 1}.$$

By applying this result to  $MV(d, \hat{v})$  we obtain

$$w^*(d, v, p, X) = \frac{(d - v + p - \pi_0(X))^2}{e^{\theta^2 T} - 1}$$

provided that  $d > v - p + \pi_0(X)$ . Assume that  $p > \pi_0(X)$ . Then  $d > v - p + \pi_0(X)$  and thus  $w^*(d, v, p, X) > w^*(d, v)$  since manifestly  $(d - v)^2 > (d - v + p - \pi_0(X))^2$  in this case. This shows

that  $p^{d,v}(X) \leq \pi_0(X)$ . Of course, for  $p = \pi_0(X)$  we have the equality of minimal variances. We conclude that  $p^{d,v}(X) = \pi_0(X)$  provided that  $\pi_0(X) \in [0, v]$ . This completes the proof of part (i).

To prove part (ii), let us assume that  $\pi_0(X) > v$ . In this case, it suffices to take  $p = v$  and to check that  $w^*(d, v, v, X) = w^*(d, \pi_0(X)) \leq w^*(d, v)$ . This is again rather obvious since for  $v < \pi_0(X) < d$  we have  $(d - \pi_0(X))^2 < (d - v)^2$ , and for  $\pi_0(X) \geq d$  we have  $w^*(d, \pi_0(X)) = 0$ .  $\square$

### 3.2.2 Case of a Generic Claim

Consider an arbitrary  $\mathcal{G}_T$ -measurable claim  $X$ , which is square-integrable under  $\mathbb{P}$ . Recall that our goal is to solve the following problem for  $0 \leq p \leq v$ .

**Problem MV**( $d, v, p, X$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$  over all trading strategies  $\phi \in \Phi(\mathbb{F})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$ .

Let us denote by  $\tilde{X}$  the conditional expectation  $\mathbb{E}_{\mathbb{P}}(X | \mathcal{F}_T)$ . Then, of course,  $\mathbb{E}_{\mathbb{P}}\tilde{X} = \mathbb{E}_{\mathbb{P}}X$ . Moreover,  $\tilde{X}$  is an attainable claim and its arbitrage price at time 0 equals

$$\pi_0(\tilde{X}) = \mathbb{E}_{\mathbb{Q}}\tilde{X} = \mathbb{E}_{\mathbb{P}}(\eta_T \mathbb{E}_{\mathbb{P}}(X | \mathcal{F}_T)) = \mathbb{E}_{\mathbb{P}}(\eta_T X) = \mathbb{E}_{\mathbb{Q}}X,$$

where  $\mathbb{Q}$  is the martingale measure introduced in Section 3.1.2. Let  $\phi^{\tilde{X}}$  stand for the replicating strategy for  $\tilde{X}$  in the class  $\Phi(\mathbb{F})$ . Arguing as in the previous case, we conclude that the problem  $\text{MV}(d, v, p, X)$  is equivalent to the following problem. We set here  $\tilde{p} = p - \pi_0(\tilde{X})$ .

**Problem MV**( $d, v, \tilde{p}, X - \tilde{X}$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^{v-\tilde{p}}(\phi) + X - \tilde{X})$  over all trading strategies  $\phi \in \Phi(\mathbb{F})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^{v-\tilde{p}}(\phi) + X - \tilde{X}) \geq d$ .

Recall that  $\mathbb{E}_{\mathbb{P}}\tilde{X} = \mathbb{E}_{\mathbb{P}}X$  and denote  $\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \tilde{X})$ . Observe that for any  $\phi \in \Phi(\mathbb{F})$  we have

$$\mathbb{V}_{\mathbb{P}}(V_T^{v-\tilde{p}}(\phi) + X - \tilde{X}) = \mathbb{V}_{\mathbb{P}}(V_T^{v-\tilde{p}}(\phi)) + \mathbb{V}_{\mathbb{P}}(X - \tilde{X}) = \mathbb{V}_{\mathbb{P}}(V_T^{v-\tilde{p}}(\phi)) + \gamma_X.$$

The problem  $\text{MV}(d, v, \tilde{p}, X - \tilde{X})$  can thus be represented as follows. We denote  $\tilde{v} = v - \tilde{p}$ .

**Problem MV**( $d, \tilde{v}; \gamma_X$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^{\tilde{v}}(\phi)) + \gamma_X$  over all trading strategies  $\phi \in \Phi(\mathbb{F})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^{\tilde{v}}(\phi)) \geq d$ .

Problem  $\text{MV}(d, \tilde{v}; \gamma_X)$  is formally equivalent to the original problem  $\text{MV}(d, v, p, X)$  in the following sense: first, the minimal variances for both problems are identical, more precisely, we have

$$w^*(d, v, p, X) = w^*(d, \tilde{v}) + \gamma_X.$$

where  $w^*(d, \tilde{v})$  is the minimal variance for  $\text{MV}(d, \tilde{v})$ . Second, if a strategy  $\psi^*$  is a solution to problem  $\text{MV}(d, \tilde{v})$ , then  $\phi^* = \psi^* - \phi^{\tilde{X}}$  is a solution to  $\text{MV}(d, v, p, X)$ .

**Remarks.** It is interesting to notice that a solution  $\text{MV}(d, \tilde{v}; \gamma_X)$  does not depend explicitly on the expected value of  $X$  under  $\mathbb{P}$ . Hence, the minimal variance for the problem  $\text{MV}(d, v, p, X)$  is independent of  $\mu_X$  as well, but, of course, it depends on the price  $\pi_0(\tilde{X}) = \mathbb{E}_{\mathbb{Q}}X$ , which may in fact coincide with  $\mu_X$  under some circumstances.

In view of the arguments above, it suffices to consider the problem  $\text{MV}(d, \tilde{v})$ , where  $\tilde{v} = v - p + \mathbb{E}_{\mathbb{Q}}X$ . Since the problem of this form has been already solved in Section 3.1, we are in the position to state the following result, which is an immediate consequence of Proposition 3.3. Recall that  $\rho(\theta) = e^{\theta^2 T} (e^{\theta^2 T} - 1)^{-1}$  and  $\eta_t(\theta) = \eta_t e^{\theta^2 t}$ , so that  $\eta_0(\theta) = 1$ . Finally,  $\tilde{v} = v - p + \mathbb{E}_{\mathbb{Q}}X = v - p + \mathbb{E}_{\mathbb{Q}}\tilde{X}$ .

**Proposition 3.4** *Assume that  $\theta \neq 0$ . (i) Suppose that  $d > \tilde{v}$ . Then a solution  $\phi^*(d, v, p, X)$  to  $MV(d, v, p, X)$  is given as  $\phi^*(d, v, p, X) = \psi^*(d, \tilde{v}) - \phi^{\tilde{X}}$ , where  $\psi^*(d, \tilde{v}) = (\psi^{1*}(d, \tilde{v}), \psi^{2*}(d, \tilde{v}))$  is such that  $\psi^{1*}(d, \tilde{v})$  equals*

$$\psi_t^{1*}(d, \tilde{v}) = (d - \tilde{v})\rho(\theta) \frac{\nu\eta_t(\theta)}{\sigma^2 Z_t^1} \quad (14)$$

and  $\psi^{2*}(d, \tilde{v})$  satisfies  $\psi_t^{*1}(d, \tilde{v})Z_t^1 + \psi_t^{*2}(d, \tilde{v}) = V_t^*(d, \tilde{v})$  for  $t \in [0, T]$ , where in turn

$$V_t^*(d, \tilde{v}) = \tilde{v} + (d - \tilde{v})\rho(\theta)(1 - \eta_t(\theta)). \quad (15)$$

Thus the optimal wealth for the problem  $MV(d, v, p, X)$  equals

$$V_t^*(d, v, p, X) = v - p + (d - \tilde{v})\rho(\theta)(1 - \eta_t(\theta)) + \mathbb{E}_{\mathbb{Q}}\tilde{X} - \mathbb{E}_{\mathbb{Q}}(\tilde{X} | \mathcal{F}_t) \quad (16)$$

and the minimal variance  $w^*(d, v, p, X)$  is given by

$$w^*(d, v, p, X) = \frac{(d - \tilde{v})^2}{e^{\theta^2 T} - 1} + \gamma_X. \quad (17)$$

(ii) If  $d \leq \tilde{v}$  then the optimal wealth process equals

$$V_t^*(d, v, p, X) = v - p + \mathbb{E}_{\mathbb{Q}}\tilde{X} - \mathbb{E}_{\mathbb{Q}}(\tilde{X} | \mathcal{F}_t)$$

and the minimal variance equals  $\gamma_X$ .

**Remarks.** Let us comment briefly on the assumption  $\theta \neq 0$ . Recall that if it fails to hold, the problem  $MV(d, \tilde{v})$  has no solution, unless  $d = \tilde{v}$ . Hence, for  $\theta = 0$  we need to postulate that  $d = v - p + \mathbb{E}_{\mathbb{P}}X$  (recall that  $\theta = 0$  if and only if  $\mathbb{Q} = \mathbb{P}$ ). The optimal strategy  $\phi^* = (0, 1)$  and thus the solution to  $MV(d, v, p, X)$  is exactly the same as in part (ii) of Proposition 3.4.

### 3.2.3 Mean-Variance Pricing and Hedging of a Generic Claim

Our next goal is to provide explicit representations for the mean-variance price of  $X$ . We maintain here the assumption that the problem  $MV(d, v, p, X)$  is examined in the class  $\Phi(\mathbb{F})$ . Thus, the mean-variance price considered in this section, denoted as  $p_{\mathbb{F}}^{d,v}(X)$ , is in fact relative to the reference filtration  $\mathbb{F}$ .

Assume that  $d > \tilde{v} = v - p + \mathbb{E}_{\mathbb{Q}}X$  (recall that  $\mathbb{E}_{\mathbb{Q}}X = \mathbb{E}_{\mathbb{Q}}\tilde{X} = \pi_0(\tilde{X})$ ). Then, by virtue of Proposition 3.4, the minimal variance for the problem  $MV(d, v, p, X)$  equals

$$w^*(d, v, p, X) = \frac{(d - v + p - \mathbb{E}_{\mathbb{Q}}X)^2}{e^{\theta^2 T} - 1} + \gamma_X$$

where

$$\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \tilde{X}).$$

Of course, if  $d \leq \tilde{v} = v - p + \mathbb{E}_{\mathbb{Q}}X$  then we have  $w^*(d, v, p, X) = \gamma_X$ . Recall that we postulate that  $d > v$ , and thus the minimal variance for the problem  $MV(d, v)$  equals

$$w^*(d, v) = \frac{(d - v)^2}{e^{\theta^2 T} - 1}.$$

Let us denote

$$\kappa = d - v - \mathbb{E}_{\mathbb{Q}}X, \quad \rho = (d - v)^2 - \gamma_X(e^{\theta^2 T} - 1).$$

**Proposition 3.5** (i) Suppose that  $\pi_0(\tilde{X}) \geq d$  so that  $\kappa \leq -v$ . If  $\gamma_X \leq w^*(d, v)$  then the mean variance price equals  $p_{\mathbb{F}}^{d,v}(X) = v$ . Otherwise,  $p_{\mathbb{F}}^{d,v}(X) = -\infty$ .

(ii) Suppose that  $d - v \leq \pi_0(\tilde{X}) < d$  so that  $-v < \kappa \leq 0$ . If, in addition,  $\rho \geq 0$  then we have

$$p_{\mathbb{F}}^{d,v}(X) = \min\{-\kappa + \sqrt{\rho}, v\} \vee 0. \quad (18)$$

Otherwise, i.e., when  $\rho < 0$ , we have  $p_{\mathbb{F}}^{d,v}(X) = -\kappa$  if  $\gamma_X \leq w^*(d, v)$ , and  $p_{\mathbb{F}}^{d,v}(X) = -\infty$  if  $\gamma_X > w^*(d, v)$ .

(iii) Suppose that  $\pi_0(\tilde{X}) < d - v$  so that  $\kappa > 0$ . If  $\rho \geq 0$  then  $p_{\mathbb{F}}^{d,v}(X)$  is given by (18). Otherwise, we have  $p_{\mathbb{F}}^{d,v}(X) = -\infty$ .

*Proof.* In case (i), we have  $d - v - \mathbb{E}_{\mathbb{Q}}X \leq -p$  for every  $p \in [0, v]$ . Thus  $d \leq v - p + \mathbb{E}_{\mathbb{Q}}X$ , so that  $w^*(d, v, p, X) = \gamma_X$ . Therefore, if  $\gamma_X \leq w^*(d, v)$  it is clear that  $p_{\mathbb{F}}^{d,v}(X) = v$ . Otherwise, for every  $p \in [0, v]$  we have  $w^*(d, v, p, X) = \gamma_X > w^*(d, v)$  and thus  $p_{\mathbb{F}}^{d,v}(X) = -\infty$ .

In case (ii), it suffices to notice that  $d \leq v - p + \mathbb{E}_{\mathbb{Q}}X$  for any  $p \in [0, -\kappa]$ , and  $d > v - p + \mathbb{E}_{\mathbb{Q}}X$  for any  $p \in (-\kappa, v]$ . Thus the maximal  $p \in [0, v]$  for which  $w^*(d, v, p, X) \leq w^*(d, v)$  can be found from the equation

$$(\kappa + p)^2 + \gamma_X(e^{\theta^2 T} - 1) = (d - v)^2,$$

which admits the solution  $p = -\kappa + \sqrt{\rho}$  provided that  $\rho \geq 0$ . If  $\rho < 0$ , then we need to examine the case  $p \in [0, -\kappa]$ , and we see that  $p_{\mathbb{F}}^{d,v}(X)$  equals either  $-\kappa$  or  $-\infty$ , depending on whether  $\gamma_X \leq w^*(d, v)$  or  $\gamma_X > w^*(d, v)$ .

In case (iii), we have  $d - v - \mathbb{E}_{\mathbb{Q}}X > 0$ , which yields  $d > v - p + \mathbb{E}_{\mathbb{Q}}X$  for any  $p \in [0, v]$ . Inequality  $w^*(d, v, p, X) \leq w^*(d, v)$  becomes

$$(d - v + p - \mathbb{E}_{\mathbb{Q}}X)^2 + \gamma_X(e^{\theta^2 T} - 1) \leq (d - v)^2$$

If  $\rho \geq 0$  then  $p_{\mathbb{F}}^{d,v}(X)$  is given by (18). Otherwise, we have  $p_{\mathbb{F}}^{d,v}(X) = -\infty$ .  $\square$

The mean variance hedging strategy is now obtained as  $\phi^{\text{MV}} = \phi^*(d, v, p_{\mathbb{F}}^{d,v}(X), X)$  for all cases above when  $p_{\mathbb{F}}^{d,v}(X) \neq -\infty$ .

### 3.3 Defaultable Claims

In order to provide a better intuition, we shall now examine in some detail two special cases. First, we shall assume that  $X$  is independent of the  $\sigma$ -field  $\mathcal{F}_T$ . Since  $X$  is  $\mathcal{G}_T$ -measurable, but obviously it is not  $\mathcal{G}_T$ -measurable, we shall refer to  $X$  as a *defaultable claim* (a more general interpretation of  $X$  is possible, however).

Although this case may look rather trivial at the first glance, we shall see that some interesting conclusions can be obtained. Second, we shall analyze the case of a defaultable zero-coupon bond with fractional recovery of Treasury value. Of course, both examples are merely simple illustrations of Proposition 3.4, and thus they should not be considered as real-life applications.

#### 3.3.1 Claim Independent of the Reference Filtration

Consider a  $\mathcal{G}_T$ -measurable contingent claim  $X$ , such that  $X$  is independent of the  $\sigma$ -field  $\mathcal{F}_T$ . Then for any strategy  $\phi \in \Phi(\mathbb{F})$ , the terminal wealth  $V_T(\phi)$  and the payoff  $X$  are independent random variables, so that

$$\mathbb{V}_{\mathbb{P}}(V_T(\phi) + X) = \mathbb{V}_{\mathbb{P}}(V_T(\phi)) + \mathbb{V}_{\mathbb{P}}(X).$$

It is clear that if the variance  $\mathbb{V}_{\mathbb{P}}(X)$  satisfies  $\mathbb{V}_{\mathbb{P}}(X) > w^*(d, v)$ , then  $p_{\mathbb{F}}^{d,v}(X) = -\infty$  for every  $v > 0$ . Moreover, if  $\mathbb{V}_{\mathbb{P}}(X) \leq w^*(d, v)$  and  $\mathbb{E}_{\mathbb{P}}X \geq d$ , then  $p_{\mathbb{F}}^{d,v}(X) = v$  for every  $v > 0$ .

It thus remains to examine the case when  $\mathbb{V}_{\mathbb{P}}(X) \leq w^*(d, v)$  and  $\mathbb{E}_{\mathbb{P}}X < d$ . Notice that  $\tilde{X} = \mathbb{E}_{\mathbb{P}}X$  and thus  $\pi_0(\tilde{X}) = \mathbb{E}_{\mathbb{P}}X$ . In particular, since  $\tilde{X}$  is constant, its replicating strategy is trivial, i.e.  $\phi^{\tilde{X}} = 0$ .

In view of Proposition 3.4, if  $d > v - p + \mathbb{E}_{\mathbb{P}}X$  then the minimal variance for the problem  $MV(d, v, p, X)$  equals

$$w^*(d, v, p, X) = \frac{(d - v + p - \mu_X)^2}{e^{\theta^2 T} - 1} + \sigma_X^2,$$

where  $\mu_X = \mathbb{E}_{\mathbb{P}}X$  and  $\sigma_X^2 = \mathbb{V}_{\mathbb{P}}(X) = \gamma_X$ . Let us denote

$$\tilde{p}^{d,v}(X) = -d + v + \mu_X + \sqrt{(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1)}.$$

**Proposition 3.6** *The mean variance price of the claim  $X$  equals*

$$p_{\mathbb{F}}^{d,v}(X) = \min\{\tilde{p}^{d,v}(X), v\} \vee 0$$

if  $(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1) \geq 0$ , and  $-\infty$  otherwise. The mean-variance hedging strategy  $\phi^{MV} = \psi^*$  where  $\psi^*$  is such that

$$\psi_t^{1*} = e^{\theta^2(T-t)} \frac{d - v + p^{d,v}(X) - \mu_X}{e^{\theta^2 T} - 1} \frac{\nu}{\sigma^2} \frac{\eta_t}{Z_t^1}, \quad \forall t \in [0, T].$$

The mean-variance price depends, of course, on the initial value  $v$  of the investor's capital. This dependence has very intuitive and natural properties, though. Let us denote

$$k = d - \sqrt{(d - \mu_X)^2 + \sigma_X^2(e^{\theta^2 T} - 1)}, \quad l = d - \sigma_X \sqrt{e^{\theta^2 T} - 1}.$$

We fix all parameters, except for  $v$ . Notice that the function  $p(v) = p_{\mathbb{F}}^{d,v}(X)$  is non-negative and finite for  $v \in [0, l \vee 0]$ . Moreover, the function  $p(v)$  is increasing for  $v \in [0, k \vee 0]$ , and it is decreasing on the interval  $[k \vee 0, l \vee 0]$ . Specifically,

$$p(v) = \begin{cases} v, & \text{if } 0 \leq v < k \vee 0, \\ \mu_X - d + v + \sqrt{(d - v)^2 - \sigma_X^2(e^{\theta^2 T} - 1)}, & \text{if } k \vee 0 \leq v \leq l \vee 0. \end{cases}$$

This conclusion is quite intuitive: once the initial level of investor's capital is big enough (that is,  $v \geq l$ ) the investor is less and less interested in purchasing the claim  $X$ . This is because when the initial capital level is sufficiently close to the expected terminal wealth level, the investor has enough leverage to meet this terminal objective at minimum risk; therefore, the investor is increasingly reluctant to purchase the claim  $X$  as this would introduce unwanted additional risk (unless of course  $\sigma_X = 0$ ). For example, if  $v = d$  then the investor is not at all interested in purchasing the claim ( $p_{\mathbb{F}}^{d,v}(X) = -\infty$  if  $\sigma_X > 0$  and  $\theta \neq 0$ ). For further properties of the mean-variance price of a claim  $X$  independent of  $\mathcal{F}_T$ , we refer to Bielecki and Jeanblanc (2003).

### 3.3.2 Defaultable Bond

Let  $\tau$  be a random time on the underlying probability space  $(\Omega, \mathcal{G}, \mathbb{P})$ . We define the indicator process  $H$  associated with  $\tau$  by setting  $H_t = \mathbb{1}_{\{\tau \leq t\}}$  for  $t \in \mathbb{R}_+$ , and we denote by  $\mathbb{H}$  the natural filtration of  $H$  ( $\mathbb{P}$ -completed). We take  $\mathbb{H}$  to serve as the auxiliary filtration, so that  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . We assume that the default time  $\tau$  is defined as follows:

$$\tau = \inf\{t \in \mathbb{R}_+ : \Gamma_t > \zeta\}, \quad (19)$$

where  $\Gamma$  is an increasing,  $\mathbb{F}$ -adapted process, with  $\Gamma_0 = 0$ , and  $\zeta$  is an exponentially distributed random variable with parameter 1, independent of  $\mathbb{F}$ . It is well known that any Brownian motion

$W$  with respect to  $\mathbb{F}$  is also a Brownian motion with respect to  $\mathbb{G}$  within the present setup (the latter property is closely related to the so-called Hypothesis (H) frequently used in the modelling of default event, see Jeanblanc and Rutkowski (2000) or Bielecki et al. (2004)).

Now, suppose that a new investment opportunity becomes available for the agent. Namely, the agent may purchase a defaultable bond that matures at time  $T \in (0, T^*]$ . We postulate that the terminal payoff at time  $T$  of the bond is  $X = L\mathbb{1}_{\{\tau > T\}} + \delta L\mathbb{1}_{\{\tau \leq T\}}$ , where  $L > 0$  is the bond's notional amount and  $\delta \in [0, 1)$  is the (constant) recovery rate. In other words, we deal with a defaultable zero-coupon bond that is subject to the fractional recovery of Treasury value.

Notice that the payoff  $X$  can be represented as follows  $X = \delta L + Y$ , where  $Y = L(1 - \delta)\mathbb{1}_{\{\tau > T\}}$ . According to our general definition, we associate to  $X$  an  $\mathcal{F}_T$ -measurable random variable  $\tilde{X}$  by setting

$$\tilde{X} = \mathbb{E}_{\mathbb{P}}(X | \mathcal{F}_T) = \delta L + \mathbb{E}_{\mathbb{P}}(Y | \mathcal{F}_T).$$

In view of (19), we have

$$\mathbb{E}_{\mathbb{P}}(Y | \mathcal{F}_T) = \mathbb{P}\{\tau > T | \mathcal{F}_T\} = e^{-\Gamma_T},$$

and thus the arbitrage price at time 0 of the attainable claim  $\tilde{X}$  equals (recall that we have reduced our problem to the case  $r = 0$ )

$$\pi_0(\tilde{X}) = \mathbb{E}_{\mathbb{Q}}\tilde{X} = \delta L + \mathbb{E}_{\mathbb{P}}(\eta_T e^{-\Gamma_T}).$$

Since clearly

$$X - \tilde{X} = L(1 - \delta)(\mathbb{1}_{\{\tau > T\}} - \mathbb{P}\{\tau > T | \mathcal{F}_T\}),$$

we obtain

$$\gamma_X = \mathbb{V}_{\mathbb{P}}(X - \tilde{X}) = L^2(1 - \delta)^2 \mathbb{E}_{\mathbb{P}}(\mathbb{1}_{\{\tau > T\}} - e^{-\Gamma_T})^2.$$

In order to find the mean-variance price  $p_{\mathbb{F}}^{d,v}(X)$  at time 0 of a defaultable bond with respect to the reference filtration  $\mathbb{F}$ , it suffices to make use of Proposition 3.4 (or Proposition 3.5). If we wish to describe the mean-variance hedging strategy with respect to  $\mathbb{F}$ , we need also to know an explicit representation for the replicating strategy  $\phi^{\tilde{X}}$  for the claim  $\tilde{X}$ . To this end, it suffices to find the integral representation of the random variable  $\mathbb{E}_{\mathbb{P}}(Y | \mathcal{F}_T)$  with respect to the price process  $Z^1$  or, equivalently, to find a process  $\phi^{\tilde{X}}$  for which

$$\tilde{X} = \pi_0(\tilde{X}) + \int_0^T \phi_t^{\tilde{X}} dZ_t^1.$$

**Example 3.1** In practical applications of the reduced-form approach, it is fairly common to postulate that the  $\mathbb{F}$ -hazard process  $\Gamma$  is given as  $\Gamma_t = \int_0^t \gamma_t dt$ , where  $\gamma$  is a non-negative process, progressively measurable with respect to  $\mathbb{F}$ , referred to as the  $\mathbb{F}$ -intensity of default. Suppose, for the sake of simplicity, that the intensity of default  $\gamma$  is deterministic, and let us set

$$p_{\gamma} = \mathbb{P}\{\tau > T\} = \tilde{\mathbb{Q}}\{\tau > T\} = \exp\left(-\int_0^T \gamma(t) dt\right).$$

Then we get

$$\pi_0(\tilde{X}) = \mathbb{E}_{\mathbb{Q}}\tilde{X} = \delta L + p_{\gamma}$$

and

$$\gamma_X = L^2(1 - \delta)^2 p_{\gamma}(1 - p_{\gamma}).$$

Of course, in the case of a deterministic default intensity  $\gamma$ , in order to replicate the claim  $\tilde{X}$ , it suffices to invest the amount  $\pi_0(\tilde{X})$  in the savings account. For a more detailed analysis of the mean-variance price of a defaultable bond, the reader may consult Bielecki and Jeanblanc (2003).

## 4 Strategies Adapted to the Full Filtration

In this section, the mean-variance hedging and pricing is examined in the case of trading strategies adapted to the full filtration. Recall that  $W$  is assumed to be a one-dimensional Brownian motion with respect to  $\mathbb{F}$  under  $\mathbb{P}$ . We postulated, in addition, that  $W$  is also a Brownian motion with respect to the filtration  $\mathbb{G}$  under the probability  $\mathbb{P}$ . We define a new probability  $\tilde{\mathbb{Q}}$  on  $(\Omega, \mathcal{G}_{T^*})$  by setting

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{P}} \Big|_{\mathcal{G}_t} = \eta_t, \quad \forall t \in [0, T^*],$$

where the process  $\eta$  is given by (2). Clearly,  $\tilde{\mathbb{Q}}$  is an equivalent martingale probability for our primary market and the process  $\eta$  is a  $\mathbb{G}$ -martingale under  $\mathbb{P}$ . Moreover, we have (cf. (3))

$$\mathbb{E}_{\mathbb{P}}(\eta_T^2 | \mathcal{G}_t) = \eta_t^2 e^{\theta^2(T-t)},$$

and thus  $\mathbb{E}_{\mathbb{P}}(\eta_t^2) = \exp(\theta^2 t)$  for every  $t \in [0, T^*]$ . It is easy to check that the process  $\tilde{W} = W_t - \theta t$  is a martingale, and thus a Brownian motion, with respect to  $\mathbb{G}$  under  $\tilde{\mathbb{Q}}$ .

From the square-integrability of  $\eta_T$  under  $\mathbb{P}$ , it follows for any strategy  $\phi \in \Phi(\mathbb{G})$  the terminal wealth  $V_T(\phi)$  is integrable under  $\tilde{\mathbb{Q}}$ . In fact, we have the following useful result. Recall that a  $\mathbb{G}$ -predictable process  $\phi^1$  uniquely determines a self-financing strategy  $\phi = (\phi^1, \phi^2)$ , and thus we may formally identify  $\phi^1$  with the associated strategy  $\phi$  (and vice versa). The following lemma will prove useful.

**Lemma 4.1** *Let  $\mathcal{A}(\tilde{\mathbb{Q}})$  be the linear space of all  $\mathbb{G}$ -predictable processes  $\psi$  such that the process  $\int_0^t \psi_u dZ_u^1$  is a  $\tilde{\mathbb{Q}}$ -martingale and the integral  $\int_0^T \psi_u dZ_u^1$  is in  $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ . Then  $\mathcal{A}(\tilde{\mathbb{Q}}) = \Phi(\mathbb{G})$ .*

*Proof.* It is clear that  $\mathcal{A}(\tilde{\mathbb{Q}}) \subseteq \Phi(\mathbb{G})$ . For the proof of the inclusion  $\Phi(\mathbb{G}) \subseteq \mathcal{A}(\tilde{\mathbb{Q}})$ , see Lemma 9 in Rheinländer and Schweizer (1997).  $\square$

Let us observe that the class  $\mathcal{A}(\tilde{\mathbb{Q}})$  corresponds to the set  $\Theta_{\text{GLP}}(\tilde{\Theta})$ , respectively) considered in Schweizer (2001) (in Rheinländer and Schweizer (1997), respectively). The class  $\Phi(\mathbb{G})$  corresponds with the class  $\Theta_S(\Theta)$ , respectively) considered in Schweizer (2001) (in Rheinländer and Schweizer (1997), respectively).

Let us denote by  $\mathbb{G}^1$  the filtration generated by all wealth processes:

$$V_t^v(\phi) = v + \int_0^t \phi_u^1 dZ_u^1,$$

where  $v \in \mathbb{R}$  and  $\phi = (\phi^1, \phi^2)$  belongs to  $\Phi(\mathbb{G})$ . Equivalently,  $\mathbb{G}^1$  is generated by the processes

$$x + \int_0^t \psi_u dZ_u^1$$

with  $x \in \mathbb{R}$  and  $\psi \in \mathcal{A}(\tilde{\mathbb{Q}})$ . Also, we denote by  $\mathcal{P}^0$  the following set of random variables:

$$\mathcal{P}^0 = \left\{ \xi \in L^2(\Omega, \mathcal{G}_T^1, \mathbb{P}) \mid \xi = \int_0^T \psi_u dZ_u^1, \psi \in \mathcal{A}(\tilde{\mathbb{Q}}) \right\}.$$

Let  $\Pi_{\mathbb{P}}^0$  stand for the orthogonal projection (in the norm of the space  $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ ) from  $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$  on the space  $\mathcal{P}^0$ . A similar notation will be also used for orthogonal projections on  $\mathcal{P}^0$  under  $\tilde{\mathbb{Q}}$ . Let us mention that, in general, we shall have  $\Pi_{\mathbb{P}}^0(Y) \neq \mathbb{E}_{\mathbb{P}}(Y | \mathcal{G}_T^1)$  for  $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$  and  $\Pi_{\tilde{\mathbb{Q}}}^0(Y) \neq \mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_T^1)$  for  $Y \in L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}})$  (see Section 4.3 for more details).

#### 4.1 Solution to $MV(d, v)$ in the class $\Phi(\mathbb{G})$

Recall that our basic mean-variance problem has the following form:

**Problem  $MV(d, v)$ :** Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$  over all strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) \geq d$ .

As in Section 3.1, we postulate that  $d > v$ , since otherwise the problem is trivial. We shall argue that it suffices to solve a simpler problem:

**Problem  $MVA(d, v)$ :** Minimize  $\mathbb{E}_{\mathbb{P}}(V_T^v(\phi))^2$  over all strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) = d$ .

In view of the definition of class  $\mathcal{A}(\tilde{\mathbb{Q}})$ , Lemma 4.1, and the fact that  $\mathbb{E}_{\mathbb{Q}}\xi = 0$  for any  $\xi \in \mathcal{P}^0$ , we see that it suffices to solve the problem

**Problem  $MVB(d, v)$ :** Minimize  $\mathbb{E}_{\mathbb{P}}(v + \xi)^2$  over all random variables  $\xi \in \mathcal{P}^0$ , subject to  $\mathbb{E}_{\mathbb{P}}\xi = d - v$ .

Solution to the last problem is exactly the same as in the case of strategies from  $\Phi(\mathbb{F})$ . Indeed, by solving the last problem in the class  $L^2(\Omega, \mathcal{G}_T, \mathbb{P})$  (rather than in  $\mathcal{P}^0$ ), and with additional constraint  $\mathbb{E}_{\mathbb{Q}}\xi = 0$ , we see that the optimal solution, given by (7), is in fact  $\mathcal{F}_T$ -measurable, and thus it belongs to the class  $\mathcal{P}^0$  as well. In view of (8), the same random variable is a solution to  $MV(d, v)$ , that is, it represents the optimal terminal wealth. We conclude that a solution to  $MV(d, v)$  in the class  $\Phi(\mathbb{G})$  is given by the formulae (11)-(13) of Proposition 3.3, i.e., it coincides with a solution in the class  $\Phi(\mathbb{F})$ .

Assume that  $X$  is an attainable contingent claim, in the sense that there exists a trading strategy  $\phi \in \Phi(\mathbb{G})$  which replicates  $X$ . Then, arguing along the same lines as in Section 3.2.1, we get the following result.

**Corollary 4.1** *Let a  $\mathcal{G}_T$ -measurable random variable  $X$  represent an attainable contingent claim. Then*

- (i) *If the arbitrage price  $\pi_0(X)$  satisfies  $\pi_0(X) \in [0, v]$  then  $p^{d,v}(X) = \pi_0(X)$ .*
- (ii) *If the arbitrage price  $\pi_0(X)$  is strictly greater than  $v$  then  $p^{d,v}(X) = v$ .*

#### 4.2 Solution to $MV(d, v, p, X)$ in the class $\Phi(\mathbb{G})$

We shall study the problem  $MV(d, v, p, X)$  for an arbitrary  $\mathcal{G}_T$ -measurable claim  $X$ , which is square-integrable under  $\mathbb{P}$ . Recall that we deal with the following problem:

**Problem  $MV(d, v, p, X)$ :** Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^{v-p}(\phi) + X)$  over all trading strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^{v-p}(\phi) + X) \geq d$ .

Basic idea of solving the problem  $MV(d, v, p, X)$  with respect to  $\mathbb{G}$ -predictable strategies is similar to the method used in the case of  $\mathbb{F}$ -predictable strategies. The main difference is that the auxiliary random variable  $\tilde{X}$  will now be defined as the orthogonal projection  $\Pi_{\mathbb{P}}(X)$  of  $X$  on  $\mathcal{P}^0$ , rather than the conditional expectation  $\mathbb{E}_{\mathbb{P}}(X | \mathcal{G}_T)$ .

Let us denote  $\hat{d} = d - v + p$ . The problem  $MV(d, v, p, X)$  can be reformulated as follows:

**Problem  $MV(\hat{d}, 0, 0, X)$ :** Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X)$  over all trading strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi) + X) \geq \hat{d}$ .

That is, if  $V_T^{0,*}$  is the optimal wealth in problem  $MV(\hat{d}, 0, 0, X)$  then  $V_T^{v-p,*} = V_T^{0,*} + v - p$  is the optimal wealth in Problem  $MV(d, v, p, X)$ , and the optimal strategies as well as the optimal variances are the same in both problems.

Let  $\tilde{X}^0 = \Pi_{\mathbb{P}}^0(X)$  stand for the orthogonal projection of  $X$  on  $\mathcal{P}^0$ , so that  $\psi^{\tilde{X}^0}$  is a process from

$\mathcal{A}(\tilde{\mathbb{Q}}) = \Phi(\mathbb{G})$ , for which

$$\tilde{X}^0 = \int_0^T \psi_t^{1, \tilde{X}^0} dZ_t^1 \quad (20)$$

and  $X - \tilde{X}^0 = X - \Pi_{\mathbb{P}}^0(X)$  is orthogonal to  $\mathcal{P}^0$ . The price of  $\tilde{X}^0$  equals

$$\pi_t(\tilde{X}^0) = \int_0^t \psi_u^{1, \tilde{X}^0} dZ_u^1 = \mathbb{E}_{\tilde{\mathbb{Q}}}(\tilde{X}^0 | \mathcal{G}_t), \quad \forall t \in [0, T]. \quad (21)$$

Let  $\psi^{\tilde{X}^0} \in \Phi(\mathbb{G})$  be a replicating strategy for  $\tilde{X}^0$ . Explicitly,  $\psi^{\tilde{X}^0} = (\psi^{1, \tilde{X}^0}, \psi^{2, \tilde{X}^0})$ , where  $\psi^{2, \tilde{X}^0}$  satisfies  $\psi_t^{1, \tilde{X}^0} Z_t^1 + \psi_t^{2, \tilde{X}^0} = \pi_t(\tilde{X}^0)$ . Notice that  $\pi_0(\tilde{X}^0) = \mathbb{E}_{\tilde{\mathbb{Q}}}\tilde{X}^0 = 0$  and, of course,  $\pi_T(\tilde{X}^0) = \tilde{X}^0$ . It thus suffices to consider the following problem:

**Problem MV**( $\hat{d}, 0, 0, X - \tilde{X}^0$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X - \tilde{X}^0)$  over all trading strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi) + X - \tilde{X}^0) \geq \hat{d}$ .

Since  $X - \tilde{X}^0$  is orthogonal to  $\mathcal{P}^0$ , for any strategy  $\phi \in \Phi(\mathbb{G})$  we have

$$\mathbb{V}_{\mathbb{P}}(V_T^0(\phi) + X - \tilde{X}^0) = \mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \mathbb{V}_{\mathbb{P}}(X - \tilde{X}^0) = \mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \gamma_X^0,$$

where  $\gamma_X^0 = \mathbb{V}_{\mathbb{P}}(X - \tilde{X}^0)$ . Let us denote  $\tilde{d} = d - v + p - \mathbb{E}_{\mathbb{P}}X + \mathbb{E}_{\mathbb{P}}\tilde{X}^0$ . Then the problem MV( $\hat{d}, 0, 0, X - \tilde{X}^0$ ) can thus be simplified as follows:

**Problem MV**( $\tilde{d}, 0, \gamma_X^0$ ): Minimize  $\mathbb{V}_{\mathbb{P}}(V_T^0(\phi)) + \gamma_X^0$  over all trading strategies  $\phi \in \Phi(\mathbb{G})$ , subject to  $\mathbb{E}_{\mathbb{P}}(V_T^0(\phi)) \geq \tilde{d} = d - v + p - \mathbb{E}_{\mathbb{P}}X + \mathbb{E}_{\mathbb{P}}\tilde{X}^0$ .

Let us write  $\tilde{v} = v - p - \mathbb{E}_{\mathbb{P}}X + \mathbb{E}_{\mathbb{P}}\tilde{X}^0$ , so that  $\tilde{d} = d - \tilde{v}$ . Then the minimal variance for the problem MV( $d, v, p, X$ ) equals

$$w^*(d, v, p, X) = w^*(\tilde{d}, 0) + \gamma_X^0 = w^*(d, \tilde{v}) + \gamma_X^0.$$

Moreover, if  $\psi^*$  is an optimal strategy to MV( $\tilde{d}, 0$ ), then  $\phi^{1*} = \psi^{1*} - \psi^{\tilde{X}^0}$  is a solution to MV( $d, v, p, X$ ). The proof of the next proposition is based on the considerations above, combined with Proposition 3.3. We use the standard notation  $\rho(\theta) = e^{\theta^2 T} (e^{\theta^2 T} - 1)^{-1}$  and  $\eta_t(\theta) = \eta_t e^{\theta^2 t}$ , so that  $\eta_0(\theta) = 1$ . Recall that  $\mathbb{E}_{\tilde{\mathbb{Q}}}\tilde{X}^0 = 0$ .

**Proposition 4.1** *Assume that  $\theta \neq 0$  and let  $\psi^{\tilde{X}^0} \in \Phi(\mathbb{G})$  be a replicating strategy for  $\tilde{X}^0 = \Pi_{\mathbb{P}}^0(X)$ . (i) Suppose that  $d > \tilde{v}$ . Then an optimal strategy  $\phi^*(d, v, p, X)$  for the problem MV( $d, v, p, X$ ) is given as  $\phi^{1*}(d, v, p, X) = \psi^{1*}(\tilde{d}, 0) - \psi^{1, \tilde{X}^0}$  with  $\psi^*(\tilde{d}, 0) = (\psi^{1*}(\tilde{d}, 0), \psi^{2*}(\tilde{d}, 0))$  such that  $\psi^{1*}(\tilde{d}, 0)$  equals*

$$\psi_t^{1*}(\tilde{d}, 0) = (d - \tilde{v})\rho(\theta) \frac{\nu\eta_t(\theta)}{\sigma^2 Z_t^1} \quad (22)$$

and  $\psi^{2*}(\tilde{d}, 0)$  satisfies  $\psi_t^{*1}(\tilde{d}, 0)Z_t^1 + \psi_t^{*2}(\tilde{d}, 0) = V_t^*(\tilde{d}, 0)$ , where in turn

$$V_t^*(\tilde{d}, 0) = (d - \tilde{v})\rho(\theta)(1 - \eta_t(\theta)). \quad (23)$$

Thus the optimal wealth for the problem MV( $d, v, p, X$ ) equals

$$V_t^*(d, v, p, X) = v - p + (d - \tilde{v})\rho(\theta)(1 - \eta_t(\theta)) - \mathbb{E}_{\tilde{\mathbb{Q}}}(\tilde{X}^0 | \mathcal{G}_t). \quad (24)$$

The minimal variance  $w^*(d, v, p, X)$  is given by

$$w^*(d, v, p, X) = \frac{(d - \tilde{v})^2}{e^{\theta^2 T} - 1} + \gamma_X^0. \quad (25)$$

(ii) If  $d \leq \tilde{v}$  then the optimal wealth process equals

$$V_t^*(d, v, p, X) = v - p - \mathbb{E}_{\tilde{\mathbb{Q}}}(\tilde{X}^0 | \mathcal{G}_t)$$

and the minimal variance equals  $\gamma_X^0$ .

**Remarks.** It is natural to expect that the optimal variance given in (25) is not greater than the optimal variance given in (17). In fact, this is the case (see Proposition 5.4 in Bielecki and Jeanblanc (2003)).

Of course, the practical relevance of the last result hinges on the availability of explicit representation for the orthogonal projection  $\tilde{X}^0 = \Pi_{\tilde{\mathbb{P}}}^0(X)$  of  $X$  on the space  $\mathcal{P}^0$ . This important issue will be examined in the next section in a general setup. We shall continue the study of this question in the framework of defaultable claims in Section 4.5.

### 4.3 Projection of a Generic Claim

Let us first recall two well-known results concerning the decomposition of a  $\mathcal{G}_T$ -measurable random variable, which represents a generic contingent claim in our financial model.

**Galtchouk-Kunita-Watanabe decomposition under  $\tilde{\mathbb{Q}}$ .** Suppose first that we work under  $\tilde{\mathbb{Q}}$ , so that the process  $Z^1$  is a continuous martingale. Recall that by assumption  $W$  is a Brownian motion with respect to  $\mathbb{G}$  under  $\mathbb{P}$ ; hence, the process  $\tilde{W}$  is a Brownian motion with respect to  $\mathbb{G}$  under  $\tilde{\mathbb{Q}}$ .

It is well known that any random variable  $Y \in L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}})$  can be represented by means of the *Galtchouk-Kunita-Watanabe decomposition* with respect to the martingale  $Z^1$  under  $\tilde{\mathbb{Q}}$ . To be more specific, for any random variable  $Y \in L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}})$  there exists a  $\mathbb{G}$ -martingale  $N^{Y, \tilde{\mathbb{Q}}}$ , which is strongly orthogonal in the martingale sense to  $Z^1$  under  $\tilde{\mathbb{Q}}$ , and a  $\mathbb{G}$ -adapted process  $\psi^{Y, \tilde{\mathbb{Q}}}$ , such that  $Y$  can be represented as follows:

$$Y = \mathbb{E}_{\tilde{\mathbb{Q}}} Y + \int_0^T \psi_t^{Y, \tilde{\mathbb{Q}}} dZ_t^1 + N_T^{Y, \tilde{\mathbb{Q}}}. \quad (26)$$

Furthermore, the process  $\psi^{Y, \tilde{\mathbb{Q}}}$  can be represented as follows:

$$\psi_t^{Y, \tilde{\mathbb{Q}}} = \frac{d\langle \mathcal{Y}, Z^1 \rangle_t}{d\langle Z^1 \rangle_t}, \quad (27)$$

where the  $\mathbb{G}$ -martingale  $\mathcal{Y}$  is defined as  $\mathcal{Y}_t = \mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_t)$ .

**Föllmer-Schweizer decomposition under  $\mathbb{P}$ .** Let us now consider the same issue, but under the original probability  $\mathbb{P}$ . The process  $Z^1$  is a (continuous) semimartingale with respect to  $\mathbb{G}$  under  $\mathbb{P}$ , and thus it admits a unique continuous martingale part under  $\mathbb{P}$ .

Any random variable  $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$  can be represented by means of the *Föllmer-Schweizer decomposition*. Specifically, there exists a  $\mathbb{G}$ -adapted process  $\psi^{Y, \mathbb{P}}$ , a  $(\mathbb{G}, \mathbb{P})$ -martingale  $N^{Y, \mathbb{P}}$ , strongly orthogonal in the martingale sense to the continuous martingale part of  $Z^1$ , and a constant  $y^{Y, \mathbb{P}}$ , so that

$$Y = y^{Y, \mathbb{P}} + \int_0^T \psi_t^{Y, \mathbb{P}} dZ_t^1 + N_T^{Y, \mathbb{P}}. \quad (28)$$

We shall see that it will be not necessary to compute the process  $\psi^{Y, \mathbb{P}}$  for the purpose of finding a hedging strategy for the problem considered in this section.

**Projection on  $\mathcal{P}^0$ .** As already mentioned,  $\Pi_{\mathbb{Q}}^0(Y) \neq \mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_T^1)$  for  $Y \in L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}})$ , as well as  $\Pi_{\mathbb{P}}^0(Y) \neq \mathbb{E}_{\mathbb{P}}(Y | \mathcal{G}_T^1)$  for  $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ , in general. For instance, for any random variable  $Y$  as in (26) we get  $\Pi_{\mathbb{Q}}^0(Y) = \int_0^T \psi_t^{Y, \tilde{\mathbb{Q}}} dZ_t^1$ , whereas

$$\mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_T^1) = Y = \Pi_{\mathbb{Q}}^0(Y) - \mathbb{E}_{\tilde{\mathbb{Q}}}Y.$$

The projection  $\Pi_{\mathbb{Q}}^0(Y)$  differs here from the conditional expectation just by the expected value  $\mathbb{E}_{\tilde{\mathbb{Q}}}Y$ . Consequently, we have  $\Pi_{\mathbb{Q}}^0(Y) = \mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_T^1)$  for any  $Y \in L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}})$  with  $\mathbb{E}_{\tilde{\mathbb{Q}}}Y = 0$ . More importantly, observe that for  $Y$  as in (28) we shall have, in general,

$$\Pi_{\mathbb{P}}^0(Y) \neq \int_0^T \psi_t^{Y, \mathbb{P}} dZ_t^1,$$

so that, in particular,  $\Pi_{\mathbb{P}}^0(Y) \neq \mathbb{E}_{\mathbb{P}}(Y | \mathcal{G}_T^1)$  even if  $\mathbb{E}_{\mathbb{P}}Y = 0$ .

Our next goal is to compute the projection  $\Pi_{\mathbb{P}}^0(Y)$  for any random variable  $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{Q})$ . We know that any such  $Y$  can be represented as in (26). Due to linearity of the projection, it is enough to compute the projection of each component in the right-hand side of (26). Let us set  $\tilde{\eta}_t = \mathbb{E}_{\tilde{\mathbb{Q}}}(\eta_T | \mathcal{G}_t)$  for every  $t \in [0, T]$ , so that, in particular,  $\tilde{\eta}_T = \eta_T$ . Since  $\tilde{\eta}$  is a square-integrable  $\mathbb{G}$ -martingale under  $\tilde{\mathbb{Q}}$ , there exists a process  $\tilde{\psi}$  in  $\mathcal{A}(\tilde{\mathbb{Q}})$  such that

$$\tilde{\eta}_t = \mathbb{E}_{\tilde{\mathbb{Q}}}\tilde{\eta}_T + \int_0^t \tilde{\psi}_u dZ_u^1 = \mathbb{E}_{\tilde{\mathbb{Q}}}\tilde{\eta}_T + Z_t^\eta, \quad \forall t \in [0, T], \quad (29)$$

where we denote

$$Z_t^\eta = \int_0^t \tilde{\psi}_u dZ_u^1.$$

**Lemma 4.2** *We have*

$$\tilde{\psi}_t = -\frac{\theta \tilde{\eta}_t}{\sigma Z_t^1} = -\frac{\theta e^{\theta^2 T}}{\sigma Z_t^1} \exp\left(-\theta \tilde{W}_t - \frac{1}{2} \theta^2 (t - 2T)\right) \quad (30)$$

and the process  $\tilde{W}_t = W_t + \theta t$  is a Brownian motion under  $\tilde{\mathbb{Q}}$ .

*Proof.* Direct calculations show that for every  $t \in [0, T]$

$$\tilde{\eta}_t = \exp\left(-\frac{\theta}{\sigma} \int_0^t \frac{dZ_u^1}{Z_u^1} - \frac{1}{2} \theta^2 (t - 2T)\right) = e^{\theta^2 T} \exp\left(-\theta \tilde{W}_t - \frac{1}{2} \theta^2 t\right). \quad (31)$$

Hence,  $\tilde{\eta}$  solves the SDE

$$d\tilde{\eta}_t = -\theta \tilde{\eta}_t d\tilde{W}_t = -\frac{\theta}{\sigma} \frac{\tilde{\eta}_t}{Z_t^1} dZ_t^1$$

with the initial condition  $\tilde{\eta}_0 = \mathbb{E}_{\tilde{\mathbb{Q}}}\tilde{\eta}_T = \mathbb{E}_{\tilde{\mathbb{Q}}}\eta_T = e^{\theta^2 T}$ .  $\square$

In the next result, we provide a general representation for the projection  $\Pi_{\mathbb{P}}^0(Y)$  for a  $\mathcal{G}_T$ -measurable random variable  $Y$ , which is square-integrable under  $\mathbb{P}$ .

**Proposition 4.2** *Let  $Y \in L^2(\Omega, \mathcal{G}_T, \mathbb{P})$ . Then we have*

$$\Pi_{\mathbb{P}}^0(Y) = \int_0^T \tilde{\psi}_t^{Y, \mathbb{P}} dZ_t^1,$$

where

$$\tilde{\psi}_t^{Y,\mathbb{P}} = \psi_t^{Y,\tilde{\mathbb{Q}}} - \tilde{\psi}_t \left( \tilde{\eta}_0^{-1} \mathbb{E}_{\tilde{\mathbb{Q}}} Y + \int_0^t \tilde{\eta}_u^{-1} dN_u^{Y,\tilde{\mathbb{Q}}} \right) \quad (32)$$

and where processes  $\psi^{Y,\tilde{\mathbb{Q}}}$  and  $N^{Y,\tilde{\mathbb{Q}}}$  are given by the Galtchouk-Kunita-Watanabe decomposition (26) of  $Y$  under  $\tilde{\mathbb{Q}}$ .

*Proof.* First, we compute projection of the constant  $c = \mathbb{E}_{\tilde{\mathbb{Q}}} Y$ . To this end, recall that  $\tilde{\eta}_T = \eta_T$  and by virtue of (29) we have  $\tilde{\eta}_T = \tilde{\eta}_0 + Z_T^\eta$ . Hence, for any  $\psi \in \mathcal{A}(\tilde{\mathbb{Q}})$  we obtain

$$\mathbb{E}_{\mathbb{P}} \left( (1 + \tilde{\eta}_0^{-1} Z_T^\eta) \int_0^T \psi_t dZ_t^1 \right) = \tilde{\eta}_0^{-1} \mathbb{E}_{\mathbb{P}} \left( \eta_T \int_0^T \psi_t dZ_t^1 \right) = \tilde{\eta}_0^{-1} \mathbb{E}_{\tilde{\mathbb{Q}}} \left( \int_0^T \psi_t dZ_t^1 \right) = 0,$$

and thus  $\Pi_{\mathbb{P}}^0(1) = -\tilde{\eta}_0^{-1} Z_T^\eta$ . We conclude that for any  $c \in \mathbb{R}$

$$\Pi_{\mathbb{P}}^0(c) = c \Pi_{\mathbb{P}}^0(1) = -c \tilde{\eta}_0^{-1} Z_T^\eta = -c \tilde{\eta}_0^{-1} \int_0^T \tilde{\psi}_t dZ_t^1 = -c \tilde{\eta}_0^{-1} \int_0^T \tilde{\psi}_t dZ_t^1. \quad (33)$$

Next, it is obvious that the projection of the second term, that is, the projection of  $\int_0^T \psi_t^{Y,\tilde{\mathbb{Q}}} dZ_t^1$ , on  $\mathcal{P}^0$  is equal to itself, so that

$$\Pi_{\mathbb{P}}^0 \left( \int_0^T \psi_t^{Y,\tilde{\mathbb{Q}}} dZ_t^1 \right) = \int_0^T \psi_t^{Y,\tilde{\mathbb{Q}}} dZ_t^1. \quad (34)$$

Finally, we shall compute the projection  $\Pi_{\mathbb{P}}^0(N_T^{Y,\tilde{\mathbb{Q}}})$ . Recall that the process  $N^{Y,\tilde{\mathbb{Q}}}$  is a  $\tilde{\mathbb{Q}}$ -martingale strongly orthogonal to  $Z^1$  under  $\tilde{\mathbb{Q}}$ . Hence, for any  $N^{Y,\tilde{\mathbb{Q}}}$ -integrable process  $\nu$  and any process  $\psi \in \mathcal{A}(\tilde{\mathbb{Q}})$  we have

$$\mathbb{E}_{\mathbb{P}} \left( \eta_T \int_0^T \nu_t dN_t^{Y,\tilde{\mathbb{Q}}} \int_0^T \psi_t dZ_t^1 \right) = 0.$$

Thus, it remains to find processes  $\hat{\nu}$  and  $\hat{\psi} \in \mathcal{A}(\tilde{\mathbb{Q}})$  for which

$$\eta_T \int_0^T \hat{\nu}_t dN_t^{Y,\tilde{\mathbb{Q}}} = N_T^{Y,\tilde{\mathbb{Q}}} - \int_0^T \hat{\psi}_t dZ_t^1, \quad (35)$$

in which case we shall have that  $\Pi_{\mathbb{P}}^0(N_T^{Y,\tilde{\mathbb{Q}}}) = \int_0^T \hat{\psi}_t dZ_t^1$ .

Let us set  $U_t = \tilde{\eta}_t \int_0^t \nu_u dN_u^{Y,\tilde{\mathbb{Q}}}$  for every  $t \in [0, T]$ . Recall that (see (29)) there exists a process  $\tilde{\psi}$  in  $\Phi(\mathbb{G}) = \mathcal{A}(\tilde{\mathbb{Q}})$  such that  $d\tilde{\eta}_t = \tilde{\psi}_t dZ_t^1$ . Using the product rule, and taking into account the orthogonality of  $\tilde{\eta}$  and  $N^{Y,\tilde{\mathbb{Q}}}$  under  $\tilde{\mathbb{Q}}$ , we find that  $U$  is a local martingale under  $\tilde{\mathbb{Q}}$ , and it satisfies

$$U_t = \int_0^t \tilde{\eta}_{u-} \nu_u dN_u^{Y,\tilde{\mathbb{Q}}} + \int_0^t \left( \int_0^u \nu_s dN_s^{Y,\tilde{\mathbb{Q}}} \right) \tilde{\psi}_u dZ_u^1. \quad (36)$$

Consequently, upon letting

$$\hat{\nu}_t = (\tilde{\eta}_{t-})^{-1}, \quad \forall t \in [0, T], \quad (37)$$

we obtain from (36)

$$U_t = N_t^{Y,\tilde{\mathbb{Q}}} + \int_0^t \tilde{\psi}_u \left( \int_0^u \hat{\nu}_s dN_s^{Y,\tilde{\mathbb{Q}}} \right) dZ_u^1. \quad (38)$$

Note that the left-hand side of (35) is equal to  $U_T$ . Thus, comparing (35) and (38), we see that we may take

$$\hat{\psi}_t = -\tilde{\psi}_t \int_0^t \hat{\nu}_u dN_u^{Y,\tilde{\mathbb{Q}}} = -\tilde{\psi}_t \int_0^t (\tilde{\eta}_{u-})^{-1} dN_u^{Y,\tilde{\mathbb{Q}}}. \quad (39)$$

It is clear that with  $\widehat{v}$  defined in (37) the integral  $\int_0^t \widehat{v}_u dN_u^{Y, \widetilde{\mathbb{Q}}}$  is a  $\widetilde{\mathbb{Q}}$ -martingale. Thus, the process  $U$  is a martingale, rather than a local martingale, under  $\widetilde{\mathbb{Q}}$ . Together with (38) this implies that the process

$$\int_0^t \widetilde{\psi}_u \left( \int_0^u \widehat{v}_s dN_s^{Y, \widetilde{\mathbb{Q}}} \right) dZ_u^1$$

is a  $\widetilde{\mathbb{Q}}$ -martingale. Consequently, the process  $\widehat{\psi}$  defined in (39) belongs to the class  $\mathcal{A}(\widetilde{\mathbb{Q}})$ . To complete the proof, it suffices to combine (33), (34) and (39).  $\square$

**Remarks.** It should be acknowledged that the last result is not new. In fact, it is merely a special case of Theorem 6 in Rheinländer and Schweizer (1997). We believe, however, that our derivation of the result sheds a new light on the structure of the orthogonal projection computed above.

**Remarks.** Although the above proposition provides us with the structure of the projection  $\Pi_{\mathbb{P}}^0(Y)$ , it is not easy in general to obtain closed-form expressions for the components on the right-hand side of (32) in terms of the initial data for the problem. Thus, one may need to resort to numerical approximations, which in principle can be obtained by solving the following problem

$$\min_{\xi \in \mathcal{P}^0} \mathbb{E}_{\mathbb{P}}(Y - \xi)^2. \quad (40)$$

An approximate solution to the last problem yields a process, say  $\psi^{Y, \mathbb{P}}$ , so that  $\Pi_{\mathbb{P}}^0(Y) \approx \int_0^T \psi_t^{Y, \mathbb{P}} dZ_t^1$ .

#### 4.4 Mean-Variance Pricing and Hedging of a Generic Claim

Let us define  $\widetilde{\kappa} = \widetilde{d} - v = d - v - \mathbb{E}_{\mathbb{P}}X + \mathbb{E}_{\mathbb{P}}\widetilde{X}^0$ . For simplicity, we shall only consider the case when  $\widetilde{\kappa} > 0$ . This is equivalent to assuming that  $\widetilde{d} > v - p$  for all  $p \in [0, v]$ . Thus, the results of Proposition 4.1 (i) apply. Consequently, denoting  $\widetilde{\rho} = (d - v)^2 - \gamma_X^0(e^{\theta^2 T} - 1)$ , we obtain the following result,

**Proposition 4.3** *Suppose that  $\gamma_X^0 \leq (d - v)^2(e^{\theta^2 T} - 1)^{-1}$ . Then the buyer's mean variance price is*

$$p^{d, v}(X) = \min\{-\widetilde{\kappa}_1 + \sqrt{\widetilde{\rho}}, v\} \vee 0. \quad (41)$$

Otherwise,  $p^{d, v}(X) = -\infty$ .

In case when  $\gamma_X^0 \leq (d - v)^2(e^{\theta^2 T} - 1)^{-1}$ , the mean-variance hedging strategy for a generic claim  $X$  is given by  $\phi^*(d, v, p^{d, v}(X), X)$ , where the process  $\phi^*$  is defined in Proposition 4.1. The projection part of the strategy  $\phi^*(d, v, p^{d, v}(X), X)$ , that is, the process  $\psi^{1, \widetilde{X}^0}$ , can be computed according to (32).

#### 4.5 Defaultable Claims

In this section, we adopt the framework of Section 3.3.2. In particular, the default time  $\tau$  is a random time on  $(\Omega, \mathcal{G}, \mathbb{P})$  given by formula (19), and the process  $H$  is given as  $H_t = \mathbb{1}_{\{\tau \leq t\}}$  for every  $t \in [0, T]$ . The natural filtration  $\mathbb{H}$  of  $H$  is an auxiliary filtration, so that  $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ . Recall that we have assumed that  $\tau$  admits the  $\mathbb{F}$ -hazard process  $\Gamma$  under  $\mathbb{P}$  and thus also, in view of the construction (19), under  $\widetilde{\mathbb{Q}}$ . Suppose, in addition, that the hazard process  $\Gamma$  is an increasing continuous process. Then the process  $M_t = H_t - \Gamma_{t \wedge \tau}$  is known to be a  $\mathbb{G}$ -martingale under  $\widetilde{\mathbb{Q}}$ . Any  $\mathcal{G}_T$ -measurable random variable  $X$  is referred to as a *defaultable claim*.

Recall that the process  $\widetilde{W}_t = W_t + \theta t$  is a Brownian motion with respect to  $\mathbb{F}$  under  $\widetilde{\mathbb{Q}}$ , and thus the process  $Z^1$  is a square-integrable  $\mathbb{G}$ -martingale under  $\widetilde{\mathbb{Q}}$ , since

$$dZ_t^1 = Z_t^1 \sigma d\widetilde{W}_t, \quad Z_0^1 > 0.$$

The following proposition is an important technical result.

**Proposition 4.4** *The filtration  $\mathbb{G}^1$  is equal to the filtration  $\mathbb{G}$ , that is,  $\mathcal{G}_t^1 = \mathcal{G}_t$  for every  $t \in \mathbb{R}_+$ .*

*Proof.* It is clear that  $\mathbb{G}^1 \subseteq \mathbb{G}$ . For a fixed  $T > 0$ , let  $y_1, y_2 \in \mathbb{R}$  and let the processes  $\psi^1, \psi^2$  belong to  $\mathcal{A}(\tilde{\mathbb{Q}})$ . Thus the processes

$$Y_t^1 = y_1 + \int_0^t \psi_u^1 dZ_u^1, \quad Y_t^2 = y_2 + \int_0^t \psi_u^2 dZ_u^1$$

be  $\mathbb{G}^1$ -adapted processes. Then the process

$$Y_t^1 Y_t^2 = y_1 y_2 + \int_0^t Y_u^1 \psi_u^2 dZ_u^1 + \int_0^t Y_u^2 \psi_u^1 dZ_u^1 + \int_0^t \psi_u^1 \psi_u^2 d\langle Z^1 \rangle_u$$

is also  $\mathbb{G}^1$ -adapted. It is easy to check that the processes

$$\int_0^t Y_u^1 \psi_u^2 dZ_u^1, \quad \int_0^t Y_u^2 \psi_u^1 dZ_u^1$$

are  $\mathbb{G}^1$ -adapted. We thus conclude that for any processes  $\phi$  and  $\psi$  from  $\mathcal{A}(\tilde{\mathbb{Q}})$ , the process

$$\int_0^t \psi_u^1 \psi_u^2 d\langle Z^1 \rangle_u = \int_0^t \psi_u^1 \psi_u^2 (Z_u^1)^2 \sigma^2 du$$

is  $\mathbb{G}^1$ -adapted as well. In particular, it follows that for any bounded  $\mathbb{G}$ -adapted process  $\zeta$  the integral  $\int_0^t \zeta_u du$  defines a  $\mathbb{G}^1$ -adapted process. Let us take  $\zeta_u = H_u$ . Then we obtain that the process  $\tau \wedge t$  is  $\mathbb{G}^1$ -adapted. Hence, it is easily seen that  $\mathcal{G}_t \subseteq \mathcal{G}_t^1$  for  $t \in [0, T]$ . Since  $T$  was an arbitrary positive number, we have shown that  $\mathbb{G} = \mathbb{G}^1$ .  $\square$

#### 4.5.1 Projection of a Survival Claim

In this section, we shall compute the process  $\psi^{Y, \mathbb{P}}$  which occurs in the projection  $\Pi_{\mathbb{P}}^0(Y)$  for a random variable  $Y = Z \mathbb{1}_{\{\tau > T\}}$ , where  $Z \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{Q}})$ . It is well known that any random variable  $Y$  from  $L^2(\Omega, \mathcal{G}_T, \tilde{\mathbb{Q}}) = L^2(\Omega, \mathcal{G}_T^1, \tilde{\mathbb{Q}})$ , which vanishes on the set  $\{\tau > T\}$ , can indeed be represented in this way. Any random variable  $Y$  of the form  $Z \mathbb{1}_{\{\tau > T\}}$  is referred to as a *survival claim* with maturity date  $T$ , and a random variable  $Z$  is said to be the *promised payoff* associated with  $Y$ .

It is known (see, e.g., Bielecki and Rutkowski (2002)) that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}(Y | \mathcal{G}_t) = \mathbb{E}_{\tilde{\mathbb{Q}}}(Z \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t) = \mathbb{E}_{\tilde{\mathbb{Q}}}(Z \mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t^1) = \mathbb{1}_{\{\tau > t\}} e^{\Gamma t} \mathbb{E}_{\tilde{\mathbb{Q}}}(Z e^{-\Gamma T} | \mathcal{F}_t) = L_t m_t^Z,$$

where  $L_t := \mathbb{1}_{\{\tau > t\}} e^{\Gamma t}$  is a  $\mathbb{G}$ -martingale and  $m_t^Z = \mathbb{E}_{\tilde{\mathbb{Q}}}(Z e^{-\Gamma T} | \mathcal{F}_t)$  is an  $\mathbb{F}$ -martingale. From the predictable representation theorem for a Brownian motion (or since the default-free market is complete), it follows that there exists an  $\mathbb{F}$ -adapted process  $\mu^Z$  such that

$$m_t^Z = m_0^Z + \int_0^t \mu_u^Z dZ_u^1. \quad (42)$$

In Proposition 4.2, we have already described the structure of the process  $\psi^{Y, \mathbb{P}}$  that specifies the projection of  $Y$  on  $\mathcal{P}^0$ . In the next two results, we shall give more explicit formulae for  $\psi^{Y, \tilde{\mathbb{Q}}}$  and  $N^{Y, \tilde{\mathbb{Q}}}$  within the present setup.

**Lemma 4.3** Consider a survival claim  $Y = Z\mathbb{1}_{\{\tau > T\}}$  with the promised payoff  $Z \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{Q}})$ . It holds that  $\psi_t^{Y, \tilde{\mathbb{Q}}} = L_{t-}\mu_t^Z$  for every  $t \in [0, T]$ , where by convention  $L_{0-} = 0$ .

*Proof.* It is easy to check that  $dL_t = -L_{t-}dM_t$ . Since  $\Gamma$  is increasing, the process  $L$  is of finite variation, and thus

$$d(L_t m_t^Z) = L_{t-} dm_t^Z + m_t^Z dL_t = L_{t-}\mu_t^Z dZ_t^1 + m_t^Z dL_t,$$

and thus we obtain

$$d\langle \mathcal{Y}, Z^1 \rangle_t = L_{t-}\mu_t^Z d\langle Z^1 \rangle_t$$

and  $\psi_t^{Y, \tilde{\mathbb{Q}}} = L_{t-}\mu_t^Z$ , which proves the result.  $\square$

For the proof of the next auxiliary result, the reader is referred, for instance, to Jeanblanc and Rutkowski (2000) or Bielecki and Rutkowski (2002).

**Lemma 4.4** Consider a survival claim  $Y = Z\mathbb{1}_{\{\tau > T\}}$  with the promised payoff  $Z \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{Q}})$ . The process  $N^{Y, \tilde{\mathbb{Q}}}$  in the Galtchouk-Kunita-Watanabe decomposition of  $Y$  with respect to  $Z^1$  under  $\tilde{\mathbb{Q}}$  is given by the expression

$$N_t^{Y, \tilde{\mathbb{Q}}} = \int_{[0, t)} n_u^Z dM_u,$$

where the process  $M_t = H_t - \Gamma_{t \wedge \tau}$  is a  $\mathbb{G}$ -martingale, strongly orthogonal in the martingale sense to  $\tilde{W}$  under  $\tilde{\mathbb{Q}}$ , and where

$$n_t^Z = -\mathbb{E}_{\tilde{\mathbb{Q}}}^{\sim}(Z e^{\Gamma_t - \Gamma_T} | \mathcal{F}_t). \quad (43)$$

By combining Proposition 4.2 with the last two results, we obtain the following corollary, which furnishes an almost explicit representation for the process  $\tilde{\psi}^{Y, \mathbb{P}}$  associated with the projection on  $\mathcal{P}^0$  of a survival claim.

**Corollary 4.2** Let  $Y = Z\mathbb{1}_{\{\tau > T\}}$  be a survival claim, where  $Z \in L^2(\Omega, \mathcal{F}_T, \tilde{\mathbb{Q}})$ . Then  $\Pi_{\mathbb{P}}^0(Y)$  is given by the following expression

$$\Pi_{\mathbb{P}}^0(Y) = \int_0^T \tilde{\psi}_t^{Y, \mathbb{P}} dZ_t^1,$$

where for every  $t \in [0, T]$

$$\tilde{\psi}_t^{Y, \mathbb{P}} = L_{t-}\mu_t^Z - \tilde{\psi}_t \left( \tilde{\eta}_0^{-1} \mathbb{E}_{\tilde{\mathbb{Q}}}^{\sim} Y + \int_0^t \tilde{\eta}_u^{-1} n_u^Z dM_u \right) \quad (44)$$

where in turn  $L_t = \mathbb{1}_{\{\tau > t\}} e^{\Gamma t}$  and the processes  $\tilde{\psi}, \tilde{\eta}, \mu^Z$  and  $n^Z$  are given by (30), (31), (42) and (43), respectively.

#### 4.5.2 Projection of a Defaultable Bond

According to the adopted convention regarding the recovery scheme, the terminal payoff at time  $T$  of a defaultable bond equals  $X = L\mathbb{1}_{\{\tau > T\}} + \delta L\mathbb{1}_{\{\tau \leq T\}}$  for some  $L > 0$  and  $\delta \in [0, 1)$ . Notice that the payoff  $X$  can be represented as follows  $X = \delta L + (1 - \delta)L Y$ , where  $Y = \mathbb{1}_{\{\tau > T\}}$  is a simple survival claim, with the promised payoff  $Z = 1$ . Using the linearity of the projection  $\Pi_{\mathbb{P}}^0$ , we notice that  $\Pi_{\mathbb{P}}^0(X)$  can be evaluated as follows

$$\Pi_{\mathbb{P}}^0(X) = \delta L \Pi_{\mathbb{P}}^0(1) + (1 - \delta)L \Pi_{\mathbb{P}}^0(Y).$$

By virtue of Corollary 4.2, we conclude that

$$\Pi_{\mathbb{P}}^0(X) = -\delta L e^{\theta^2 T} \Pi_{\mathbb{Q}}^0(\eta_T) + (1 - \delta)L \int_0^T \psi_t dZ_t^1,$$

where (cf. (44))

$$\psi_t = \mathbb{1}_{\{\tau > t\}} e^{\Gamma t} \mu_t - \tilde{\psi}_t \left( e^{-\theta^2 T} \mathbb{E}_{\tilde{\mathbb{Q}}}(e^{-\Gamma T}) + \int_0^t \tilde{\eta}_u^{-1} n_u dM_u \right), \quad (45)$$

where in turn the process  $\tilde{\psi}$  is given by (30), the process  $n$  equals  $n_t = -\mathbb{E}_{\tilde{\mathbb{Q}}}(e^{\Gamma t - \Gamma T} | \mathcal{F}_t)$ , and the process  $\mu$  is such that

$$\mathbb{E}_{\tilde{\mathbb{Q}}}(e^{-\Gamma T} | \mathcal{F}_t) = \mathbb{E}_{\tilde{\mathbb{Q}}} Y + \int_0^t \mu_u dZ_u^1, \quad \forall t \in [0, T].$$

**Example 4.1** Consider the special case when  $\Gamma$  is deterministic. It is easily seen that we now have  $\mu = 0$  and  $n_t = -e^{\Gamma t - \Gamma T}$ . Consequently, (45) becomes

$$\psi_t = -\tilde{\psi}_t e^{-\Gamma T} \left( e^{-\theta^2 T} - \int_0^t \tilde{\eta}_u^{-1} e^{\Gamma t} dM_u \right),$$

and thus

$$\Pi_{\mathbb{P}}^0(X) = -\delta L e^{\theta^2 T} \Pi_{\mathbb{Q}}^0(\eta_T) - (1 - \delta)L \int_0^T \tilde{\psi}_t e^{-\Gamma T} \left( e^{-\theta^2 T} - \int_0^t \tilde{\eta}_u^{-1} e^{\Gamma t} dM_u \right) dZ_t^1,$$

where the processes  $\tilde{\psi}$  and  $\tilde{\eta}$  are given by (30) and (31), respectively.

## 5 Risk-Return Portfolio Selection

In the preceding sections, we have examined the Markowitz-type mean-variance hedging problem from the particular perspective of valuation of non-attainable contingent claims. In view of the dependence of the mean-variance price obtained through this procedure on agent's preferences, (formally reflected, among others, by the values of parameters  $d$  and  $v$ ), this specific application of Markowitz-type methodology suffers from deficiencies, which may undermine its practical implementations.

In this section, we shall take a totally different perspective, and we shall assume that a given claim  $X$  can be purchased by an agent (an asset management fund, say) for some pre-specified price. For instance, the price of  $X$  can be given by an investment bank that is able to hedge this claim using some arbitrage-free model, or it can be simply given by the OTC market. Let us emphasize that an agent is now considered as a price-taker. We assume that an agent would like to invest in  $X$ , but will not be able (or willing) to hedge this claim using the underlying primary assets (if any such assets are available). As a consequence, an agent will only have in its portfolio standard instruments that are widely available for trading. The two important issues we would like to address in this section are:

- What proportion of the initial endowment  $v$  should an agent invest in the claim  $X$  if the goal is to lower the standard deviation (or, equivalently, the variance) of return, and to keep the expected rate of return at the desired level.
- How much should an agent invest in  $X$  in order to enhance the expected rate of return, and to preserve at the same time the pre-specified level of risk, as measured by the standard deviation of the rate of return.

We shall argue that mathematical tools and results presented in the previous sections are sufficient to solve both these problems. It seems to us that this alternative application of the mean-variance methodology can be of practical importance as well.

For the sake of simplicity, we shall solve the optimization problems formulated above in the class  $\Phi(\mathbb{F})$  of  $\mathbb{F}$ -admissible trading strategies. A similar study can be conducted for the case of  $\mathbb{G}$ -admissible strategies. For any  $v > 0$  and any trading strategy  $\phi \in \Phi(\mathbb{F})$ , let  $r(\phi)$  be the simple rate of return, defined as

$$r(\phi) = \frac{V_T^v(\phi) - v}{v}.$$

The minimization of the standard deviation of the rate of return, which equals

$$\sigma(r(\phi)) = \sqrt{\mathbb{V}_{\mathbb{P}}\left(\frac{V_T^v(\phi) - v}{v}\right)} = v^{-1} \sqrt{\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))},$$

is, of course, equivalent to the minimization of the variance  $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$ . Within the present context, it is natural to introduce the constraint  $\mathbb{E}_{\mathbb{P}}(v^{-1}V_T^v(\phi)) \geq d = 1 + d_r$ , where  $d_r > 0$  represents the desired minimal level of the expected rate of return.

## 5.1 Auxiliary Problems

The following auxiliary problem  $MV(dv, v)$  is merely a version of the previously considered problem  $MV(d, v)$ :

**Problem  $MV(dv, v)$ :** For a fixed  $v > 0$  and  $d > 1$ , minimize the variance  $\mathbb{V}_{\mathbb{P}}(V_T^v(\phi))$  over all strategies  $\phi \in \Phi(\mathbb{F})$ , subject to  $\mathbb{E}_{\mathbb{P}}V_T^v(\phi) \geq dv$ .

We assume from now on that  $\theta \neq 0$ , and we denote  $\Theta = (e^{\theta^2 T} - 1)^{-1} > 0$ . Recall that for the problem  $MV(dv, v)$ , the risk-return trade-off can be summarized by the minimal variance curve  $w^*(dv, v)$ . By virtue of Proposition 3.3, we have

$$w^*(dv, v) = \Theta v^2 (d - 1)^2 = \Theta v^2 d_r^2. \quad (46)$$

Equivalently, the minimal standard deviation of the rate of return satisfies

$$\sigma_r^* = \sigma(r(\phi^*(dv, v))) = \sqrt{w^*(dv, v)} = \sqrt{\Theta} d_r,$$

so that it does not depend, as expected, on the value of the initial endowment  $v$ .

Suppose now that a claim  $X$  is available for some price  $p_X \neq 0$ , referred to as the *market price*. It is convenient to introduce the normalized claim  $\bar{X} = X p_X^{-1}$  so that, by the postulated linearity property of the market price, the price  $p_{\bar{X}}$  of one unit of  $\bar{X}$  equals 1.

The next auxiliary problem we wish to solve reads: find  $p \in \mathbb{R}$  such that the solution to the problem  $MV(dv, v, p, p\bar{X})$  has the minimal variance. This means, of course, that we are looking for  $p \in \mathbb{R}$  for which  $w^*(dv, v, p, p\bar{X})$  is minimal. Notice that the constraint on the expected rate of return becomes

$$\mathbb{E}_{\mathbb{P}}(v^{-1}V_T^{v-p}(\phi) + p\bar{X}) \geq d = 1 + d_r,$$

where  $d_r > 0$ . It is clear that the curve  $w^*(dv, v, p, p\bar{X})$  can be derived from the general expression for  $w^*(d, v, p, X)$ , which was established in Proposition 3.4. Let us denote

$$\gamma_{\bar{X}} = \mathbb{V}_{\mathbb{P}}(\bar{X} - \mathbb{E}_{\mathbb{P}}(\bar{X} | \mathcal{F}_T)), \quad \nu_{\bar{X}} = \mathbb{E}_{\mathbb{Q}}\bar{X} - 1.$$

Let us notice that the condition  $d - v + p - \mathbb{E}_{\mathbb{Q}}X > 0$ , which was imposed in part (i) of Proposition 3.4, now corresponds to the following inequality:  $vd_r > p\nu_{\bar{X}}$ . We shall assume from now on that  $\bar{X} \neq 1$  (i.e., that the claim  $\bar{X}$  does not represent the savings account). Recall that  $v > 0$  and  $d_r = d - 1 > 1$ .

**Proposition 5.1** (i) Let  $\gamma_{\bar{X}} > 0$  and  $\nu_{\bar{X}} \neq 0$ . Then the problem  $MV(dv, v, p, p\bar{X})$  has a solution with the minimal variance with respect to  $p$ . The minimal variance equals

$$w^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2 \left( 1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2} \right) \quad (47)$$

and the optimal value of  $p$  equals

$$p^* = \frac{vd_r\nu_{\bar{X}}}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2}. \quad (48)$$

(ii) Let  $\gamma_{\bar{X}} > 0$  and  $\nu_{\bar{X}} = 0$ . Then we have  $p^* = 0$  and  $w^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2$ .

(iii) Let  $\gamma_{\bar{X}} = 0$  and  $\nu_{\bar{X}} \neq 0$ . If the inequality  $\nu_{\bar{X}} > 0$  ( $\nu_{\bar{X}} < 0$ , respectively) holds then for any  $p \geq vd_r\nu_{\bar{X}}^{-1}$  ( $p \leq vd_r\nu_{\bar{X}}^{-1}$ , respectively) the minimal variance  $w^*(dv, v, p, p\bar{X})$  is minimal with respect to  $p$  and it equals 0.

(iv) Let  $\gamma_{\bar{X}} = \nu_{\bar{X}} = 0$ . Then  $\bar{X}$  is an attainable claim and  $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ . In this case, for any  $p \in \mathbb{R}$  the minimal variance equals

$$w^*(dv, v, p, p\bar{X}) = \Theta v^2 d_r^2.$$

*Proof.* Let us first prove parts (i)-(ii). It suffices to observe that, by virtue of Proposition 3.4, the minimal variance for the problem  $MV(dv, v, p, p\bar{X})$  is given by the expression:

$$w^*(dv, v, p, p\bar{X}) = \Theta(d_r v - p\nu_{\bar{X}})^2 + p^2\gamma_{\bar{X}} \quad (49)$$

provided that  $vd_r > p\nu_{\bar{X}}$ . A simple argument shows that the minimal value for the right-hand side in (49) is obtained by setting  $p = p^*$ , where  $p^*$  is given by (48), and the minimal variance is given by (47). Moreover, it is easily seen that for  $p^*$  given by (48) the inequality  $vd_r > p^*\nu_{\bar{X}}$  is indeed satisfied, provided that  $\gamma_{\bar{X}} > 0$ . Notice also that if  $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ , we obviously have  $vd_r > p\nu_{\bar{X}} = 0$  for any  $p \in \mathbb{R}_+$ , and thus we obtain the optimal values  $p^* = 0$  and  $w^*(dv, v, p^*, p^*\bar{X}) = \Theta v^2 d_r^2$ .

Assume now that  $vd_r \leq p\nu_{\bar{X}}$ , so that the case  $\nu_{\bar{X}} = 0$  (i.e., the case  $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ ) is excluded. Then, by virtue of part (ii) in Proposition 3.4, the minimal variance equals  $p^2\gamma_{\bar{X}}$  (notice that the assumption that  $\gamma_{\bar{X}}$  is strictly positive is not needed here). Assume first that  $\mathbb{E}_{\mathbb{Q}}\bar{X} < 1$ , so that  $\nu_{\bar{X}} < 0$ . Then the condition  $vd_r \leq p\nu_{\bar{X}}$  becomes  $p \leq vd_r\nu_{\bar{X}}^{-1}$ , and thus  $p$  is necessarily negative. The minimal variance corresponds to  $p^* = vd_r\nu_{\bar{X}}^{-1}$ , and it equals

$$w_2^* = w^*(dv, v, p^*, p^*\bar{X}) = (p^*)^2\gamma_{\bar{X}} = v^2 d_r^2 \nu_{\bar{X}}^{-2} \gamma_{\bar{X}}. \quad (50)$$

In, on the contrary,  $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1$ , then  $\nu_{\bar{X}} > 0$  and we obtain  $p \geq vd_r\nu_{\bar{X}}^{-1}$ , so that  $p$  is strictly positive. Again, the minimal variance corresponds to  $p^* = vd_r\nu_{\bar{X}}^{-1}$ , and it is given by (50). It is easy to check that the following inequality holds:

$$\Theta v^2 d_r^2 \left( 1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2} \right) < v^2 d_r^2 \nu_{\bar{X}}^{-2} \gamma_{\bar{X}}.$$

By combining the considerations above, we conclude that (i)-(ii) is valid. The proof of part (iii) is also based on the analysis above. We thus proceed to the proof of part (iv).

Notice that  $\Theta^{-1}\gamma_{\bar{X}} + \nu_{\bar{X}}^2 = 0$  if and only if  $\gamma_{\bar{X}} = 0$  and  $\nu_{\bar{X}} = 0$ . This means that  $\bar{X}$  is  $\mathcal{F}_T$ -adapted (and thus  $\mathbb{F}$ -attainable) and  $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$  (so that the arbitrage price of  $\bar{X}$  coincides with its market price  $p_{\bar{X}}$ ). Condition  $vd_r - p\nu_{\bar{X}} > 0$  is now satisfied, and thus the minimal variance is given by (49), which now becomes

$$w^*(dv, v, p, p\bar{X}) = \Theta v^2 d_r^2, \quad \forall p \in \mathbb{R}_+.$$

Obviously, the result does not depend on  $p$ . This proves part (iv).  $\square$

**Remarks.** In the last proposition, no a priori restriction on the value of the parameter  $p$  was imposed. Of course, one can also consider a related constrained problem by postulating, for instance, that  $p \in [0, v]$ .

## 5.2 Minimization of Risk

We are in the position to examine the first question, which reads: how much to invest in the new opportunity in order to minimize the risk and to preserve at the same time the pre-specified level  $d_r > 0$  of the expected rate of return.

**Case of an attainable claim.** Assume first that  $\bar{X}$  is an  $\mathbb{F}$ -attainable contingent claim, so that  $\mathbb{E}_{\mathbb{P}}(\bar{X} | \mathcal{F}_T) = \bar{X}$ , and thus  $\gamma_{\bar{X}} = 0$ . If the claim  $\bar{X}$  is correctly priced by the market, i.e., if  $\mathbb{E}_{\mathbb{Q}}\bar{X} = p_{\bar{X}} = 1$  then, by virtue of part (iv) in Proposition 5.1, for any choice of  $p$  the minimal variance is the same as in the problem  $MV(dv, v)$ . Hence, as expected, the possibility of investing in the claim  $\bar{X}$  has no bearing on the efficiency of trading.

Let us now consider the case where  $\mathbb{E}_{\mathbb{Q}}\bar{X} \neq 1$ , that is, the market price  $p_{\bar{X}}$  does not coincide with the arbitrage price  $\pi_0(\bar{X})$ . Suppose first that  $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1$ , that is,  $\bar{X}$  is underpriced by the market. Then, in view of part (iii) in Proposition 5.1, the variance of the rate of return can be reduced to 0 by choosing  $p \geq vd_r(\mathbb{E}_{\mathbb{Q}}\bar{X} - 1)^{-1} > 0$ . Similarly, if  $\mathbb{E}_{\mathbb{Q}}\bar{X} < 1$  then for any  $p \leq vd_r(\mathbb{E}_{\mathbb{Q}}\bar{X} - 1)^{-1} < 0$  the variance equals 0. Of course, this feature is due to the presence of arbitrage opportunities in the market. We conclude that, as expected, the case of an attainable claim is of no practical interest.

**Case of a non-attainable claim.** We now assume that  $\gamma_{\bar{X}} > 0$ . Suppose first that  $\mathbb{E}_{\mathbb{Q}}\bar{X} = 1$ . By virtue of part (ii) in Proposition 5.1, under this assumption it is optimal not to invest in  $\bar{X}$ . To better appreciate this result, notice that for the conditional expectation  $\tilde{X} = \mathbb{E}_{\mathbb{P}}(\bar{X} | \mathcal{F}_T)$  we have  $\mathbb{E}_{\mathbb{Q}}\tilde{X} = \mathbb{E}_{\mathbb{Q}}\bar{X} = 1$  and  $\mathbb{E}_{\mathbb{P}}\tilde{X} = \mathbb{E}_{\mathbb{P}}\bar{X}$  (cf. Section 3.2.2). Therefore, trading in  $\bar{X}$  is essentially equivalent to trading in an attainable claim  $\tilde{X}$ , but trading in  $\bar{X}$  results in the residual variance  $p^2\gamma_{\bar{X}}$ . This explains why the solution  $p^* = 0$  is optimal.

Suppose now that  $\mathbb{E}_{\mathbb{Q}}\bar{X} \neq 1$ . Then part (i) of Proposition 5.1 shows that the variance of the rate of return can always be reduced by trading in  $\bar{X}$ . Specifically,  $p^*$  is strictly positive provided that  $\mathbb{E}_{\mathbb{Q}}\bar{X} > 1 = p_{\bar{X}}$ , that is, the expected value of  $\bar{X}$  under the martingale measure  $\mathbb{Q}$  for the underlying market is greater than its market price.

**Case of an independent claim.** Assume that the claim  $\bar{X}$  is independent of  $\mathcal{F}_T$ , so that  $\gamma_{\bar{X}} > 0$  is the variance of  $\bar{X}$ . In this case  $\mathbb{E}_{\mathbb{Q}}\bar{X} = \mathbb{E}_{\mathbb{P}}\bar{X}$  and thus (47) becomes

$$w^* = \Theta v^2 d_r^2 \left( 1 - \frac{(\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2}{\Theta^{-1}\mathbb{V}_{\mathbb{P}}(\bar{X}) + (\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2} \right).$$

From the last formula, it is clear that an agent should always to invest either a positive or negative amount of initial endowment  $v$  in an independent claim  $X$ , except for the case where  $\mathbb{E}_{\mathbb{P}}\bar{X} = 1$ . If  $\mathbb{E}_{\mathbb{P}}\bar{X} \neq 1$  then the optimal value of  $p$  equals (cf. (48))

$$p^* = \frac{vd_r(\mathbb{E}_{\mathbb{P}}\bar{X} - 1)}{\Theta^{-1}\mathbb{V}_{\mathbb{P}}(\bar{X}) + (\mathbb{E}_{\mathbb{P}}\bar{X} - 1)^2}$$

so that it is positive if and only if  $\mathbb{E}_{\mathbb{P}}\bar{X} > 1$ .

**Case of a claim with zero market price.** The case when the market price of  $X$  is zero (that is,  $p_X = 0$ ) is also of practical interest, since it is a typical feature of forward contracts. This particular case is not covered by Proposition 5.1, however. In fact, we deal here with the following variant of the mean-variance problem: find  $\alpha \in \mathbb{R}$  such that the solution to the problem  $MV(dv, v, 0, \alpha X)$  has the minimal variance. Under the assumption  $vd_r > \alpha\mathbb{E}_{\mathbb{Q}}X$ , we have

$$w^*(dv, v, 0, \alpha X) = \Theta(vd_r - \alpha\mathbb{E}_{\mathbb{Q}}X)^2 + \alpha^2\gamma_X.$$

If, on the contrary,  $vd_r \leq \alpha\mathbb{E}_{\mathbb{Q}}X$ , then the minimal variance equals  $\alpha^2\gamma_X$ . Of course, we necessarily have  $\alpha \neq 0$  here (since  $vd_r > 0$ ).

### 5.3 Maximization of Expected Return

Let us focus on part (i) in Proposition 5.1, that is, let us assume that  $\gamma_{\bar{X}} > 0$  and  $\nu_{\bar{X}} \neq 0$  (as was explained above, other cases examined in Proposition 5.1 are of minor practical interest). The question of maximization of the expected rate return for a pre-specified level of risk, can be easily solved by comparing (46) with (47). Indeed, for a given level  $d_r$  of the expected rate of return, and thus a given level  $w^*(dv, v)$  of the minimal variance, it suffices to find a number  $\hat{d}_r$  which solves the following equation

$$\Theta v^2 d_r^2 = \Theta v^2 \hat{d}_r^2 \left( 1 - \frac{\nu_{\bar{X}}^2}{\Theta^{-1} \gamma_{\bar{X}} + \nu_{\bar{X}}^2} \right)$$

It is obvious that the last equation has the unique solution

$$\hat{d}_r = d_r \sqrt{1 + \frac{\nu_{\bar{X}}^2}{\Theta^{-1} \gamma_{\bar{X}}}} > d_r.$$

The corresponding value of  $p^*$  is given by (48) with  $d_r$  substituted with  $\hat{d}_r$ . It is thus clear that, under the present assumptions, a new investment opportunity can be used to enhance the expected rate of return. If we insist, in addition, that  $p > 0$ , then the latter statement remains valid, provided that  $\mathbb{E}_{\mathbb{Q}} \bar{X} > 1$ .

### References

- Bielecki, T.R., and Jeanblanc, M. (2003) Genuine mean-variance hedging of credit risk: A case study. Working paper.
- Bielecki, T.R., Jeanblanc, M. and Rutkowski M. (2004a) Modeling and valuation of credit risk. In: *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone, July 6-12, 2003*. Springer-Verlag, Berlin Heidelberg New York.
- Bielecki, T.R., and Rutkowski, M. (2002) *Credit Risk: Modeling, Valuation and Hedging*. Springer-Verlag, Berlin Heidelberg New York.
- Bielecki, T.R., Jin, H., Pliska, S.R., and Zhou, X.Z. (2004b) Dynamic mean-variance portfolio selection with bankruptcy prohibition. To appear in *Mathematical Finance*.
- Collin-Dufresne, P. and Hugonnier, J.-N. (2002) On the pricing and hedging of contingent claims in the presence of extraneous risks. To appear in *Finance and Stochastics*.
- Davis, M.H.A. (1997) Option pricing in incomplete markets. In: *Mathematics of Derivative Securities*, M.A.H. Dempster and S.R. Pliska, editors, Cambridge University Press, pp. 216–226.
- ElKaroui, N. and Quenez, M.-C. (1995) Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM J. Control and Optimization* 33, 29–66.
- ElKaroui, N., Peng, S. and Quenez, M.C. (1997) Backward stochastic differential equations in finance. *Mathematical Finance* 7, 1–71.
- ElKaroui, N. and Rouge, R. (2000) Pricing via utility maximization and entropy. *Mathematical Finance* 10, 259–276.
- Gouriéroux, C., Laurent, J.-P. and Pham, N. (1998) Mean-variance hedging and numéraire. *Mathematical Finance* 8, 179–200.
- Jeanblanc, M. and Rutkowski, M. (2000) Modelling of default risk: Mathematical tools. Working paper, Université d'Evry and Politechnika Warszawska.
- Kramkov, D. (1996) Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields* 105, 459–479.

- Pham, N., Rheinländer, Th. and Schweizer, M. (1998) Mean-variance hedging for continuous processes: new results and examples. *Finance and Stochastics* 2, 173–98.
- Pliska, S.R. (2001) Dynamic Markowitz problems with no bankruptcy. Presentation at JAFEE Conference, 2001.
- Rheinländer, Th. (1999) Optimal martingale measures and their applications in mathematical finance. *Ph.D. Thesis*, TU-Berlin.
- Rheinländer, Th. and Schweizer, M. (1997) On  $L^2$ -projections on a space of stochastic integrals. *Annals of Probability* 25, 1810–1831.
- Schweizer, M. (2001) A guided tour through quadratic hedging approaches. In: *Option Pricing, Interest Rates and Risk Management*, E. Jouini, J. Cvitanić and M. Musiela, editors, Cambridge University Press, pp. 538–574.