

Completeness of a Reduced-Form Credit Risk Model with Discontinuous Asset Prices

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1 Introduction

The goal of this work is to examine the issue of attainability of a generic defaultable claim within the reduced-form approach to credit risk modelling, and the closely related issue of completeness of a credit risk model. In contrast to our previous work Bielecki et al. (2004a), we consider here the case where the prices of default-free assets and the pre-default values of defaultable assets follow general (that is, not necessarily continuous) semimartingales.

In Section 2, we concentrate on trading in primary assets with discontinuous prices. Our main goal is to analyze the issue of replication of a generic contingent claim by means a self-financing trading strategy that is additionally subject to an algebraic constraint, interpreted as the balance condition. We examine the relationships between the completeness of a market model with unconstrained strategies and the corresponding market with strategies satisfying the balance condition. Though in this section we do not deal with specific issues related to the modelling of credit risk, it is essential for a full appreciation of results of Section 3. For the proofs of all results of Section 2, the interested reader is referred to Bielecki et al. (2004c).

In Section 3, we also include defaultable assets in our portfolio. In this case, our primary goal is to examine the issue of replication of a generic defaultable contingent claim. In addition, we establish basic results on completeness of a credit risk model within the reduced-form set-up. We conclude by analyzing particular examples of survival claims under deterministic and stochastic intensities. Let us emphasize that we use throughout a purely probabilistic approach. A PDE approach to the valuation and hedging of credit derivatives will be studied in a separate paper.

2 Generic Market Model

Our goal in this section is to present some auxiliary results related to the concept of a self-financing trading strategy for a generic market model. As already mentioned, these results were established in a companion paper Bielecki et al. (2004c). They are summarized here for the sake of a reader's convenience.

Let $Y_t^1, Y_t^2, \dots, Y_t^k$ represent cash values at time t of k assets. We postulate that the prices Y^1, Y^2, \dots, Y^k follow semimartingales on some probability space $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions. As it is usually done, we set $X_{0-} = X_0$ for any stochastic process X , and we only consider semimartingales with càdlàg sample paths. We assume, in addition that at least one of the processes Y^1, Y^2, \dots, Y^k , say process Y^1 , is strictly positive, so that it can be chosen as a numeraire asset. Unless explicitly stated otherwise, we do not assume that this market is complete.

2.1 Unconstrained Trading Strategies

We consider trading within the time interval $[0, T]$ for some finite horizon date $T > 0$. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a generic trading strategy; in particular, the processes ϕ^i are predictable with respect to the reference filtration \mathbb{F} . The wealth $V_t(\phi)$ at time t of the trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$V_t(\phi) = \sum_{i=1}^k \phi_t^i Y_t^i, \quad \forall t \in [0, T], \quad (1)$$

and ϕ is said to be a *self-financing strategy* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^k \int_0^t \phi_u^i dY_u^i, \quad \forall t \in [0, T]. \quad (2)$$

Let Φ be the class of all self-financing trading strategies. By combining the last two formulae, we obtain the following expression for the dynamics of the wealth process of a strategy $\phi \in \Phi$

$$dV_t(\phi) = \left(V_t(\phi) - \sum_{i=2}^k \phi_t^i Y_t^i \right) (Y_t^1)^{-1} dY_t^1 + \sum_{i=2}^k \phi_t^i dY_t^i.$$

The representation above shows that the wealth process $V(\phi)$ depends only on $k - 1$ components of ϕ . Choosing Y^1 as a numeraire asset, and denoting $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$, $Y_t^{i,1} = Y_t^i(Y_t^1)^{-1}$, we get the following well-known result showing that the self-financing feature of a trading strategy is invariant with respect to the choice of a numeraire asset. The proof of Lemma 2.1 is well known in the case of continuous semi-martingale. In the case of discontinuous processes, the proof of part (i) is given in Protter (2001) and the proof of part (ii) in Bielecki et al. (2004c).

Lemma 2.1 (i) *For any $\phi \in \Phi$, we have*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^k \int_0^t \phi_u^i dY_u^{i,1}, \quad \forall t \in [0, T]. \quad (3)$$

(ii) *Conversely, let X be a \mathcal{F}_T -measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and \mathbb{F} -predictable processes ϕ^i , $i = 2, 3, \dots, k$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^k \int_0^T \phi_t^i dY_t^{i,1} \right).$$

Then there exists a \mathbb{F} -predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is self-financing and replicates X . Moreover, the wealth process of ϕ satisfies $V_t(\phi) = V_t^1 Y_t^1$, where the process V^1 is given by

$$V_t^1 = x + \sum_{i=2}^k \int_0^t \phi_u^i dY_u^{i,1}, \quad \forall t \in [0, T]. \quad (4)$$

2.2 Constrained Trading Strategies

In this section, we make an additional assumption that the price process Y^k is strictly positive. Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=l+1}^k \phi_t^i Y_{t-}^i = Z_t, \quad \forall t \in [0, T], \quad (5)$$

for some $1 \leq l \leq k - 1$ and a predetermined, \mathbb{F} -predictable process Z . In the financial interpretation, equality (5) means that the portfolio ϕ should be rebalanced in such a way that the total wealth invested in securities $Y^{l+1}, Y^{l+2}, \dots, Y^k$ should match a predetermined stochastic process (for instance, we may assume that it is constant over time or follows a deterministic function of time). For this reason, the constraint (5) will be referred to as the *balance condition*. In the next section, a suitable version of this condition will be used to ensure that a properly chosen portfolio of default-free assets insures against the liability associated with a premature default.

Our first goal is to extend part (i) in Lemma 2.1 to the case of constrained strategies. Let $\Phi_l(Z)$ stand for the class of all self-financing trading strategies satisfying the balance condition (5). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi_l(Z)$ is self-financing, we have

$$\Delta V_t(\phi) = \sum_{i=1}^k \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^k \phi_t^i Y_{t-}^i,$$

and thus we deduce from (5) that

$$V_{t-}(\phi) = \sum_{i=1}^k \phi_t^i Y_{t-}^i = \sum_{i=1}^l \phi_t^i Y_{t-}^i + Z_t.$$

Let us write $Y_t^{i,1} = Y_t^i (Y_t^1)^{-1}$, $Y_t^{i,k} = Y_t^i (Y_t^k)^{-1}$, $Z_t^1 = Z_t (Y_t^1)^{-1}$. It is apparent from Proposition 2.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi_l(Z)$ depends only on $k-2$ components of ϕ .

Proposition 2.1 *The relative wealth $V_t^1(\phi) = V_t(\phi)(Y_t^1)^{-1}$ of a strategy $\phi \in \Phi_l(Z)$ satisfies*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^l \int_0^t \phi_u^i dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} dY_u^{k,1} \right) + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} dY_u^{k,1}. \quad (6)$$

2.3 Replication with Constrained Strategies

The next result is essentially a converse to Proposition 2.1. Also, it extends part (ii) of Lemma 2.1 to the case of constrained trading strategies. As in Section 2.2, we assume that $1 \leq l \leq k-1$, and Z is a predetermined, \mathbb{F} -predictable process. For the sake of notational simplicity, we shall write

$$Y_t^{i,k,1} = \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} dY_u^{k,1} \right). \quad (7)$$

Proposition 2.2 *Let a \mathcal{F}_T -measurable random variable X represent a contingent claim that settles at time T . Assume that there exist \mathbb{F} -predictable processes ϕ^i , $i = 2, 3, \dots, k-1$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^l \int_0^T \phi_t^i dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i dY_t^{i,k,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} dY_t^{k,1} \right). \quad (8)$$

Then there exist the \mathbb{F} -predictable processes ϕ^1 and ϕ^k such that the strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ belongs to $\Phi_l(Z)$ and replicates X . The wealth process of ϕ equals, for every $t \in [0, T]$,

$$V_t(\phi) = Y_t^1 \left(x + \sum_{i=2}^l \int_0^t \phi_u^i dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i dY_u^{i,k,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} dY_u^{k,1} \right). \quad (9)$$

Note that equality (8) is a necessary (by Proposition 2.1) and sufficient (by Proposition 2.2) condition for the existence of a constrained strategy replicating a given contingent claim X .

2.4 Synthetic Assets

Let us fix i , and let us analyze the auxiliary process $Y^{i,k,1}$ given by formula (7). We claim that this process can be interpreted as the relative wealth of a specific self-financing trading strategy associated with Y^1, Y^2, \dots, Y^k . Specifically, we will show that for any $i = 2, 3, \dots, k-1$ the process $\bar{Y}^{i,k,1}$, given by the formula

$$\bar{Y}_t^{i,k,1} = Y_t^1 Y_t^{i,k,1} = Y_t^1 \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} dY_u^{k,1} \right),$$

represents the price of a *synthetic asset*. For brevity, we shall frequently write \bar{Y}^i instead of $\bar{Y}^{i,k,1}$.

2.4.1 Equivalence of Primary and Synthetic Assets

Our goal is to show that trading in primary assets is formally equivalent to trading in synthetic assets. The first result shows that the process \bar{Y}^i can be obtained from primary assets Y^1, Y^i and Y^k through a simple self-financing strategy. This justifies the name synthetic asset given to \bar{Y}^i .

Lemma 2.2 *For any fixed $i = 2, 3, \dots, k-1$, let a \mathcal{F}_T -measurable random variable \bar{Y}_T^i be given as*

$$\bar{Y}_T^i = Y_T^1 Y_T^{i,k,1} = Y_T^1 \int_0^T \left(dY_t^{i,1} - \frac{Y_{t-}^{i,1}}{Y_{t-}^{k,1}} dY_t^{k,1} \right). \quad (10)$$

Then there exists a strategy $\phi \in \Phi_1(0)$ that replicates the claim \bar{Y}_T^i . Moreover, we have, for every $t \in [0, T]$,

$$V_t(\phi) = Y_t^1 Y_t^{i,k,1} = Y_t^1 \int_0^t \left(dY_u^{i,1} - \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} dY_u^{k,1} \right) = \bar{Y}_t^i. \quad (11)$$

Note that to replicate the claim $\bar{Y}_T^i = \bar{Y}_T^{i,k,1}$, it suffices to invest in primary assets Y^1, Y^i and Y^k . Essentially, we start with zero initial endowment, we keep at any time one unit of the i th asset, we rebalance the portfolio in such a way that the total wealth invested in the i th and k th assets is always zero, and we put the residual wealth in the first asset. Hence, we deal here with a specific strategy such that the risk of the i th asset is perfectly offset by rebalancing the investment in the k th asset, and our trades are financed by taking positions in the first asset.

Note that the process $Y^{i,1}$ satisfies the following SDE

$$Y_t^{i,1} = Y_0^{i,1} + \bar{Y}_t^{i,1} + \int_0^t \frac{Y_{u-}^{i,1}}{Y_{u-}^{k,1}} dY_u^{k,1}, \quad (12)$$

which is known to possess a unique strong solution. Hence, the relative price $Y_t^{i,1}$ at time t is uniquely determined by the initial value $Y_0^{i,1}$ and processes $\bar{Y}^{i,1}$ and $Y^{k,1}$. Consequently, the price Y_t^i at time t of the i th primary asset is uniquely determined by the initial value Y_0^i , the prices Y^1, Y^k of primary assets, and the price \bar{Y}^i of the i th synthetic asset.

Lemma 2.3 *Filtrations generated by the primary assets Y^1, Y^2, \dots, Y^k and by the price processes $Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$ coincide.*

Lemma 2.3 suggests that for any choice of the underlying filtration \mathbb{F} (such that $\mathbb{F}^Y \subseteq \mathbb{F}$), trading in assets Y^1, Y^2, \dots, Y^k is essentially equivalent to trading in $Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$. Let us first formally define the equivalence of market models.

Definition 2.1 We say that the two models, \mathcal{M} and $\widetilde{\mathcal{M}}$ say, are *equivalent with respect to a filtration* \mathbb{F} if both models are defined on a common probability space and primary assets in \mathcal{M} can be obtained by trading in primary assets in $\widetilde{\mathcal{M}}$ and vice versa, under the assumption that trading strategies are \mathbb{F} -predictable.

Note that we do not assume that models \mathcal{M} and $\widetilde{\mathcal{M}}$ have the same number of primary assets. The next result justifies our claim of equivalence of primary and synthetic assets.

Corollary 2.1 *Models $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ and $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ are equivalent.*

2.4.2 Replicating Strategies with Synthetic Assets

In view of Lemma 2.2, the replicating trading strategy for a contingent claim X , for which (8) holds, can be conveniently expressed in terms of primary securities Y^1, Y^2, \dots, Y^l and Y^k , and synthetic assets $\bar{Y}^{l+1}, \bar{Y}^{l+2}, \dots, \bar{Y}^{k-1}$. To this end, we represent (8)-(9) in the following way:

$$X = Y_T^1 \left(x + \sum_{i=2}^l \int_0^T \phi_t^i dY_t^{i,1} + \sum_{i=l+1}^{k-1} \int_0^T \phi_t^i d\bar{Y}_t^{i,1} + \int_0^T \frac{Z_t^1}{Y_{t-}^{k,1}} dY_t^{k,1} \right) \quad (13)$$

where $\bar{Y}_t^{i,1} = \bar{Y}_t^i / Y_t^1 = Y_t^{i,k,1}$, and

$$V_t(\phi) = Y_t^1 \left(x + \sum_{i=2}^l \int_0^t \phi_u^i dY_u^{i,1} + \sum_{i=l+1}^{k-1} \int_0^t \phi_u^i d\bar{Y}_u^{i,1} + \int_0^t \frac{Z_u^1}{Y_{u-}^{k,1}} dY_u^{k,1} \right). \quad (14)$$

Corollary 2.2 *Let X be a \mathcal{F}_T -measurable random variable such that (13) holds for some \mathbb{F} -predictable process Z and some \mathbb{F} -predictable processes $\phi^2, \phi^3, \dots, \phi^{k-1}$. Let $\psi^i = \phi^i$ for $i = 2, 3, \dots, k-1$,*

$$\psi_t^k = \frac{Z_t^1}{Y_{t-}^{k,1}} = \frac{Z_t}{Y_{t-}^k},$$

and

$$\psi_t^1 = V_t^1 - \sum_{i=2}^l \psi_t^i Y_t^{i,1} - \sum_{i=l+1}^{k-1} \psi_t^i \bar{Y}_t^{i,1} - \psi_t^k Y_t^{k,1} = V_{t-}^1 - \sum_{i=2}^l \psi_t^i Y_{t-}^{i,1} - \sum_{i=l+1}^{k-1} \psi_t^i \bar{Y}_{t-}^{i,1} - \psi_t^k Y_{t-}^{k,1}.$$

Then $\psi = (\psi^1, \psi^2, \dots, \psi^k)$ is a self-financing trading strategy in assets $Y^1, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$. Moreover, ψ satisfies $\psi_t^k Y_{t-}^k = Z_t$, $t \in [0, T]$, and it replicates X .

2.5 Model Completeness

We shall now examine the relationship between the arbitrage-free property and completeness of a market model in which trading is restricted a priori to self-financing strategies satisfying the balance condition. A good understanding of these features is crucial from the viewpoint of further applications to replication of defaultable claims.

2.5.1 Minimal Completeness of an Unconstrained Model

Let $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be an arbitrage-free market model. Unless explicitly stated otherwise, Φ stands for the class of all \mathbb{F} -predictable, self-financing strategies. Note, however, that the number of traded assets and their selection may be different for each particular model. Consequently, the dimension of a strategy $\phi \in \Phi$ will depend on the number of traded assets in a given model. For the sake of brevity, this feature is not reflected in our notation.

Definition 2.2 We say that a model \mathcal{M} is *complete* with respect to \mathbb{F} if any bounded \mathcal{F}_T -measurable contingent claim X is attainable in \mathcal{M} . Otherwise, a model \mathcal{M} is said to be *incomplete* with respect to \mathbb{F} .

Definition 2.3 A model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ is *minimally complete* with respect to \mathbb{F} if \mathcal{M} is complete, and for any $i = 1, 2, \dots, k$ the *reduced model* $\mathcal{M}^i = (Y^1, Y^2, \dots, Y^{i-1}, Y^{i+1}, \dots, Y^k; \Phi)$ is incomplete with respect to \mathbb{F} , so that for each i there exists a bounded, \mathcal{F}_T -measurable contingent claim, which is not attainable in the model \mathcal{M}^i . In this case, we say that the *degree of completeness* of \mathcal{M} equals k .

Let us stress that trading strategies in the reduced model \mathcal{M}^i are predictable with respect to \mathbb{F} , rather than with respect to the filtration generated by price processes $Y^1, Y^2, \dots, Y^{i-1}, Y^{i+1}, \dots, Y^k$. Hence, when we move from \mathcal{M} to \mathcal{M}^i , we reduce the number of traded asset, but we preserve the original information structure \mathbb{F} . Minimal completeness of a model \mathcal{M} means that all primary assets Y^1, Y^2, \dots, Y^k are needed if we wish to generate the class of all (bounded) \mathcal{F}_T -measurable claims through \mathbb{F} -predictable trading strategies. The following lemma is thus an immediate consequence of Definition 2.3.

Lemma 2.4 *Assume that a model \mathcal{M} is complete, but not minimally complete, with respect to \mathbb{F} . Then there exists at least one primary asset Y^i , which is redundant in \mathcal{M} , in the sense that it corresponds to the wealth process of some trading strategy in the reduced model \mathcal{M}^i .*

Lemma 2.5 shows that the degree of completeness is a well-defined notion, in the sense that it does not depend on the choice of traded assets, provided that the model completeness is preserved.

Lemma 2.5 *Let a model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be minimally complete with respect to \mathbb{F} . Let $\widetilde{\mathcal{M}} = (\widetilde{Y}^1, \widetilde{Y}^2, \dots, \widetilde{Y}^k; \Phi)$, where the processes $\widetilde{Y}^i = V(\phi_i)$, $i = 1, 2, \dots, k$ represent the wealth processes of some trading strategies $\phi_1, \phi_2, \dots, \phi_k \in \Phi$. If a model $\widetilde{\mathcal{M}}$ is complete with respect to \mathbb{F} then it is also minimally complete with respect to \mathbb{F} , and thus its degree of completeness equals k .*

By combining Lemma 2.5 with Corollary 2.1, we obtain the following result.

Corollary 2.3 *A model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ is minimally complete if and only if a model $\bar{\mathcal{M}} = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi)$ has this property.*

2.5.2 Completeness of a Constrained Model

Let $\mathcal{M} = (Y^1, Y^2, \dots, Y^k; \Phi)$ be an arbitrage-free market model, and let us denote by $\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$ the associated model in which the class Φ is replaced by the class $\Phi_l(Z)$ of constrained strategies. We claim that if \mathcal{M} is arbitrage-free and minimally complete with respect to the filtration $\mathbb{F} = \mathbb{F}^Y$, where $Y = (Y^1, Y^2, \dots, Y^k)$, then the constrained model $\mathcal{M}_l(Z)$ is arbitrage-free, but it is incomplete with respect to \mathbb{F} . Conversely, if the model $\mathcal{M}_l(Z)$ is arbitrage-free and complete with respect to \mathbb{F} , then the original model \mathcal{M} is not minimally complete. To prove these claims, we need some preliminary results.

The following definition extends the notion of equivalence of security market models to the case of constrained trading.

Definition 2.4 We say that the two constrained models are *equivalent with respect to a filtration \mathbb{F}* if they are defined on a common probability space and the class of all wealth processes of \mathbb{F} -predictable constrained trading strategies is the same in both models.

In view of the next result, the constrained models $\bar{\mathcal{M}}_{k-1}(Z)$ and $\mathcal{M}_l(Z)$ are equivalent. Typically, the latter model is easier to handle than the former.

Corollary 2.4 *A constrained model*

$$\mathcal{M}_l(Z) = (Y^1, Y^2, \dots, Y^k; \Phi_l(Z))$$

is equivalent to a constrained model

$$\bar{\mathcal{M}}_{k-1}(Z) = (Y^1, Y^2, \dots, Y^l, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k; \Phi_{k-1}(Z)).$$

Proposition 2.3 (i) *Assume that a model \mathcal{M} is arbitrage-free and minimally complete. Then for any \mathbb{F} -predictable process Z and any $l = 1, 2, \dots, k - 1$ a constrained model $\mathcal{M}_l(Z)$ is arbitrage-free and incomplete.*

(ii) *Assume that a constrained model $\mathcal{M}_l(Z)$ associated with \mathcal{M} is arbitrage-free and complete. Then \mathcal{M} is either not arbitrage-free or not minimally complete.*

3 Credit Risk Model

Our goal is to extend the results of the previous section to the case of a market model with default-free and defaultable primary assets. As mentioned in the introduction, replication of defaultable claim in a credit risk model in which prices of default-free assets and pre-default values of defaultable assets follow continuous semimartingales was examined in Chapter 1 of Bielecki et al. (2004a).

3.1 Self-financing Trading Strategies

Let τ be a strictly positive random variable on a probability space $(\Omega, \mathcal{G}, \mathbb{Q})$, referred to as a *default time*. In order to exclude trivial cases, we assume that $\mathbb{Q}\{\tau > 0\} = 1$ and $\mathbb{Q}\{\tau \leq T\} > 0$. Let us introduce the jump process $H_t = \mathbb{1}_{\{\tau \leq t\}}$ and denote by \mathbb{H} the filtration generated by this process.

Assume that we are given, in addition, a *reference filtration* \mathbb{F} such that $\mathcal{F}_t \subseteq \mathcal{G}$ for every $t \in [0, T]$. We set $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every $t \in \mathbb{R}_+$. The filtration \mathbb{G} is referred to as to the *full filtration*; it includes the observations of default event. We assume that any \mathbb{F} -martingale is also a \mathbb{G} -martingale. Such an assumption is sometimes called the *H hypothesis*. For more details on this assumption, we refer to Bielecki et al. (2004a).

We assume that we are given a family Y^1, Y^2, \dots, Y^k of semimartingales defined on the filtered probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{Q})$. We interpret Y^1, Y^2, \dots, Y^m as price processes of m defaultable assets with a common default time τ . Processes $Y^{m+1}, Y^{m+2}, \dots, Y^k$ represent the prices of $k - m$ default-free assets; it is thus natural to postulate that they are \mathbb{F} -adapted.

Definition of a *self-financing* trading strategy ϕ is exactly the same as before (see formula (2)), except that now the components $\phi^1, \phi^2, \dots, \phi^k$ are \mathbb{G} -predictable, rather than \mathbb{F} -predictable. However, in this note we shall never deal with a trading strategy after the random time $\tau \wedge T$ representing the *effective maturity* of a defaultable claim (see, in particular, Definition 3.2 below). Hence, we may and do assume, without loss of generality, that the components $\phi^1, \phi^2, \dots, \phi^k$ of a trading strategy ϕ are \mathbb{F} -predictable processes (cf. Bielecki et al. (2004b)). We assume throughout that the market model $\mathcal{M} = (Y^1, Y^2, \dots, Y^k, \Phi)$ is arbitrage-free in the usual sense.

For simplicity, we shall assume that the defaultable assets Y^1, Y^2, \dots, Y^m are subject to the zero recovery at default, so that $Y_t^i = 0$ for every $t \geq \tau$ and $i = 1, 2, \dots, m$. Let \tilde{Y}_t^i stand for the *pre-default value* of the i th defaultable asset at time t , so that we have $Y_t^i = (1 - H_t)\tilde{Y}_t^i$ for $i = 1, 2, \dots, m$, where \tilde{Y}^i is a \mathbb{F} -adapted process. The reader is referred to Bielecki et al. (2004b) for the definition of the pre-default value and methods of computation of this process. It appears that within the reduced-form framework the pre-default values $\tilde{Y}^i, i = 1, 2, \dots, m$ follow \mathbb{F} -semimartingales.

Note that, under the present assumptions, the value of every defaultable asset jumps to zero after default, and thus it makes no sense to assume that these assets are also traded after default. Hence, the pre-default value can be interpreted as the *traded value* of a defaultable asset. Let us stress, however, that the concept of a pre-default value is useful also in the case of non-zero recovery. In fact, a pre-default value is a convenient and natural object to work with if we are interested in hedging a defaultable claim up to default time. With this motivation in mind, we introduce the notion of a pre-default wealth of a trading strategy.

Definition 3.1 The *pre-default wealth* $\tilde{V}(\phi)$ of a trading strategy $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ equals

$$\tilde{V}_t(\phi) = \sum_{i=1}^m \phi_t^i \tilde{Y}_t^i + \sum_{i=m+1}^k \phi_t^i Y_t^i, \quad \forall t \in [0, T].$$

A strategy ϕ is said to be *self-financing prior to default time* if

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \sum_{i=1}^m \int_0^t \phi_u^i d\tilde{Y}_u^i + \sum_{i=m+1}^k \int_0^t \phi_u^i dY_u^i, \quad \forall t \in [0, T].$$

Note that $\tilde{V}_0(\phi) = V_0(\phi)$, since $\mathbb{Q}\{\tau > 0\} = 1$. Let us stress that if a trading strategy ϕ is self-financing prior to default time in the sense of Definition 3.1, then it may be easily extended to a self-financing strategy on the interval $[0, T]$. Indeed, we may and do postulate that all trading activities are stopped at the effective maturity of a claim, and thus the terminal wealth at time $\tau \wedge T$ equals

$$V_{\tau \wedge T}(\phi) = \sum_{i=1}^k \phi_{\tau \wedge T}^i Y_{\tau \wedge T}^i.$$

In particular, on the event $\{\tau > T\}$ we have

$$V_{\tau \wedge T}(\phi) = V_T(\phi) = \tilde{V}_T(\phi) = \sum_{i=1}^m \phi_T^i \tilde{Y}_T^i + \sum_{i=m+1}^k \phi_T^i Y_T^i.$$

Hence, we shall not distinguish in what follows between the concepts of a self-financing trading strategy and a trading strategy that is self-financing prior to default time.

3.2 Replication of a Defaultable Claim

We are interested in determining replicating strategies for a generic defaultable claim. According to our convention, it is specified by a \mathcal{F}_T -measurable random variable interpreted as the promised payoff, and a \mathbb{F} -predictable process Z that determines the recovery payoff at the time of default. All necessary integrability conditions are implicitly assumed to hold with regard to X and Z . Formally, a generic defaultable claim can thus be represented as a triple (X, Z, τ) . We denote by \tilde{U} the pre-default value of the claim (X, Z, τ) , so that $\tilde{U}_t = \tilde{U}_t(X) + \tilde{U}_t(Z)$ (for computations of $\tilde{U}(X)$ and $\tilde{U}(Z)$ in terms of the hazard process of τ see, for instance, Bielecki et al. (2004a)).

Definition 3.2 A self-financing trading strategy ϕ is a *replicating strategy* for a defaultable claim (X, Z, τ) if and only if the following hold:

- (i) $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \leq T\}$,
- (ii) $V_T(\phi) = X$ on the set $\{\tau > T\}$.

We say that a defaultable claim is *attainable* if it admits at least one replicating strategy.

Note that if a self-financing trading strategy ϕ satisfies conditions (i)-(ii) of Definition 3.2, then we also have: (iii) $\tilde{V}_t(\phi) = \tilde{U}_t(X) + \tilde{U}_t(Z)$ on the random interval $\llbracket 0, \tau \wedge T \rrbracket$. In other words, the pre-default wealth of ϕ and the pre-default value of (X, Z, τ) coincide on this interval.

We assume throughout that the primary defaultable securities Y^i , $i = 1, 2, \dots, m$ are subject to zero recovery, that is, $Y_t^i = 0$ for $t \geq \tau$. It is thus natural to introduce the following counterpart of the balance condition (5):

$$\sum_{i=m+1}^k \phi_t^i Y_{t-}^i = Z_t, \quad \forall t \in [0, T], \quad (15)$$

where Z is a predetermined recovery process. We now make the following standing assumption.

Assumption (A): For every $i = m + 1, m + 2, \dots, k$, we have $\Delta Y_\tau^i = Y_\tau^i - Y_{\tau-}^i = 0$.

Remarks. It is important to stress that the notion of a ‘defaultable asset’ should not be taken literally. For instance, in the case of the so-called *flight to quality* the price of a default-free bond jumps at the moment τ associated with some ‘default event’ (see, e.g., Collin-Dufresne et al. (2003)). Let us stress that a default-free bond can be formally classified as a ‘defaultable asset’ if its price has a jump at default time τ . Generally speaking, the distinction between the default-risk-sensitive securities Y^1, Y^2, \dots, Y^m and the default-risk-insensitive securities $Y^{m+1}, Y^{m+2}, \dots, Y^k$ is done on the basis of the *no-jump at τ condition* of Assumption (A). Hence, Assumption (A) is not restrictive

at all; it should be rather seen as a convention that serves to distinguish between the two classes of assets that exhibit a different type of behavior at time τ .

The prices Y^i , $i = 1, 2, \dots, m$ of defaultable assets vanish for $t \geq \tau$ and, typically, $\Delta Y_\tau^i = -Y_{\tau-}^i = -\tilde{Y}_{\tau-}^i \neq 0$. Note also that Assumption (A), combined with condition (15), imply that on the event $\{\tau \leq T\}$ we have

$$V_\tau(\phi) = \sum_{i=m+1}^k \phi_\tau^i Y_\tau^i = Z_\tau. \quad (16)$$

Equality (16) coincides with the property (ii) of Definition 3.2. This means that, under Assumption (A), the balance condition (15) ensures that a strategy ϕ satisfies the property (ii) of Definition 3.2. It thus suffices to concentrate on the property (iii) in this definition.

We find it convenient to denote $\tilde{Y}^i = Y^i$ for $i = m+1, m+2, \dots, k$ in what follows. This notational convention can be justified by a simple observation that the pre-default value of a default-free asset coincides with its value. We assume that at least two processes, \tilde{Y}^l and \tilde{Y}^p say, are strictly positive, where $l \in \{1, 2, \dots, m\}$ and $p \in \{m+1, m+2, \dots, k\}$. The proper choice of \tilde{Y}^l and \tilde{Y}^p depends on the hedging problem at hand. We shall argue in what follows that in order to replicate a defaultable claim one needs to have $l \in \{1, 2, \dots, m\}$. Hence, we adopt the convention that $l = 1$ and $p = k$. Let us denote

$$\tilde{V}_t^1(\phi) = \frac{V_t(\phi)}{\tilde{Y}_t^1}, \quad \tilde{Y}_t^{k,1} = \frac{\tilde{Y}_t^k}{\tilde{Y}_t^1}, \quad \tilde{Z}_t^1 = \frac{Z_t}{\tilde{Y}_t^1}, \quad d\tilde{Y}_t^{i,k,1} = d\tilde{Y}_t^{i,1} - \frac{\tilde{Y}_t^{i,1}}{\tilde{Y}_t^{k,1}} d\tilde{Y}_t^{k,1}.$$

Also let $\bar{Y}^i = \bar{Y}^{i,k,1} = \tilde{Y}^1 \tilde{Y}^{i,k,1} = \tilde{Y}^1 Y^{i,k,1}$ represents the price of the i th synthetic asset. Note that the synthetic assets are virtual objects, but we can always re-express a constrained trading strategy in terms of primary assets Y^1, Y^2, \dots, Y^k .

Proposition 3.1 (i) *Let $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ be a self-financing strategy satisfying (15). Then the pre-default wealth process $\tilde{V}(\phi)$ is a \mathbb{F} -adapted process satisfying, for every $t \in [0, T]$,*

$$\tilde{V}_t(\phi) = \tilde{Y}_t^1 \left(\tilde{V}_0^1(\phi) + \sum_{i=2}^m \int_0^t \phi_u^i d\tilde{Y}_u^{i,1} + \sum_{i=m+1}^{k-1} \int_0^t \phi_u^i d\tilde{Y}_u^{i,k,1} + \int_0^t \frac{\tilde{Z}_u^1}{\tilde{Y}_u^{k,1}} d\tilde{Y}_u^{k,1} \right). \quad (17)$$

In addition, we also have $\tilde{V}_{t-}(\phi) = \sum_{i=1}^m \phi_t^i \tilde{Y}_{t-}^i + Z_t$ for every $t \in [0, T]$

(ii) *Suppose that there exist \mathbb{F} -predictable processes ϕ^i , $i = 2, 3, \dots, k-1$, such that*

$$X = \tilde{Y}_T^1 \left(x + \sum_{i=2}^m \int_0^T \phi_t^i d\tilde{Y}_t^{i,1} + \sum_{i=m+1}^{k-1} \int_0^T \phi_t^i d\tilde{Y}_t^{i,k,1} + \int_0^T \frac{\tilde{Z}_t^1}{\tilde{Y}_t^{k,1}} d\tilde{Y}_t^{k,1} \right). \quad (18)$$

Then there exist the \mathbb{F} -predictable processes ϕ^1 and ϕ^k such that $\phi = (\phi^1, \phi^2, \dots, \phi^k)$ is a self-financing strategy in assets $Y^1, \dots, Y^m, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$. The pre-default wealth of ϕ satisfies the equality $\tilde{V}_T(\phi) = X$, and ϕ satisfies the balance condition $\sum_{i=m+1}^k \phi_t^i \tilde{Y}_t^i = Z_t$. Hence, in view of part (i) above, it holds that $\tilde{V}_{\tau-}(\phi) = \sum_{i=1}^m \phi_\tau^i \tilde{Y}_{\tau-}^i + Z_\tau$ on the event $\{\tau \leq T\}$.

Proof. The proofs of parts (i) and (ii) rely on exactly the same arguments as the proofs of Propositions 2.1 and 2.2, respectively, (see Bielecki et al. (2004c)). \square

Corollary 3.1 *Let ϕ be a self-financing trading strategy in the assets $Y^1, \dots, Y^m, \bar{Y}^{l+1}, \dots, \bar{Y}^{k-1}, Y^k$ introduced in part (ii) of Proposition 3.1. Then ϕ replicates a defaultable claim (X, Z, τ) and the equality $\tilde{U}_0(X) + \tilde{U}_0(Z) = \tilde{Y}_0^1 x$ is valid.*

Proof. Since the strategy ϕ is self-financing, we need only to verify is that: $V_T(\phi) = X$ on the event $\{\tau > T\}$, and $V_\tau(\phi) = Z_\tau$ on the event $\{\tau \leq T\}$. For the former equality, observe that it

follows from part (ii) of Proposition 3.1 that the predefault wealth process of the strategy satisfies $\tilde{V}_T(\phi) = X$; consequently, $V_T(\phi) = X$ on the set $\{\tau > T\}$. The latter equality is an immediate consequence of the equality: $\tilde{V}_{\tau-}(\phi) = V_{\tau-}(\phi) = \sum_{i=1}^m \phi_\tau^i \tilde{Y}_{\tau-}^i + Z_\tau$ on the event $\{\tau \leq T\}$, and the assumed zero recovery scheme for all defaultable assets Y^1, Y^2, \dots, Y^m . \square

3.3 Replication of a Survival Claim

The goal of this section is to provide concrete, but still fairly general, examples of defaultable claims and their replication strategies.

Let us take $k = 3$ and $m = 1$, so that the asset Y^1 is defaultable and Y^2 and Y^3 are default-free assets. According to the convention set forth above, we denote by \tilde{Y}^1 the pre-default value of the first asset. For the sake of concreteness and computational convenience, we assume that $Y_t^1 = D^0(t, T)$ is the price process of a defaultable unit zero-coupon bond subject to zero recovery, and $Y_t^3 = B(t, T)$ represents the price of a default-free unit zero-coupon bond. Note that both bonds are assumed to mature at time T . According to the convention set forth above, we denote by \tilde{Y}^1 the pre-default value of the defaultable asset Y^1 . Hence, we have $\tilde{Y}_T^1 = Y_T^3 = 1$, since obviously $\tilde{D}^0(T, T) = B(T, T) = 1$. Moreover, we make the natural assumption that the processes \tilde{Y}^1 and Y^3 are strictly positive.

Remarks. Note that the bond $B(t, T)$ falls here in the category of default-free securities. Hence, in view of the standing assumption (A), credit risk models in which the price of a default-free bond jumps at the default time τ are not covered by results of this section.

We shall consider a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim with the promised payoff X and zero recovery at default. We postulate from now on that the random variable $X \neq 0$, representing a default-free claim associated with a survival claim, can be replicated by trading in the primary assets Y^2 and Y^3 . This is formalized in the following assumption.

Assumption (B): The promised payoff X represents an attainable claim in the default-free market model $\mathcal{M}^1 = (Y^2, Y^3; \Phi)$. Hence, there exists a constant y and a \mathbb{F} -predictable process α such that

$$\frac{X}{Y_T^3} = y + \int_0^T \alpha_t dY_t^{2,3}. \quad (19)$$

We denote by $\pi_t(X)$ the arbitrage price of X at time $t \in [0, T]$ in \mathcal{M}^1 . Note that $\pi_0(X) = Y_0^3 y$.

It is rather clear that, under Assumption (A), it is not possible to replicate a survival claim using a self-financing trading strategy of the form $\phi = (0, \phi^2, \phi^3)$. In other words, a survival claim is not attainable in the reduced model \mathcal{M}^1 .

Lemma 3.1 *A survival claim $(X, 0, \tau)$ is not attainable in the default-free market \mathcal{M}^1 .*

Proof. Recall that we have assumed that $\mathbb{Q}\{\tau \leq T\} > 0$. Assumption (A) implies that the wealth $V_t(\phi) = \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3$ is continuous at the default time τ , since under this assumption we have $\Delta V_\tau(\phi) = \phi_\tau^2 \Delta Y_\tau^2 + \phi_\tau^3 \Delta Y_\tau^3 = 0$. The value of a survival claim jumps to zero at default time, and thus it cannot be replicated by means of a strategy ϕ of the form $(0, \phi^2, \phi^3)$. \square

Obviously, the price Y^1 of a defaultable bond drops from the positive pre-default value to zero at the time of default. Thus, in order to replicate a survival claim $(X, 0, \tau)$, it is natural to include a defaultable bond in hedging portfolio.

We shall find a replicating strategy for a survival claim in the market model $\bar{\mathcal{M}} = (Y^1, \bar{Y}^2, Y^3; \Phi)$, and we shall establish a relationship between the arbitrage price $\pi_t(X)$ of the promised payoff and the pre-default value $\tilde{U}_t(X)$ of a survival claim. By virtue of part (ii) in Proposition 3.1, it suffices

to examine the existence of a constant x and a \mathbb{F} -predictable process ϕ^2 such that

$$\frac{X}{\widetilde{Y}_T^1} = x + \int_0^T \phi_t^2 d\widetilde{Y}_t^{2,3,1}. \quad (20)$$

It is easily seen that $\widetilde{U}_0(X) = \widetilde{Y}_0^1 x$, so that formally

$$\widetilde{U}_0(X) = \pi_0(X) \frac{\widetilde{Y}_0^1 x}{\widetilde{Y}_0^3 y} = \pi_0(X) \widetilde{Y}_0^{1,3} \frac{x}{y},$$

provided that $\pi_0(X) \neq 0$ (so that $y \neq 0$). We wish to make the last equality more explicit, and to extend it to any $t \in [0, T]$.

3.3.1 Case of a Deterministic Default Intensity

A quite explicit pricing result for a survival claim can be obtained if we assume that the relative value $\widetilde{Y}^{3,1} = \widetilde{Y}^3/\widetilde{Y}^1 = Y^3/Y^1$ follows a continuous process of finite variation. Recall that we formally set $\widetilde{Y}_t^i = Y_t^i$ for $i = 2, 3$, but for the defaultable zero-coupon bond, we have $Y_t^1 = (1 - H_t)\widetilde{Y}_t^1$. Also, recall that $\widetilde{Y}^2 = \widetilde{Y}^{2,3,1} = \widetilde{Y}^1 \widetilde{Y}^{2,3,1} = \widetilde{Y}^1 Y^{2,3,1}$. The next proposition is an essential generalization of a result obtained in Bielecki et al. (2004a).

Proposition 3.2 *Assume that $\widetilde{Y}^{3,1} = \widetilde{Y}^3/\widetilde{Y}^1$ is a continuous process of finite variation. Then a survival claim $(X, 0, \tau)$ can be replicated in the market model \mathcal{M} by means of a strategy ϕ such that $\phi^2 = \alpha \widetilde{Y}^{1,3}$ and the processes ϕ^1, ϕ^3 are given by (24) below. The pre-default value of $(X, 0, \tau)$ equals*

$$\widetilde{U}_t(X) = \widetilde{Y}_t^{1,3} \pi_t(X) = \widetilde{Y}_t^1 F_X(t, T), \quad \forall t \in [0, T],$$

where $F_X(t, T) = \pi_t(X)/B(t, T)$ is the forward price of X .

Proof. It suffices to consider the case $t = 0$. Let α be the first component of a replicating strategy for the promised payoff X in \mathcal{M}^1 (see formula (19)), so that we have, for every $t \in [0, T]$,

$$\frac{\pi_t(X)}{\widetilde{Y}_t^3} = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^t \alpha_u dY_u^{2,3} = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^t \alpha_u d\widetilde{Y}_u^{2,3}. \quad (21)$$

Let us set $\phi_t^2 = \alpha_t \widetilde{Y}_t^{1,3}$. We claim that

$$\frac{X}{\widetilde{Y}_T^1} = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^T \phi_t^2 \left(d\widetilde{Y}_t^{2,1} - \frac{\widetilde{Y}_{t-}^{2,1}}{\widetilde{Y}_{t-}^{3,1}} d\widetilde{Y}_t^{3,1} \right) = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^T \phi_t^2 d\widetilde{Y}_t^{2,3,1}. \quad (22)$$

To establish (22), it suffices to note that we have

$$d\widetilde{Y}_t^{2,3,1} = d\widetilde{Y}_t^{2,1} - \frac{\widetilde{Y}_{t-}^{2,1}}{\widetilde{Y}_{t-}^{3,1}} d\widetilde{Y}_t^{3,1} = \widetilde{Y}_t^{3,1} d\widetilde{Y}_t^{2,3} + d[\widetilde{Y}^{2,3}, \widetilde{Y}^{3,1}]_t = \widetilde{Y}_t^{3,1} d\widetilde{Y}_t^{2,3}, \quad (23)$$

since $\widetilde{Y}^{3,1}$ is assumed to be a continuous process of finite variation. Combining (21) with (23), we obtain

$$d(\pi_t(X)/\widetilde{Y}_t^3) = \alpha_t d\widetilde{Y}_t^{2,3} = \alpha_t \widetilde{Y}_t^{1,3} d\widetilde{Y}_t^{2,3,1}.$$

Finally, since $\widetilde{Y}_T^1 = \widetilde{Y}_T^3 = 1$, we obtain

$$\frac{X}{\widetilde{Y}_T^1} = \frac{X}{\widetilde{Y}_T^3} = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^T \alpha_t d\widetilde{Y}_t^{2,3} = \frac{\pi_0(X)}{\widetilde{Y}_0^3} + \int_0^T \alpha_t \widetilde{Y}_t^{1,3} d\widetilde{Y}_t^{2,3,1},$$

so that (22) holds with $\phi_t^2 = \alpha_t \tilde{Y}_t^{1,3}$. Note that (22) is a special case of (18) with $k = 3$ and $Z^1 = 0$. From Corollary 3.1, it follows that the replicating strategy $\phi = (\phi^1, \phi^2, \phi^3)$ for a survival claim $(X, 0, \tau)$ can be constructed as in part (ii) of Proposition 3.1. Specifically, we define the process \tilde{V} by the formula

$$\tilde{V}_t = \tilde{Y}_t^1 \left(\frac{\pi_0(X)}{\tilde{Y}_0^3} + \int_0^t \alpha_u \tilde{Y}_u^{1,3} d\tilde{Y}_u^{2,3,1} \right),$$

and we set (cf. Proposition 2.2)

$$\phi_t^1 = (\tilde{Y}_t^1)^{-1} \left(\tilde{V}_t - \phi_t^2 \tilde{Y}_t^2 - \phi_t^3 \tilde{Y}_t^3 \right) = \tilde{V}_t^1, \quad \phi_t^3 = -\phi_t^2 \tilde{Y}_t^{2,3}. \quad (24)$$

Then the pre-default wealth of the strategy ϕ coincides with \tilde{V} . In particular, the price at time 0 of a survival claim equals $\tilde{U}_0(X) = \tilde{V}_0 = \tilde{Y}_0^{1,3} \pi_0(X)$. \square

Remarks. Recall that Y^1 and Y^3 are prices of defaultable and default-free zero-coupon bonds respectively. Within the reduced-form set-up, the assumption that $\tilde{Y}^{3,1} = \tilde{Y}^3 / \tilde{Y}^1 = Y^3 / Y^1$ follows a continuous process of finite variation is satisfied only when the hazard process of the default time τ with respect to \mathbb{F} is deterministic (see Bielecki et al. (2004b)). Hence, if we wish to deal with the case of stochastic intensity process, we need to relax this assumption.

3.3.2 Case of a Stochastic Default Intensity

In this section, we no longer assume that the process $\tilde{Y}^{3,1}$ is a continuous process of finite variation. As mentioned above, our goal is to give results that cover the case of stochastic intensity of default. For the sake of concreteness, we postulate that \tilde{Y}^i , $i = 1, 2, 3$ follow Itô processes:

$$d\tilde{Y}_t^i = \tilde{Y}_t^i (\mu_t^i dt + \sigma_t^i dW_t), \quad (25)$$

where μ^i and $\sigma^i = (\sigma^{i1}, \sigma^{i2}, \dots, \sigma^{id})$ are \mathbb{F} -adapted processes and the underlying filtration \mathbb{F} is generated by a d -dimensional Brownian motion $W = (W^1, W^2, \dots, W^d)$ on $(\Omega, \mathcal{F}, \mathbb{Q})$.

Before proceeding to an analysis of particular models, let us recall a well-known auxiliary result (see, e.g., Lemma 1.6.7 in Karatzas and Shreve (1998)) that will prove useful in what follows.

Lemma 3.2 *Let \widehat{W}_t , $t \in [0, T]$, be a d -dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \widehat{\mathbb{Q}})$ and let \mathbb{F} be the natural filtration of \widehat{W} . Let*

$$\eta_t = \exp \left(\int_0^t \theta_u d\widehat{W}_u - \frac{1}{2} \int_0^t \|\theta_u\|^2 du \right),$$

where $\theta = (\theta^1, \theta^2, \dots, \theta^d)$ is a \mathbb{F} -progressively measurable process, such that $\mathbb{E}_{\widehat{\mathbb{Q}}}(\eta_T) = 1$. Let $\widetilde{\mathbb{Q}}$ be a probability measure on (Ω, \mathcal{F}_T) given by $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, so that the process $\widetilde{W}_t = \widehat{W}_t - \int_0^t \theta_u du$, $t \in [0, T]$, follows a Brownian motion under $\widetilde{\mathbb{Q}}$. Let X be a \mathcal{F}_T -measurable random variable integrable with respect to $\widetilde{\mathbb{Q}}$. Let us set $\widetilde{X}_t = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X | \mathcal{F}_t)$ and $\widehat{X}_t = \mathbb{E}_{\widehat{\mathbb{Q}}}(X \eta_T | \mathcal{F}_t)$, so that $\widetilde{X}_0 = \widehat{X}_0 = \mathbb{E}_{\widetilde{\mathbb{Q}}}(X)$. Then there exists a \mathbb{F} -progressively measurable process $\widehat{\phi}$ such that

$$\widehat{X}_t = \widehat{X}_0 + \int_0^t \widehat{\phi}_u d\widehat{W}_u, \quad \forall t \in [0, T].$$

Moreover

$$\widetilde{X}_t = \widetilde{X}_0 + \int_0^t \widetilde{\phi}_u d\widetilde{W}_u, \quad \forall t \in [0, T],$$

where $\widetilde{\phi}_t = \eta_t^{-1} (\widehat{\phi}_t - \theta_t \widehat{X}_t)$.

Complete case. We first consider the case of a stochastic intensity model driven by a one-dimensional Brownian motion W . Under the assumption that $\sigma_t^2 \neq \sigma_t^3$ for every $t \in [0, T]$, the default-free market model $\mathcal{M}^1 = (Y^2, Y^3; \Phi)$ is complete, and thus any contingent claim X is attainable in \mathcal{M}^1 (hence, the assumption (B) is satisfied). Before stating the next result, we find it convenient to introduce some notation. Using Itô's formula, it is easy to check that

$$d\widetilde{Y}_t^{2,3} = \widetilde{Y}_t^{2,3}(\sigma_t^2 - \sigma_t^3) d\widehat{W}_t,$$

where

$$d\widehat{W}_t = dW_t - \sigma_t^3 dt + \frac{(\mu_t^2 - \mu_t^3)}{(\sigma_t^2 - \sigma_t^3)} dt. \quad (26)$$

Similarly, we obtain

$$d\widetilde{Y}_t^{2,3,1} = \widetilde{Y}_t^{2,1}(\sigma_t^2 - \sigma_t^3) d\widetilde{W}_t,$$

where

$$d\widetilde{W}_t = dW_t - \sigma_t^1 dt + \frac{(\mu_t^2 - \mu_t^3)}{(\sigma_t^2 - \sigma_t^3)} dt. \quad (27)$$

Under mild regularity assumptions, there exist unique equivalent probability measures $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ on (Ω, \mathcal{F}_T) such that \widehat{W} and \widetilde{W} are Brownian motions under $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ respectively. It is easy to check that $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, where the process η is given by the formula

$$\eta_t = \exp\left(\int_0^t (\sigma_u^1 - \sigma_u^3) d\widehat{W}_u - \frac{1}{2} \int_0^t (\sigma_u^1 - \sigma_u^3)^2 du\right). \quad (28)$$

Let us denote by \widehat{X} an auxiliary contingent claim given as $\widehat{X} = X\eta_T$.

Proposition 3.3 *A survival claim $(X, 0, \tau)$ is attainable in \mathcal{M} and its pre-default value $\widetilde{U}(X)$ satisfies*

$$\frac{\widetilde{U}_t(X)}{\widetilde{Y}_t^1} = x + \int_0^t \phi_u^2 d\widetilde{Y}_u^{2,3,1},$$

where

$$\phi_t^2 = \eta_t^{-1} \left(\widehat{\alpha}_t \widetilde{Y}_t^{1,3} - \frac{\sigma_t^1 - \sigma_t^3}{\sigma_t^2 - \sigma_t^3} \widetilde{Y}_t^{1,2} F_{\widehat{X}}(t, T) \right), \quad (29)$$

the process η is given by (28), the process $\widehat{\alpha}$ is such that

$$\widehat{X} = \frac{\widehat{X}}{\widetilde{Y}_T^3} = \widehat{y} + \int_0^T \widehat{\alpha}_t d\widetilde{Y}_t^{2,3},$$

and $F_{\widehat{X}}(t, T)$ is the forward price of a default-free claim \widehat{X} , that is,

$$F_{\widehat{X}}(t, T) = \frac{\pi_t(\widehat{X})}{B(t, T)} = \frac{\pi_t(\widehat{X})}{\widetilde{Y}_t^3}.$$

The components ϕ^1 and ϕ^3 of a replicating strategy for $(X, 0, \tau)$ are given by (24), and its pre-default value equals, for every $t \in [0, T]$,

$$\widetilde{U}_t(X) = \widetilde{Y}_t^{1,3} \pi_t(\widehat{X}) = \widetilde{Y}_t^1 F_{\widehat{X}}(t, T). \quad (30)$$

Proof. The main tool used in the proof of the proposition is Lemma 3.2. First, we observe that there exists a process $\widehat{\phi}$ such that

$$\frac{X\eta_T}{\widetilde{Y}_T^3} = \widehat{X} = \widehat{y} + \int_0^T \widehat{\phi}_t d\widehat{W}_t = \widehat{y} + \int_0^T \widehat{\phi}_t \widetilde{Y}_t^{3,2} (\sigma_t^2 - \sigma_t^3)^{-1} d\widetilde{Y}_t^{2,3} = \widehat{y} + \int_0^T \widehat{\alpha}_t d\widetilde{Y}_t^{2,3},$$

where $\hat{y} = \mathbb{E}_{\hat{\mathbb{Q}}}(\hat{X}) = \mathbb{E}_{\hat{\mathbb{Q}}}(X)$, and where we denote $\hat{\alpha}_t = \hat{\phi}_t \tilde{Y}_t^{3,2} (\sigma_t^2 - \sigma_t^3)^{-1}$. Note that the process $\hat{X}_t = \mathbb{E}_{\hat{\mathbb{Q}}}(\hat{X} | \mathcal{F}_t)$ coincides with the forward price of \hat{X} , i.e., $\hat{X}_t = F_{\hat{X}}(t, T)$. In view of Lemma 3.2, for the process $\tilde{\phi}$ given by the formula

$$\tilde{\phi}_t = \eta_t^{-1} (\hat{\phi}_t - (\sigma_t^1 - \sigma_t^3) F_{\hat{X}}(t, T)),$$

we obtain

$$\frac{X}{\tilde{Y}_T^1} = X = x + \int_0^T \tilde{\phi}_t d\tilde{W}_t = x + \int_0^T \tilde{\phi}_t \tilde{Y}_t^{1,2} (\sigma_t^2 - \sigma_t^3)^{-1} d\tilde{Y}_t^{2,3,1} = x + \int_0^T \phi_t^2 d\tilde{Y}_t^{2,3,1},$$

where $x = \mathbb{E}_{\hat{\mathbb{Q}}}(X) = \hat{y}$ and where we set $\phi_t^2 = \tilde{\phi}_t \tilde{Y}_t^{1,2} (\sigma_t^2 - \sigma_t^3)^{-1}$. To obtain (29), it suffices to combine the formulae above, and to use Corollary 3.1. For equality (30), note that

$$\tilde{U}_0(X) = \tilde{Y}_0^1 x = \tilde{Y}_0^1 \hat{y} = \tilde{Y}_0^{1,3} \pi_0(\hat{X}),$$

as expected. \square

Incomplete case. We shall now examine a more general situation in which a credit risk model is associated with an incomplete default-free market model. Results of this paragraph cover the case of a reduced-form model in which the default time admits a stochastic intensity adapted to the filtration generated by $W = (W^1, W^2, \dots, W^d)$. We postulate that \tilde{Y}^i , $i = 1, 2, 3$ are given by (25) for some $d \geq 2$. Using Itô's formula, we find that

$$d\tilde{Y}_t^{2,3} = \tilde{Y}_t^{2,3} (\sigma_t^2 - \sigma_t^3) d\widehat{W}_t,$$

where the process $\widehat{W} = (\widehat{W}^1, \widehat{W}^2, \dots, \widehat{W}^d)$ satisfies

$$(\sigma_t^2 - \sigma_t^3) d\widehat{W}_t = (\sigma_t^2 - \sigma_t^3) dW_t - \sigma_t^3 (\sigma_t^2 - \sigma_t^3) dt + (\mu_t^2 - \mu_t^3) dt. \quad (31)$$

Similarly, we obtain

$$d\tilde{Y}_t^{2,3,1} = \tilde{Y}_t^{2,1} (\sigma_t^2 - \sigma_t^3) d\widetilde{W}_t,$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2, \dots, \widetilde{W}^d)$ is such that $d\widetilde{W}_t = d\widehat{W}_t - (\sigma_t^1 - \sigma_t^3) dt$, so that

$$(\sigma_t^2 - \sigma_t^3) d\widetilde{W}_t = (\sigma_t^2 - \sigma_t^3) dW_t - \sigma_t^1 (\sigma_t^2 - \sigma_t^3) dt + (\mu_t^2 - \mu_t^3) dt. \quad (32)$$

Under mild technical assumptions, one can show the existence of the two probability measures, $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$, equivalent to \mathbb{Q} on (Ω, \mathcal{F}_T) , and such that the processes \widehat{W} and \widetilde{W} that satisfy (31) and (32) are Brownian motions under $\widehat{\mathbb{Q}}$ and $\widetilde{\mathbb{Q}}$ respectively. Note that $d\widetilde{\mathbb{Q}}/d\widehat{\mathbb{Q}} = \eta_T$, where the Radon-Nikodým density process η is defined by the formula

$$\eta_t = \exp \left(\int_0^t (\sigma_u^1 - \sigma_u^3) d\widehat{W}_u - \frac{1}{2} \int_0^t \|\sigma_u^1 - \sigma_u^3\|^2 du \right). \quad (33)$$

Since $d \geq 2$, it is easy to see that the reduced model \mathcal{M}^1 is not necessarily complete, in general. To proceed further, we shall postulate that the claim $\hat{X} = X\eta_T$ is attainable in \mathcal{M}^1 .

Proposition 3.4 *Assume that there exists a \mathbb{F} -predictable process β such that*

$$\frac{\sigma_t^{1i} - \sigma_t^{3i}}{\sigma_t^{2i} - \sigma_t^{3i}} = \beta_t, \quad \forall i = 1, 2, \dots, d. \quad (34)$$

Let the claim $\hat{X} = X\eta_T$ be attainable in \mathcal{M}^1 . Then a survival claim $(X, 0, \tau)$ is attainable in the market model $\widetilde{\mathcal{M}}$ and its pre-default value satisfies

$$\frac{\tilde{U}_t(X)}{\tilde{Y}_t^1} = x + \int_0^t \phi_u^2 d\tilde{Y}_u^{2,3,1},$$

where

$$\phi_t^2 = \eta_t^{-1} \left(\hat{\alpha}_t \tilde{Y}_t^{1,3} - \beta_t \tilde{Y}_t^{1,2} F_{\hat{X}}(t, T) \right), \quad (35)$$

the process η is given by (33), the process $\hat{\alpha}$ is such that

$$\hat{X} = \frac{\hat{X}}{\tilde{Y}_T^3} = \hat{y} + \int_0^T \hat{\alpha}_t d\tilde{Y}_t^{2,3},$$

and $F_{\hat{X}}(t, T)$ is the forward price of a default-free claim \hat{X} . The components ϕ^1 and ϕ^3 of a replicating strategy for a survival claim are given by (24), and its the pre-default value equals, for every $t \in [0, T]$,

$$\tilde{U}_t(X) = \tilde{Y}_t^{1,3} \pi_t(\hat{X}) = \tilde{Y}_t^1 F_{\hat{X}}(t, T). \quad (36)$$

Proof. Once again, the proof relies on Lemma 3.2. By assumption, there exists a \mathbb{F} -predictable process $\hat{\alpha}$ such that

$$\frac{\hat{X}}{\tilde{Y}_T^3} = \hat{X} = \hat{y} + \int_0^T \hat{\alpha}_t d\tilde{Y}_t^{2,3} = \hat{y} + \int_0^T \hat{\alpha}_t \tilde{Y}_t^{2,3} (\sigma_t^2 - \sigma_t^3) d\tilde{W}_t = \hat{y} + \int_0^T \hat{\phi}_t d\tilde{W}_t,$$

where we denote $\hat{\phi}_t = \hat{\alpha}_t \tilde{Y}_t^{2,3} (\sigma_t^2 - \sigma_t^3)$. Note that $\hat{y} = \mathbb{E}_{\hat{\mathbb{Q}}}(\hat{X})$ and the process $\hat{X}_t = \mathbb{E}_{\hat{\mathbb{Q}}}(\hat{X} | \mathcal{F}_t)$ coincides with the forward price of \hat{X} , that is, $\hat{X}_t = F_{\hat{X}}(t, T)$. Using Lemma 3.2, we conclude that for the process

$$\tilde{\phi}_t = \eta_t^{-1} (\hat{\phi}_t - (\sigma_t^1 - \sigma_t^3) F_{\hat{X}}(t, T)), \quad (37)$$

we have

$$\frac{X}{\tilde{Y}_T^1} = X = x + \int_0^T \tilde{\phi}_t d\tilde{W}_t,$$

where $x = \mathbb{E}_{\hat{\mathbb{Q}}}(X) = \mathbb{E}_{\hat{\mathbb{Q}}}(\hat{X}) = \hat{y}$. In view of (34), for the process ϕ^2 given by (35), we obtain

$$\int_0^T \phi_t^2 d\tilde{Y}_t^{2,3,1} = \int_0^T \tilde{\phi}_t d\tilde{W}_t.$$

To complete the proof, it suffices to combine the formulae above, and to use Corollary 3.1. \square

Remarks. (a) If $\sigma_t^1 = \sigma_t^3 = 0$, we are back in the set-up of Section 3.3.1. In this case, we may assume that W is a one-dimensional Brownian motion. In particular, the equality $\hat{\mathbb{Q}} = \tilde{\mathbb{Q}}$ (equivalently, $\eta_T = 1$) holds, and

$$d\tilde{W}_t = d\hat{W}_t = dW_t + \frac{\mu_t^2 - \mu_t^3}{\sigma_t^2} dt.$$

Moreover, in this case we have that

$$d\tilde{Y}_t^{2,3,1} = -\sigma_t^2 \tilde{Y}_t^{2,1} d\tilde{W}_t = \tilde{Y}_t^{2,1} ((\mu_t^2 - \mu_t^3) dt + \sigma_t^2 dW_t) = \tilde{Y}_t^{3,1} d\tilde{Y}_t^{2,3},$$

and the formulae of Proposition 3.4 (or Proposition 3.3) reduce to those of Proposition 3.2.

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