HEDGING OF CREDIT DERIVATIVES IN MODELS WITH TOTALLY UNEXPECTED DEFAULT*

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Introduction

This paper presents some methods to hedge defaultable derivatives under the assumption that there exist tradeable assets with dynamics allowing for elimination of default risk of derivative securities. We investigate hedging strategies in alternative frameworks with different degrees of generality, an abstract semimartingale framework and a more specific Markovian set-up, and we use two alternative approaches.

On the one hand, we use the stochastic calculus approach in order to establish rather abstract characterization results for hedgeable contingent claims in a fairly general set-up. We subsequently apply these results to derive closed-form solutions for prices and replicating strategies in particular models.

On the other hand, we examine the PDE approach in a Markovian setting. In this method, the arbitrage price and the hedging strategy for an attainable contingent claim are described in terms of solutions of a pair of coupled PDEs. Again, for some standard examples of defaultable claims, we provide explicit formulae for prices and hedging strategies (for further examples of trading strategies involving tradeable credit derivatives, we refer to Laurent [30] or Bielecki et al. [7]). As expected, both methods yield identical results for some special cases considered in this work.

For the sake of simplicity, we only deal with financial models with no more than three primary assets (models with an arbitrary number of primary assets were studied in Bielecki et al. [5]). Also, it is postulated throughout that the default time is the same for all defaultable securities. An extension of our results to the case of several (possibly dependent) default times is crucial if someone wishes to cover the so-called basket credit derivatives (in this regard, see Section 6 in Bielecki et al. [6]).

Let us comment briefly on the terminology used in this work. Traditionally, credit risk models are classified either as *structural models* (also known as *value-of-the-firm models*) or as *reduced-form models* (also termed *intensity-based models*). In their original forms, the two approaches, structural and reduced-form, are extreme cases, in the sense that the default time is modelled either as a predictable stopping time (the first moment when the firm's value hits some barrier, as in Black and Cox [8]), or by a totally inaccessible stopping time (defined via its intensity, as in Jarrow and Turnbull [24]). However, as argued by several authors (see, for instance, Duffie and Lando [16], Giesecke [20]-[21], Jarrow and Protter [23], Jeanblanc and Valchev [27], or Guo at al. [22]), probabilistic properties of default time are directly related to the publicly available information (it is important, for instance, whether the value of the firm and/or the default triggering barrier are observed by the market with absolute accuracy).

In fact, in several structural models the default time is no longer predictable, as it was the case in classic models with deterministic default triggering barrier and full observation of the firm value process (see, Merton [31] or Black and Cox [8]). For this reason, we decided to refer to credit risk models considered in this work as models with totally unexpected default (the strict mathematical term, *totally inaccessible stopping time*, seems to be rather cumbersome for a frequent use). For a more exhaustive presentation of mathematical theory of credit risk, we refer to Arvanitis and Gregory [1], Bielecki and Rutkowski [2], Bielecki et al. [3], Cossin and Pirotte [14], Duffie and Singleton [17], Lando [29], or Schönbucher [35].

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1 Totally Unexpected Default

In this section, we describe briefly the fundamental features of the credit risk models with unexpected default. Also, we collect here few technical results that are used in subsequent sections.

1.1 General Set-up

We assume that we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and a nonnegative random variable τ on this space. We always postulate that τ is strictly positive with probability 1. Note that the probability measure $\mathbb{P}$ represents the historical probability reflecting the real-life dynamics of prices of primary traded assets, rather than some martingale measure for our financial model. We first focus on different definitions of default intensity encountered in the literature.

1.1.1 Intensity of a Stopping Time

Suppose that $(\Omega, \mathcal{G}, \mathbb{P})$ is endowed with some filtration $\widetilde{\mathbb{G}}$ such that τ is a $\widetilde{\mathbb{G}}$-stopping time. Let H be the *default process*, defined as $H_t = \mathbb{1}_{\{t \geq \tau\}}$ (note that H is a bounded $\widetilde{\mathbb{G}}$-submartingale). We say that τ admits a $\widetilde{\mathbb{G}}$-*intensity* if there exists a $\widetilde{\mathbb{G}}$-adapted, nonnegative process $\widetilde{\lambda}$ such that the process

$$M_t = H_t - \int_0^t \widetilde{\lambda}_u \, du = H_t - \int_0^{t \wedge \tau} \widetilde{\lambda}_u \, du \tag{1}$$

is a $\widetilde{\mathbb{G}}$-martingale (the second equality in (1) follows from the fact that the process H is stopped at τ). Then M is called the compensated $\widetilde{\mathbb{G}}$-martingale of the default process H. In order for a $\widetilde{\mathbb{G}}$-stopping time τ to admit a $\widetilde{\mathbb{G}}$-intensity $\widetilde{\lambda}$, it has to be *totally inaccessible* with respect to $\widetilde{\mathbb{G}}$, so that $\mathbb{P}(\tau = \theta) = 0$ for any $\widetilde{\mathbb{G}}$-predictable stopping time θ. The simplest example is the moment of the first jump a Poisson process. Note that the intensity $\widetilde{\lambda}$ necessarily vanishes after default.

Remark 1.1 Some authors define the intensity as the process $\widetilde{\lambda}$ such that $H_t - \int_0^{t \wedge \tau} \widetilde{\lambda}_u \, du$ is a $\widetilde{\mathbb{G}}$-martingale. In that case, the process $\widetilde{\lambda}$ is not uniquely defined after time τ.

1.1.2 $\mathbb{F}$-Intensity of a Random Time

We change the perspective, and we no longer assume that the filtration $\widetilde{\mathbb{G}}$ is given a priori. We assume instead that τ is a positive random variable on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Let $\mathbb{H} = (\mathcal{H}_t, \, t \geq 0)$ be the natural filtration generated by the default process $(H_t, \, t \geq 0)$, and let $\mathbb{F} = (\mathcal{F}_t, \, t \geq 0)$ be some *reference filtration* in $(\Omega, \mathcal{G}, \mathbb{P})$.

We assume throughout that the information available to an investor is modeled by the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Consequently, we can reduce our study to the case where the default intensity (if it exists) is $\mathbb{G}$-adapted, meaning that the process M given by (1) is a $\mathbb{G}$-martingale for some $\mathbb{G}$-adapted process $\widetilde{\lambda}$. In this setting, there exists a process $\lambda = (\lambda_t, \, t \geq 0)$, called the $\mathbb{F}$-*intensity* of τ, which is $\mathbb{F}$-adapted and equal to $\widetilde{\lambda}$ before default, so that $\widetilde{\lambda}_t \mathbb{1}_{\{t \leq \tau\}} = \lambda_t \mathbb{1}_{\{t \leq \tau\}}$ for every $t \in \mathbb{R}_+$. The existence of $\widetilde{\lambda}$ (and its uniqueness under some technical conditions) follows from the following result (see Dellacherie et al. [15], Page 186).

Lemma 1.1 *Let $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Then for any $\mathbb{G}$-predictable process ζ there exists an $\mathbb{F}$-predictable process $\widetilde{\zeta}$ such that*

$$\widetilde{\zeta}_t \mathbb{1}_{\{t \leq \tau\}} = \zeta_t \mathbb{1}_{\{t \leq \tau\}}, \quad \forall \, t \in \mathbb{R}_+. \tag{2}$$

If, in addition, the inequality $F_t := \mathbb{P}(\tau \leq t \,|\, \mathcal{F}_t) < 1$ holds for every $t \in \mathbb{R}_+$ then the process $\widetilde{\zeta}$ satisfying (2) is unique.

Of course, we have that

$$M_t = H_t - \int_0^{t\wedge\tau} \widetilde{\lambda}_u \, du = H_t - \int_0^{t\wedge\tau} \lambda_u \, du.$$

Suppose that the reference filtration is chosen in such a way that the default events $\{\tau \le t\}$ are not in $\mathbb{F}$. Then the $\mathbb{F}$-intensity λ is uniquely defined after τ and, typically, does not vanish after τ.

1.2 Hypothesis (H)

In this section, we focus on the invariance property of the so-called hypothesis (H) under an equivalent change of a probability measure.

Definition 1.1 We say that filtrations $\mathbb{F}$ and $\mathbb{G}$, with $\mathbb{F} \subseteq \mathbb{G}$, satisfy the *hypothesis* (H) under $\mathbb{P}$ whenever any $\mathbb{F}$-local martingale L follows also a $\mathbb{G}$-local martingale.

Remark 1.2 We emphasize that, in general, an $\mathbb{F}$-martingale may fail to follow a $\mathbb{G}$-martingale. As a trivial example, consider a fixed date $T > 0$ and take $\mathcal{G}_t = \mathcal{F}_T$ for every $t \in [0, T]$. Then any $\mathbb{F}$-martingale L satisfies $\mathbb{E}_{\mathbb{P}}(L_t \,|\, \mathcal{G}_s) = L_t$ for $s \le t$, and thus L is not a $\mathbb{G}$-martingale, in general. It is even possible, but more difficult, to produce an example of an $\mathbb{F}$-martingale, which is not a semimartingale with respect to $\mathbb{G}$. For other counter-examples, in particular those involving progressive enlargement of filtrations, we refer interested reader to Protter [34], or Mansuy and Yor [32].

The original formulations of the hypothesis (H) refer to martingales (or even square-integrable martingales), rather than local martingales. We shall show that in our set-up the definition given above is equivalent to the original definition. In fact, the hypothesis (H) postulates a certain form of conditional independence of σ-fields associated with $\mathbb{F}$ and $\mathbb{G}$, rather than a specific property of $\mathbb{F}$-(local) martingales. In particular the following well known result is valid.

Lemma 1.2 *Assume that $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the process $H_t = \mathbb{1}_{\{\tau \le t\}}$. Then the following conditions are equivalent to the hypothesis* (H).
(i) For any $t, h \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_{t+h}). \tag{3}$$

(i') For any $t \in \mathbb{R}_+$, we have

$$\mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_\infty). \tag{4}$$

(ii) For any $t \in \mathbb{R}_+$, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$ under $\mathbb{P}$, that is,

$$\mathbb{E}_{\mathbb{P}}(\xi\, \eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)\, \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t)$$

for any bounded, $\mathcal{F}_\infty$-measurable random variable ξ and bounded, $\mathcal{G}_t$-measurable random variable η.
(iii) For any $t \in \mathbb{R}_+$, and any $u \ge t$ the σ-fields $\mathcal{F}_u$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$.
(iv) For any $t \in \mathbb{R}_+$ and any bounded, $\mathcal{F}_\infty$-measurable random variable ξ: $\mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$.
(v) For any $t \in \mathbb{R}_+$, and any bounded, $\mathcal{G}_t$-measurable random variable η: $\mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(\eta \,|\, \mathcal{F}_\infty)$.

Proof. The proof of equivalence of conditions (i')-(v) can be found, for instance, in Section 6.1.1 of Bielecki and Rutkowski [2] (for related results, see Elliott et al. [19]). Using monotone class theorem it can be shown that conditions (i) and (i') are equivalent. Hence, we shall only show that condition (iv) and the hypothesis (H) are equivalent.

Assume first that the hypothesis (H) holds. Consider any bounded, $\mathcal{F}_\infty$-measurable random variable ξ. Let $L_t = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{F}_t)$ be the martingale associated with ξ. Then, (H) implies that L is also a local martingale with respect to $\mathbb{G}$, and thus a $\mathbb{G}$-martingale, since L is bounded (recall that any bounded local martingale is a martingale). We conclude that $L_t = \mathbb{E}_{\mathbb{P}}(\xi \,|\, \mathcal{G}_t)$ and thus (iv) holds.

Suppose now that (iv) holds. First, we note that the standard truncation argument shows that the boundedness of ξ in (iv) can be replaced by the assumption that ξ is $\mathbb{P}$-integrable. Hence, any $\mathbb{F}$-martingale L is an $\mathbb{G}$-martingale, since L is clearly $\mathbb{G}$-adapted and we have, for every $t \le s$,

$$L_t = \mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}(L_s \,|\, \mathcal{G}_t).$$

Now, suppose that L is an $\mathbb{F}$-local martingale so that there exists an increasing sequence of $\mathbb{F}$-stopping times τ_n such that $\lim_{n\to\infty} \tau_n = \infty$, for any n the stopped process L^{τ_n} follows a uniformly integrable $\mathbb{F}$-martingale. Hence, L^{τ_n} is also a uniformly integrable $\mathbb{G}$-martingale, and this means that L follows a $\mathbb{G}$-local martingale. □

1.2.1 Hazard Process

Let τ be a random time on a space $(\Omega, \mathcal{G}, \mathbb{P})$ such that the filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ satisfy the hypothesis (H). Then, from (4), the process $F_t = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$ is increasing.

We make the standing assumption that $F_t < 1$ for every $t \in \mathbb{R}_+$, and we define the $\mathbb{F}$-*hazard process* Γ by setting $\Gamma_t = -\ln(1 - F_t)$. Let, in addition, the process F be absolutely continuous with respect to the Lebesgue measure, so that

$$F_t = \int_0^t f_u \, du, \quad \forall\, t \in \mathbb{R}_+,$$

for some $\mathbb{F}$-progressively measurable (or $\mathbb{F}$-predictable) process f. Then the $\mathbb{F}$-hazard process satisfies

$$\Gamma_t = \int_0^t \gamma_u \, du, \quad \forall\, t \in \mathbb{R}_+,$$

where the $\mathbb{F}$-*intensity* γ is given by

$$\gamma_t = \frac{f_t}{1 - F_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{5}$$

From now on, we take (5) as the definition of the $\mathbb{F}$-intensity γ, and we make the standing assumption that the hypothesis (H) holds under $\mathbb{P}$. The following auxiliary result is standard (see, for instance, Elliott et al. [19] or Blanchet-Scalliet and Jeanblanc [9]).

Lemma 1.3 *For any $\mathbb{P}$-integrable, $\mathcal{F}_T$-measurable random variable X we have, for $t \in [0, T]$,*

$$\mathbb{E}_{\mathbb{P}}(X \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\, e^{\Gamma_t} \mathbb{E}_{\mathbb{P}}(X e^{-\Gamma_T} \,|\, \mathcal{F}_t).$$

1.2.2 Canonical Construction

We now describe the canonical construction of a random time with a given $\mathbb{F}$-hazard process. Let Ψ be an $\mathbb{F}$-adapted, increasing, nonnegative process with $\Psi_0 = 0$ and $\lim_{t\to\infty} \Psi_t = \infty$. We define a nonnegative random variable τ by setting

$$\tau = \inf\{\, t \in \mathbb{R}_+ : \Psi_t \ge \Theta\},$$

where Θ is a random variable independent of $\mathbb{F}$, with the exponential distribution of parameter 1. Of course, the existence of Θ on the original probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is not guaranteed, so that we allow for an extension of the underlying probability space.

We shall now find the process $F_t = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}$. Since clearly $\{\tau > t\} = \{\Theta > \Psi_t\}$, we get

$$\mathbb{P}\{\tau > t \,|\, \mathcal{F}_\infty\} = \mathbb{P}\{\Theta > \Psi_t \,|\, \mathcal{F}_\infty\} = e^{-\Psi_t}.$$

Consequently,

$$1 - F_t = \mathbb{P}\{\tau > t \,|\, \mathcal{F}_t\} = \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}\{\tau > t \,|\, \mathcal{F}_\infty\} \,|\, \mathcal{F}_t\big) = e^{-\Psi_t},$$

and so F is an $\mathbb{F}$-adapted, continuous, increasing process. We conclude that for every $t \in \mathbb{R}_+$

$$F_t = 1 - e^{-\Psi_t} = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_\infty\} = \mathbb{P}\{\tau \le t \,|\, \mathcal{F}_t\}, \tag{6}$$

and thus Ψ coincides with the $\mathbb{F}$-hazard process Γ of τ and the hypothesis (H) is valid. It is also not difficult to show that the process $M_t = H_t - \Gamma_{t\wedge\tau} = H_t - \Psi_{t\wedge\tau}$ follows a $\mathbb{G}$-martingale.

The following result shows that under the hypothesis (H), for any random time τ with continuous hazard process Γ, the auxiliary random variable Θ can be constructed on the original probability space, using τ and Γ (see El Karoui [18] or Blanchet-Scalliet and Jeanblanc [9]).

Lemma 1.4 *Let τ be a random time on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ such that the $\mathbb{F}$-hazard process Γ of τ under $\mathbb{P}$ is continuous and the hypothesis (H) holds. Then there exists a random variable Θ on $(\Omega, \mathcal{G}, \mathbb{P})$, independent of $\mathbb{F}$ and with the exponential distribution of parameter 1, such that*

$$\tau = \inf\{\, t \in \mathbb{R}_+ : \Gamma_t \ge \Theta\}. \tag{7}$$

Proof. It suffices to check that the random variable $\Theta = \Gamma_\tau$ has the desired properties. Indeed, we have, for every $t \in \mathbb{R}_+$,

$$\mathbb{P}(\Theta \ge t \,|\, \mathcal{F}_\infty) = \mathbb{P}(\Gamma_\tau \ge t \,|\, \mathcal{F}_\infty) = \mathbb{P}(\tau \ge A_t \,|\, \mathcal{F}_\infty) = \exp(-\Gamma_{A_t}) = e^{-t},$$

where A is the left inverse of Γ, so that $\Gamma_{A_t} = t$ for every $t \in \mathbb{R}_+$. □

1.3 Change of a Probability Measure

Kusuoka [28] shows, by means of a counter-example, that the hypothesis (H) is not invariant with respect to an equivalent change of the underlying probability measure, in general. It is worth noting that his counter-example is based on two filtrations, $\mathbb{H}^1$ and $\mathbb{H}^2$, generated by the two random times τ^1 and τ^2, and he chooses $\mathbb{H}^1$ to play the role of the reference filtration $\mathbb{F}$. We shall argue that in the case where $\mathbb{F}$ is generated by a Brownian motion (or, more generally, by some martingale orthogonal to M under $\mathbb{P}$), the above-mentioned invariance property is valid under mild technical assumptions.

1.3.1 Preliminary Lemma

Let us first examine a general set-up in which $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the default process H. We say that $\mathbb{Q}$ is locally equivalent to $\mathbb{P}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$. Then there exists the Radon-Nikodým density process η such that

$$d\mathbb{Q}\,|_{\mathcal{G}_t} = \eta_t \, d\mathbb{P}\,|_{\mathcal{G}_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{8}$$

Part (i) in the next lemma is well known (see Jamshidian [26]). We assume that the hypothesis (H) holds under $\mathbb{P}$.

Lemma 1.5 *(i) Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process η. If the density process η is $\mathbb{F}$-adapted then we have $\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. Hence, the hypothesis* (H) *is also valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.*
(ii) Assume that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ and $d\mathbb{Q} = \eta_\infty \, d\mathbb{P}$, so that $\eta_t = \mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{G}_t)$. Then the hypothesis (H) *is valid under $\mathbb{Q}$ whenever we have, for every $t \in \mathbb{R}_+$,*

$$\frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t H_t \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)}. \tag{9}$$

Proof. To prove (i), assume that the density process η is $\mathbb{F}$-adapted. We have for each $t, h \in \mathbb{R}_+$

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \le t \,|\, \mathcal{F}_{t+h}) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_{t+h}),$$

where the last equality follows by another application of the Bayes formula. The assertion now follows from part (i) in Lemma 1.2.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t := \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+. \tag{10}$$

Note that since the random variables $\eta_t \mathbb{1}_{\{\tau \le t\}}$ and η_t are $\mathbb{P}$-integrable and $\mathcal{G}_t$-measurable, using the Bayes formula, part (v) in Lemma 1.2, and assumed equality (9), we obtain the following chain of equalities

$$\mathbb{Q}(\tau \le t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_t \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_t \,|\, \mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(\eta_\infty \mathbb{1}_{\{\tau \le t\}} \,|\, \mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(\eta_\infty \,|\, \mathcal{F}_\infty)} = \mathbb{Q}(\tau \le t \,|\, \mathcal{F}_\infty).$$

We conclude that the hypothesis (H) holds under $\mathbb{Q}$ if and only if (9) is valid. □

Unfortunately, straightforward verification of condition (9) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of the hypothesis (H) under a locally equivalent probability measure.

1.3.2 Case of the Brownian Filtration

Let W be a Brownian motion under $\mathbb{P}$ with respect to its natural filtration $\mathbb{F}$. Since we work under the hypothesis (H), the process W is also a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Hence, W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$. Our goal is to show that the hypothesis (H) is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures on $(\Omega, \mathcal{G})$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. Kusuoka [28] (see also Section 5.2.2 in Bielecki and Rutkowski [2]) proved that, under the hypothesis (H), any $\mathbb{G}$-martingale under $\mathbb{P}$ can be represented as the sum of stochastic integrals with respect to the Brownian motion W and the jump martingale M. In our set-up, Kusuoka's representation theorem implies that there exist $\mathbb{G}$-predictable processes θ and $\zeta > -1$, such that the Radon-Nikodým density η of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$d\eta_t = \eta_{t-}\big(\theta_t \, dW_t + \zeta_t \, dM_t\big) \tag{11}$$

with the initial value $\eta_0 = 1$. More explicitly, the process η equals

$$\eta_t = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \eta_t^1 \eta_t^2, \tag{12}$$

where we write

$$\eta_t^1 = \mathcal{E}_t\left(\int_0^{\cdot} \theta_u \, dW_u\right) = \exp\left(\int_0^t \theta_u \, dW_u - \frac{1}{2}\int_0^t \theta_u^2 \, du\right), \tag{13}$$

and

$$\eta_t^2 = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u \, dM_u\right) = \exp\left(\int_0^t \ln(1+\zeta_u) \, dH_u - \int_0^{t\wedge\tau} \zeta_u \gamma_u \, du\right). \tag{14}$$

Moreover, by virtue of a suitable version of Girsanov's theorem, the following processes $\widehat{W}$ and $\widehat{M}$ are $\mathbb{G}$-martingales under $\mathbb{Q}$

$$\widehat{W}_t = W_t - \int_0^t \theta_u \, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}} \gamma_u \zeta_u \, du. \tag{15}$$

Proposition 1.1 *Assume that the hypothesis* (H) *holds under* $\mathbb{P}$*. Let* $\mathbb{Q}$ *be a probability measure locally equivalent to* $\mathbb{P}$ *with the associated Radon-Nikodým density process* η *given by formula* (12)*. If the process* θ *is* $\mathbb{F}$*-adapted then the hypothesis* (H) *is valid under* $\mathbb{Q}$ *and the* $\mathbb{F}$*-intensity of* τ *under* $\mathbb{Q}$ *equals* $\widehat{\gamma}_t = (1+\widetilde{\zeta}_t)\gamma_t$*, where* $\widetilde{\zeta}$ *is the unique* $\mathbb{F}$*-predictable process such that the equality* $\widetilde{\zeta}_t \mathbb{1}_{\{t\le\tau\}} = \zeta_t \mathbb{1}_{\{t\le\tau\}}$ *holds for every* $t\in\mathbb{R}_+$*.*

Proof. Let $\widetilde{\mathbb{P}}$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$, given by

$$d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\Big(\int_0^{\cdot} \zeta_u\, dM_u\Big)\, d\mathbb{P}\,|\,_{\mathcal{G}_t} = \eta_t^2\, d\mathbb{P}\,|\,_{\mathcal{G}_t}. \tag{16}$$

We claim that the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. From Girsanov's theorem, the process W follows a Brownian motion under $\widetilde{\mathbb{P}}$ with respect to both $\mathbb{F}$ and $\mathbb{G}$. Moreover, from the predictable representation property of W under $\widetilde{\mathbb{P}}$, we deduce that any $\mathbb{F}$-local martingale L under $\widetilde{\mathbb{P}}$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbb{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u\, dW_u.$$

This shows that L is also a $\mathbb{G}$-local martingale, and thus the hypothesis (H) holds under $\widetilde{\mathbb{P}}$. Since

$$d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\Big(\int_0^{\cdot} \theta_u\, dW_u\Big)\, d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t},$$

by virtue of part (i) in Lemma 1.5, the hypothesis (H) is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that the hypothesis (H) holds under $\mathbb{Q}$ and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\, du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}}(1+\widetilde{\zeta}_u)\gamma_u\, du$$

is a $\mathbb{Q}$-martingale. □

We claim that the equality $\widetilde{\mathbb{P}} = \mathbb{P}$ holds on the filtration $\mathbb{F}$. Indeed, we have $d\widetilde{\mathbb{P}}\,|\,_{\mathcal{F}_t} = \widetilde{\eta}_t\, d\mathbb{P}\,|\,_{\mathcal{F}_t}$, where we write $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^2\,|\,\mathcal{F}_t)$, and

$$\mathbb{E}_{\mathbb{P}}(\eta_t^2\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\Big(\mathcal{E}_t\Big(\int_0^{\cdot} \zeta_u\, dM_u\Big)\,\Big|\,\mathcal{F}_\infty\Big) = 1, \quad \forall\, t\in\mathbb{R}_+, \tag{17}$$

where the first equality follows from part (v) in Lemma 1.2.

To establish the second equality in (17), we first note that since the process M is stopped at τ, we may assume, without loss of generality, that $\zeta = \widetilde{\zeta}$ where the process $\widetilde{\zeta}$ is $\mathbb{F}$-predictable (see Lemma 1.1). Moreover, in view of (7) the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1-\exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes ζ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (14)),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\Big(\mathcal{E}_t\Big(\int_0^{\cdot} \widetilde{\zeta}_u\, dM_u\Big)\,\Big|\,\mathcal{F}_\infty\Big) &= \mathbb{E}_{\mathbb{P}}\Big((1+\mathbb{1}_{\{t\ge\tau\}}\widetilde{\zeta}_\tau)\exp\Big(-\int_0^{t\wedge\tau}\widetilde{\zeta}_u\gamma_u\,du\Big)\,\Big|\,\mathcal{F}_\infty\Big)\\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_0^\infty (1+\mathbb{1}_{\{t\ge u\}}\widetilde{\zeta}_u)\exp\Big(-\int_0^{t\wedge u}\widetilde{\zeta}_v\gamma_v\,dv\Big)\gamma_u e^{-\int_0^u \gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\Big)\\
&= \mathbb{E}_{\mathbb{P}}\Big(\int_0^t (1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u (1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\,\Big|\,\mathcal{F}_\infty\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\,\mathbb{E}_{\mathbb{P}}\Big(\int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\Big)\\
&=\int_0^t (1+\widetilde{\zeta}_u)\gamma_u \exp\Big(-\int_0^u (1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\int_t^\infty \gamma_u e^{-\int_0^u \gamma_v\,dv}du\\
&=1-\exp\Big(-\int_0^t (1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)+\exp\Big(-\int_0^t \widetilde{\zeta}_v\gamma_v\,dv\Big)\exp\Big(-\int_0^t \gamma_v\,dv\Big)=1,
\end{aligned}
$$

where the second last equality follows by an application of the chain rule.

1.3.3 Extension to Orthogonal Martingales

Equality (17) suggests that Proposition 1.1 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$. It is also a $\mathbb{G}$-local martingale, since we maintain the assumption that the hypothesis (H) holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$. We assume that the Radon-Nikodým density process η of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$
d\eta_t = \eta_{t-}\big(\theta_t\,dN_t + \zeta_t\,dM_t\big) \tag{18}
$$

for some $\mathbb{G}$-predictable processes θ and $\zeta > -1$ (the properties of the process θ depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, so that the product MN follows a $\mathbb{G}$-local martingale.

Proposition 1.2 *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbb{F}$, in the sense that any $\mathbb{F}$-local martingale L under $\mathbb{P}$ can be written as

$$
L_t = L_0 + \int_0^t \xi_u\,dN_u, \quad \forall\, t\in\mathbb{R}_+,
$$

for some $\mathbb{F}$-predictable process ξ,
(c) $\widetilde{\mathbb{P}}$ is a probability measure on $(\Omega,\mathcal{G})$ such that (16) *holds.*
Then we have:
(i) the hypothesis (H) *is valid under $\widetilde{\mathbb{P}}$,*
(ii) if the process θ is $\mathbb{F}$-adapted then the hypothesis (H) *is valid under $\mathbb{Q}$.*

The proof of the proposition hinges on the following simple lemma.

Lemma 1.6 *Under the assumptions of Proposition 1.2, we have:*
(i) N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$,
(ii) N has the predictable representation property for $\mathbb{F}$-local martingales under $\widetilde{\mathbb{P}}$.

Proof. In view of (c), we have $d\widetilde{\mathbb{P}}\,|_{\mathcal{G}_t} = \eta_t^2\,d\mathbb{P}\,|_{\mathcal{G}_t}$, where the density process η^2 is given by (14), so that $d\eta_t^2 = \eta_{t-}^2\zeta_t\,dM_t$. From the assumed orthogonality of N and M, it follows that N and η^2 are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, and thus $N\eta^2$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{\eta}$ by setting $\widetilde{\eta}_t = \mathbb{E}_{\mathbb{P}}(\eta_t^2\,|\,\mathcal{F}_t)$. Then manifestly $d\widetilde{\mathbb{P}}\,|_{\mathcal{F}_t} = \widetilde{\eta}_t\,d\mathbb{P}\,|_{\mathcal{F}_t}$, and thus in order to show that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ follows an $\mathbb{F}$-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t\in\mathbb{R}_+$, so that $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$. To this end, we note that

$$
\mathbb{E}_{\mathbb{P}}(\eta_t^2\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\Big(\mathcal{E}_t\Big(\int_0^\cdot \zeta_u\,dM_u\Big)\,\Big|\,\mathcal{F}_\infty\Big) = 1, \quad \forall\, t\in\mathbb{R}_+,
$$

where the first equality follows from part (v) in Lemma 1.2, and the second one can established similarly as the second equality in (17).

We are in a position to prove (ii). Let L be an $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$. Then it follows also an $\mathbb{F}$-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. This shows that N has the predictable representation property with respect to $\mathbb{F}$ under $\widetilde{\mathbb{P}}$. □

We now proceed to the proof of Proposition 1.2.

Proof of Proposition 1.2. We shall argue along the similar lines as in the proof of Proposition 1.1. To prove (i), note that by part (ii) in Lemma 1.6 we know that any $\mathbb{F}$-local martingale under $\widetilde{\mathbb{P}}$ admits the integral representation with respect to N. But, by part (i) in Lemma 1.6, N is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$. We conclude that L is a $\mathbb{G}$-local martingale under $\widetilde{\mathbb{P}}$, and thus the hypothesis (H) is valid under $\widetilde{\mathbb{P}}$. Assertion (ii) now follows from part (i) in Lemma 1.5. □

Remark 1.3 It should be stressed that Proposition 1.2 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modelling dependent default times through an equivalent change of a probability measure (see Kusuoka [28]).

Example 1.1 Kusuoka [28] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M^i_t = H^i_t - \int_0^{t\wedge\tau_i} \gamma_i(u)\, du$, where $H^i_t = \mathbb{1}_{\{t\geq\tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \eta_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where $\eta_t = \eta^1_t\eta^2_t$ and, for $i = 1, 2$ and every $t \in \mathbb{R}_+$,

$$\eta^i_t = 1 + \int_0^t \eta^i_{u-}\zeta^i_u\, dM^i_u = \mathcal{E}_t\left(\int_0^{\cdot} \zeta^i_u\, dM^i_u\right)$$

for some $\mathbb{G}$-predictable processes ζ^i, $i = 1, 2$, where $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Manifestly, the hypothesis (H) holds under $\mathbb{P}$. Moreover, in view of Proposition 1.2, it is still valid under the equivalent probability measure $\widetilde{\mathbb{P}}$ given by

$$d\widetilde{\mathbb{P}}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \zeta^2_u\, dM^2_u\right) d\mathbb{P}\,|\,_{\mathcal{G}_t}.$$

It is clear that $\widetilde{\mathbb{P}} = \mathbb{P}$ on $\mathbb{F}$, since

$$\mathbb{E}_{\mathbb{P}}(\eta^2_t\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta^2_u\, dM^2_u\right)\Big|\,\mathcal{H}^1_t\right) = 1, \quad \forall\, t \in \mathbb{R}_+.$$

However, the hypothesis (H) is not necessarily valid under $\mathbb{Q}$ if the process ζ^1 fails to be $\mathbb{F}$-adapted. In Kusuoka's counter-example, the process ζ^1 was chosen to be explicitly dependent on both random times, and it was shown that the hypothesis (H) does not hold under $\mathbb{Q}$. For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne et al. [12].

2 Semimartingale Model with a Common Default

In what follows, we fix a finite horizon date $T > 0$. For the purpose of this work, it is enough to formally define a generic defaultable claim through the following definition.

Definition 2.1 A *defaultable claim* with maturity date T is represented by a triplet (X, Z, τ), where:

(i) the *default time* τ specifies the random time of default, and thus also the default events $\{\tau \leq t\}$ for every $t \in [0,T]$,
(ii) the *promised payoff* $X \in \mathcal{F}_T$ represents the random payoff received by the owner of the claim at time T, provided that there was no default prior to or at time T; the actual payoff at time T associated with X thus equals $X\mathbb{1}_{\{T<\tau\}}$,
(iii) the $\mathbb{F}$-adapted *recovery process* Z specifies the recovery payoff Z_τ received by the owner of a claim at time of default (or at maturity), provided that the default occurred prior to or at maturity date T.

In practice, hedging of a credit derivative after default time is usually of minor interest. Also, in a model with a single default time, hedging after default reduces to replication of a non-defaultable claim. It is thus natural to define the replication of a defaultable claim in the following way.

Definition 2.2 We say that a self-financing strategy ϕ replicates a defaultable claim (X, Z, τ) if its wealth process $V(\phi)$ satisfies $V_T(\phi)\mathbb{1}_{\{T<\tau\}} = X\mathbb{1}_{\{T<\tau\}}$ and $V_\tau(\phi)\mathbb{1}_{\{T\geq\tau\}} = Z_\tau\mathbb{1}_{\{T\geq\tau\}}$.

When dealing with replicating strategies, in the sense of Definition 2.2, we will always assume, without loss of generality, that the components of the process ϕ are $\mathbb{F}$-predictable processes.

2.1 Dynamics of Asset Prices

We assume that we are given a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with a (possibly multi-dimensional) standard Brownian motion W and a random time τ admitting an $\mathbb{F}$-intensity γ under $\mathbb{P}$, where $\mathbb{F}$ is the filtration generated by W. In addition, we assume that τ satisfies (4), so that the hypothesis (H) is valid under $\mathbb{P}$ for filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. Since the default time admits an $\mathbb{F}$-intensity, it is not an $\mathbb{F}$-stopping time. Indeed, any stopping time with respect to a Brownian filtration is known to be predictable.

We interpret τ as the common default time for all defaultable assets in our model. For simplicity, we assume that only three primary assets are traded in the market, and the dynamics under the historical probability $\mathbb{P}$ of their prices are, for $i = 1, 2, 3$ and $t \in [0,T]$,

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dM_t\big), \tag{19}$$

or equivalently,

$$dY^i_t = Y^i_{t-}\big((\mu_{i,t} - \kappa_{i,t}\gamma_t\mathbb{1}_{\{t\leq\tau\}})\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dH_t\big). \tag{20}$$

The processes $(\mu_i, \sigma_i, \kappa_i) = (\mu_{i,t}, \sigma_{i,t}, \kappa_{i,t},\, t \geq 0)$, $i = 1, 2, 3$, are assumed to be $\mathbb{G}$-adapted, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. In addition, we assume that $\kappa_i \geq -1$ for any $i = 1, 2, 3$, so that Y^i are nonnegative processes, and they are strictly positive prior to τ.

Note that, according to Definition 2.2, replication refers to the behavior of the wealth process $V(\phi)$ on the random interval $[\![0, \tau \wedge T]\!]$ only. Hence, for the purpose of replication of defaultable claims of the form (X, Z, τ), it is sufficient to consider prices of primary assets stopped at $\tau \wedge T$. This implies that instead of dealing with $\mathbb{G}$-adapted coefficients in (19), it suffices to focus on $\mathbb{F}$-adapted coefficients of stopped price processes. However, for the sake of completeness, we shall also deal with T-maturity claims of the form $Y = G(Y^1_T, Y^2_T, Y^3_T, H_T)$ (see Section 5 below).

2.1.1 Pre-default Values

As will become clear in what follows, when dealing with defaultable claims of the form (X, Z, τ), we will be mainly concerned with the so-called pre-default prices. The *pre-default price* $\widetilde{Y}^i$ of the ith asset is an $\mathbb{F}$-adapted, continuous process, given by the equation, for $i = 1, 2, 3$ and $t \in [0,T]$,

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t\big((\mu_{i,t} - \kappa_{i,t}\gamma_t)\, dt + \sigma_{i,t}\, dW_t\big) \tag{21}$$

with $\widetilde{Y}^i_0 = Y^i_0$. Put another way, $\widetilde{Y}^i$ is the unique $\mathbb{F}$-predictable process such that (see Lemma 1.1) $\widetilde{Y}^i_t \mathbb{1}_{\{t\le\tau\}} = Y^i_t \mathbb{1}_{\{t\le\tau\}}$ for $t \in \mathbb{R}_+$. When dealing with the pre-default prices, we may and do assume, without loss of generality, that the processes μ_i, σ_i and κ_i are $\mathbb{F}$-predictable.

It is worth stressing that the historically observed drift coefficient equals $\mu_{i,t} - \kappa_{i,t}\gamma_t$, rather than $\mu_{i,t}$. The drift coefficient denoted by $\mu_{i,t}$ is already credit-risk adjusted in the sense of our model, and it is not directly observed. This convention was chosen here for the sake of simplicity of notation. It also lends itself to the following intuitive interpretation: if ϕ^i is the number of units of the ith asset held in our portfolio at time t then the gains/losses from trades in this asset, prior to default time, can be represented by the differential

$$\phi^i_t \, d\widetilde{Y}^i_t = \phi^i_t \widetilde{Y}^i_t \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t\big) - \phi^i_t \widetilde{Y}^i_t \kappa_{i,t}\gamma_t \, dt.$$

The last term may be here separated, and formally treated as an effect of continuously paid dividends at the dividend rate $\kappa_{i,t}\gamma_t$. However, this interpretation may be misleading, since this quantity is not directly observed. In fact, the mere estimation of the drift coefficient in dynamics (21) is not practical.

Still, if this formal interpretation is adopted, it is sometimes possible make use of the standard results concerning the valuation of derivatives of dividend-paying assets. It is, of course, a delicate issue how to separate in practice both components of the drift coefficient. We shall argue below that although the dividend-based approach is formally correct, a more pertinent and simpler way of dealing with hedging relies on the assumption that only the effective drift $\mu_{i,t} - \kappa_{i,t}\gamma_t$ is observable. In practical approach to hedging, the values of drift coefficients in dynamics of asset prices play no essential role, so that they are considered as market observables.

2.1.2 Market Observables

To summarize, we assume throughout that the *market observables* are: the pre-default market prices of primary assets, their volatilities and correlations, as well as the jump coefficients $\kappa_{i,t}$ (the financial interpretation of jump coefficients is examined in the next subsection). To summarize we postulate that under the statistical probability $\mathbb{P}$ we have

$$dY^i_t = Y^i_{t-}\big(\widetilde{\mu}_{i,t}\, dt + \sigma_{i,t}\, dW_t + \kappa_{i,t}\, dH_t\big) \tag{22}$$

where the drift terms $\widetilde{\mu}_{i,t}$ are not observable, but we can observe the volatilities $\sigma_{i,t}$ (and thus the assets correlations), and we have an a priori assessment of jump coefficients $\kappa_{i,t}$. In this general set-up, the most natural assumption is that the dimension of a driving Brownian motion W equals the number of tradable assets. However, for the sake of simplicity of presentation, we shall frequently assume that W is one-dimensional. One of our goals will be to derive closed-form solutions for replicating strategies for derivative securities in terms of market observables only (whenever replication of a given claim is actually feasible). To achieve this goal, we shall combine a general theory of hedging defaultable claims within a continuous semimartingale set-up, with a judicious specification of particular models with deterministic volatilities and correlations.

2.1.3 Recovery Schemes

It is clear that the sample paths of price processes Y^i are continuous, except for a possible discontinuity at time τ. Specifically, we have that

$$\Delta Y^i_\tau := Y^i_\tau - Y^i_{\tau-} = \kappa_{i,\tau} Y^i_{\tau-},$$

so that $Y^i_\tau = Y^i_{\tau-}(1 + \kappa_{i,\tau}) = \widetilde{Y}^i_{\tau-}(1 + \kappa_{i,\tau})$.

A primary asset Y^i is termed a *default-free asset* (*defaultable asset*, respectively) if $\kappa_i = 0$ ($\kappa_i \neq 0$, respectively). In the special case when $\kappa_i = -1$, we say that a defaultable asset Y^i is subject to a *total default*, since its price drops to zero at time τ and stays there forever. Such an asset ceases to exist after default, in the sense that it is no longer traded after default. This feature makes the case of a total default quite different from other cases, as we shall see in our study below.

In market practice, it is common for a credit derivative to deliver a positive recovery (for instance, a *protection payment*) in case of default. Formally, the value of this recovery at default is determined as the value of some underlying process, that is, it is equal to the value at time τ of some $\mathbb{F}$-adapted recovery process Z.

For example, the process Z can be equal to δ, where δ is a constant, or to $g(t, \delta Y_t)$ where g is a deterministic function and $(Y_t,\, t \geq 0)$ is the price process of some default-free asset. Typically, the recovery is paid at default time, but it may also happen that it is postponed to the maturity date.

Let us observe that the case where a defaultable asset Y^i pays a pre-determined recovery at default is covered by our set-up defined in (19). For instance, the case of a constant recovery payoff $\delta_i \geq 0$ at default time τ corresponds to the process $\kappa_{i,t} = \delta_i (Y^i_{t-})^{-1} - 1$. Under this convention, the price Y^i is governed under $\mathbb{P}$ by the SDE

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + (\delta_i (Y^i_{t-})^{-1} - 1)\, dM_t\big). \tag{23}$$

If the recovery is proportional to the pre-default value $Y^i_{\tau-}$, and is paid at default time τ (this scheme is known as the *fractional recovery of market value*), we have $\kappa_{i,t} = \delta_i - 1$ and

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t + (\delta_i - 1)\, dM_t\big). \tag{24}$$

2.2 Risk-Neutral Valuation

To provide a partial justification for the postulated dynamics of the price of a defaultable asset delivering a recovery, let us study a toy example with two assets: a savings account with constant interest rate r and a defaultable asset Y represented by a defaultable claim (X, Z, τ). In this toy model, the only source of noise is the default time, hence, the only relevant filtration is $\mathbb{H}$ (in other words, the reference filtration $\mathbb{F}$ is trivial). We assume that by choosing today's prices of a large family liquidly traded defaultable assets, the market implicitly specifies a martingale measure $\mathbb{Q}$, equivalent to the historical probability $\mathbb{P}$. More precisely, the probability distribution of τ under an equivalent martingale measure (e.m.m.) $\mathbb{Q}$ can be inferred from market data. We are thus interested in the dynamics of the price process of (X, Z, τ) under $\mathbb{Q}$.

It is worth noting that in this subsection we adopt a totally different perspective than in the rest of the present paper. In fact, no attempt to replicate a defaultable claim is done in this section. We assume instead that the risk-neutral default intensity can be uniquely determined from prices of traded assets, and we postulate that the price of (X, Z, τ) is defined by the standard risk-neutral valuation formula. The argument that formally justifies the use of this pricing rule is that we obtain in this way an arbitrage-free market model in which $\mathbb{Q}$ is a martingale measure, and a defaultable claim can be considered to be an additional traded asset. Since we do not assume here that a defaultable claim is attainable, its spot price (that is, the price expressed in units of cash) depends explicitly on the risk-neutral default intensity. As was mentioned above, the arbitrage price of a defaultable claim, when expressed in terms of tradeable assets used for its replication, will be shown to not depend directly on real-world (or risk-neutral) default intensity.

To conclude, the rationale for the calculations given below, is that we strive here to justify the dynamics of prices of primary assets in our model. The risk-neutral valuation considered in this subsection is not supported by replication-based arguments, and thus it is not surprising that it exhibits specific features that are not present in the replication-based valuation.

We make the standing assumption that τ admits a continuous cumulative distribution function $\widehat{F}$ under $\mathbb{Q}$. Hence, the hazard function $\widehat{\Gamma}$ is also continuous, and the process $\widehat{M}_t = H_t - \widehat{\Gamma}(t \wedge \tau)$ is an $\mathbb{H}$-martingale under $\mathbb{Q}$. The following result is standard (see, e.g., Proposition 4.3.2 in Bielecki and Rutkowski [2]).

Proposition 2.1 *Assume that the cumulative distribution function F of τ is continuous. Let M^h be an $\mathbb{H}$-martingale given by $M^h_t = \mathbb{E}_{\mathbb{Q}}(h(\tau) \,|\, \mathcal{H}_t)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $h(\tau)$ is $\mathbb{Q}$-integrable. Then*

$$M^h_t = M^h_0 + \int_0^t (h(u) - g(u))\, d\widehat{M}_u = M^h_0 + \int_0^t (h(u) - M^h_{u-})\, d\widehat{M}_u, \tag{25}$$

where we write

$$g(t) = e^{\widehat{\Gamma}(t)}\, \mathbb{E}_{\mathbb{Q}}\big(1\!\!1_{\{t<\tau\}} h(\tau)\big).$$

Remark 2.1 Using the above proposition, it can be easily shown that on $(\Omega, \mathcal{G}_T)$ we have

$$d\mathbb{P} = \mathcal{E}_T\left(-\int_0^{\cdot} \zeta(u)\, d\widehat{M}_u \right) d\mathbb{Q},$$

for some $\mathbb{H}$-predictable process ζ.

2.2.1 Price Dynamics of a Survival Claim $(X, 0, \tau)$.

In what follows, we shall refer to a defaultable claim of the form $(X, 0, \tau)$ as a *survival claim*. By virtue of the risk-neutral valuation formula, the price of the payoff $1\!\!1_{\{T<\tau\}} X$ that settles at time T equals, for every $t \in [0, T]$,

$$Y_t = e^{rt}\, \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\{T<\tau\}} e^{-rT} X \,|\, \mathcal{H}_t).$$

Note that X is $\mathcal{F}_T$-measurable, and thus constant since the σ-field $\mathcal{F}_T$ is trivial. To find the dynamics of the price process, it suffices to apply Proposition 2.1 to the function $h(u) = 1\!\!1_{\{u>T\}} e^{-rT} X$. For the $\mathbb{Q}$-martingale $M^h_t = e^{-rt} Y_t$, we thus get, for every $t \in [0, T]$,

$$e^{-rt} Y_t = Y_0 - \int_0^t e^{-ru} Y_{u-}\, d\widehat{M}_u.$$

Suppose that $\widehat{\Gamma}(t) = \int_0^t \widehat{\gamma}(u)\, du$. Then an application of Itô's formula yields

$$dY_t = rY_t\, dt - Y_{t-}\, d\widehat{M}_t = \big(r + 1\!\!1_{\{t<\tau\}} \widehat{\gamma}(t)\big) Y_t\, dt - Y_{t-}\, dH_t. \tag{26}$$

We deal here with an example of a defaultable asset that is subject to the *total default*, meaning that its price vanishes at and after default.

2.2.2 Price Dynamics of a Recovery Claim $(0, Z, \tau)$.

Recall that our standard convention stipulates that the recovery Z is paid at the time of default. Hence, the price process Y of $(0, Z, \tau)$ is given by the expression

$$Y_t = e^{rt}\, \mathbb{E}_{\mathbb{Q}}(1\!\!1_{\{T\geq\tau\}} e^{-r\tau} Z(\tau) \,|\, \mathcal{H}_t).$$

We now have $h(u) = 1\!\!1_{\{u\leq T\}} e^{-ru} Z(u)$. Consequently,

$$e^{-rt} Y_t = Y_0 + \int_0^t \big(e^{-ru} Z(u) - e^{-ru} Y_{u-}\big)\, d\widehat{M}_u.$$

By applying Itô's formula, we conclude that that the dynamics under $\mathbb{Q}$ of an asset that delivers $Z(\tau)$ at time τ are

$$\begin{aligned} dY_t &= rY_{t-}\, dt + (Z(t) - Y_{t-})\, d\widehat{M}_t \\ &= \big(r + 1\!\!1_{\{t<\tau\}} \widehat{\gamma}(t)\big) Y_t\, dt - 1\!\!1_{\{t<\tau\}} Z(t) \widehat{\gamma}(t)\, dt + (Z(t) - Y_{t-})\, dH_t. \end{aligned}$$

2.2.3 Price Dynamics of a Defaultable Claim (X, Z, τ).

By combining the formula above with (26), and using Remark 2.1 together with Girsanov's theorem, we arrive at the following result.

Proposition 2.2 *The price process Y of a defaultable claim (X, Z, τ) satisfies under $\mathbb{Q}$*

$$dY_t = rY_{t-}\,dt + (Z(t) - Y_{t-})\,d\widehat{M}_t$$

with the initial condition

$$Y_0 = \mathbb{E}_{\mathbb{Q}}\big(1\!\!1_{\{T<\tau\}}e^{-rT}X + 1\!\!1_{\{T\geq\tau\}}e^{-r\tau}Z(\tau)\big) = e^{-(rT+\widehat{\Gamma}(T))}X + \int_0^T Z(u)\widehat{\gamma}(u)e^{-\widehat{\Gamma}(u)}\,du.$$

Under the statistical probability $\mathbb{P}$, the price process Y satisfies

$$dY_t = \big(rY_{t-} + 1\!\!1_{\{t<\tau\}}(Z(t) - Y_{t-})\widehat{\gamma}(t)\zeta(t)\big)\,dt + (Z(t) - Y_{t-})\,dM_t,$$

where the $\mathbb{G}$-martingale M under $\mathbb{P}$ equals

$$M_t = \widehat{M}_t + \int_0^t 1\!\!1_{\{u<\tau\}}\widehat{\gamma}(u)\zeta(u)\,du.$$

Remark 2.2 Proposition 2.2 can be extended to the case when the recovery is random, and is given in the feedback form as $Z(t) = g(t, Y_{t-})$ for some function $g(t, y)$, which is Lipschitz continuous with respect to y. Assume, for instance, that the claim is subject to the fractional recovery of market value, so that $Z(t) = \delta Y_{t-}$ for some constant δ. If, in addition, ζ and $\widehat{\gamma}$ are constant, then we obtain (cf. (24))

$$dY_t = Y_{t-}\Big((r + 1\!\!1_{\{t<\tau\}}(\delta - 1)\widehat{\gamma}\zeta)\,dt + (\delta - 1)\,dM_t\Big).$$

Note that here the drift coefficient $\mu_t = r + 1\!\!1_{\{t<\tau\}}(\delta - 1)\widehat{\gamma}\zeta$ in dynamics of Y follows a $\mathbb{G}$-predictable process, but it is not $\mathbb{F}$-predictable. However, the drift of the pre-default value $\widetilde{Y}$ is simply r.

3 Trading Strategies in a Semimartingale Set-up

We consider trading within the time interval $[0, T]$ for some finite horizon date $T > 0$. For the sake of expositional clarity, we restrict our attention to the case where only three primary assets are traded. The general case of k traded assets was examined by Bielecki et al. [4]. We first recall some general properties, which do not depend on the choice of specific dynamics of asset prices.

In this section, we consider a fairly general set-up. In particular, processes Y^i, $i = 1, 2, 3$, are assumed to be nonnegative semi-martingales on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ endowed with some filtration $\mathbb{G}$. We assume that they represent spot prices of traded assets in our model of the financial market. Neither the existence of a savings account, nor the market completeness are assumed, in general.

Our goal is to characterize contingent claims which are *hedgeable*, in the sense that they can be replicated by continuously rebalanced portfolios consisting of primary assets. Here, by a contingent claim we mean an arbitrary $\mathcal{G}_T$-measurable random variable. We work under the standard assumptions of a frictionless market.

3.1 Unconstrained Strategies

Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a trading strategy; in particular, each process ϕ^i is predictable with respect to the filtration $\mathbb{G}$. The wealth of ϕ equals

$$V_t(\phi) = \sum_{i=1}^{3} \phi^i_t Y^i_t, \quad \forall\, t \in [0, T],$$

and a trading strategy ϕ is said to be *self-financing* if

$$V_t(\phi) = V_0(\phi) + \sum_{i=1}^{3} \int_0^t \phi_u^i \, dY_u^i, \quad \forall \, t \in [0,T].$$

Let Φ stand for the class of all self-financing trading strategies. We shall first prove that a self-financing strategy is determined by its initial wealth and the two components ϕ^2, ϕ^3. To this end, we postulate that the price of Y^1 follows a strictly positive process, and we choose Y^1 as a numéraire asset. We shall now analyze the relative values:

$$V_t^1(\phi) := V_t(\phi)(Y_t^1)^{-1}, \quad Y_t^{i,1} := Y_t^i (Y_t^1)^{-1}.$$

Lemma 3.1 (i) *For any $\phi \in \Phi$, we have*

$$V_t^1(\phi) = V_0^1(\phi) + \sum_{i=2}^{3} \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall \, t \in [0,T].$$

(ii) *Conversely, let X be a $\mathcal{G}_T$-measurable random variable, and let us assume that there exists $x \in \mathbb{R}$ and $\mathbb{G}$-predictable processes ϕ^i, $i = 2,3$ such that*

$$X = Y_T^1 \left(x + \sum_{i=2}^{3} \int_0^T \phi_u^i \, dY_u^{i,1} \right). \tag{27}$$

Then there exists a $\mathbb{G}$-predictable process ϕ^1 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and replicates X. Moreover, the wealth process of ϕ (i.e. the time-t price of X) satisfies $V_t(\phi) = V_t^1 Y_t^1$, where

$$V_t^1 = x + \sum_{i=2}^{3} \int_0^t \phi_u^i \, dY_u^{i,1}, \quad \forall \, t \in [0,T]. \tag{28}$$

Proof. The proof of part (i) is given, for instance, in Protter [33]. In the case of continuous semimartingales, this is a well-known result; for discontinuous processes, the proof is not much different. We reproduce it here for the reader's convenience.

Let us first introduce some notation. As usual, $[X,Y]$ stands for the *quadratic covariation* of the two semi-martingales X and Y, as defined by the integration by parts formula:

$$X_t Y_t = X_0 Y_0 + \int_0^t X_{u-} \, dY_u + \int_0^t Y_{u-} \, dX_u + [X,Y]_t.$$

For any càdlàg (i.e., RCLL) process Y, we denote by $\Delta Y_t = Y_t - Y_{t-}$ the size of the jump at time t. Let $V = V(\phi)$ be the value of a self-financing strategy, and let $V^1 = V^1(\phi) = V(\phi)(Y^1)^{-1}$ be its value relative to the numéraire Y^1. The integration by parts formula yields

$$dV_t^1 = V_{t-} d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1} dV_t + d[(Y^1)^{-1}, V]_t.$$

From the self-financing condition, we have $dV_t = \sum_{i=1}^{3} \phi_t^i \, dY_t^i$. Hence, using elementary rules to compute the quadratic covariation $[X,Y]$ of the two semi-martingales X, Y, we obtain

$$\begin{aligned}
dV_t^1 &= \phi_t^1 Y_{t-}^1 \, d(Y_t^1)^{-1} + \phi_t^2 Y_{t-}^2 \, d(Y_t^1)^{-1} + \phi_t^3 Y_{t-}^3 \, d(Y_t^1)^{-1} \\
&\quad + (Y_{t-}^1)^{-1} \phi_t^1 \, dY_t^1 + (Y_{t-}^1)^{-1} \phi_t^2 \, dY_t^1 + (Y_{t-}^1)^{-1} \phi_t^3 \, dY_t^1 \\
&\quad + \phi_t^1 \, d[(Y^1)^{-1}, Y^1]_t + \phi_t^2 \, d[(Y^1)^{-1}, Y^2]_t + \phi_t^3 \, d[(Y^1)^{-1}, Y^1]_t \\
&= \phi_t^1 \big(Y_{t-}^1 \, d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1} \, dY_t^1 + d[(Y^1)^{-1}, Y^1]_t \big) \\
&\quad + \phi_t^2 \big(Y_{t-}^2 \, d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1} \, dY_{t-}^1 + d[(Y^1)^{-1}, Y^2]_t \big) \\
&\quad + \phi_t^3 \big(Y_{t-}^3 \, d(Y_t^1)^{-1} + (Y_{t-}^1)^{-1} \, dY_{t-}^1 + d[(Y^1)^{-1}, Y^3]_t \big).
\end{aligned}$$

We now observe that

$$Y^1_{t-}\, d(Y^1_t)^{-1} + (Y^1_{t-})^{-1}\, dY^1_t + d[(Y^1)^{-1}, Y^1]_t = d(Y^1_t(Y^1_t)^{-1}) = 0$$

and

$$Y^i_{t-}\, d(Y^1_t)^{-1} + (Y^1_{t-})^{-1}\, dY^i_t + d[(Y^1)^{-1}, Y^i]_t = d((Y^1_t)^{-1}Y^i_t).$$

Consequently,

$$dV^1_t = \phi^2_t\, dY^{2,1}_t + \phi^3_t\, dY^{3,1}_t,$$

as was claimed in part (i). We now proceed to the proof of part (ii). We assume that (27) holds for some constant x and processes ϕ^2, ϕ^3, and we define the process V^1 by setting (cf. (28))

$$V^1_t = x + \sum_{i=2}^{3} \int_0^t \phi^i_u\, dY^{i,1}_u, \quad \forall\, t \in [0,T].$$

Next, we define the process ϕ^1 as follows:

$$\phi^1_t = V^1_t - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_t = (Y^1_t)^{-1}\Big(V_t - \sum_{i=2}^{3} \phi^i_t Y^i_t\Big),$$

where $V_t = V^1_t Y^1_t$. Since $dV^1_t = \sum_{i=2}^3 \phi^i_t\, dY^{i,1}_t$, we obtain

$$\begin{aligned} dV_t &= d(V^1_t Y^1_t) = V^1_{t-} dY^1_t + Y^1_{t-} dV^1_t + d[Y^1, V^1]_t \\ &= V^1_{t-} dY^1_t + \sum_{i=2}^{3} \phi^i_t\big(Y^1_{t-}\, dY^{i,1}_t + d[Y^1, Y^{i,1}]_t\big). \end{aligned}$$

From the equality

$$dY^i_t = d(Y^{i,1}_t Y^1_t) = Y^{i,1}_{t-} dY^1_t + Y^1_{t-} dY^{i,1}_t + d[Y^1, Y^{i,1}]_t,$$

it follows that

$$dV_t = V^1_{t-} dY^1_t + \sum_{i=2}^{3} \phi^i_t\big(dY^i_t - Y^{i,1}_{t-} dY^1_t\big) = \Big(V^1_{t-} - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_{t-}\Big) dY^1_t + \sum_{i=2}^{3} \phi^i_t\, dY^i_t,$$

and our aim is to prove that $dV_t = \sum_{i=1}^3 \phi^i_t\, dY^i_t$. The last equality holds if

$$\phi^1_t = V^1_t - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_t = V^1_{t-} - \sum_{i=2}^{3} \phi^i_t Y^{i,1}_{t-}, \tag{29}$$

i.e., if $\Delta V^1_t = \sum_{i=2}^3 \phi^i_t \Delta Y^{i,1}_t$, which is the case from the definition (28) of V^1. Note also that from the second equality in (29) it follows that the process ϕ^1 is indeed $\mathbb{G}$-predictable. Finally, the wealth process of ϕ satisfies $V_t(\phi) = V^1_t Y^1_t$ for every $t \in [0,T]$, and thus $V_T(\phi) = X$. □

We say that a self-financing strategy ϕ *replicates* a claim $X \in \mathcal{G}_T$ if

$$X = \sum_{i=1}^{3} \phi^i_T Y^i_T = V_T(\phi),$$

or equivalently,

$$X = V_0(\phi) + \sum_{i=1}^{3} \int_0^T \phi^i_t\, dY^i_t.$$

Suppose that there exists an e.m.m. for some choice of a numéraire asset, and let us restrict our attention to the class of all *admissible* trading strategies, so that our model is arbitrage-free.

Assume that a claim X can be replicated by some admissible trading strategy, so that it is *attainable* (or *hedgeable*). Then, by definition, the *arbitrage price* at time t of X, denoted as $\pi_t(X)$, equals $V_t(\phi)$ for any admissible trading strategy ϕ that replicates X.

In the context of Lemma 3.1, it is natural to choose as an e.m.m. a probability measure $\mathbb{Q}^1$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ and such that the prices $Y^{i,1}$, $i = 2, 3$, are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. If a contingent claim X is hedgeable, then its arbitrage price satisfies

$$\pi_t(X) = Y_t^1 \mathbb{E}_{\mathbb{Q}^1}(X(Y_T^1)^{-1} \,|\, \mathcal{G}_t).$$

We emphasize that even if an e.m.m. $\mathbb{Q}^1$ is not unique, the price of any hedgeable claim X is given by this conditional expectation. That is to say, in case of a hedgeable claim these conditional expectations under various equivalent martingale measures coincide.

In the special case where $Y_t^1 = B(t,T)$ is the price of a default-free zero-coupon bond with maturity T (abbreviated as ZC-bond in what follows), $\mathbb{Q}^1$ is called *T-forward martingale measure*, and it is denoted by $\mathbb{Q}_T$. Since $B(T,T) = 1$, the price of any hedgeable claim X now equals $\pi_t(X) = B(t,T)\, \mathbb{E}_{\mathbb{Q}_T}(X \,|\, \mathcal{G}_t)$.

3.2 Constrained Strategies

In this section, we make an additional assumption that the price process Y^3 is strictly positive. Let $\phi = (\phi^1, \phi^2, \phi^3)$ be a self-financing trading strategy satisfying the following constraint:

$$\sum_{i=1}^{2} \phi_t^i Y_{t-}^i = Z_t, \quad \forall\, t \in [0,T], \tag{30}$$

for a predetermined, $\mathbb{G}$-predictable process Z. In the financial interpretation, equality (30) means that a portfolio ϕ is rebalanced in such a way that the total wealth invested in assets Y^1, Y^2 matches a predetermined stochastic process Z. For this reason, the constraint given by (30) is referred to as the *balance condition*.

Our first goal is to extend part (i) in Lemma 3.1 to the case of constrained strategies. Let $\Phi(Z)$ stand for the class of all (admissible) self-financing trading strategies satisfying the balance condition (30). They will be sometimes referred to as *constrained strategies*. Since any strategy $\phi \in \Phi(Z)$ is self-financing, from $dV_t(\phi) = \sum_{i=1}^3 \phi_t^i \, dY_t^i$, we obtain

$$\Delta V_t(\phi) = \sum_{i=1}^{3} \phi_t^i \Delta Y_t^i = V_t(\phi) - \sum_{i=1}^{3} \phi_t^i Y_{t-}^i.$$

By combining this equality with (30), we deduce that

$$V_{t-}(\phi) = \sum_{i=1}^{3} \phi_t^i Y_{t-}^i = Z_t + \phi_t^3 Y_{t-}^i.$$

Let us write $Y_t^{i,3} = Y_t^i (Y_t^3)^{-1}$, $Z_t^3 = Z_t (Y_t^3)^{-1}$. The following result extends Lemma 1.7 in Bielecki et al. [3] from the case of continuous semi-martingales to the general case (see also [4]). It is apparent from Proposition 3.1 that the wealth process $V(\phi)$ of a strategy $\phi \in \Phi(Z)$ depends only on a single component of ϕ, namely, ϕ^2.

Proposition 3.1 *The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of any trading strategy $\phi \in \Phi(Z)$ satisfies*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \phi_u^2 \left(dY_u^{2,3} - \frac{Y_{u-}^{2,3}}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right) + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}. \tag{31}$$

Proof. Let us consider discounted values of price processes Y^1, Y^2, Y^3, with Y^3 taken as a numéraire asset. By virtue of part (i) in Lemma 3.1, we thus have

$$V_t^3(\phi) = V_0^3(\phi) + \sum_{i=1}^{2} \int_0^t \phi_u^i \, dY_u^{i,3}. \tag{32}$$

The balance condition (30) implies that

$$\sum_{i=1}^{2} \phi_t^i Y_{t-}^{i,3} = Z_t^3,$$

and thus

$$\phi_t^1 = (Y_{t-}^{1,3})^{-1} \Big(Z_t^3 - \phi_t^2 Y_{t-}^{2,3} \Big). \tag{33}$$

By inserting (33) into (32), we arrive at the desired formula (31). □

The next result will prove particularly useful for deriving replicating strategies for defaultable claims.

Proposition 3.2 *Let a $\mathcal{G}_T$-measurable random variable X represent a contingent claim that settles at time T. Assume that there exists a $\mathbb{G}$-predictable process ϕ^2, such that*

$$X = Y_T^3 \left(x + \int_0^T \phi_t^2 \, dY_t^* + \int_0^T \frac{Z_t^3}{Y_{t-}^{1,3}} \, dY_t^{1,3} \right). \tag{34}$$

Then there exist $\mathbb{G}$-predictable processes ϕ^1 and ϕ^3 such that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ belongs to $\Phi(Z)$ and replicates X. The wealth process of ϕ equals, for every $t \in [0, T]$,

$$V_t(\phi) = Y_t^3 \left(x + \int_0^t \phi_u^2 \, dY_u^* + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3} \right). \tag{35}$$

Proof. As expected, we first set (note that the process ϕ^1 is a $\mathbb{G}$-predictable process)

$$\phi_t^1 = \frac{1}{Y_{t-}^1} \Big(Z_t - \phi_t^2 Y_{t-}^2 \Big) \tag{36}$$

and

$$V_t^3 = x + \int_0^t \phi_u^2 \, dY_u^* + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}.$$

Arguing along the same lines as in the proof of Proposition 3.1, we obtain

$$V_t^3 = V_0^3 + \sum_{i=1}^{2} \int_0^t \phi_u^i \, dY_u^{i,3}.$$

Now, we define

$$\phi_t^3 = V_t^3 - \sum_{i=1}^{2} \phi_t^i Y_t^{i,3} = (Y_t^3)^{-1} \Big(V_t - \sum_{i=1}^{2} \phi_t^i Y_t^i \Big),$$

where $V_t = V_t^3 Y_t^3$. As in the proof of Lemma 3.1, we check that

$$\phi_t^3 = V_{t-}^3 - \sum_{i=1}^{2} \phi_t^i Y_{t-}^{i,3},$$

and thus the process ϕ^3 is $\mathbb{G}$-predictable. It is clear that the strategy $\phi = (\phi^1, \phi^2, \phi^3)$ is self-financing and its wealth process satisfies $V_t(\phi) = V_t$ for every $t \in [0, T]$. In particular, $V_T(\phi) = X$, so that ϕ replicates X. Finally, equality (36) implies (30), and thus ϕ belongs to the class $\Phi(Z)$. □

Note that equality (34) is a necessary (by Lemma 3.1) and sufficient (by Proposition 3.2) condition for the existence of a constrained strategy that replicates a given contingent claim X.

3.2.1 Synthetic Asset

Let us take $Z = 0$, so that $\phi \in \Phi(0)$. Then the balance condition becomes $\sum_{i=1}^{2} \phi_t^i Y_{t-}^i = 0$, and formula (31) reduces to

$$dV_t^3(\phi) = \phi_t^2 \left(dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}} \, dY_t^{1,3} \right). \tag{37}$$

We set

$$dY_t^* = dY_t^{2,3} - \frac{Y_{t-}^{2,3}}{Y_{t-}^{1,3}} \, dY_t^{1,3} = dY_t^{2,3} - Y_{t-}^{2,1} \, dY_t^{1,3},$$

where, by convention, $Y_0^* = 0$. The process $\bar{Y}^2 = Y^3 Y^*$ is called a *synthetic asset*. It corresponds to a particular self-financing portfolio, with the long position in Y^2 and the short position of $Y_{t-}^{2,1}$ number of shares of Y^1, and suitably re-balanced positions in the third asset so that the portfolio is self-financing, as in Lemma 3.1.

It can be shown (see Bielecki et al. [4]) that trading in primary assets Y^1, Y^2, Y^3 is formally equivalent to trading in assets $Y^1, \bar{Y}^2, Y^3$. This observation supports the name synthetic asset attributed to the process $\bar{Y}^2$. Note, however, that the synthetic asset process may take negative values.

3.2.2 Case of Continuous Asset Prices

In the case of continuous asset prices, the relative price $Y^* = \bar{Y}^2 (Y^3)^{-1}$ of the synthetic asset can be given an alternative representation, as the following result shows. Recall that the *predictable bracket* of the two continuous semi-martingales X and Y, denoted as $\langle X, Y \rangle$, coincides with their quadratic covariation $[X, Y]$.

Proposition 3.3 *Assume that the price processes Y^1 and Y^2 are continuous. Then the relative price of the synthetic asset satisfies*

$$Y_t^* = \int_0^t (Y_u^{3,1})^{-1} e^{\alpha_u} \, d\widehat{Y}_u,$$

where $\widehat{Y}_t := Y_t^{2,1} e^{-\alpha_t}$ and

$$\alpha_t := \langle \ln Y^{2,1}, \ln Y^{3,1} \rangle_t = \int_0^t (Y_u^{2,1})^{-1} (Y_u^{3,1})^{-1} \, d\langle Y^{2,1}, Y^{3,1} \rangle_u. \tag{38}$$

In terms of the auxiliary process $\widehat{Y}$, formula (31) *becomes*

$$V_t^3(\phi) = V_0^3(\phi) + \int_0^t \widehat{\phi}_u \, d\widehat{Y}_u + \int_0^t \frac{Z_u^3}{Y_{u-}^{1,3}} \, dY_u^{1,3}, \tag{39}$$

where $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Proof. It suffices to give the proof for $Z = 0$. The proof relies on the integration by parts formula stating that for any two continuous semi-martingales, say X and Y, we have

$$Y_t^{-1} \big(dX_t - Y_t^{-1} d\langle X, Y \rangle_t \big) = d(X_t Y_t^{-1}) - X_t \, dY_t^{-1},$$

provided that Y is strictly positive. An application of this formula to processes $X = Y^{2,1}$ and $Y = Y^{3,1}$ leads to

$$(Y_t^{3,1})^{-1} \big(dY_t^{2,1} - (Y_t^{3,1})^{-1} d\langle Y^{2,1}, Y^{3,1} \rangle_t \big) = d(Y_t^{2,1} (Y_t^{3,1})^{-1}) - Y_t^{2,1} \, d(Y^{3,1})_t^{-1}.$$

The relative wealth $V_t^3(\phi) = V_t(\phi)(Y_t^3)^{-1}$ of a strategy $\phi \in \Phi(0)$ satisfies

$$\begin{aligned} V_t^3(\phi) &= V_0^3(\phi) + \int_0^t \phi_u^2 \, dY_u^* \\ &= V_0^3(\phi) + \int_0^t \phi_u^2 (Y_u^{3,1})^{-1} e^{\alpha_u} \, d\widehat{Y}_u, \\ &= V_0^3(\phi) + \int_0^t \widehat{\phi}_u \, d\widehat{Y}_u \end{aligned}$$

where we denote $\widehat{\phi}_t = \phi_t^2 (Y_t^{3,1})^{-1} e^{\alpha_t}$.

Remark 3.1 The financial interpretation of the auxiliary process $\widehat{Y}$ will be studied in Sections 4.1.6 and 4.1.8 below. Let us only observe here that if Y^* is a local martingale under some probability $\mathbb{Q}^*$ then $\widehat{Y}$ is a $\mathbb{Q}^*$-local martingale (and vice-versa, if $\widehat{Y}$ is a $\widehat{\mathbb{Q}}$-local martingale under some probability $\widehat{\mathbb{Q}}$ then Y^* is a $\widehat{\mathbb{Q}}$-local martingale). Nevertheless, for the reader's convenience, we shall use two symbols $\mathbb{Q}^*$ and $\widehat{\mathbb{Q}}$, since this equivalence holds for continuous processes only.

It is thus worth stressing that we will apply Proposition 3.3 to pre-default values of assets, rather than directly to asset prices, within the set-up of a semimartingale model with a common default, as described in Section 2.1. In this model, the asset prices may have discontinuities, but their pre-default values follow continuous processes.

4 Martingale Approach to Valuation and Hedging

Our goal is to derive quasi-explicit conditions for replicating strategies for a defaultable claim in a fairly general set-up introduced in Section 2.1. In this section, we only deal with trading strategies based on the reference filtration $\mathbb{F}$, and the underlying price processes (that is, prices of default-free assets and pre-default values of defaultable assets) are assumed to be continuous. Hence, our arguments will hinge on Proposition 3.3, rather than on a more general Proposition 3.1. We shall also adapt Proposition 3.2 to our current purposes.

To simplify the presentation, we make a standing assumption that all coefficient processes are such that the SDEs appearing below admit unique strong solutions, and all stochastic exponentials (used as Radon-Nikodým derivatives) are true martingales under respective probabilities.

4.1 Defaultable Asset with Total Default

In this section, we shall examine in some detail a particular model where the two assets, Y^1 and Y^2, are default-free and satisfy

$$dY_t^i = Y_t^i \big(\mu_{i,t} \, dt + \sigma_{i,t} \, dW_t \big), \quad i = 1, 2,$$

where W is a one-dimensional Brownian motion. The third asset is a defaultable asset with total default, so that

$$dY_t^3 = Y_{t-}^3 \big(\mu_{3,t} \, dt + \sigma_{3,t} \, dW_t - dM_t \big).$$

Since we will be interested in replicating strategies in the sense of Definition 2.2, we may and do do assume, without loss of generality, that the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-predictable, rather than $\mathbb{G}$-predictable. Recall that, in general, there exist $\mathbb{F}$-predictable processes $\widetilde{\mu}_3$ and $\widetilde{\sigma}_3$ such that

$$\widetilde{\mu}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \mu_{3,t} \mathbb{1}_{\{t \le \tau\}}, \quad \widetilde{\sigma}_{3,t} \mathbb{1}_{\{t \le \tau\}} = \sigma_{3,t} \mathbb{1}_{\{t \le \tau\}}. \tag{40}$$

We assume throughout that $Y_0^i > 0$ for every i, so that the price processes Y^1, Y^2 are strictly positive, and the process Y^3 is nonnegative, and has strictly positive pre-default value.

4.1.1 Default-Free Market

It is natural to postulate that the default-free market with the two traded assets, Y^1 and Y^2, is arbitrage-free. More precisely, we choose Y^1 as a numéraire, and we require that there exists a probability measure $\mathbb{P}^1$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$, and such that the process $Y^{2,1}$ is a $\mathbb{P}^1$-martingale. The dynamics of processes $(Y^1)^{-1}$ and $Y^{2,1}$ are

$$d(Y^1_t)^{-1} = (Y^1_t)^{-1}\big((\sigma^2_{1,t} - \mu_{1,t})\,dt - \sigma_{1,t}\,dW_t\big), \tag{41}$$

and

$$dY^{2,1}_t = Y^{2,1}_t\big((\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t}))\,dt + (\sigma_{2,t} - \sigma_{1,t})\,dW_t\big),$$

respectively. Hence, the necessary condition for the existence of an e.m.m. $\mathbb{P}^1$ is the inclusion $A \subseteq B$, where $A = \{(t,\omega) \in [0,T] \times \Omega : \sigma_{1,t}(\omega) = \sigma_{2,t}(\omega)\}$ and $B = \{(t,\omega) \in [0,T] \times \Omega : \mu_{1,t}(\omega) = \mu_{2,t}(\omega)\}$. The necessary and sufficient condition for the existence and uniqueness of an e.m.m. $\mathbb{P}^1$ reads

$$\mathbb{E}_{\mathbb{P}}\left\{\mathcal{E}_T\left(\int_0^{\cdot} \theta_u\,dW_u\right)\right\} = 1 \tag{42}$$

where the process θ is given by the formula (by convention, $0/0 = 0$)

$$\theta_t = \sigma_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}, \quad \forall\, t \in [0,T]. \tag{43}$$

Note that in the case of constant coefficients, if $\sigma_1 = \sigma_2$ then the model is arbitrage-free only in the trivial case when $\mu_2 = \mu_1$.

Remark 4.1 Since the martingale measure $\mathbb{P}^1$ is unique, the default-free model (Y^1, Y^2) is complete. However, this is not a necessary assumption and thus it can be relaxed. As we shall see in what follows, it is typically more natural to assume that the driving Brownian motion W is multi-dimensional.

4.1.2 Arbitrage-Free Property

Let us now consider also a defaultable asset Y^3. Our goal is now to find a martingale measure $\mathbb{Q}^1$ (if it exists) for relative prices $Y^{2,1}$ and $Y^{3,1}$. Recall that we postulate that the hypothesis (H) holds under $\mathbb{P}$ for filtrations $\mathbb{F}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$. The dynamics of $Y^{3,1}$ under $\mathbb{P}$ are

$$dY^{3,1}_t = Y^{3,1}_{t-}\Big\{\big(\mu_{3,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{3,t})\big)dt + (\sigma_{3,t} - \sigma_{1,t})\,dW_t - dM_t\Big\}.$$

Let $\mathbb{Q}^1$ be any probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and let η be the associated Radon-Nikodým density process, so that

$$d\mathbb{Q}^1\,|_{\mathcal{G}_t} = \eta_t\,d\mathbb{P}\,|_{\mathcal{G}_t}, \tag{44}$$

where the process η satisfies

$$d\eta_t = \eta_{t-}(\theta_t\,dW_t + \zeta_t\,dM_t) \tag{45}$$

for some $\mathbb{G}$-predictable processes θ and ζ, and η is a $\mathbb{G}$-martingale under $\mathbb{P}$.

From Girsanov's theorem, the processes $\widehat{W}$ and $\widehat{M}$, given by

$$\widehat{W}_t = W_t - \int_0^t \theta_u\,du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\,du, \tag{46}$$

are $\mathbb{G}$-martingales under $\mathbb{Q}^1$. To ensure that $Y^{2,1}$ is a $\mathbb{Q}^1$-martingale, we postulate that (42) and (43) are valid. Consequently, for the process $Y^{3,1}$ to be a $\mathbb{Q}^1$-martingale, it is necessary and sufficient that ζ satisfies

$$\gamma_t\zeta_t = \mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}}(\sigma_{3,t} - \sigma_{1,t}).$$

To ensure that $\mathbb{Q}^1$ is a probability measure equivalent to $\mathbb{P}$, we require that $\zeta_t > -1$. The unique martingale measure $\mathbb{Q}^1$ is then given by the formula (44) where η solves (45), so that

$$\eta_t = \mathcal{E}_t \left(\int_0^\cdot \theta_u \, dW_u \right) \mathcal{E}_t \left(\int_0^\cdot \zeta_u \, dM_u \right).$$

We are in a position to formulate the following result.

Proposition 4.1 *Assume that the process θ given by (43) satisfies (42), and*

$$\zeta_t = \frac{1}{\gamma_t} \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right) > -1. \tag{47}$$

Then the model $\mathcal{M} = (Y^1, Y^2, Y^3; \Phi)$ is arbitrage-free and complete. The dynamics of relative prices under the unique martingale measure $\mathbb{Q}^1$ are

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1} (\sigma_{2,t} - \sigma_{1,t}) \, d\widehat{W}_t, \\ dY_t^{3,1} &= Y_{t-}^{3,1} \big((\sigma_{3,t} - \sigma_{1,t}) \, d\widehat{W}_t - d\widehat{M}_t \big). \end{aligned}$$

Since the coefficients $\mu_{i,t}$, $\sigma_{i,t}$, $i = 1, 2$, are $\mathbb{F}$-adapted, the process $\widehat{W}$ is an $\mathbb{F}$-martingale (hence, a Brownian motion) under $\mathbb{Q}^1$. Hence, by virtue of Proposition 1.1, the hypothesis (H) holds under $\mathbb{Q}^1$, and the $\mathbb{F}$-intensity of default under $\mathbb{Q}^1$ equals

$$\widehat{\gamma}_t = \gamma_t (1 + \zeta_t) = \gamma_t + \left(\mu_{3,t} - \mu_{1,t} - \frac{\mu_{1,t} - \mu_{2,t}}{\sigma_{1,t} - \sigma_{2,t}} (\sigma_{3,t} - \sigma_{1,t}) \right).$$

Example 4.1 We present an example where the condition (47) does not hold, and thus arbitrage opportunities arise. Assume the coefficients are constant and satisfy: $\mu_1 = \mu_2 = \sigma_1 = 0$, $\mu_3 < -\gamma$ for a constant default intensity $\gamma > 0$. Then

$$Y_t^3 = \mathbb{1}_{\{t < \tau\}} Y_0^3 \exp \left(\sigma_3 W_t - \frac{1}{2} \sigma_3^2 t + (\mu_3 + \gamma) t \right) \leq Y_0^3 \exp \left(\sigma_3 W_t - \frac{1}{2} \sigma_3^2 t \right) = V_t(\phi),$$

where $V(\phi)$ represents the wealth of a self-financing strategy $(\phi^1, \phi^2, 0)$ with $\phi^2 = \frac{\sigma_3}{\sigma_2}$. Hence, the arbitrage strategy would be to sell the asset Y^3, and to follow the strategy ϕ.

Remark 4.2 Let us stress once again, that the existence of an e.m.m. is a necessary condition for viability of a financial model, but the uniqueness of an e.m.m. is not always a convenient condition to impose on a model. In fact, when building a model, we should be mostly concerned with its flexibility and ability to reflect the pertinent risk factors, rather than with its mathematical completeness. In the present context, it is natural to postulate that the dimension of the underlying Brownian motion equals the number of tradeable risky assets. In addition, each particular model should be tailored to provide intuitive and handy solutions for a predetermined family of contingent claims that will be priced and hedged within its framework.

4.1.3 Hedging a Survival Claim

We first focus on replication of a *survival claim* $(X, 0, \tau)$, that is, a defaultable claim represented by the terminal payoff $X \mathbb{1}_{\{T < \tau\}}$, where X is an $\mathcal{F}_T$-measurable random variable. For the moment, we maintain the simplifying assumption that W is one-dimensional. As we shall see in what follows, it may lead to certain pathological features of a model. If, on the contrary, the driving noise is multi-dimensional, most of the analysis remains valid, except that model completeness is no longer ensured, in general.

Recall that $\widetilde{Y}^3$ stands for the pre-default price of Y^3, defined as (see (21))

$$d\widetilde{Y}^3_t = \widetilde{Y}^3_t \big((\widetilde{\mu}_{3,t} + \gamma_t)\, dt + \widetilde{\sigma}_{3,t}\, dW_t \big) \tag{48}$$

with $\widetilde{Y}^3_0 = Y^3_0$. This strictly positive, continuous, $\mathbb{F}$-adapted process enjoys the property that $Y^3_t = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}^3_t$. Let us denote the pre-default values in the numéraire $\widetilde{Y}^3$ by $\widetilde{Y}^{i,3}_t = Y^i_t(\widetilde{Y}^3_t)^{-1}$, $i = 1, 2$, and let us introduce the pre-default relative price $\widetilde{Y}^*$ of the synthetic asset $\bar{Y}^2$ by setting

$$d\widetilde{Y}^*_t := d\widetilde{Y}^{2,3}_t - \frac{\widetilde{Y}^{2,3}_t}{\widetilde{Y}^{1,3}_t}\, d\widetilde{Y}^{1,3}_t = \widetilde{Y}^{2,3}_t \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big),$$

and let us assume that $\sigma_{1,t} - \sigma_{2,t} \neq 0$. It is also useful to note that the process $\widehat{Y}$, defined in Proposition 3.3, satisfies

$$d\widehat{Y}_t = \widehat{Y}_t \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{3,t}(\sigma_{1,t} - \sigma_{2,t})\big)\, dt + (\sigma_{2,t} - \sigma_{1,t})\, dW_t \Big).$$

In Sections 4.1.6 and 4.1.8, we shall show that in the case, where α given by (38) is deterministic, the process $\widehat{Y}$ has a pertinent financial interpretation as a credit-risk adjusted forward price of Y^2 relative to Y^1. Therefore, it is more convenient to work with the process $\widetilde{Y}^*$ when dealing with the general case, but to use the process $\widehat{Y}$ when analyzing a model with deterministic volatilities.

Consider an $\mathbb{F}$-predictable self-financing strategy ϕ satisfying the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$, and the corresponding wealth process

$$V_t(\phi) := \sum_{i=1}^3 \phi^i_t Y^i_t = \phi^3_t Y^3_t.$$

Let $\widetilde{V}_t(\phi) := \phi^3_t \widetilde{Y}^3_t$. Since the process $\widetilde{V}(\phi)$ is $\mathbb{F}$-adapted, we see that this is the *pre-default price* process of the portfolio ϕ, that is, we have $\mathbb{1}_{\{\tau>t\}}V_t(\phi) = \mathbb{1}_{\{\tau>t\}}\widetilde{V}_t(\phi)$; we shall call this process the *pre-default wealth* of ϕ. Consequently, the process $\widetilde{V}^3_t(\phi) := \widetilde{V}_t(\phi)(\widetilde{Y}^3_t)^{-1} = \phi^3_t$ is termed the relative pre-default wealth.

Using Proposition 3.1, with suitably modified notation, we find that the $\mathbb{F}$-adapted process $\widetilde{V}^3(\phi)$ satisfies, for every $t \in [0, T]$,

$$\widetilde{V}^3_t(\phi) = \widetilde{V}^3_0(\phi) + \int_0^t \phi^2_u\, d\widetilde{Y}^*_u.$$

Define a new probability on $(\Omega, \mathcal{F}_T)$ by setting

$$d\mathbb{Q}^* = \eta^*_T\, d\mathbb{P},$$

where $d\eta^*_t = \eta^*_t \theta^*_t\, dW_t$, and

$$\theta^*_t = \frac{\mu_{2,t} - \mu_{1,t} + \widetilde{\sigma}_{3,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}}. \tag{49}$$

The process $\widetilde{Y}^*_t$, $t \in [0, T]$, is a (local) martingale under $\mathbb{Q}^*$. We shall require that this process is in fact a true martingale; a sufficient condition for this is that

$$\int_0^T \mathbb{E}_{\mathbb{Q}^*} \left(\widetilde{Y}^{2,3}_t(\sigma_{2,t} - \sigma_{1,t}) \right)^2 dt < \infty.$$

From the predictable representation theorem, it follows that for any $X \in \mathcal{F}_T$, such that $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$, there exists a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$X = \widetilde{Y}^3_T \left(x + \int_0^T \phi^2_u\, d\widetilde{Y}^*_u \right). \tag{50}$$

We now deduce from Proposition 3.2 that there exists a self-financing strategy ϕ with the pre-default wealth $\widetilde{V}_t(\phi) = \widetilde{Y}^3_t \widetilde{V}^3_t$ for every $t \in [0,T]$, where we set

$$\widetilde{V}^3_t = x + \int_0^t \phi^2_u \, d\widetilde{Y}^*_u. \tag{51}$$

Moreover, it satisfies the balance condition $\phi^1_t Y^1_t + \phi^2_t Y^2_t = 0$ for every $t \in [0,T]$. Since clearly $\widetilde{V}_T(\phi) = X$, we have that

$$V_T(\phi) = \phi^3_T Y^3_T = \mathbb{1}_{\{T<\tau\}} \phi^3_T \widetilde{Y}^3_T = \mathbb{1}_{\{T<\tau\}} \widetilde{V}_T(\phi) = \mathbb{1}_{\{T<\tau\}} X,$$

and thus this strategy replicates the survival claim $(X,0,\tau)$. In fact, we have that $V_t(\phi) = 0$ on the random interval $[\![\tau, T]\!]$.

Definition 4.1 We say that a survival claim $(X,0,\tau)$ is *attainable* if the process $\widetilde{V}^3$ given by (51) is a martingale under $\mathbb{Q}^*$.

The following result is an immediate consequence of (50) and (51).

Corollary 4.1 *Let $X \in \mathcal{F}_T$ be such that $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$. Then the survival claim $(X,0,\tau)$ is attainable. Moreover, the pre-default price $\widetilde{\pi}_t(X,0,\tau)$ of the claim $(X,0,\tau)$ is given by the conditional expectation*

$$\widetilde{\pi}_t(X,0,\tau) = \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t), \quad \forall\, t \in [0,T]. \tag{52}$$

The process $\widetilde{\pi}(X,0,\tau)(\widetilde{Y}^3)^{-1}$ is an $\mathbb{F}$-martingale under $\mathbb{Q}^*$.

Proof. Since $X(\widetilde{Y}^3_T)^{-1}$ is square-integrable under $\mathbb{Q}^*$, we know from the predictable representation theorem that ϕ^2 in (50) is such that $\mathbb{E}_{\mathbb{Q}^*}\left(\int_0^T (\phi^2_t)^2 \, d\langle \widetilde{Y}^* \rangle_t\right) < \infty$, so that the process $\widetilde{V}^3$ given by (51) is a true martingale under $\mathbb{Q}^*$. We conclude that $(X,0,\tau)$ is attainable.

Now, let us denote by $\pi_t(X,0,\tau)$ the time-t price of the claim $(X,0,\tau)$. Since ϕ is a hedging portfolio for $(X,0,\tau)$ we thus have $V_t(\phi) = \pi_t(X,0,\tau)$ for each $t \in [0,T]$. Consequently,

$$\mathbb{1}_{\{\tau>t\}} \widetilde{\pi}_t(X,0,\tau) = \mathbb{1}_{\{\tau>t\}} \widetilde{V}_t(\phi) = \mathbb{1}_{\{\tau>t\}} \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(\widetilde{V}^3_T \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}} \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}(X(\widetilde{Y}^3_T)^{-1} \,|\, \mathcal{F}_t)$$

for each $t \in [0,T]$. This proves equality (52). □

In view of the last result, it is justified to refer to $\mathbb{Q}^*$ as the *pricing measure relative to* Y^3 for attainable survival claims.

Remark 4.3 It can be proved that there exists a unique absolutely continuous probability measure $\bar{\mathbb{Q}}$ on $(\Omega, \mathcal{G}_T)$ such that we have

$$Y^3_t \, \mathbb{E}_{\bar{\mathbb{Q}}}\left(\frac{\mathbb{1}_{\{\tau>T\}} X}{Y^3_T} \,\Big|\, \mathcal{G}_t \right) = \mathbb{1}_{\{\tau>t\}} \widetilde{Y}^3_t \, \mathbb{E}_{\mathbb{Q}^*}\left(\frac{X}{\widetilde{Y}^3_T} \,\Big|\, \mathcal{F}_t \right).$$

However, this probability measure is not equivalent to $\mathbb{Q}^*$, since its Radon-Nikodým density vanishes after τ (for a related result, see Collin-Dufresne et al. [12]).

Example 4.2 We provide here an explicit calculation of the pre-default price of a survival claim. For simplicity, we assume that $X = 1$, so that the claim represents a defaultable zero-coupon bond. Also, we set $\gamma_t = \gamma = const$, $\mu_{i,t} = 0$, and $\sigma_{i,t} = \sigma_i$, $i = 1,2,3$. Straightforward calculations yield the following pricing formula

$$\widetilde{\pi}_0(1,0,\tau) = Y^3_0 e^{-(\gamma + \frac{1}{2}\sigma^2_3)T}.$$

We see that here the pre-default price $\widetilde{\pi}_0(1,0,\tau)$ depends explicitly on the intensity γ, or rather, on the drift term in dynamics of pre-default value of defaultable asset. Indeed, from the practical viewpoint, the interpretation of the drift coefficient in dynamics of Y^2 as the real-world default intensity is questionable, since within our set-up (and in practice) the default intensity never appears as an independent variable, but is merely a component of the drift term in dynamics of pre-default value of Y^3.

Note also that we deal here with a model with three tradeable assets driven by a one-dimensional Brownian motion. No wonder that the model enjoys completeness, but as a downside, it has an undesirable property that the pre-default values of all three assets are perfectly correlated. Consequently, the drift terms in dynamics of traded assets are closely linked to each other, in the sense, that their behavior under an equivalent change of a probability measure is quite specific.

As we shall see later, if traded primary assets are judiciously chosen then, typically, the pre-default price (and hence the price) of a survival claim will not explicitly depend on the intensity process.

Remark 4.4 Generally speaking, we believe that one can classify a financial model as 'realistic' if its implementation does not require estimation of drift parameters in (pre-default) prices, at least for the purpose of hedging and valuation of a sufficiently large class of (defaultable) contingent claims of interest. It is worth recalling that the drift coefficients are not assumed to be market observables. Since the default intensity can formally interpreted as a component of the drift term in dynamics of pre-default prices, in a realistic model there is no need to estimate this quantity. From this perspective, the model considered in Example 4.2 may serve as an example of an 'unrealistic' model, since its implementation requires the knowledge of the drift parameter in the dynamics of Y^3. We do not pretend here that it is always possible to hedge derivative assets without using the drift coefficients in dynamics of tradeable assets, but it seems to us that a good idea is to develop models in which this knowledge is not of primary importance.

Of course, a generic semimartingale model considered until now provides only a framework for a construction of realistic models for hedging of default risk. A choice of tradeable assets and specification of their dynamics should be examined on a case-by-case basis, rather than in a general semimartingale set-up. We shall address this important issue in the foregoing sections, in which we shall deal with particular examples of practically interesting defaultable claims.

4.1.4 Hedging a Recovery Process

Let us now briefly study the situation where the promised payoff equals zero, and the recovery payoff is paid at time τ and equals Z_τ for some $\mathbb{F}$-adapted process Z. Put another way, we consider a defaultable claim of the form $(0, Z, \tau)$. Once again, we make use of Propositions 3.1 and 3.2. In view of (34), we need to find a constant x and an $\mathbb{F}$-predictable process ϕ^2 such that

$$\psi_T := -\int_0^T \frac{Z_t}{Y_t^1}\, d\widetilde{Y}_t^{1,3} = x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^{*}. \tag{53}$$

Similarly as in Section 4.1.3 we conclude that, under suitable integrability conditions on ψ_T, there exists ϕ^2 such that $d\psi_t = \phi_t^2\, dY_t^*$, where $\psi_t = \mathbb{E}_{\mathbb{Q}^*}(\psi_T \,|\, \mathcal{F}_t)$. We now set

$$\widetilde{V}_t^3 = x + \int_0^t \phi_u^2\, dY_u^* + \int_0^T \frac{\widetilde{Z}_u^3}{\widetilde{Y}_u^{1,3}}\, d\widetilde{Y}_u^{1,3},$$

so that, in particular, $\widetilde{V}_T^3 = 0$. Then it is possible to find processes ϕ^1 and ϕ^3 such that the strategy ϕ is self-financing and it satisfies: $\widetilde{V}_t(\phi) = \widetilde{V}_t^3 \widetilde{Y}_t^3$ and $V_t(\phi) = Z_t + \phi_t^3 Y_t^3$ for every $t \in [0,T]$. It is thus clear that $V_\tau(\phi) = Z_\tau$ on the set $\{\tau \le T\}$ and $V_T(\phi) = 0$ on the set $\{\tau > T\}$.

4.1.5 Bond Market

For the sake of concreteness, we assume that $Y_t^1 = B(t,T)$ is the price of a default-free ZC-bond with maturity T, and $Y_t^3 = D(t,T)$ is the price of a defaultable ZC-bond with zero recovery, that is, an asset with the terminal payoff $Y_T^3 = \mathbb{1}_{\{T<\tau\}}$. We postulate that the dynamics under $\mathbb{P}$ of the default-free ZC-bond are

$$dB(t,T) = B(t,T)\big(\mu(t,T)\,dt + b(t,T)\,dW_t\big) \tag{54}$$

for some $\mathbb{F}$-predictable processes $\mu(t,T)$ and $b(t,T)$. We choose the process $Y_t^1 = B(t,T)$ as a numéraire. Since the prices of the other two assets are not given a priori, we may choose any probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$ to play the role of $\mathbb{Q}^1$.

In such a case, an e.m.m. $\mathbb{Q}^1$ is referred to as the *forward martingale measure* for the date T, and is denoted by $\mathbb{Q}_T$. Hence, the Radon-Nikodým density of $\mathbb{Q}_T$ with respect to $\mathbb{P}$ is given by (45) for some $\mathbb{F}$-predictable processes θ and ζ, and the process

$$W_t^T = W_t - \int_0^t \theta_u\,du, \quad \forall\, t \in [0,T],$$

is a Brownian motion under $\mathbb{Q}_T$. Under $\mathbb{Q}_T$ the default-free ZC-bond is governed by

$$dB(t,T) = B(t,T)\big(\widehat{\mu}(t,T)\,dt + b(t,T)\,dW_t^T\big)$$

where $\widehat{\mu}(t,T) = \mu(t,T) + \theta_t b(t,T)$. Let $\widehat{\Gamma}$ stand for the $\mathbb{F}$-hazard process of τ under $\mathbb{Q}_T$, so that $\widehat{\Gamma}_t = -\ln(1-\widehat{F}_t)$, where $\widehat{F}_t = \mathbb{Q}_T(\tau \le t \,|\, \mathcal{F}_t)$. Assume that the hypothesis (H) holds under $\mathbb{Q}_T$ so that, in particular, the process $\widehat{\Gamma}$ is increasing. We define the price process of a defaultable ZC-bond with zero recovery by the formula

$$D(t,T) := B(t,T)\mathbb{E}_{\mathbb{Q}_T}(\mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big),$$

where the second equality follows from Lemma 1.3. It is then clear that $Y_t^{3,1} = D(t,T)(B(t,T))^{-1}$ is a $\mathbb{Q}_T$-martingale, and the pre-default price $\widetilde{D}(t,T)$ equals

$$\widetilde{D}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big).$$

The next result examines the basic properties of the auxiliary process $\widehat{\Gamma}(t,T)$ given as, for every $t \in [0,T]$,

$$\widehat{\Gamma}(t,T) = \widetilde{Y}_t^{3,1} = \widetilde{D}(t,T)(B(t,T))^{-1} = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,\big|\, \mathcal{F}_t\big).$$

The quantity $\widehat{\Gamma}(t,T)$ can be interpreted as the conditional probability (under $\mathbb{Q}_T$) that default will not occur prior to the maturity date T, given that we observe $\mathcal{F}_t$ and we know that the default has not yet happened. We will be more interested, however, in its volatility process $\beta(t,T)$ as defined in the following result.

Lemma 4.1 *Assume that the $\mathbb{F}$-hazard process $\widehat{\Gamma}$ of τ under $\mathbb{Q}_T$ is continuous. Then the process $\widehat{\Gamma}(t,T)$, $t \in [0,T]$, is a continuous $\mathbb{F}$-submartingale and*

$$d\widehat{\Gamma}(t,T) = \widehat{\Gamma}(t,T)\big(d\widehat{\Gamma}_t + \beta(t,T)\,dW_t^T\big) \tag{55}$$

for some $\mathbb{F}$-predictable process $\beta(t,T)$. The process $\widehat{\Gamma}(t,T)$ is of finite variation if and only if the hazard process $\widehat{\Gamma}$ is deterministic. In this case, we have $\widehat{\Gamma}(t,T) = e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T}$.

Proof. We have

$$\widehat{\Gamma}(t,T) = \mathbb{E}_{\mathbb{Q}_T}\big(e^{\widehat{\Gamma}_t - \widehat{\Gamma}_T} \,|\, \mathcal{F}_t\big) = e^{\widehat{\Gamma}_t} L_t,$$

where we set $L_t = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T} \,|\, \mathcal{F}_t\big)$. Hence, $\widehat{\Gamma}(t,T)$ is equal to the product of a strictly positive, increasing, right-continuous, $\mathbb{F}$-adapted process $e^{\widehat{\Gamma}_t}$, and a strictly positive, continuous $\mathbb{F}$-martingale L. Furthermore, there exists an $\mathbb{F}$-predictable process $\widehat{\beta}(t,T)$ such that L satisfies

$$dL_t = L_t \widehat{\beta}(t,T)\, dW_t^T$$

with the initial condition $L_0 = \mathbb{E}_{\mathbb{Q}_T}\big(e^{-\widehat{\Gamma}_T}\big)$. Formula (55) now follows by an application of Itô's formula, by setting $\beta(t,T) = e^{-\widehat{\Gamma}_t}\widehat{\beta}(t,T)$. To complete the proof, it suffices to recall that a continuous martingale is never of finite variation, unless it is a constant process. □

Remark 4.5 It can be checked that $\beta(t,T)$ is also the volatility of the process

$$\Gamma(t,T) = \mathbb{E}_{\mathbb{P}}\big(e^{\Gamma_t - \Gamma_T} \,\big|\, \mathcal{F}_t\big).$$

Assume that $\widehat{\Gamma}_t = \int_0^t \widehat{\gamma}_u \, du$ for some $\mathbb{F}$-predictable, nonnegative process $\widehat{\gamma}$. Then we have the following auxiliary result, which gives, in particular, the volatility of the defaultable ZC-bond.

Corollary 4.2 *The dynamics under $\mathbb{Q}_T$ of the pre-default price $\widetilde{D}(t,T)$ equals*

$$d\widetilde{D}(t,T) = \widetilde{D}(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T) + \widehat{\gamma}_t\big)\, dt + \big(b(t,T) + \beta(t,T)\big)\widetilde{d}(t,T)\, dW_t^T\Big).$$

Equivalently, the price $D(t,T)$ of the defaultable ZC-bond satisfies under $\mathbb{Q}_T$

$$dD(t,T) = D(t,T)\Big(\big(\widehat{\mu}(t,T) + b(t,T)\beta(t,T)\big)\, dt + \widetilde{d}(t,T)\, dW_t^T - dM_t\Big).$$

where we set $\widetilde{d}(t,T) = b(t,T) + \beta(t,T)$.

Note that the process $\beta(t,T)$ can be expressed in terms of market observables, since it is simply the difference of volatilities $\widetilde{d}(t,T)$ and $b(t,T)$ of pre-default prices of tradeable assets.

4.1.6 Credit-Risk-Adjusted Forward Price

Assume that the price Y^2 satisfies under the statistical probability $\mathbb{P}$

$$dY_t^2 = Y_t^2\big(\mu_{2,t}\, dt + \sigma_t\, dW_t\big) \tag{56}$$

with $\mathbb{F}$-predictable coefficients μ and σ. Let $F_{Y^2}(t,T) = Y_t^2(B(t,T))^{-1}$ be the forward price of Y_T^2. For an appropriate choice of θ (see 49), we shall have that

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\, dW_t^T.$$

Therefore, the dynamics of the pre-default synthetic asset $\widetilde{Y}_t^*$ under $\mathbb{Q}^T$ are

$$d\widetilde{Y}_t^* = \widetilde{Y}_t^{2,3}\big(\sigma_t - b(t,T)\big)\, \big(dW_t^T - \beta(t,T)\, dt\big),$$

and the process $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ satisfies

$$d\widehat{Y}_t = \widehat{Y}_t\big(\sigma_t - b(t,T)\big)\, \big(dW_t^T - \beta(t,T)\, dt\big).$$

Let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that $\widehat{Y}$ (or, equivalently, $\widetilde{Y}^*$) is a $\widehat{\mathbb{Q}}$-martingale. By virtue of Girsanov's theorem, the process $\widehat{W}$ given by the formula

$$\widehat{W}_t = W_t^T - \int_0^t \beta(u,T)\, du, \quad \forall\, t \in [0,T],$$

is a Brownian motion under $\widehat{\mathbb{Q}}$. Thus, the forward price $F_{Y^2}(t,T)$ satisfies under $\widehat{\mathbb{Q}}$

$$dF_{Y^2}(t,T) = F_{Y^2}(t,T)\big(\sigma_t - b(t,T)\big)\big(d\widehat{W}_t + \beta(t,T)\,dt\big). \tag{57}$$

It appears that the valuation results are easier to interpret when they are expressed in terms of forward prices associated with vulnerable forward contracts, rather than in terms of spot prices of primary assets. For this reason, we shall now examine credit-risk-adjusted forward prices of default-free and defaultable assets.

Definition 4.2 Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the *credit-risk-adjusted forward price* of Y if the pre-default value at time t of the vulnerable forward contract represented by the claim $\mathbb{1}_{\{T<\tau\}}(Y-K)$ equals 0.

Lemma 4.2 *The credit-risk-adjusted forward price $\widehat{F}_Y(t,T)$ of an attainable survival claim $(X,0,\tau)$, represented by a $\mathcal{G}_T$-measurable claim $Y = X\mathbb{1}_{\{T<\tau\}}$, equals $\widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$, where $\widetilde{\pi}_t(X,0,\tau)$ is the pre-default price of $(X,0,\tau)$. The process $\widehat{F}_Y(t,T)$, $t\in[0,T]$, is an $\mathbb{F}$-martingale under $\widehat{\mathbb{Q}}$.*

Proof. The forward price is defined as an $\mathcal{F}_t$-measurable random variable K such that the claim

$$\mathbb{1}_{\{T<\tau\}}(X\mathbb{1}_{\{T<\tau\}} - K) = X\mathbb{1}_{\{T<\tau\}} - KD(T,T)$$

is worthless at time t on the set $\{t<\tau\}$. It is clear that the pre-default value at time t of this claim equals $\widetilde{\pi}_t(X,0,\tau) - K\widetilde{D}(t,T)$. Consequently, we obtain $\widetilde{F}_Y(t,T) = \widetilde{\pi}_t(X,0,\tau)(\widetilde{D}(t,T))^{-1}$. □

Let us now focus on default-free assets. Manifestly, the credit-risk-adjusted forward price of the bond $B(t,T)$ equals 1. To find the credit-risk-adjusted forward price of Y^2, let us write

$$\widehat{F}_{Y^2}(t,T) := F_{Y^2}(t,T)\,e^{\alpha_T-\alpha_t} = Y_t^{2,1}e^{\alpha_T-\alpha_t},$$

where α is given by (see (38))

$$\alpha_t = \int_0^t \big(\sigma_u - b(u,T)\big)\beta(u,T)\,du = \int_0^t \big(\sigma_u - b(u,T)\big)\big(\widetilde{d}(u,T) - b(u,T)\big)\,du. \tag{58}$$

Lemma 4.3 *Assume that α given by (58) is a deterministic function. Then the credit-risk-adjusted forward price of Y^2 equals $\widehat{F}_{Y^2}(t,T)$ for every $t\in[0,T]$.*

Proof. According to Definition 4.2, the price $\widehat{F}_{Y^2}(t,T)$ is an $\mathcal{F}_t$-measurable random variable K, which makes the forward contract represented by the claim $D(T,T)(Y_T^2-K)$ worthless on the set $\{t<\tau\}$. Assume that the claim Y_T^2-K is attainable.[1] Since $\widetilde{D}(T,T)=1$, from equation (52) it follows that the pre-default value of this claim is given by the conditional expectation

$$\widetilde{D}(t,T)\,\mathbb{E}_{\mathbb{Q}^*}\big(Y_T^2 - K \,\big|\, \mathcal{F}_t\big).$$

Consequently,

$$\widehat{F}_{Y^2}(t,T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2 \,\big|\, \mathcal{F}_t\big) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(F_{Y^2}(T,T) \,\big|\, \mathcal{F}_t\big) = F_{Y^2}(t,T)\,e^{\alpha_T-\alpha_t},$$

as was claimed. □

It is worth noting that the process $\widehat{F}_{Y^2}(t,T)$ is a (local) martingale under the pricing measure $\widehat{\mathbb{Q}}$, since it satisfies

$$d\widehat{F}_{Y^2}(t,T) = \widehat{F}_{Y^2}(t,T)(\sigma_t - b(t,T))\,d\widehat{W}_t. \tag{59}$$

Under the present assumptions, the auxiliary process $\widehat{Y}$ introduced in Proposition 3.3 and the credit-risk-adjusted forward price $\widehat{F}_{Y^2}(t,T)$ are closely related to each other. Indeed, we have $\widehat{F}_{Y^2}(t,T) = \widehat{Y}_t e^{\alpha_T}$, so that thus the two processes are proportional.

[1] Attainability of this claim can be shown in a similar way as the attainability of a vulnerable call option considered in Section 4.1.7.

4.1.7 Vulnerable Option on a Default-Free Asset

We shall now analyze a vulnerable call option with the payoff

$$C^d_T = \mathbb{1}_{\{T<\tau\}}(Y^2_T - K)^+.$$

Our goal is to find a replicating strategy for this claim, interpreted as a survival claim $(X, 0, \tau)$ with the promised payoff $X = C_T = (Y^2_T - K)^+$, where C_T is the payoff of an equivalent non-vulnerable option. The method presented below is quite general, however, so that it can be applied to any survival claim with the promised payoff $X = G(Y^2_T)$ for some function $G : \mathbb{R} \to \mathbb{R}$ satisfying the usual integrability assumptions.

We assume that $Y^1_t = B(t,T)$, $Y^3_t = D(t,T)$ and the price of a default-free asset Y^2 is governed by (56). Then

$$C^d_T = \mathbb{1}_{\{T<\tau\}}(Y^2_T - K)^+ = \mathbb{1}_{\{T<\tau\}}(Y^2_T - KY^1_T)^+.$$

We are going to apply Proposition 3.3. In the present set-up, we have $Y^{2,1}_t = F_{Y^2}(t,T)$ and $\widehat{Y}_t = F_{Y^2}(t,T)e^{-\alpha_t}$. Since a vulnerable option is an example of a survival claim, in view of Lemma 4.2, its credit-risk-adjusted forward price satisfies $\widehat{F}_{C^d}(t,T) = \widetilde{C}^d_t(\widetilde{D}(t,T))^{-1}$.

Proposition 4.2 *Suppose that the volatilities σ, b and β are deterministic functions. Then the credit-risk-adjusted forward price of a vulnerable call option written on a default-free asset Y^2 equals*

$$\widehat{F}_{C^d}(t,T) = \widehat{F}_{Y^2}(t,T)N(d_+(\widehat{F}_{Y^2}(t,T),t,T)) - KN(d_-(\widehat{F}_{Y^2}(t,T),t,T)) \tag{60}$$

where

$$d_\pm(\widehat{f},t,T) = \frac{\ln \widehat{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_u - b(u,T))^2\,du.$$

The replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi^1_t B(t,T) = -\phi^2_t Y^2_t,\ \ \phi^2_t = \widetilde{D}(t,T)(B(t,T))^{-1}N(d_+(t,T))e^{\alpha_T - \alpha_t},\ \ \phi^3_t \widetilde{D}(t,T) = \widetilde{C}^d_t,$$

where $d_+(t,T) = d_+(\widehat{F}_{Y^2}(t,T),t,T)$.

Proof. In the first step, we compute the valuation formula. Assume for the moment that the option is attainable. Then the pre-default value of the option equals, for every $t \in [0,T]$,

$$\widetilde{C}^d_t = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((F_{Y^2}(T,T) - K)^+ \,\big|\, \mathcal{F}_t\big) = \widetilde{D}(t,T)\,\mathbb{E}_{\widehat{\mathbb{Q}}}\big((\widehat{F}_{Y^2}(T,T) - K)^+ \,\big|\, \mathcal{F}_t\big). \tag{61}$$

In view of (59), the conditional expectation above can be computed explicitly, yielding the valuation formula (60).

To find the replicating strategy, and establish attainability of the option, we consider the Itô differential $d\widehat{F}_{C^d}(t,T)$ and we identify terms in (51). It appears that

$$d\widehat{F}_{C^d}(t,T) = N(d_+(t,T))\,d\widehat{F}_{Y^2}(t,T) = N(d_+(t,T))e^{\alpha_T}\,d\widehat{Y}_t = N(d_+(t,T))\widetilde{Y}^{3,1}_t e^{\alpha_T - \alpha_t}\,d\widetilde{Y}^*_t \tag{62}$$

so that the process ϕ^2 in (50) equals

$$\phi^2_t = \widetilde{Y}^{3,1}_t N(d_+(t,T))e^{\alpha_T - \alpha_t}.$$

Moreover, ϕ^1 is such that $\phi^1_t B(t,T) + \phi^2_t Y^2_t = 0$ and $\phi^3_t = \widetilde{C}^d_t(\widetilde{D}(t,T))^{-1}$. It is easily seen that this proves also the attainability of the option. □

Let us examine the financial interpretation of the last result.

First, equality (62) shows that it is easy to replicate the option using vulnerable forward contracts. Indeed, we have

$$\widehat{F}_{C^d}(T,T) = X = \frac{\widetilde{C}_0^d}{\widetilde{D}(0,T)} + \int_0^T N(d_+(t,T))\, d\widehat{F}_{Y^2}(t,T)$$

and thus it is enough to invest the premium $\widetilde{C}_0^d = C_0^d$ in defaultable ZC-bonds of maturity T, and take at any instant t prior to default $N(d_+(t,T))$ positions in vulnerable forward contracts. It is understood that if default occurs prior to T all outstanding vulnerable forward contracts become void.

Second, it is worth stressing that neither the arbitrage price, nor the replicating strategy for a vulnerable option, depend explicitly on the default intensity. This remarkable feature is due to the fact that the default risk of the writer of the option can be completely eliminated by trading in defaultable zero-coupon bond with the same exposure to credit risk as a vulnerable option.

In fact, since the volatility β is invariant with respect to an equivalent change of a probability measure, and so are the volatilities σ and $b(t,T)$, the formulae of Proposition 4.2 are valid for any choice of a forward measure $\mathbb{Q}_T$ equivalent to $\mathbb{P}$ (and, of course, they are valid under $\mathbb{P}$ as well). The only way in which the choice of a forward measure $\mathbb{Q}_T$ impacts these results is through the pre-default value of a defaultable ZC-bond.

We conclude that we deal here with the volatility based relative pricing a defaultable claim. This should be contrasted with more popular intensity-based risk-neutral pricing, which is commonly used to produce an arbitrage-free model of tradeable defaultable assets. Recall, however, that if tradeable assets are not chosen carefully for a given class of survival claims, then both hedging strategy and pre-default price may depend explicitly on values of drift parameters, which can be linked in our set-up to the default intensity (see Example 4.2).

Remark 4.6 Assume that $X = G(Y_T^2)$ for some function $G : \mathbb{R} \to \mathbb{R}$. Then the credit-risk-adjusted forward price of a survival claim satisfies $\widehat{F}_X(t,T) = v(t, \widehat{F}_{Y^2}(t,T))$, where the pricing function v solves the PDE

$$\partial_t v(t,\widehat{f}) + \frac{1}{2}(\sigma_t - b(t,T))^2 \widehat{f}^2 \partial_{\widehat{f}\widehat{f}} v(t,\widehat{f}) = 0$$

with the terminal condition $v(T,\widehat{f}) = G(\widehat{f})$. The PDE approach is studied in Section 5 below.

Remark 4.7 Proposition 4.2 is still valid if the driving Brownian motion is two-dimensional, rather than one-dimensional. In an extended model, the volatilities $\sigma_t, b(t,T)$ and $\beta(t,T)$ take values in $\mathbb{R}^2$ and the respective products are interpreted as inner products in $\mathbb{R}^3$. Equivalently, one may prefer to deal with real-valued volatilities, but with correlated one-dimensional Brownian motions.

4.1.8 Vulnerable Swaption

In this section, we relax the assumption that Y^1 is the price of a default-free bond. We now let Y^1 and Y^2 to be arbitrary default-free assets, with dynamics

$$dY_t^i = Y_t^i\big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t\big), \quad i = 1,2.$$

We still take $D(t,T)$ to be the third asset, and we maintain the assumption that the model is arbitrage-free, but we no longer postulate its completeness. In other words, we postulate the existence an e.m.m. $\mathbb{Q}^1$, as defined in Section 4.1.2, but not the uniqueness of $\mathbb{Q}^1$.

We take the first asset as a numéraire, so that all prices are expressed in units of Y^1. In particular, $Y_t^{1,1} = 1$ for every $t \in \mathbb{R}_+$, and the relative prices $Y^{2,1}$ and $Y^{3,1}$ satisfy under $\mathbb{Q}^1$ (cf. Proposition 4.1)

$$dY_t^{2,1} = Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\, d\widehat{W}_t,$$
$$dY_t^{3,1} = Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\, d\widehat{W}_t - d\widehat{M}_t\big).$$

It is natural to postulate that the driving Brownian noise is two-dimensional. In such a case, we may represent the joint dynamics of $Y^{2,1}$ and $Y^{3,1}$ under $\mathbb{Q}^1$ as follows

$$\begin{aligned} dY_t^{2,1} &= Y_t^{2,1}(\sigma_{2,t} - \sigma_{1,t})\, dW_t^1, \\ dY_t^{3,1} &= Y_{t-}^{3,1}\big((\sigma_{3,t} - \sigma_{1,t})\, dW_t^2 - d\widehat{M}_t\big), \end{aligned}$$

where W^1, W^2 are one-dimensional Brownian motions under $\mathbb{Q}^1$, such that $d\langle W^1, W^2\rangle_t = \rho_t\, dt$ for a deterministic instantaneous correlation coefficient ρ taking values in $[-1, 1]$.

We assume from now on that the volatilities σ_i, $i = 1, 2, 3$ are determnistic. Let us set

$$\alpha_t = \langle \ln \widetilde{Y}^{2,1}, \ln \widetilde{Y}^{3,1}\rangle_t = \int_0^t \rho_u(\sigma_{2,u} - \sigma_{1,u})(\sigma_{3,u} - \sigma_{1,u})\, du, \tag{63}$$

and let $\widehat{\mathbb{Q}}$ be an equivalent probability measure on $(\Omega, \mathcal{G}_T)$ such that the process $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale. To clarify the financial interpretation of the auxiliary process $\widehat{Y}$ in the present context, we introduce the concept of credit-risk-adjusted forward price relative to the numéraire Y^1.

Definition 4.3 Let Y be a $\mathcal{G}_T$-measurable claim. An $\mathcal{F}_t$-measurable random variable K is called the time-t *credit-risk-adjusted* Y^1*-forward price* of Y if the pre-default value at time t of a vulnerable forward contract, represented by the claim

$$\mathbb{1}_{\{T<\tau\}}(Y_T^1)^{-1}(Y - KY_T^1) = \mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K),$$

equals 0.

The credit-risk-adjusted Y^1-forward price of Y is denoted by $\widehat{F}_{Y|Y^1}(t, T)$, and it is also interpreted as an abstract defaultable swap rate. The following auxiliary results are easy to establish, along the same lines as Lemmas 4.2 and 4.3.

Lemma 4.4 *The credit-risk-adjusted Y^1-forward price of a survival claim $Y = (X, 0, \tau)$ equals*

$$\widehat{F}_{Y|Y^1}(t, T) = \widetilde{\pi}_t(X^1, 0, \tau)(\widetilde{D}(t, T))^{-1}$$

where $X^1 = X(Y_T^1)^{-1}$ is the price of X in the numéraire Y^1, and $\widetilde{\pi}_t(X^1, 0, \tau)$ is the pre-default value of a survival claim with the promised payoff X^1.

Proof. It suffices to note that for $Y = \mathbb{1}_{\{T<\tau\}}X$, we have

$$\mathbb{1}_{\{T<\tau\}}(Y(Y_T^1)^{-1} - K) = \mathbb{1}_{\{T<\tau\}}X^1 - KD(T, T),$$

where $X^1 = X(Y_T^1)^{-1}$, and to consider the pre-default values. □

Lemma 4.5 *The credit-risk-adjusted Y^1-forward price of the asset Y^2 equals*

$$\widehat{F}_{Y^2|Y^1}(t, T) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t e^{\alpha_T}, \tag{64}$$

where α is given by (63).

Proof. It suffices to find an $\mathcal{F}_t$-measurable random variable K for which

$$\widetilde{D}(t, T)\, \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^2(Y_T^1)^{-1} - K \,\big|\, \mathcal{F}_t\big) = 0.$$

Consequently, $K = \widehat{F}_{Y^2|Y^1}(t, T)$, where

$$\widehat{F}_{Y^2|Y^1}(t, T) = \mathbb{E}_{\widehat{\mathbb{Q}}}\big(Y_T^{2,1} \,\big|\, \mathcal{F}_t\big) = Y_t^{2,1}\, e^{\alpha_T - \alpha_t} = \widehat{Y}_t\, e^{\alpha_T},$$

where we have used the facts that $\widehat{Y}_t = Y_t^{2,1} e^{-\alpha_t}$ is a $\widehat{\mathbb{Q}}$-martingale, and α is deterministic. □

We are in a position to examine a vulnerable option to exchange default-free assets with the payoff

$$C_T^d = \mathbb{1}_{\{T<\tau\}} (Y_T^1)^{-1} (Y_T^2 - K Y_T^1)^+ = \mathbb{1}_{\{T<\tau\}} (Y_T^{2,1} - K)^+. \tag{65}$$

The last expression shows that the option can be interpreted as a vulnerable swaption associated with the assets Y^1 and Y^2. It is useful to observe that

$$\frac{C_T^d}{Y_T^1} = \frac{\mathbb{1}_{\{T<\tau\}}}{Y_T^1} \left(\frac{Y_T^2}{Y_T^1} - K \right)^+,$$

so that, when expressed in the numéraire Y^1, the payoff becomes

$$C_T^{1,d} = D^1(T,T)(Y_T^{2,1} - K)^+,$$

where $C_t^{1,d} = C_t^d (Y_t^1)^{-1}$ and $D^1(t,T) = D(t,T)(Y_t^1)^{-1}$ stand for the prices relative to Y^1.

It is clear that we deal here with a model analogous to the model examined in Sections 4.1.5 and 4.1.7 in which, however, all prices are now relative to the numéraire Y^1. This observation allows us to directly derive the valuation formula from Proposition 4.2.

Proposition 4.3 *The credit-risk-adjusted Y^1-forward price of a vulnerable call option written with the payoff given by* (65) *equals*

$$\widehat{F}_{C^d|Y^1}(t,T) = \widehat{F}_{Y^2|Y^1}(t,T) N\big(d_+(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big) - K N\big(d_-(\widehat{F}_{Y^2|Y^1}(t,T),t,T)\big)$$

where

$$d_\pm(\widehat{f},t,T) = \frac{\ln \widehat{f} - \ln K \pm \frac{1}{2} v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{2,u} - \sigma_{1,u})^2 \, du.$$

The replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 Y_t^1 = -\phi_t^2 Y_t^2, \quad \phi_t^2 = \widetilde{D}(t,T)(Y_t^1)^{-1} N(d_+(t,T)) e^{\alpha_T - \alpha_t}, \quad \phi_t^3 \widetilde{D}(t,T) = \widetilde{C}_t^d,$$

where $d_+(t,T) = d_+\big(\widehat{F}_{Y^2}(t,T),t,T\big)$.

Proof. The proof is analogous to that of Proposition 4.2, and thus it is omitted. □

It is worth noting that the payoff (65) was judiciously chosen. Suppose instead that the option payoff is not defined by (65), but it is given by an apparently simpler expression

$$C_T^d = \mathbb{1}_{\{T<\tau\}} (Y_T^2 - K Y_T^1)^+. \tag{66}$$

Since the payoff C_T^d can be represented as follows

$$C_T^d = \widehat{G}(Y_T^1, Y_T^2, Y_T^3) = Y_T^3 (Y_T^2 - K Y_T^1)^+,$$

where $\widehat{G}(y_1,y_2,y_3) = y_3(y_2 - K y_1)^+$, the option can be seen an option to exchange the second asset for K units of the first asset, but with the payoff expressed in units of the defaultable asset. When expressed in relative prices, the payoff becomes

$$C_T^{1,d} = \mathbb{1}_{\{T<\tau\}} (Y_T^{2,1} - K)^+.$$

where $\mathbb{1}_{\{T<\tau\}} = D^1(T,T) Y_T^1$. It is thus rather clear that it is not longer possible to apply the same method as in the proof of Proposition 4.2.

4.2 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. This case is also examined by Carr [11], who studies some imperfect hedging of digital options. Note that here we present results for perfect hedging.

We shall briefly outline the analysis of hedging of a survival claim. Under the present assumptions, we have, for $i = 1, 2$,

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t - dM_t \big), \tag{67}$$

where W is a one-dimensional Brownian motion, so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}} \widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i \big((\mu_{i,t} + \gamma_t)\, dt + \sigma_{i,t}\, dW_t \big). \tag{68}$$

The wealth process V associated with the self-financing trading strategy (ϕ^1, ϕ^2) satisfies, for every $t \in [0, T]$,

$$V_t = Y_t^1 \left(V_0^1 + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1} \right),$$

where $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2 / \widetilde{Y}_t^1$. Since both primary traded assets are subject to total default, it is clear that the present model is incomplete, in the sense, that not all defaultable claims can be replicated. We shall check in Section 4.2.1 that, under the assumption that the driving Brownian motion W is one-dimensional, all survival claims satisfying natural technical conditions are hedgeable, however. In the more realistic case of a two-dimensional noise, we will still be able to hedge a large class of survival claims, including options on a defaultable asset (see Section 4.2.2) and options to exchange defultable assets (see Section 4.2.3).

4.2.1 Hedging a Survival Claim

For the sake of expositional simplicity, we assume in this section that the driving Brownian motion W is one-dimensional. This is definitely not the right choice, since we deal here with two risky assets, and thus they will be perfectly correlated. However, this assumption is convenient for the expositional purposes, since it will ensure the model completeness with respect to survival claims, and it will be later relaxed anyway.

We shall argue that in a model with two defaultable assets governed by (67), replication of a survival claim $(X, 0, \tau)$ is in fact equivalent to replication of the promised payoff X using the pre-default processes.

Lemma 4.6 *If a strategy ϕ^i, $i = 1, 2$, based on pre-default values $\widetilde{Y}^i$, $i = 1, 2$, is a replicating strategy for an $\mathcal{F}_T$-measurable claim X, that is, if ϕ is such that the process $\widetilde{V}_t(\phi) = \phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2$ satisfies, for every $t \in [0, T]$,*

$$\begin{aligned} d\widetilde{V}_t(\phi) &= \phi_t^1\, d\widetilde{Y}_t^1 + \phi_t^2\, d\widetilde{Y}_t^2, \\ \widetilde{V}_T(\phi) &= X, \end{aligned}$$

then for the process $V_t(\phi) = \phi_t^1 Y_t^1 + \phi_t^2 Y_t^2$ we have, for every $t \in [0, T]$,

$$\begin{aligned} dV_t(\phi) &= \phi_t^1\, dY_t^1 + \phi_t^2\, dY_t^2, \\ V_T(\phi) &= X \mathbb{1}_{\{T<\tau\}}. \end{aligned}$$

This means that a strategy ϕ replicates a survival claim $(X, 0, \tau)$.

Proof. It is clear that $V_t(\phi) = \mathbb{1}_{\{t<\tau\}} V_t(\phi) = \mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)$. From

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = -(\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2) \, dH_t + (1 - H_{t-})(\phi_t^1 \, d\widetilde{Y}_t^1 + \phi_t^2 \, d\widetilde{Y}_t^2),$$

it follows that

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = -\widetilde{V}_t(\phi) \, dH_t + (1 - H_{t-}) d\widetilde{V}_t(\phi),$$

that is,

$$\phi_t^1 \, dY_t^1 + \phi_t^2 \, dY_t^2 = d(\mathbb{1}_{\{t<\tau\}} \widetilde{V}_t(\phi)) = dV_t(\phi).$$

It is also obvious that $V_T(\phi) = X \mathbb{1}_{\{T<\tau\}}$. □

Combining the last result with Lemma 3.1, we see that a strategy (ϕ^1, ϕ^2) replicates a survival claim $(X, 0, \tau)$ whenever we have

$$\widetilde{Y}_T^1 \Big(x + \int_0^T \phi_t^2 \, d\widetilde{Y}_t^{2,1} \Big) = X$$

for some constant x and some $\mathbb{F}$-predictable process ϕ^2, where, in view of (68),

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1} \Big(\big(\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t})\big) \, dt + (\sigma_{2,t} - \sigma_{1,t}) \, dW_t \Big).$$

We introduce a probability measure $\widetilde{\mathbb{Q}}$, equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_T)$, and such that $\widetilde{Y}^{2,1}$ is an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$. It is easily seen that the Radon-Nikodým density η satisfies, for $t \in [0, T]$,

$$d\widetilde{\mathbb{Q}} \,|\, _{\mathcal{G}_t} = \eta_t \, d\mathbb{P} \,|\, _{\mathcal{G}_t} = \mathcal{E}_t \left(\int_0^{\cdot} \theta_s \, dW_s \right) d\mathbb{P} \,|\, _{\mathcal{G}_t} \tag{69}$$

with

$$\theta_t = \frac{\mu_{2,t} - \mu_{1,t} + \sigma_{1,t}(\sigma_{1,t} - \sigma_{2,t})}{\sigma_{1,t} - \sigma_{2,t}},$$

provided, of course, that the process θ is well defined and satisfies suitable integrability conditions. We shall show that a survival claim is attainable if the random variable $X(\widetilde{Y}_T^1)^{-1}$ is $\widetilde{\mathbb{Q}}$-integrable. Indeed, the pre-default value $\widetilde{V}_t$ at time t of a survival claim equals

$$\widetilde{V}_t = \widetilde{Y}_t^1 \, \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big),$$

and from the predictable representation theorem, we deduce that there exists a process ϕ^2 such that

$$\mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \,|\, \mathcal{F}_t \big) = \mathbb{E}_{\widetilde{\mathbb{Q}}} \big(X(\widetilde{Y}_T^1)^{-1} \big) + \int_0^t \phi_u^2 \, d\widetilde{Y}_u^{2,1}.$$

The component ϕ^1 of the self-financing trading strategy $\phi = (\phi^1, \phi^2)$ is then chosen in such a way that

$$\phi_t^1 \widetilde{Y}_t^1 + \phi_t^2 \widetilde{Y}_t^2 = \widetilde{V}_t, \quad \forall \, t \in [0, T].$$

To conclude, by focusing on pre-default values, we have shown that the replication of survival claims can be reduced here to classic results on replication of (non-defaultable) contingent claims in a default-free market model.

4.2.2 Option on a Defaultable Asset

In order to get a complete model with respect to survival claims, we postulated in the previous section that the driving Brownian motion in dynamics (67) is one-dimensional. This assumption is questionable, since it implies the perfect correlation of risky assets. However, we may relax this restriction, and work instead with the two correlated one-dimensional Brownian motions. The model will no longer be complete, but options on a defaultable assets will be still attainable. The payoff of a (non-vulnerable) call option written on the defaultable asset Y^2 equals

$$C_T = (Y_T^2 - K)^+ = \mathbb{1}_{\{T<\tau\}}(\widetilde{Y}_T^2 - K)^+,$$

so that it is natural to interpret this contract as a survival claim with the promised payoff $X = (\widetilde{Y}_T^2 - K)^+$.

To deal with this option in an efficient way, we consider a model in which

$$dY_t^i = Y_{t-}^i \big(\mu_{i,t}\, dt + \sigma_{i,t}\, dW_t^i - dM_t\big), \tag{70}$$

where W^1 and W^2 are two one-dimensional correlated Brownian motions with the instantaneous correlation coefficient ρ_t. More specifically, we assume that $Y_t^1 = D(t,T) = \mathbb{1}_{\{t<\tau\}}\widetilde{D}(t,T)$ represents a defaultable ZC-bond with zero recovery, and $Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2$ is a generic defaultable asset with total default. Within the present set-up, the payoff can also be represented as follows

$$C_T = G(Y_T^1, Y_T^2) = (Y_T^2 - KY_T^1)^+,$$

where $g(y_1, y_2) = (y_2 - Ky_1)^+$, and thus it can also be seen as an option to exchange the second asset for K units of the first asset.

The requirement that the process $\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^2(\widetilde{Y}_t^1)^{-1}$ follows an $\mathbb{F}$-martingale under $\widetilde{\mathbb{Q}}$ implies that

$$d\widetilde{Y}_t^{2,1} = \widetilde{Y}_t^{2,1}\big((\sigma_{2,t}\rho_t - \sigma_{1,t})\, d\widetilde{W}_t^1 + \sigma_{2,t}\sqrt{1-\rho_t^2}\, d\widetilde{W}_t^2\big), \tag{71}$$

where $\widetilde{W} = (\widetilde{W}^1, \widetilde{W}^2)$ follows a two-dimensional Brownian motion under $\widetilde{\mathbb{Q}}$. Since $\widetilde{Y}_T^1 = 1$, replication of the option reduces to finding a constant x and an $\mathbb{F}$-predictable process ϕ^2 satisfying

$$x + \int_0^T \phi_t^2\, d\widetilde{Y}_t^{2,1} = (\widetilde{Y}_T^2 - K)^+.$$

To obtain closed-form expressions for the option price and replicating strategy, we postulate that the volatilities $\sigma_{1,t}, \sigma_{2,t}$ and the correlation coefficient ρ_t are deterministic. Let $\widehat{F}_{Y^2}(t,T) = \widetilde{Y}_t^2(\widetilde{D}(t,T))^{-1}$ ($\widehat{F}_C(t,T) = \widetilde{C}_t(\widetilde{D}(t,T))^{-1}$, respectively) stand for the credit-risk-adjusted forward price of the second asset (the option, respectively). The proof of the following valuation result is fairly standard, and thus it is omitted.

Proposition 4.4 *The credit-risk-adjusted forward price of the option written on Y^2 equals*

$$\widehat{F}_C(t,T) = \widehat{F}_{Y^2}(t,T)N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

Equivalently, the pre-default price of the option equals

$$\widetilde{C}_t = \widetilde{Y}_t^2 N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big) - K\widetilde{D}(t,T)N\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big),$$

where

$$d_\pm(\widetilde{f},t,T) = \frac{\ln \widetilde{f} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T (\sigma_{1,u}^2 + \sigma_{2,u}^2 - 2\rho_u\sigma_{1,u}\sigma_{2,u})\, du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi_t^1 = -KN\big(d_-(\widehat{F}_{Y^2}(t,T),t,T)\big), \quad \phi_t^2 = N\big(d_+(\widehat{F}_{Y^2}(t,T),t,T)\big).$$

4.2.3 Option to Exchange Defaultable Assets

We work here with the two correlated one-dimensional Brownian motions, so that

$$dY^i_t = Y^i_{t-}\big(\mu_{i,t}\,dt + \sigma_{i,t}\,dW^i_t - dM_t\big), \quad i=1,2, \tag{72}$$

where $d\langle W^1, W^2\rangle_t = \rho_t\,dt$ for some function ρ with values in $[-1,1]$. The model is no longer complete, but it is still not difficult to establish a direct counterpart of Proposition 4.4 for the exchange option with the payoff $(Y^2_T - KY^1_T)^+$. In fact, the next result shows that the pricing formula expressed in terms of pre-default prices has the same shape as the standard formula for the option to exchange non-defaultable assets with dynamics (67). It is notable that we do not need to make any assumption about the behavior of the default intensity.

We only assume that the coefficients in (72) are such that there exist an e.m.m. for the process $\widetilde{Y}^{2,1}$, where

$$d\widetilde{Y}^i_t = \widetilde{Y}^i_t\big((\mu_{i,t}+\gamma_t)\,dt + \sigma_{i,t}\,dW^i_t\big), \quad i=1,2, \tag{73}$$

so that we implicitly impose mild technical conditions on drift coefficients.

Proposition 4.5 *Assume that the volatilities σ_1, σ_2 and the instantaneous correlation coefficient ρ are deterministic. Then the pre-default price of the exchange option equals*

$$\widetilde{C}_t = \widetilde{Y}^2_t N\big(d_+(\widetilde{Y}^{2,1}_t, t, T)\big) - K\widetilde{Y}^1_t N\big(d_-(\widetilde{Y}^{2,1}_t, t, T)\big),$$

where

$$d_\pm(\widetilde{y}, t, T) = \frac{\ln \widetilde{y} - \ln K \pm \frac{1}{2}v^2(t,T)}{v(t,T)}$$

and

$$v^2(t,T) = \int_t^T \big(\sigma^2_{1,u} + \sigma^2_{2,u} - 2\rho_u\sigma_{1,u}\sigma_{2,u}\big)\,du.$$

Moreover the replicating strategy ϕ in the spot market satisfies for every $t \in [0,T]$, on the set $\{t < \tau\}$,

$$\phi^1_t = -KN\big(d_-(\widetilde{Y}^{2,1}_t, t, T)\big), \quad \phi^2_t = N\big(d_+(\widetilde{Y}^{2,1}_t, t, T)\big).$$

The pricing formula for the option on a defaultable asset (see Proposition 4.4) can be seen as a special case of the formula established in Proposition 4.5.

Similarly as in Sections 4.1.7 and 4.1.8, we conclude that the pricing and hedging of any attainable survival claim with the promised payoff $X = g(\widetilde{Y}^1_T, \widetilde{Y}^2_T)$ depends on the choice of a default intensity only through the pre-default prices $\widetilde{Y}^1_t$ and $\widetilde{Y}^2_t$. This property shows that we have correctly specified the hedging instruments for a claim at hand. Of course, the model considered in this section is not complete, even if the concept of completeness is reduced to survival claims. Basically, a survival claim can be hedged if its promised payoff can be represents as $X = \widetilde{Y}^1_T h(Y^{2,1}_T)$.

5 PDE Approach to Valuation and Hedging

In the remaining part of the paper, we take a different perspective, and we assume that trading occurs on the time interval $[0,T]$ and our goal is to replicate a contingent claim of the form

$$Y = \mathbb{1}_{\{T\geq\tau\}}g_1(Y^1_T, Y^2_T, Y^3_T) + \mathbb{1}_{\{T<\tau\}}g_0(Y^1_T, Y^2_T, Y^3_T) = G(Y^1_T, Y^2_T, Y^3_T, H_T),$$

which settles at time T. We do not need to assume here that the coefficients in dynamics of primary assets are $\mathbb{F}$-predictable. Since our goal is to develop the PDE approach, it will be essential, however, to postulate a Markovian character of a model. For the sake of simplicity, we assume that the coefficients are constant, so that

$$dY^i_t = Y^i_{t-}\big(\mu_i\,dt + \sigma_i\,dW_t + \kappa_i\,dM_t\big), \quad i=1,2,3.$$

The assumption of constancy of coefficients is rarely, if ever, satisfied in practically relevant models of credit risk. It is thus important to note that it was postulated here mainly for the sake of notational convenience, and the general results established in this section can be easily extended to a non-homogeneous Markov case in which $\mu_{i,t} = \mu_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-}), \sigma_{i,t} = \sigma_i(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$, etc.

5.1 Defaultable Asset with Total Default

We first assume that Y^1 and Y^2 are default-free, so that $\kappa_1 = \kappa_2 = 0$, and the third asset is subject to total default, i.e. $\kappa_3 = -1$,

$$dY^3_t = Y^3_{t-}\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big).$$

We work throughout under the assumptions of Proposition 4.1. This means that any $\mathbb{Q}^1$-integrable contingent claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ is attainable, and its arbitrage price equals

$$\pi_t(Y) = Y^1_t\, \mathbb{E}_{\mathbb{Q}^1}(Y(Y^1_T)^{-1}\,|\,\mathcal{G}_t), \quad \forall\, t \in [0,T]. \tag{74}$$

The following auxiliary result is thus rather obvious.

Lemma 5.1 *The process (Y^1, Y^2, Y^3, H) has the Markov property with respect to the filtration $\mathbb{G}$ under the martingale measure $\mathbb{Q}^1$. For any attainable claim $Y = G(Y^1_T, Y^2_T, Y^3_T; H_T)$ there exists a function $v : [0,T] \times \mathbb{R}^3 \times \{0,1\} \to \mathbb{R}$ such that $\pi_t(Y) = v(t, Y^1_t, Y^2_t, Y^3_t; H_t)$.*

We find it convenient to introduce the *pre-default* pricing function $v(\cdot\,;0) = v(t, y_1, y_2, y_3; 0)$ and the *post-default* pricing function $v(\cdot\,;1) = v(t, y_1, y_2, y_3; 1)$. In fact, since $Y^3_t = 0$ if $H_t = 1$, it suffices to study the post-default function $v(t, y_1, y_2; 1) = v(t, y_1, y_2, 0; 1)$. Also, we write

$$\alpha_i = \mu_i - \sigma_i \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2}, \quad b = (\mu_3 - \mu_1)(\sigma_1 - \sigma_2) - (\mu_1 - \mu_3)(\sigma_1 - \sigma_3).$$

Let $\gamma > 0$ be the constant default intensity under $\mathbb{P}$, and let $\zeta > -1$ be given by formula (47).

Proposition 5.1 *Assume that the functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ belong to the class $\mathrm{C}^{1,2}([0,T] \times \mathbb{R}^3_+, \mathbb{R})$. Then $v(t, y_1, y_2, y_3; 0)$ satisfies the PDE*

$$\begin{aligned}\partial_t v(\cdot\,;0) + \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,;0) + (\alpha_3 + \zeta) y_3 \partial_3 v(\cdot\,;0) + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,;0) - \alpha_1 v(\cdot\,;0)\\ + \left(\gamma - \frac{b}{\sigma_1 - \sigma_2}\right)\big[v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0)\big] = 0\end{aligned}$$

subject to the terminal condition $v(T, y_1, y_2, y_3; 0) = G(y_1, y_2, y_3; 0)$, and $v(t, y_1, y_2; 1)$ satisfies the PDE

$$\partial_t v(\cdot\,;1) + \sum_{i=1}^{2} \alpha_i y_i \partial_i v(\cdot\,;1) + \frac{1}{2} \sum_{i,j=1}^{2} \sigma_i \sigma_j y_i y_j \partial_{ij} v(\cdot\,;1) - \alpha_1 v(\cdot\,;1) = 0$$

subject to the terminal condition $v(T, y_1, y_2; 1) = G(y_1, y_2, 0; 1)$.

Proof. For simplicity, we write $C_t = \pi_t(Y)$. Let us define

$$\Delta v(t, y_1, y_2, y_3) = v(t, y_1, y_2; 1) - v(t, y_1, y_2, y_3; 0).$$

Then the jump $\Delta C_t = C_t - C_{t-}$ can be represented as follows:

$$\Delta C_t = \mathbb{1}_{\{\tau = t\}}\big(v(t, Y^1_t, Y^2_t; 1) - v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0)\big) = \mathbb{1}_{\{\tau = t\}}\,\Delta v(t, Y^1_t, Y^2_t, Y^3_{t-}).$$

We write ∂_i to denote the partial derivative with respect to the variable y_i, and we typically omit the variables $(t, Y^1_{t-}, Y^2_{t-}, Y^3_{t-}, H_{t-})$ in expressions $\partial_t v$, $\partial_i v$, Δv, etc. We shall also make use of the fact that for a Borel function g

$$\int_0^t g(u, Y^2_u, Y^3_{u-})\, du = \int_0^t g(u, Y^2_u, Y^3_u)\, du$$

since Y^3_u and Y^3_{u-} differ only for at most one value of u (for each ω).

Let $\xi_t = \mathbb{1}_{\{t<\tau\}}\gamma$. An application of Itô's formula yields

$$\begin{aligned}
dC_t &= \partial_t v\, dt + \sum_{i=1}^3 \partial_i v\, dY^i_t + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\, dt + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\, dH_t \\
&= \partial_t v\, dt + \sum_{i=1}^3 \partial_i v\, dY^i_t + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\, dt + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\big(dM_t + \xi_t\, dt\big) \\
&= \partial_t v\, dt + \sum_{i=1}^3 Y^i_{t-}\partial_i v\big(\mu_i\, dt + \sigma_i\, dW_t\big) + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v\, dt \\
&\quad + \Delta v\, dM_t + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\, dt \\
&= \left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\} dt \\
&\qquad + \Big(\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\Big)\, dW_t + \Delta v\, dM_t.
\end{aligned}$$

We now use the integration by parts formula together with (41) to derive dynamics of the relative price $\widehat{C}_t = C_t(Y^1_t)^{-1}$. In view of (46), we find that

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\Big((-\mu_1 + \sigma_1^2)\, dt - \sigma_1\, dW_t\Big) \\
&\quad + (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\} dt \\
&\quad + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\, dW_t + (Y^1_{t-})^{-1}\Delta v\, dM_t - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\, dt \\
&= \widehat{C}_{t-}\big(-\mu_1 + \sigma_1^2\big)\, dt + \widehat{C}_{t-}\Big(-\sigma_1\, d\widehat{W}_t - \sigma_1\theta\, dt\Big) \\
&\quad + (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\} dt \\
&\quad + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\partial_i v\, d\widehat{W}_t + (Y^1_{t-})^{-1}\sum_{i=1}^3 \sigma_i Y^i_{t-}\theta\partial_i v\, dt \\
&\quad + (Y^1_{t-})^{-1}\Delta v\, d\widehat{M}_t + (Y^1_{t-})^{-1}\zeta\xi_t\Delta v\, dt - (Y^1_{t-})^{-1}\sigma_1\sum_{i=1}^3 \sigma^i Y^i_{t-}\partial_i v\, dt.
\end{aligned}$$

This yields the following decomposition for the process $\widehat{C}$

$$\begin{aligned}
d\widehat{C}_t &= \widehat{C}_{t-}\big(-\mu_1 + \sigma_1^2 - \sigma_1\theta\big)\, dt \\
&\quad + (Y^1_{t-})^{-1}\left\{\partial_t v + \sum_{i=1}^3 \mu_i Y^i_{t-}\partial_i v + \frac{1}{2}\sum_{i,j=1}^3 \sigma_i\sigma_j Y^i_{t-}Y^j_{t-}\partial_{ij}v + \Big(\Delta v + Y^3_{t-}\partial_3 v\Big)\xi_t\right\} dt
\end{aligned}$$

$$+ (Y^1_{t-})^{-1} \sum_{i=1}^{3} \sigma_i Y^i_{t-} \theta \partial_i v \, dt + (Y^1_{t-})^{-1} \zeta \xi_t \Delta v \, dt$$
$$- (Y^1_{t-})^{-1} \sigma_1 \sum_{i=1}^{3} \sigma_i Y^i_{t-} \partial_i v \, dt + \text{a } \mathbb{Q}^1\text{-martingale.}$$

From (74), it follows that the process $\widehat{C}$ is a martingale under $\mathbb{Q}^1$. Therefore, the continuous finite variation part in the above decomposition necessarily vanishes, and thus we get

$$\begin{aligned} 0 &= C_{t-}(Y^1_{t-})^{-1}\big(- \mu_1 + \sigma_1^2 - \sigma_1 \theta\big) \\ &\quad + (Y^1_{t-})^{-1} \left\{ \partial_t v + \sum_{i=1}^{3} \mu_i Y^i_{t-} \partial_i v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} v + \big(\Delta v + Y^3_{t-} \partial_3 v\big) \xi_t \right\} \\ &\quad + (Y^1_{t-})^{-1} \sum_{i=1}^{3} \sigma_i Y^i_{t-} \theta \partial_i v + (Y^1_{t-})^{-1} \zeta \xi_t \Delta v - (Y^1_{t-})^{-1} \sigma_1 \sum_{i=1}^{3} \sigma_i Y^i_{t-} \partial_i v. \end{aligned}$$

Consequently, we have that

$$\begin{aligned} 0 &= C_{t-}\big(- \mu_1 + \sigma_1^2 - \sigma_1 \theta\big) \\ &\quad + \partial_t v + \sum_{i=1}^{3} \mu_i Y^i_{t-} \partial_i v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} v + \big(\Delta v + Y^3_{t-} \partial_3 v\big) \xi_t \\ &\quad + \sum_{i=1}^{3} \sigma_i Y^i_{t-} \theta \partial_i v + \zeta \xi_t \Delta v - \sigma_1 \sum_{i=1}^{3} \sigma_i Y^i_{t-} \partial_i v. \end{aligned}$$

Finally, we obtain

$$\partial_t v + \sum_{i=1}^{2} \alpha_i Y^i_{t-} \partial_i v + (\alpha_3 + \xi_t)\, Y^3_{t-} \partial_3 v + \frac{1}{2} \sum_{i,j=1}^{3} \sigma_i \sigma_j Y^i_{t-} Y^j_{t-} \partial_{ij} v - \alpha_1 C_{t-} + (1+\zeta) \xi_t \Delta v = 0.$$

Recall that $\xi_t = 1\!\!1_{\{t<\tau\}} \gamma$. It is thus clear that the pricing functions $v(\cdot, 0)$ and $v(\cdot; 1)$ satisfy the PDEs given in the statement of the proposition. □

The next result deals with a replicating strategy for Y.

Proposition 5.2 *The replicating strategy ϕ for the claim Y is given by formulae*

$$\begin{aligned} \phi^3_t Y^3_{t-} &= -\Delta v(t, Y^1_t, Y^2_t, Y^3_{t-}) = v(t, Y^1_t, Y^2_t, Y^3_{t-}; 0) - v(t, Y^1_t, Y^2_t; 1), \\ \phi^2_t Y^2_t (\sigma_2 - \sigma_1) &= -(\sigma_1 - \sigma_3)\Delta v - \sigma_1 v + \sum_{i=1}^{3} Y^i_{t-} \sigma_i \partial_i v, \\ \phi^1_t Y^1_t &= v - \phi^2_t Y^2_t - \phi^3_t Y^3_t. \end{aligned}$$

Proof. As a by-product of our computations, we obtain

$$d\widehat{C}_t = -(Y^1_t)^{-1} \sigma_1 v \, d\widehat{W}_t + (Y^1_t)^{-1} \sum_{i=1}^{3} \sigma_i Y^i_{t-} \partial_i v \, d\widehat{W}_t + (Y^1_t)^{-1} \Delta v \, d\widehat{M}_t.$$

The self-financing strategy that replicates Y is determined by two components ϕ^2, ϕ^3 and the following relationship:

$$d\widehat{C}_t = \phi^2_t \, dY^{2,1}_t + \phi^3_t \, dY^{3,1}_t = \phi^2_t Y^{2,1}_t (\sigma_2 - \sigma_1) \, d\widehat{W}_t + \phi^3_t Y^{3,1}_{t-} \Big((\sigma_3 - \sigma_1) \, d\widehat{W}_t - d\widehat{M}_t \Big).$$

By identification, we obtain $\phi_t^3 Y_{t-}^{3,1} = (Y_t^1)^{-1}\Delta v$ and

$$\phi_t^2 Y_t^2(\sigma_2 - \sigma_1) - (\sigma_3 - \sigma_1)\Delta v = -\sigma_1 C_t + \sum_{i=1}^{3} Y_{t-}^i \sigma_i \partial_i v.$$

This yields the claimed formulae. □

Corollary 5.1 *In the case of a total default claim, the hedging strategy satisfies the balance condition.*

Proof. A total default corresponds to the assumption that $G(y_1, y_2, y_3, 1) = 0$. We now have $v(t, y_1, y_2; 1) = 0$, and thus $\phi_t^3 Y_{t-}^3 = v(t, Y_t^1, Y_t^2, Y_{t-}^3; 0)$ for every $t \in [0, T]$. Hence, the equality $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0$ holds for every $t \in [0, T]$. The last equality is the balance condition for $Z = 0$. Recall that it ensures that the wealth of a replicating portfolio jumps to zero at default time. □

5.1.1 Hedging with the Savings Account

Let us now study the particular case where Y^1 is the savings account, i.e.,

$$dY_t^1 = rY_t^1\,dt, \quad Y_0^1 = 1,$$

which corresponds to $\mu_1 = r$ and $\sigma_1 = 0$. Let us write $\widehat{r} = r + \widehat{\gamma}$, where

$$\widehat{\gamma} = \gamma(1 + \zeta) = \gamma + \mu_3 - r + \frac{\sigma_3}{\sigma_2}(r - \mu_2)$$

stands for the intensity of default under $\mathbb{Q}^1$. The quantity $\widehat{r}$ has a natural interpretation as the risk-neutral *credit-risk adjusted* short-term interest rate. Straightforward calculations yield the following corollary to Proposition 5.1.

Corollary 5.2 *Assume that* $\sigma_2 \neq 0$ *and*

$$\begin{aligned} dY_t^1 &= rY_t^1\,dt, \\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big), \\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t - dM_t\big). \end{aligned}$$

Then the function $v(\cdot\,; 0)$ *satisfies*

$$\begin{aligned} &\partial_t v(t, y_2, y_3; 0) + r y_2 \partial_2 v(t, y_2, y_3; 0) + \widehat{r} y_3 \partial_3 v(t, y_2, y_3; 0) - \widehat{r} v(t, y_2, y_3; 0) \\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} v(t, y_2, y_3; 0) + \widehat{\gamma} v(t, y_2; 1) = 0 \end{aligned}$$

with $v(T, y_2, y_3; 0) = G(y_2, y_3; 0)$, *and the function* $v(\cdot\,; 1)$ *satisfies*

$$\partial_t v(t, y_2; 1) + r y_2 \partial_2 v(t, y_2; 1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} v(t, y_2; 1) - r v(t, y_2; 1) = 0$$

with $v(T, y_2; 1) = G(y_2, 0; 1)$.

In the special case of a survival claim, the function $v(\cdot\,; 1)$ vanishes identically, and thus the following result can be easily established.

Corollary 5.3 *The pre-default pricing function $v(\cdot\,;0)$ of a survival claim $Y = \mathbb{1}_{\{T<\tau\}}G(Y_T^2, Y_T^3)$ is a solution of the following PDE:*

$$\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + \widehat{r}y_3\partial_3 v(t,y_2,y_3;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) - \widehat{r}v(t,y_2,y_3;0) = 0$$

with the terminal condition $v(T,y_2,y_3;0) = G(y_2,y_3)$. The components ϕ^2 and ϕ^3 of the replicating strategy satisfy

$$\phi_t^2\sigma_2 Y_t^2 = \sum_{i=2}^{3}\sigma_i Y_{t-}^i \partial_i v(t,Y_t^2,Y_{t-}^3;0) + \sigma_3 v(t,Y_t^2,Y_{t-}^3;0),$$
$$\phi_t^3 Y_{t-}^3 = v(t,Y_t^2,Y_{t-}^3;0).$$

Example 5.1 Consider a survival claim $Y = \mathbb{1}_{\{T<\tau\}}g(Y_T^2)$, that is, a vulnerable claim with default-free underlying asset. Its pre-default pricing function $v(\cdot\,;0)$ does not depend on y_3, and satisfies the PDE (y stands here for y_2 and σ for σ_2)

$$\partial_t v(t,y;0) + ry\partial_2 v(t,y;0) + \frac{1}{2}\sigma^2 y^2 \partial_{22} v(t,y;0) - \widehat{r}v(t,y;0) = 0 \qquad (75)$$

with the terminal condition $v(T,y;0) = \mathbb{1}_{\{t<\tau\}}g(y)$. The solution to (75) is

$$v(t,y) = e^{(\widehat{r}-r)(t-T)}\, v^{r,g,2}(t,y) = e^{\widehat{\gamma}(t-T)}\, v^{r,g,2}(t,y),$$

where the function $v^{r,g,2}$ is the Black-Scholes price of $g(Y_T)$ in a Black-Scholes model for Y_t with interest rate r and volatility σ_2.

5.2 Defaultable Asset with Non-Zero Recovery

We now assume that

$$dY_t^3 = Y_{t-}^3(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t)$$

with $\kappa_3 > -1$ and $\kappa_3 \neq 0$. We assume that $Y_0^3 > 0$, so that $Y_t^3 > 0$ for every $t \in \mathbb{R}_+$. We shall briefly describe the same steps as in the case of a defaultable asset with total default.

5.2.1 Arbitrage-Free Property

As usual, we need first to impose specific constraints on model coefficients, so that the model is arbitrage-free. Indeed, an e.m.m. $\mathbb{Q}^1$ exists if there exists a pair (θ,ζ) such that

$$\theta_t(\sigma_i - \sigma_1) + \zeta_t\xi_t\frac{\kappa_i-\kappa_1}{1+\kappa_1} = \mu_1 - \mu_i + \sigma_1(\sigma_i-\sigma_1) + \xi_t(\kappa_i-\kappa_1)\frac{\kappa_1}{1+\kappa_1}, \quad i = 2,3.$$

To ensure the existence of a solution (θ,ζ) on the set $\tau < t$, we impose the condition

$$\sigma_1 - \frac{\mu_1-\mu_2}{\sigma_1-\sigma_2} = \sigma_1 - \frac{\mu_1-\mu_3}{\sigma_1-\sigma_3},$$

that is,

$$\mu_1(\sigma_3-\sigma_2) + \mu_2(\sigma_1-\sigma_3) + \mu_3(\sigma_2-\sigma_1) = 0.$$

Now, on the set $\tau \geq t$, we have to solve the two equations

$$\begin{aligned}\theta_t(\sigma_2-\sigma_1) &= \mu_1 - \mu_2 + \sigma_1(\sigma_2-\sigma_1),\\ \theta_t(\sigma_3-\sigma_1) + \zeta_t\gamma\kappa_3 &= \mu_1 - \mu_3 + \sigma_1(\sigma_3-\sigma_1).\end{aligned}$$

If, in addition, $(\sigma_2 - \sigma_1)\kappa_3 \neq 0$, we obtain the unique solution

$$\begin{aligned}\theta &= \sigma_1 - \frac{\mu_1 - \mu_2}{\sigma_1 - \sigma_2} = \sigma_1 - \frac{\mu_1 - \mu_3}{\sigma_1 - \sigma_3},\\ \zeta &= 0 > -1,\end{aligned}$$

so that the martingale measure $\mathbb{Q}^1$ exists and is unique.

5.2.2 Pricing PDE and Replicating Strategy

We are in a position to derive the pricing PDEs. For the sake of simplicity, we assume that Y^1 is the savings account, so that Proposition 5.3 is a counterpart of Corollary 5.2. For the proof of Proposition 5.3, the interested reader is referred to Bielecki et al. [6].

Proposition 5.3 *Let $\sigma_2 \neq 0$ and let Y^1, Y^2, Y^3 satisfy*

$$\begin{aligned}dY_t^1 &= rY_t^1\,dt,\\ dY_t^2 &= Y_t^2\big(\mu_2\,dt + \sigma_2\,dW_t\big),\\ dY_t^3 &= Y_{t-}^3\big(\mu_3\,dt + \sigma_3\,dW_t + \kappa_3\,dM_t\big).\end{aligned}$$

Assume, in addition, that $\sigma_2(r-\mu_3) = \sigma_3(r-\mu_2)$ and $\kappa_3 \neq 0$, $\kappa_3 > -1$. Then the price of a contingent claim $Y = G(Y_T^2, Y_T^3, H_T)$ can be represented as $\pi_t(Y) = v(t, Y_t^2, Y_t^3, H_t)$, where the pricing functions $v(\cdot\,;0)$ and $v(\cdot\,;1)$ satisfy the following PDEs

$$\begin{aligned}&\partial_t v(t,y_2,y_3;0) + ry_2\partial_2 v(t,y_2,y_3;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v(t,y_2,y_3;0) - rv(t,y_2,y_3;0)\\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;0) + \gamma\big(v(t,y_2,y_3(1+\kappa_3);1) - v(t,y_2,y_3;0)\big) = 0\end{aligned}$$

and

$$\begin{aligned}&\partial_t v(t,y_2,y_3;1) + ry_2\partial_2 v(t,y_2,y_3;1) + ry_3\partial_3 v(t,y_2,y_3;1) - rv(t,y_2,y_3;1)\\ &\quad + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v(t,y_2,y_3;1) = 0\end{aligned}$$

subject to the terminal conditions

$$v(T,y_2,y_3;0) = G(y_2,y_3;0), \quad v(T,y_2,y_3;1) = G(y_2,y_3;1).$$

The replicating strategy ϕ equals

$$\begin{aligned}\phi_t^2 &= \frac{1}{\sigma_2\kappa_3 Y_t^2}\left(\kappa_3\sum_{i=2}^{3}\sigma_i y_i \partial_i v(t,Y_t^2,Y_{t-}^3,H_{t-}) - \sigma_3\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big)\right),\\ \phi_t^3 &= \frac{1}{\kappa_3 Y_{t-}^3}\big(v(t,Y_t^2,Y_{t-}^3(1+\kappa_3);1) - v(t,Y_t^2,Y_{t-}^3;0)\big),\end{aligned}$$

and ϕ_t^1 is given by $\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 + \phi_t^3 Y_t^3 = C_t$.

5.2.3 Hedging of a Survival Claim

We shall illustrate Proposition 5.3 by means of examples. First, consider a survival claim of the form

$$Y = G(Y_T^2, Y_T^3, H_T) = \mathbb{1}_{\{T<\tau\}}\, g(Y_T^3).$$

Then the post-default pricing function $v^g(\cdot\,;1)$ vanishes identically, and the pre-default pricing function $v^g(\cdot\,;0)$ solves the PDE

$$\partial_t v^g(\cdot\,;0) + ry_2\partial_2 v^g(\cdot\,;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v^g(\cdot\,;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot\,;0) - (r+\gamma)v^g(\cdot\,;0) = 0$$

with the terminal condition $v^g(T,y_2,y_3;0) = g(y_3)$. Denote $\alpha = r-\kappa_3\gamma$ and $\beta = \gamma(1+\kappa_3)$.

It is not difficult to check that $v^g(t,y_2,y_3;0) = e^{\beta(T-t)}v^{\alpha,g,3}(t,y_3)$ is a solution of the above equation, where the function $w(t,y) = v^{\alpha,g,3}(t,y)$ is the solution of the standard Black-Scholes PDE equation

$$\partial_t w + y\alpha\partial_y w + \frac{1}{2}\sigma_3^2 y^2 \partial_{yy} w - \alpha w = 0$$

with the terminal condition $w(T,y) = g(y)$, that is, the price of the contingent claim $g(Y_T)$ in the Black-Scholes framework with the interest rate α and the volatility parameter equal to σ_3.

Let C_t be the current value of the contingent claim Y, so that

$$C_t = 1\!\!1_{\{t<\tau\}} e^{\beta(T-t)} v^{\alpha,g,3}(t,Y_t^3).$$

The hedging strategy of the survival claim is, on the event $\{t<\tau\}$,

$$\begin{aligned}
\phi_t^3 Y_t^3 &= -\frac{1}{\kappa_3} e^{-\beta(T-t)} v^{\alpha,g,3}(t,Y_t^3) = -\frac{1}{\kappa_3} C_t,\\
\phi_t^2 Y_t^2 &= \frac{\sigma_3}{\sigma_2}\left(Y_t^3 e^{-\beta(T-t)}\partial_y v^{\alpha,g,3}(t,Y_t^3) - \phi_t^3 Y_t^3\right).
\end{aligned}$$

5.2.4 Hedging of a Recovery Payoff

As another illustration of Proposition 5.3, we shall now consider the contingent claim $G(Y_T^2, Y_T^3, H_T) = 1\!\!1_{\{T\geq\tau\}} g(Y_T^2)$, that is, we assume that recovery is paid at maturity and equals $g(Y_T^2)$. Let v^g be the pricing function of this claim. The post-default pricing function $v^g(\cdot\,;1)$ does not depend on y_3. Indeed, the equation (we write here $y_2 = y$)

$$\partial_t v^g(\cdot\,;1) + ry\partial_y v^g(\cdot\,;1) + \frac{1}{2}\sigma_2^2 y^2 \partial_{yy} v^g(\cdot\,;1) - rv^g(\cdot\,;1) = 0,$$

with $v^g(T,y;1) = g(y)$, admits a unique solution $v^{r,g,2}$, which is the price of $g(Y_T)$ in the Black-Scholes model with interest rate r and volatility σ_2.

Prior to default, the price of the claim can be found by solving the following PDE

$$\partial_t v^g(\cdot;0) + ry_2\partial_2 v^g(\cdot;0) + y_3\,(r-\kappa_3\gamma)\,\partial_3 v^g(\cdot;0) + \frac{1}{2}\sum_{i,j=2}^{3}\sigma_i\sigma_j y_i y_j \partial_{ij} v^g(\cdot;0) - (r+\gamma)v^g(\cdot;0) = -\gamma v^g(t,y_2;1)$$

with $v^g(T,y_2,y_3;0) = 0$. It is not difficult to check that

$$v^g(t,y_2,y_3;0) = (1-e^{\gamma(t-T)})v^{r,g,2}(t,y_2).$$

The reader can compare this result with the one of Example 5.1.

5.3 Two Defaultable Assets with Total Default

We shall now assume that we have only two assets, and both are defaultable assets with total default. We shall briefly outline the analysis of this case, leaving the details and the study of other relevant cases to the reader. We postulate that

$$dY_t^i = Y_{t-}^i\big(\mu_i\,dt + \sigma_i\,dW_t - dM_t\big),\ i=1,2, \tag{76}$$

so that

$$Y_t^1 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^1, \quad Y_t^2 = \mathbb{1}_{\{t<\tau\}}\widetilde{Y}_t^2,$$

with the pre-default prices governed by the SDEs

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i\big((\mu_i + \gamma)\,dt + \sigma_i\,dW_t\big),\ i = 1, 2.$$

In the case where the promised payoff X is path-independent, so that

$$X\mathbb{1}_{\{T<\tau\}} = G(Y_T^1, Y_T^2)\mathbb{1}_{\{T<\tau\}} = G(\widetilde{Y}_T^1, \widetilde{Y}_T^2)\mathbb{1}_{\{T<\tau\}}$$

for some function G, it is possible to use the PDE approach in order to value and replicate survival claims prior to default (needless to say that the valuation and hedging after default are trivial here).

We know already from the martingale approach that hedging of a survival claim $X\mathbb{1}_{\{T<\tau\}}$ is formally equivalent to replicating the promised payoff X using the pre-default values of tradeable assets

$$d\widetilde{Y}_t^i = \widetilde{Y}_t^i\big((\mu_i + \gamma)\,dt + \sigma_i\,dW_t\big), \quad i = 1, 2.$$

We need not to worry here about the balance condition, since in case of default the wealth of the portfolio will drop to zero, as it should in view of the equality $Z = 0$.

We shall find the pre-default pricing function $v(t, y_1, y_2)$, which is bound to satisfy the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$, as well as the hedging strategy (ϕ^1, ϕ^2). The replicating strategy ϕ is such that for the pre-default value $\widetilde{C}$ of our claim we have

$$\widetilde{C}_t := v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2) = \phi_t^1\widetilde{Y}_t^1 + \phi_t^2\widetilde{Y}_t^2,$$

and

$$d\widetilde{C}_t = \phi_t^1\,d\widetilde{Y}_t^1 + \phi_t^2\,d\widetilde{Y}_t^2. \tag{77}$$

Proposition 5.4 *Assume that $\sigma_1 \neq \sigma_2$. Then the pre-default pricing function v satisfies the PDE*

$$\begin{aligned}\partial_t v + y_1\left(\mu_1 + \gamma - \sigma_1\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right)\partial_1 v + y_2\left(\mu_2 + \gamma - \sigma_2\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right)\partial_2 v \\ + \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) = \left(\mu_1 + \gamma - \sigma_1\frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}\right)v\end{aligned}$$

with the terminal condition $v(T, y_1, y_2) = G(y_1, y_2)$.

Proof. We shall merely sketch the proof. By applying Itô's formula to $v(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$, and comparing the diffusion terms in (77) and in the Itô differential $dv(t, \widetilde{Y}_t^1, \widetilde{Y}_t^2)$, we find that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = \phi^1 y_1\sigma_1 + \phi^2 y_2\sigma_2, \tag{78}$$

where $\phi^i = \phi^i(t, y_1, y_2)$. Since $\phi^1 y_1 = v(t, y_1, y_2) - \phi^2 y_2$, we deduce from (78) that

$$y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v = v\sigma_1 + \phi^2 y_2(\sigma_2 - \sigma_1),$$

and thus

$$\phi^2 y_2 = \frac{y_1\sigma_1\partial_1 v + y_2\sigma_2\partial_2 v - v\sigma_1}{\sigma_2 - \sigma_1}.$$

On the other hand, by identification of drift terms in (78), we obtain

$$\begin{aligned}\partial_t v \quad &+ \quad y_1(\mu_1 + \gamma)\partial_1 v + y_2(\mu_2 + \gamma)\partial_2 v + \frac{1}{2}\Big(y_1^2\sigma_1^2\partial_{11}v + y_2^2\sigma_2^2\partial_{22}v + 2y_1y_2\sigma_1\sigma_2\partial_{12}v\Big) \\ &= \quad \phi^1 y_1(\mu_1 + \gamma) + \phi^2 y_2(\mu_2 + \gamma).\end{aligned}$$

Upon elimination of ϕ^1 and ϕ^2, we arrive at the stated PDE. □

Recall that the historically observed drift terms are $\widehat{\mu}_i = \mu_i + \gamma$, rather than μ_i. The pricing PDE can thus be simplified as follows:

$$\partial_t v + y_1 \left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_1 v + y_2 \left(\widehat{\mu}_2 - \sigma_2 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right) \partial_2 v$$
$$+ \frac{1}{2} \Big(y_1^2 \sigma_1^2 \partial_{11} v + y_2^2 \sigma_2^2 \partial_{22} v + 2 y_1 y_2 \sigma_1 \sigma_2 \partial_{12} v \Big) = v \left(\widehat{\mu}_1 - \sigma_1 \frac{\widehat{\mu}_2 - \widehat{\mu}_1}{\sigma_2 - \sigma_1} \right).$$

The pre-default pricing function v depends on the market observables (drift coefficients, volatilities, and pre-default prices), but not on the (deterministic) default intensity.

To make one more simplifying step, we make an additional assumption about the payoff function. Suppose, in addition, that the payoff function is such that $G(y_1, y_2) = y_1 g(y_2/y_1)$ for some function $g : \mathbb{R}_+ \to \mathbb{R}$ (or equivalently, $G(y_1, y_2) = y_2 h(y_1/y_2)$ for some function $h : \mathbb{R}_+ \to \mathbb{R}$). Then we may focus on relative pre-default prices $\widehat{C}_t = \widetilde{C}_t (\widetilde{Y}_t^1)^{-1}$ and $\widetilde{Y}^{2,1} = \widetilde{Y}_t^2 (\widetilde{Y}_t^1)^{-1}$. The corresponding pre-default pricing function $\widehat{v}(t, z)$, such that $\widehat{C}_t = \widehat{v}(t, Y_t^{2,1})$ will satisfy the PDE

$$\partial_t \widehat{v} + \frac{1}{2} (\sigma_2 - \sigma_1)^2 z^2 \partial_{zz} \widehat{v} = 0$$

with terminal condition $\widehat{v}(T, z) = g(z)$. If the price processes Y^1 and Y^2 in (67) are driven by the correlated Brownian motions W and $\widehat{W}$ with the constant instantaneous correlation coefficient ρ, then the PDE becomes

$$\partial_t \widehat{v} + \frac{1}{2} (\sigma_2^2 + \sigma_1^2 - 2\rho \sigma_1 \sigma_2) z^2 \partial_{zz} \widehat{v} = 0.$$

Consequently, the pre-default price $\widetilde{C}_t = \widetilde{Y}_t^1 \widehat{v}(t, \widetilde{Y}_t^{2,1})$ will not depend directly on the drift coefficients $\widehat{\mu}_1$ and $\widehat{\mu}_2$, and thus, in principle, we should be able to derive an expression the price of the claim in terms of market observables: the prices of the underlying assets, their volatilities and the correlation coefficient. Put another way, neither the default intensity nor the drift coefficients of the underlying assets appear as independent parameters in the pre-default pricing function.

Before we conclude this work, let us stress once again that the martingale approach can be used in fairly general set-up. By contrast, the PDE methodology is only suitable when dealing with a Markovian framework. In a forthcoming paper [7], we analyze a more general situation where a traded defaultable asset is a credit default swap, so that its dynamics involve also a continuous dividend stream.

References

[1] Arvanitis A. and J. Gregory (2001) *Credit: The Complete Guide to Pricing, Hedging and Risk Management.* Risk Publications.

[2] Bielecki, T.R. and M. Rutkowski (2002) *Credit Risk: Modelling, Valuation and Hedging.* Springer-Verlag, Berlin Heidelberg New York.

[3] Bielecki, T.R., M. Jeanblanc and M. Rutkowski (2004a) Hedging of defaultable claims. In: *Paris-Princeton Lectures on Mathematical Finance 2003*, R.A. Carmona, E. Cinlar, I. Ekeland, E. Jouini, J.E. Scheinkman, N. Touzi, eds., Springer-Verlag, Berlin Heidelberg New York, pp. 1–132.

[4] Bielecki, T.R., M. Jeanblanc and M. Rutkowski (2004b) Completeness of a general semimartingale market under constrained trading. To appear.

[5] Bielecki, T.R., M. Jeanblanc and M. Rutkowski (2004c) Completeness of a reduced-form credit risk model with discontinuous asset prices. To appear.

[6] Bielecki, T.R., M. Jeanblanc and M. Rutkowski (2005a) PDE approach to valuation and hedging of credit derivatives. To appear.

[7] Bielecki, T.R., M. Jeanblanc and M. Rutkowski (2005b) Pricing and trading credit default swaps. Working paper.

[8] Black, F. and J.C. Cox (1976) Valuing corporate securities: Some effects of bond indenture provisions. *Journal of Finance* 31, 351–367.

[9] Blanchet-Scalliet, C. and M. Jeanblanc (2004) Hazard rate for credit risk and hedging defaultable contingent claims. *Finance and Stochastics* 8, 145–159.

[10] Brémaud, P. (1981) *Point Processes and Queues. Martingale Dynamics.* Springer-Verlag, Berlin Heidelberg New York.

[11] Carr, P. (2005) Dynamic replication of a digital default claim. Working paper.

[12] Collin-Dufresne, P., R.S. Goldstein and J.-N. Hugonnier (2004) A general formula for valuing defaultable securities. *Econometrica* 72, 1377–1407.

[13] Collin-Dufresne, P. and J.-N. Hugonnier (1999) On the pricing and hedging of contingent claims in the presence of extraneous risks. Working paper.

[14] Cossin, D. and H. Pirotte (2000) *Advanced Credit Risk Analysis.* J. Wiley, Chichester.

[15] Dellacherie, C., B. Maisonneuve and P.-A. Meyer (1992) *Probabilités et potentiel. Chapitres XVII-XXIV.* Hermann, Paris.

[16] Duffie, D. and D. Lando (2001) The term structure of credit spreads with incomplete accounting information. *Econometrica* 69, 633–664.

[17] Duffie, D. and K. Singleton (2003) *Credit Risk: Pricing, Measurement and Management.* Princeton University Press, Princeton.

[18] El Karoui, N. (1999) Modélisation de l'information. CEA-EDF-INRIA, Ecole d'été. Unpublished manuscript.

[19] Elliott, R.J., M. Jeanblanc and M. Yor (2000) On models of default risk. *Mathematical Finance* 10, 179–195.

[20] Giesecke, K. (2001) Default and information. Working paper, Cornell University.

[21] Giesecke, K. (2004) Correlated default with incomplete information. *Journal of Banking and Finance* 28, 1521–1545.

[22] Guo, X., R.A. Jarrow and Y. Zheng (2005) Information reduction in credit risk models. Working paper, Cornell University.

[23] Jarrow, R.A. and P. Protter (2004) Structural versus reduced form models: A new information based perspective. *Journal of Investment Management* 2/2, 1–10.

[24] Jarrow, R.A. and S.M. Turnbull (1995) Pricing derivatives on financial securities subject to credit risk. *Journal of Finance* 50, 53–85.

[25] Jacod, J. and A.N. Shiryaev (1987) *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin Heidelberg New York.

[26] Jamshidian, F. (2004) Valuation of credit default swap and swaptions, *Finance and Stochastics* 8, 343–371.

[27] Jeanblanc, M. and S. Valchev (2004) Partial information, default hazard process, and default-risky bonds. To appear.

[28] Kusuoka, S. (1999) A remark on default risk models. *Advances in Mathematical Economics* 1, 69-82.

[29] Lando, D. (2004) *Credit Risk Modeling.* Princeton University Press, Princeton.

[30] Laurent, J.-P. (2001) Applying hedging techniques to credit derivatives. Credit Risk Conference, London.

[31] Merton, R.C. (1974) On the pricing of corporate debt: The risk structure of interest rates. *Journal of Finance* 29, 449–470.

[32] Mansuy, R. and Yor, M. (2005) *Random Times and Enlargement of Filtrations in a Brownian Setting.* Forthcoming, Springer-Verlag, Berlin Heidelberg New York.

[33] Protter, P. (2001) A partial introduction to financial asset pricing theory. *Stochastic Processes and Their Applications* 91, 169-203.

[34] Protter, P. (2005) *Stochastic Integration and Differential Equations.* 2nd edition, version 2.1, Springer-Verlag, Berlin Heidelberg New York.

[35] Schönbucher, P.J. (2003) *Credit Derivatives Pricing Models.* J.Wiley, Chichester.

[36] Vaillant, N. (2001) A beginner's guide to credit derivatives. Working paper, Nomura International.