

# Progressive enlargement of filtration with initial times

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## Abstract

The preservation of the semi-martingale property in a progressive enlargement of filtration has been studied by many authors. Most of them focus on progressive enlargement with a honest time, allowing for semi-martingale invariance and simple decomposition formulas. However, times allowing for semi-martingale invariance in initial enlargements preserve as well this property in progressive enlargements. This paper is devoted to the related canonical decomposition of the semi-martingales. Examples are given in credit risk modelling.

## 1 Introduction

The stability of the class of semi-martingales with respect to various operations has been a major field of research during the last decades: composition with convex functions (Tanaka-Meyer formula), change of equivalent probability (Girsanov theorem), or filtration shrinking. In the case of a filtration shrinking, for  $\mathbb{F} \subset \mathbb{G}$ , any  $\mathbb{G}$ -semi-martingale which is  $\mathbb{F}$ -adapted is a  $\mathbb{F}$ -semi-martingale (Stricker's theorem).

The situation of an enlargement of filtration is more complex, and the stability of semi-martingale sets for enlargement of filtration is not always valid. In this framework, for  $\mathbb{F} \subset \mathbb{G}$ , it is usual to say that  $(\mathcal{H}')$ -hypothesis holds between  $\mathbb{F}$  and  $\mathbb{G}$  if any  $\mathbb{F}$ -semi-martingale is a  $\mathbb{G}$ -semi-martingale.

In what follows, we denote by  $\mathcal{M}(\mathbb{F})$  (resp.  $\mathcal{M}(\mathbb{G})$ ) the set of  $\mathbb{F}$ -martingales (resp.  $\mathbb{G}$ -martingales). We start by recalling some well-known facts on initial and progressive enlargements of filtrations.

- The initial enlargement of a reference filtration  $\mathbb{F}$  by a random time  $\tau$  (a non-negative random variable) is the filtration  $\mathbb{G}^{(\tau)}$  defined as  $\mathcal{G}_t^{(\tau)} = \bigcap_{\epsilon > 0} (\mathcal{F}_{t+\epsilon} \vee \sigma(\tau))$ . In this framework, no general theorem guarantees that  $(\mathcal{H}')$ -hypothesis holds. However it is well known that if the conditional laws of the random time  $\tau$  (with respect to the reference filtration) are absolutely continuous with respect to a probability  $\eta$ , then  $(\mathcal{H}')$ -hypothesis holds. Moreover, one can reduce attention to the case where  $\eta$  is the law of  $\tau$ . This result is known as Jacod's theorem (see for example the paper of J. Jacod [8] or Chapter VI in the book of P. Protter [23]). Random times satisfying this property will be named *initial times* in the sequel (see first section).

- The progressive enlargement of a reference filtration  $\mathbb{F}$  by a random time  $\tau$  is the smallest right-continuous filtration which contains  $\mathbb{F}$  and that makes  $\tau$  a stopping time. This filtration  $\mathbb{G}$  is defined as  $\mathcal{G}_t = \bigcap_{\epsilon > 0} \mathcal{G}_{t+\epsilon}^0$  where  $\mathcal{G}_t^0 = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ . Remark that, for fixed  $t$ , the  $\sigma$ -algebra  $\mathcal{G}_t$  is generated by the set of random variables of the form  $F_t h(t \wedge \tau)$ , with  $h$  a Borel function, and

$F_t$  an  $\mathcal{F}_t$ -measurable random variable. It follows that the filtration  $\mathbb{G}$  coincides with  $\mathbb{F}$  before  $\tau$  and with  $\mathbb{G}^{(\tau)}$  after.

The study of  $(\mathcal{H}')$ -hypothesis in progressive enlargements can be split in two different time intervals: before and after the occurrence of  $\tau$ .

- On the set  $\{t < \tau\}$ ,  $(\mathcal{H}')$ -hypothesis always holds: precisely for any  $\mathbb{F}$ -martingale  $X$ , the stopped process  $X^\tau$  is a  $\mathbb{G}$ -semi-martingale (a nice and short argument of M. Yor may be found in [26]). Moreover, the canonical decomposition of the  $\mathbb{G}$ -semi-martingale  $X^\tau$  writes

$$X_t^\tau = \mu_t + \int_0^{t \wedge \tau} \frac{d\langle X, M \rangle_s + dB_s}{G_{s-}}, \text{ with } \mu \in \mathcal{M}(\mathbb{G})$$

where  $G_t := \mathbb{P}(\tau > t | \mathcal{F}_t)$  is the conditional survival process<sup>1</sup> (an  $\mathbb{F}$ -super-martingale), and  $M$  denotes the martingale part of its Doob-Meyer decomposition. The process  $B$  is the  $\mathbb{F}$ -predictable dual projection of the  $\mathbb{G}$ -adapted process  $(\varepsilon_u)_u = (H_u \Delta X_\tau)_u$ . A proof of this decomposition can be found in the books of T. Jeulin [11] and [15] or in the papers of T. Jeulin and M. Yor [13], [12] or [14]. If  $(\mathcal{A})$  is satisfied (i.e., if the random time  $\tau$  avoids the  $\mathbb{F}$ -stopping times, that is  $\mathbb{P}(\tau = T) = 0$  for any  $\mathbb{F}$ -stopping time  $T$ ), then  $\Delta X_\tau = 0$  and  $B = 0$ . Under this assumption, a proof of the decomposition can be found in Chapter VI of the book of P. Protter [23].

- For the general case of non stopped semi-martingales, semi-martingale invariance deeply depends on properties of the random time. The natural extension of the proof leading to the last result - which is based on the structure of the  $\mathbb{G}$ -predictable  $\sigma$ -field  $\mathcal{P}(\mathbb{G})$  and its links with  $\mathcal{P}(\mathbb{F})$  - lies on the study of times allowing any  $\mathbb{G}$ -predictable process  $K$  to be written as  $K = K^1 1_{[0, \tau]} + K^2 1_{\tau, \infty[}$ , where  $(K^1, K^2)$  are  $\mathbb{F}$ -predictable processes (see M.T. Barlow [1], M. Yor [26], T. Jeulin [11] or C. Dellacherie and P.A. Meyer [5]). Such times are called honest time<sup>2</sup>, precisely the time  $\tau$  is said "honest" if it is equal to an  $\mathcal{F}_t$ -measurable variable on  $\{\tau < t\}$ , for any  $t > 0$ . If the time is honest, the sequence of  $\sigma$ -algebras  $\mathcal{G}_t = \{A \in \mathcal{F}, \forall t, \exists A_t, B_t \in \mathcal{F}_t, A = (A_t \cap \{\tau > t\}) \cup (B_t \cap \{\tau \leq t\})\}$  is increasing (by honesty of the time) and forms a filtration. In that framework,  $(\mathcal{H}')$ -hypothesis holds, and if  $X \in \mathcal{M}^2(\mathbb{F})$  (a square integrable martingale, so that the brackets exist):

$$X_t = \mu_t + \int_0^{t \wedge \tau} \frac{d\langle X, M \rangle_s + dB_s}{G_{s-}} - 1_{\{\tau \leq t\}} \int_\tau^t \frac{d\langle X, M \rangle_s + dB_s}{1 - G_{s-}}, \text{ with } \mu \in \mathcal{M}(\mathbb{G})$$

(remark that the brackets exit since  $M$  is also a square integrable martingale, see [23] for example).

The theory of progressive enlargement with an honest time presents nonetheless some major drawbacks in some application fields, such as credit derivatives modelling. For the approach based on the enlargement of a reference filtration by the progressive knowledge of a credit event (see R. Elliott, M. Jeanblanc and M. Yor in [6] or M. Jeanblanc and Y. Le Cam in [10]),  $(\mathcal{H}')$ -hypothesis is fundamental. Indeed, the absence of arbitrage in finance is closely linked to the property of semi-martingale satisfied by the assets (see for example F. Delbaen and W. Schachermayer in [3]). The most important argument which makes impossible the application of the "honest" theory in credit modelling is the belonging of the honest time to  $\mathcal{F}_\infty$ : unfortunately the credit event (a change in the ranking of an obligation or an unpaid coupon for example) can

<sup>1</sup>It is well known that  $G$  does not reach zero before  $\tau$ . Indeed, if  $T = \inf\{t > 0, G_t = 0 \text{ or } G_{t-} = 0\}$ ,  $G_t$  is null after  $T$  (a non negative super-martingale sticks at zero) and  $\mathbb{P}(T < \tau) = \mathbb{E}(G_T 1_{\{T < \infty\}}) = 0$

<sup>2</sup>Honest times coincide with the end of  $\mathbb{F}$ -optional set in  $[0, \infty] \times \Omega$ , and finite honest times coincide with the end of  $\mathbb{F}$ -optional sets in  $[0, \infty[ \times \Omega$ , see [11].

neither be directly read on the market price of the asset nor on its future and modelling it through an  $\mathcal{F}_\infty$  variable is not consistent with reality. The widespread model of Cox construction of the credit event - in which  $\tau = \inf\{t : \Lambda_t \geq \Theta\}$  with  $\Lambda$  an  $\mathbb{F}$ -adapted increasing process and  $\Theta$  a variable that is not  $\mathcal{F}_\infty$ -measurable- strengthens this point (see examples in the third part).

The main goal in this paper is to present the progressive enlargement of a reference filtration with an initial time, focusing on the canonical decomposition of the  $\mathbb{F}$ -martingale as a  $\mathbb{G}$  semi-martingale. The paper is organized as follows. The first part presents the initial times and the features that make them a natural tool for the enlargement of filtration in many applications. In the second part we prove that if  $X \in \mathcal{M}(\mathbb{F})$ , then  $X$  is a  $\mathbb{G}$ -semi-martingale with canonical decomposition

$$X_t = \mu_t + \int_0^{t \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} - \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Bigg|_{\theta=\tau}, \text{ with } \mu \in \mathcal{M}(\mathbb{G}). \quad (1)$$

We generalize our study to the case of enlargement with multiple times. The last part gives examples of initial times that can be used in credit modelling, linked to Cox construction. Because of the important applications in this field, the random time  $\tau$  will sometimes be called default time or credit event.

## 2 Initial Times

As recalled in the introduction, the notion of initial times has been introduced by J. Jacod in [8] who proved that these times allow for  $(\mathcal{H}')$ -hypothesis in an initial enlargement. The name initial time is a reference to this property. For any positive random time  $\tau$ , and for any  $t$ , we write  $Q_t(\omega, dT)$  the regular conditional distribution of  $\tau$ , and

$$G_t^T(\omega) = \mathbb{P}(\tau > T | \mathcal{F}_t)(\omega) = Q_t(\omega, ]T, \infty[).$$

**Definition 1 (Initial times)** *The positive random time  $\tau$  is called an initial time if there exists a measure  $\eta$  on  $\mathcal{B}(\mathbb{R}^+)$  such that  $Q_t(\omega, dT) \ll \eta(dT)$ .*

By Doob's theorem of disintegration of measures, the definition of an initial time is equivalent to the existence of a family of positive  $\mathbb{F}$ -adapted processes  $(\alpha_t^u, t \geq 0)$  such that

$$G_t^T = \int_T^\infty \alpha_t^u \eta(du),$$

and the martingale property of  $(G_t^T, t \geq 0)$ , i.e., for any  $T$ , for any  $s \leq t$ ,  $G_s^T = \mathbb{E}(G_t^T | \mathcal{F}_s)$ , it is immediate to check that for any  $u \geq 0$ ,  $(\alpha_t^u)_t$  is a non negative  $\mathbb{F}$ -martingale. Note that, if  $f(u) = \mathbb{E}(\alpha_t^u) = \alpha_0^u$ , then  $\mathbb{P}(\tau \in du) = f(u)\eta(du)$ . We can consider without loss of generality the case where  $f(u) = 1$ . Remark that in this framework, we can write the survival process  $G_t := G_t^t$  as

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = \int_t^\infty \alpha_t^u \eta(du) = \int_0^\infty \alpha_{u \wedge t}^u \eta(du) - \int_0^t \alpha_u^u \eta(du) = M_t - \tilde{A}_t$$

where  $M$  is an  $\mathbb{F}$ -martingale (recall  $(\alpha_{u \wedge t}^u)_t$  is a stopped  $\mathbb{F}$ -martingale) and  $\tilde{A}$  an  $\mathbb{F}$ -adapted increasing process. If  $\eta$  does not have any atom,  $\tilde{A}$  is continuous hence  $\mathbb{F}$ -predictable. If not, this process can be not predictable, for example if the martingale  $\alpha$  jumps at an atom  $t$  of  $\eta$  and the size of the jump is not in  $\mathcal{F}_{t-}$  (recall a process is  $\mathbb{F}$ -predictable iff it jumps at  $\mathbb{F}$ -predictable

times and its jump at any  $\mathbb{F}$ -stopping time  $T$  belongs to  $\mathcal{F}_{T-}$ ). It is therefore necessary to compensate the finite variation process  $\tilde{A}$ , which is immediate, since as  $\eta$  is deterministic, its compensator writes  $A_t = \int_0^t ({}^p\alpha_u^u) \eta(du) = \int_0^t \alpha_{u-}^u \eta(du)$  (recall that the  $\mathbb{F}$ -predictable projection  ${}^p\alpha$  of the  $\mathbb{F}$ -martingale  $\alpha$  is  $\alpha_-$ ). It follows the compensator of  $G$  writes  $A_t = \int_0^t \alpha_{u-}^u \eta(du)$ . It is also the opposite of the  $\mathbb{F}$ -predictable dual projection of  $H_t = 1_{\tau \leq t}$ .

**Remark 1** *We have also the relation for the brackets, for any  $X$  square integrable martingale:*

$$\langle X, G \rangle_t = \int_0^\infty (\langle X, \alpha^u \rangle_{u \wedge t} + \Delta X_u \Delta \alpha_u^u 1_{u \leq t}) \eta(du), \quad (2)$$

as soon as for any  $u \geq 0$ , the martingale  $\alpha^u$  is square integrable. Remark that this property can fail to hold, as the example at the end of last section will illustrate. Remark that such a property is assumed to ensure the existence of the sharp bracket, defined by polarisation, since both  $\langle X \rangle$  and  $\langle \alpha^u \rangle$  exist. If  $X$  is only a local martingale, a stronger regularity has to be imposed to the martingale  $\alpha^u$  to allow for the existence of  $\langle X, \alpha^u \rangle$ . Recall<sup>3</sup> that any local martingale  $X$  is locally in the space  $\mathcal{H}^1$ , and consider therefore an increasing sequence of stopping times  $(T_n)$  such that  $X^{T_n} \in \mathcal{H}^1$ . If for any  $u$ , the martingale  $\alpha^u$  is in  $BMO^4$ , Fefferman's inequality implies the existence of a constant  $k$  such that

$$\mathbb{E} \int_0^\infty |d[X^{T_n}, \alpha^u]_s| \leq k \|X^{T_n}\|_{\mathcal{H}^1} \|\alpha^u\|_{BMO},$$

hence  $[X, \alpha^u]$  is locally of integrable variation and its compensator  $\langle X, \alpha^u \rangle$  exists. It follows that the results of this paper will remain true for  $X \in \mathcal{M}^{loc}(\mathbb{F})$ , is for any  $u$ , the martingale  $\alpha^u$  is in  $BMO$ . We assume until the end of this paper that when  $X$  is a square integrable martingale,  $\alpha^u$  is a square integrable martingale for any  $u \geq 0$ .

Initial times have the very interesting feature (for credit modelling for example) to allow for the existence of a probability under which the reference filtration and the random time are independent. Precisely, as proved in Grorud and Pontier [7] we have the:

**Proposition 1** *If  $\tau$  is an initial time with  $\mathbb{E}_{\mathbb{P}}(1/\alpha_\infty^\tau) < \infty$ , there exists a probability  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which  $\tau$  and  $\mathcal{F}_\infty$  are independent.*

The proof of this result is classical (define  $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{G}_t} = \mathbb{E}_{\mathbb{P}}(1/\alpha_\infty^\tau | \mathcal{G}_t) / \mathbb{E}_{\mathbb{P}}(1/\alpha_\infty^\tau)$ ) and leads to a straightforward proof of Jacod's theorem when this integrability assumption holds, since  $(H^l)$ -hypothesis is stable by a change of equivalent probability.

Under this new probability, immersion holds i.e., any  $\mathbb{F}$ -martingale remains a  $\mathbb{G}$ -martingale. There exists - at our knowledge - no such result in a "honest" expansion. Characterizations of immersion itself are also very tractable in the framework of initial times, as it will be seen in the corollary 1 in the next section.

<sup>3</sup>For any integer  $p$ , The space  $\mathcal{H}^p$  is defined as the set of the local martingales  $N$  such that  $\|N\|_{\mathcal{H}^p} < \infty$ , with  $\|N\|_{\mathcal{H}^p}^p = \mathbb{E}[N, N]_\infty^{p/2}$

<sup>4</sup>The space  $BMO$  is defined as the subspace of  $\mathcal{H}^2$  composed the local martingales  $N$  such that  $\mathbb{E} \left( (N_\infty - N_{T-})^2 | \mathcal{F}_T \right) \leq k$  for any  $\mathbb{F}$ -stopping time  $T$ .  $\|N\|_{BMO}^2$  denotes the smallest  $k$  if it exists (i.e.,  $N \in BMO$ ) or  $\infty$ .

### 3 Invariance of Semi-martingales

Let  $\tau$  be an initial time, i.e., satisfying  $G_t^T = \int_T^\infty \alpha_t^u \eta(du)$ . From Jacod's theorem, we know that  $(\mathcal{H}')$  hypothesis holds within the initial expansion  $\mathbb{F} \subset \mathbb{G}^{(\tau)}$ . If  $\mathbb{G}$  denotes the progressive expansion of  $\mathbb{F}$  by  $\tau$ ,  $\mathbb{F} \subset \mathbb{G} \subset \mathbb{G}^{(\tau)}$  and Stricker's theorem ensures that  $(\mathcal{H}')$  hypothesis holds between  $\mathbb{F}$  and  $\mathbb{G}$ . For any  $\mathbb{F}$ -martingale  $X$ , the question we address now is therefore the  $\mathbb{G}$ -semi-martingale decomposition of  $X$ . It has been recalled in the last section that the compensator of  $G$  writes  $A_t = \int_0^t \alpha_{u-}^u \eta(du)$ , and that it is also the opposite of the  $\mathbb{F}$ -predictable dual projection of  $H_t = 1_{\tau \leq t}$ . We start with a simple lemma that will be central in the following proofs.

**Lemma 1** *Let  $(K_t^u)_{t \geq 0}$  be a family of  $\mathbb{F}$ -predictable processes indexed by  $u \geq 0$  (i.e., for any  $u \geq 0$ ,  $t \rightarrow K_t^u$  is  $\mathbb{F}$ -predictable). Then*

$$\mathbb{E}(K_t^\tau | \mathcal{F}_t) = \int_0^\infty K_t^u \alpha_t^u \eta(du)$$

*Proof.* Consider  $\Lambda := \Omega \times \mathbb{R}^+ \times \mathbb{R}^+$  endowed with the  $\sigma$ -algebra  $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$ . Let  $\Delta$  be the set

$$\Delta := \left\{ k : \Lambda \longrightarrow \mathbb{R} \text{ measurable bounded, } \mathbb{E}(k(\cdot, t, \tau(\cdot)) | \mathcal{F}_t) = \int_0^\infty k(\cdot, t, u) \alpha(\cdot, t, u) \eta(du) \right\}.$$

By linearity of the conditional expectation,  $\Delta$  is a vector space that contains the unity function. For any  $\mathbb{F}$ -predictable process  $K$ , any measurable function  $h$ , and  $k = K.h$  :

$$\begin{aligned} \mathbb{E}(k(\cdot, t, \tau(\cdot)) | \mathcal{F}_t) &= \mathbb{E}(K_t h(\tau) | \mathcal{F}_t) = K_t \mathbb{E}(h(\tau) | \mathcal{F}_t) = K_t \int_0^\infty h(u) \alpha_t^u \eta(du) \\ &= \int_0^\infty K_t h(u) \alpha_t^u \eta(du) = \int_0^\infty k(\cdot, t, u) \alpha(\cdot, t, u) \eta(du). \end{aligned}$$

At last, for any increasing sequence  $k_n$  that belongs to  $\Delta$  and converges to  $k$ , we have that  $k \in \Delta$ , since for any  $F_t \in \mathcal{F}_t$  :

$$\mathbb{E}(F_t k_n(\cdot, t, \tau(\cdot))) \rightarrow \mathbb{E}(F_t k(\cdot, t, \tau(\cdot)))$$

a.s. (Beppo-Levi theorem, and existence of a regular version of the conditional expectation) and

$$\int_0^\infty k_n(\omega, t, u) \alpha(\omega, t, u) \eta(du) \rightarrow \int_0^\infty k(\omega, t, u) \alpha(\omega, t, u) \eta(du)$$

a.s. (Beppo-Levi theorem as well). The lemma follows by an application of the monotone class theorem.  $\square$

Remark that this result should not be confused with a result of the type  $\mathbb{E}(X_\tau | \mathcal{F}_t) = \int_0^\infty X_u \alpha_t^u \eta(du)$  for an  $\mathbb{F}$ -predictable process  $X$ . Such a result is not true in general (we can write for example,

$$\mathbb{E}X_\tau = \mathbb{E} \int_0^\infty X_u dH_u = \mathbb{E} \int_0^\infty X_u dA_u = \mathbb{E} \int_0^\infty X_u \alpha_u^u \eta(du) \neq \mathbb{E} \int_0^\infty X_u \alpha_t^u \eta(du).$$

Such a result is nonetheless true before default when immersion holds, i.e.,  $\mathbb{E}(X_\tau \mathbb{1}_{\tau > t} | \mathcal{F}_t) = \int_t^\infty X_u \alpha_t^u \eta(du)$ . Indeed, for any  $\mathcal{F}_t$ -measurable variable  $F_t$ ,

$$\begin{aligned} \mathbb{E}(X_\tau \mathbb{1}_{\tau > t} F_t) &= \mathbb{E} \left( \int_t^\infty X_u F_t dH_u \right) = \mathbb{E} \left( \int_t^\infty X_u F_t dA_u \right) \\ &= \mathbb{E} \left( \int_t^\infty X_u \alpha_u^u \eta(du) F_t \right) = \mathbb{E} \left( \int_t^\infty X_u \alpha_t^u \eta(du) F_t \right), \end{aligned}$$

where the last equality comes from the characterization of immersion presented in corollary 1.

Let  $X$  be a square integrable  $\mathbb{F}$ -martingale. We shall prove that there exists a family of  $\mathbb{F}$ -predictable processes  $(J, K(\theta), \theta \in \mathbb{R}^+)$  with finite variation such that

$$Y_t = X_t - \int_0^{t \wedge \tau} dJ_u - \int_{t \wedge \tau}^t dK_u(\tau)$$

is a  $\mathbb{G}$ -martingale. Recall that the  $\mathbb{F}$ -predictable process  $B$  refers to the dual  $\mathbb{F}$ -predictable projection of the  $\mathbb{G}$ -adapted process  $\varepsilon_u = \Delta X_\tau H_u$ . Let us start with two remarks:

- If such a representation holds, it is necessary that  $dJ_u = (d\langle X, G \rangle_u + dB_u) / G_{u-}$ . Indeed since  $Y \in \mathcal{M}(\mathbb{G})$  and  $\tau$  is a  $\mathbb{G}$ -stopping time, the stopped process  $Y^\tau$  is a  $\mathbb{G}$ -martingale (optional sampling theorem), hence  $X_t^\tau - \int_0^{t \wedge \tau} dJ_u \in \mathcal{M}(\mathbb{G})$ . The result follows from Jeulin's formula and uniqueness in Doob-Meyer decomposition.
- Let  $F_s$  be a bounded  $\mathcal{F}_s$ -measurable random variable and  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$ , be a bounded Borel function. Then the variable  $F_s h(\tau) 1_{\tau \leq s}$  is  $\mathcal{G}_s$ -measurable and if such a decomposition holds, the martingale property of  $Y$  should imply that  $\mathbb{E}(F_s h(\tau) 1_{\tau \leq s} (Y_t - Y_s)) = 0$ , hence

$$\mathbb{E}(F_s h(\tau) 1_{\tau \leq s} (X_t - X_s)) = \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_u(\tau)\right).$$

We can write:

$$\begin{aligned} \mathbb{E}(F_s h(\tau) 1_{\tau \leq s} (X_t - X_s)) &= \mathbb{E}\left(F_s (X_t - X_s) \int_0^s h(\theta) \alpha_t^\theta \eta(d\theta)\right) \\ &= \int_0^s h(\theta) \mathbb{E}(F_s (X_t \alpha_t^\theta - X_s \alpha_s^\theta)) \eta(d\theta) \\ &= \int_0^s h(\theta) \mathbb{E}\left(F_s \int_s^t d\langle X, \alpha^\theta \rangle_v\right) \eta(d\theta) \end{aligned}$$

where the first equality comes from a conditioning w.r.t.  $\mathcal{F}_t$ , the second from the martingale property of  $\alpha^\theta$ , and the third from the fact that both  $X$  and  $\alpha^\theta$  are  $\mathbb{F}$ -martingales (after compensation of  $\square$  with  $\langle \rangle$ , since the martingales are square integrable). Moreover:

$$\begin{aligned} \mathbb{E}\left(F_s h(\tau) 1_{\tau \leq s} \int_s^t dK_v(\tau)\right) &= \mathbb{E}\left(F_s \int_0^s h(\theta) \int_s^t dK_v(\theta) \alpha_t^\theta \eta(d\theta)\right) \\ &= \int_0^s h(\theta) \mathbb{E}\left(F_s \int_s^t \alpha_{v-}^\theta dK_v(\theta)\right) \eta(d\theta) \end{aligned}$$

where the first equality comes from the lemma applied to the  $\mathbb{F}$ -predictable process indexed by  $u$   $J_t^u = h(u) 1_{u \leq s} \int_s^t dK_v(u)$  ( $F_s$  is  $F_t$ -measurable) and the second from the martingale property of  $\alpha^\theta$  (recall that  $K_v(\theta)$  is  $\mathbb{F}$ -predictable hence  $K_u(\theta) \in F_{u-}$ ). By equalization of these two quantities, we obtain that it is necessary to have  $dK_u(\theta) = d\langle X, \alpha^\theta \rangle_u / \alpha_{u-}^\theta$ .

Remark that this result holds if  $X$  is only a local martingale, as soon as  $\alpha^u$  is in  $BMO$  for any  $u$ . From these two points, we shall prove the

**Theorem 1** *If  $X$  is a square integrable  $\mathbb{F}$ -martingale ( $X \in \mathcal{M}^2(\mathbb{F})$ ):*

$$Y_t = X_t - \int_0^{t \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} - \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Bigg|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}). \quad (3)$$

*If  $X \in \mathcal{M}^{loc}(\mathbb{F})$  and  $\forall u \geq 0$   $\alpha^u \in BMO$ ,  $Y_t \in \mathcal{M}^{loc}(\mathbb{G})$ .*

*Proof.* We prove the theorem in the square integrable case, the second case follows the same proof after localization (see remark 1). Let us consider a  $\mathcal{G}_s$ -measurable random variable of the form  $F_s h(\tau \wedge s)$  with  $F_s$  a bounded  $\mathcal{F}_s$ -measurable random variable and  $h : \mathbb{R}^+ \rightarrow \mathbb{R}$  a bounded Borel function. Then,

$$\mathbb{E}(F_s h(\tau \wedge s)(Y_t - Y_s)) = \underbrace{\mathbb{E}(F_s h(\tau)1_{\tau \leq s}(Y_t - Y_s))}_a + \underbrace{\mathbb{E}(F_s h(s)1_{s < \tau}(Y_t - Y_s))}_b$$

and we can compute each part of the right hand side member:

$a$  : On  $\{\tau \leq s\}$ ,  $t \wedge \tau = s \wedge \tau = \tau$  hence

$$1_{\tau \leq s} \left( \int_0^{t \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} - \int_0^{s \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} \right) = 0,$$

and it follows that

$$a = \mathbb{E}(F_s h(\tau)1_{\tau \leq s}(X_t - X_s)) - \mathbb{E} \left( F_s h(\tau)1_{\tau \leq s} \left( \int_s^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Big|_{\theta=\tau} \right) \right)$$

which is equal to zero, as we have seen previously.

$b$  : We rewrite  $b$  as

$$\begin{aligned} b &= \mathbb{E}(F_s h(s)1_{s < \tau}(X_t - X_{t \wedge \tau})) + \mathbb{E}(F_s h(s)1_{s < \tau}(X_{t \wedge \tau} - X_s)) \\ &\quad - \mathbb{E} \left( F_s h(s)1_{s < \tau} \int_s^{t \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} \right) - \mathbb{E} \left( F_s h(s)1_{s < \tau} \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\tau \rangle_u}{\alpha_{u-}^\tau} \right). \end{aligned}$$

Using Jeulin's formula before default, we have

$$\begin{aligned} \mathbb{E}(F_s h(s)1_{s < \tau}(X_{t \wedge \tau} - X_s)) &= \mathbb{E}(F_s h(s)1_{s < \tau}(X_{t \wedge \tau} - X_{s \wedge \tau})) \\ &= \mathbb{E} \left( F_s h(s)1_{s < \tau} \int_s^{t \wedge \tau} \frac{d\langle X, G \rangle_u + dB_u}{G_{u-}} \right), \end{aligned}$$

and it follows the expression of  $b$  :

$$\begin{aligned} b &= \mathbb{E}(F_s h(s)1_{s < \tau}(X_t - X_{t \wedge \tau})) - \mathbb{E} \left( F_s h(s)1_{s < \tau} \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\tau \rangle_u}{\alpha_{u-}^\tau} \right) \\ &= \mathbb{E}(F_s h(s)1_{s < \tau \leq t}(X_t - X_\tau)) - \mathbb{E} \left( F_s h(s)1_{s < \tau \leq t} \int_\tau^t \frac{d\langle X, \alpha^\tau \rangle_u}{\alpha_{u-}^\tau} \right). \end{aligned}$$

Moreover, we can write the decomposition:

$$\begin{aligned} \mathbb{E}(F_s h(s)1_{s < \tau \leq t} X_\tau) &= \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} X_{v-} dH_v \right) + \mathbb{E}(F_s h(s)1_{s < \tau \leq t} \Delta X_\tau) \\ &= \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} X_{v-} dA_v \right) + \mathbb{E}(F_s h(s)1_{s < \tau \leq t} \Delta X_\tau) \\ &= \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} X_{v-} \alpha_{v-}^v \eta(dv) \right) + \mathbb{E}(F_s h(s)1_{s < \tau \leq t} \Delta X_\tau) \end{aligned}$$

where the second equality comes from the definition of the predictable dual projection (recall that  $X_{v-}$  is  $\mathbb{F}$ -predictable), and the third from the computation of the Doob Meyer decomposition of  $G$  (see above). This computation is necessary, as emphasized in the remark following

the lemma. It follows

$$\begin{aligned}
b &= \mathbb{E} \left( F_s h(s) X_t \int_{v \in ]s, t]} \alpha_t^v \eta(dv) \right) - \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} X_{v-} \alpha_{v-}^v \eta(dv) \right) \\
&\quad + \mathbb{E} (F_s h(s) 1_{s < \tau \leq t} \Delta X_\tau) - \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} \int_{u \in ]v, t]} \frac{d \langle X, \alpha^v \rangle_u}{\alpha_{u-}^v} \alpha_t^v \eta(dv) \right) \\
&= \mathbb{E} \left( F_s h(s) \int_{v \in ]s, t]} \left( (X_t \alpha_t^v - X_v \alpha_v^v) - \int_{u \in ]v, t]} d \langle X, \alpha^v \rangle_u \right) \eta(dv) \right) \\
&\quad + \mathbb{E} \left( F_s h(s) \left( \int_{v \in ]s, t]} \Delta (X \alpha^v)_v \eta(dv) - 1_{s < \tau \leq t} \Delta X_\tau \right) \right) \\
&= \mathbb{E} \left( F_s h(s) \left( \int_{v \in ]s, t]} \Delta X_v \alpha_v^v \eta(dv) - 1_{s < \tau \leq t} \Delta X_\tau \right) \right)
\end{aligned}$$

where the first equality comes from an application of the lemma to the  $\mathbb{F}$ -predictable process indexed by  $u$   $J_t^u = 1_{s < u \leq t} \int_u^t dK_v(u)$  ( $F_s$  is  $\mathcal{F}_t$ -measurable), and the second equality, from the fact that a  $\mathbb{F}$ -predictable process is adapted to  $\mathbb{F}_-$  and that the expectation of a martingale conditionally to  $\mathcal{F}_{u-}$  is its value at  $u-$ .

If  $\tau$  avoids the  $\mathbb{F}$ -stopping times (( $\mathcal{A}$ )-condition), the proof is done: indeed the law  $\eta$  is therefore continuous and

$$\mathbb{E} \left( F_s h(s) \left( \int_{v \in ]s, t]} \Delta X_v \alpha_v^v \eta(dv) \right) \right) = \mathbb{E} \left( F_s h(s) \left( \int_{v \in ]s, t]} \Delta X_v \alpha_{v-}^v \eta(dv) \right) \right) = 0$$

since  $\mathbb{E}(\Delta X_v | \mathcal{F}_{v-}) = 0$  (recall  $X$  is a  $\mathbb{F}$ -martingale), whereas  $\mathbb{E}(F_s h(s) 1_{s < \tau \leq t} \Delta X_\tau) = 0$  (the  $\mathbb{F}$ -martingale  $X$  jumps at  $\mathbb{F}$ -stopping times, hence not at  $\tau$ ). Let us consider from now on that ( $\mathcal{A}$ )-condition does not hold. Assume the following result is true: for any  $\mathbb{F}$ -stopping time  $T$ ,

$$\mathbb{E} (1_{\{\tau=T\}} | \mathcal{F}_T) = \alpha_T^T \eta(T).$$

This relation will be proved in the next lemma. We write  $(T_n)_{n \geq 0}$  a sequence of  $\mathbb{F}$ -stopping times that exhausts the jumps of  $X$ . We have:

$$\begin{aligned}
\mathbb{E} (F_s h(s) 1_{s < \tau \leq t} \Delta X_\tau) &= \sum_{n \geq 0} \mathbb{E} (F_s h(s) 1_{s < T_n \leq t} \Delta X_{T_n} 1_{\{\tau=T_n\}}) \\
&= \sum_{n \geq 0} \mathbb{E} (F_s h(s) 1_{s < T_n \leq t} \Delta X_{T_n} \mathbb{E} (1_{\{\tau=T_n\}} | \mathcal{F}_{T_n})) \\
&= \sum_{n \geq 0} \mathbb{E} (F_s h(s) 1_{s < T_n \leq t} \Delta X_{T_n} \alpha_{T_n}^{T_n} \eta(T_n)) \\
&= \mathbb{E} \left( F_s h(s) \left( \int_{v \in ]s, t]} \Delta X_v \alpha_v^v \eta(dv) \right) \right)
\end{aligned}$$

where first and last equalities come from the definition of the dates  $T_n$ , the second from the fact that  $F_s h(s) 1_{s < T_n \leq t} \Delta X_{T_n} \in \mathcal{F}_{T_n}$  and the third from next lemma. This leads to the conclusion that  $\mathbb{E}(F_s h(\tau \wedge s) (Y_t - Y_s)) = 0$ . Using the monotone class theorem,  $\mathbb{E}(G_s (Y_t - Y_s)) = 0$  for any  $G_s$   $\mathcal{G}_s$ -measurable, hence  $Y$  is a  $\mathbb{G}$ -martingale.  $\square$

We used in the proof the following result:

**Lemma 2** *Let  $T$  be a finite  $\mathbb{F}$ -stopping time,*

$$\mathbb{E} (1_{\{\tau=T\}} | \mathcal{F}_T) = \alpha_T^T \eta (\{T\}) \text{ p.s.}$$

*Proof.* Let us first prove that for any  $\varepsilon \geq 0$  and  $T$  finite  $\mathbb{F}$ -stopping time:

$$\mathbb{E} (1_{\{\tau \leq T-\varepsilon\}} | \mathcal{F}_T) = \int_0^{T-\varepsilon} \alpha_T^u \eta (du).$$

Recall that  $\mathcal{F}_T = \{A \in \mathcal{F}, A \cap \{T \leq t\} \in \mathcal{F}_t\}$  also writes  $\mathcal{F}_T = \{Z_T, Z \text{ bounded } \mathbb{F}\text{-optional process}\}$ . Let  $Z$  be a bounded  $\mathbb{F}$ -optional process right continuous. We shall prove that

$$\mathbb{E} (Z_T 1_{\{\tau \leq T-\varepsilon\}}) = \mathbb{E} \left( Z_T \int_0^{T-\varepsilon} \alpha_T^u \eta (du) \right).$$

For any  $n \geq 0$ , consider  $T_n$  defined by  $T_n = (k+1)/2^n$  if  $k/2^n < T \leq (k+1)/2^n$ . This defines a sequence of  $\mathbb{F}$ -stopping times decreasing to  $T$ . We have

$$\begin{aligned} \mathbb{E} (Z_{T_n} 1_{\{\tau \leq T_n-\varepsilon\}}) &= \sum_k \mathbb{E} \left( Z_{\frac{k+1}{2^n}} 1_{\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\}} 1_{\{\tau \leq \frac{k+1}{2^n}-\varepsilon\}} \right) \\ &= \sum_k \mathbb{E} \left( Z_{\frac{k+1}{2^n}} 1_{\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\}} \mathbb{E} \left( 1_{\{\tau \leq \frac{k+1}{2^n}-\varepsilon\}} | \mathcal{F}_{\frac{k+1}{2^n}} \right) \right) \\ &= \sum_k \mathbb{E} \left( Z_{\frac{k+1}{2^n}} 1_{\{\frac{k}{2^n} < T \leq \frac{k+1}{2^n}\}} \int_0^{(k+1)/2^n-\varepsilon} \alpha_{\frac{k+1}{2^n}}^u \eta (du) \right) \\ &= \mathbb{E} \left( Z_{T_n} \int_0^{T_n-\varepsilon} \alpha_{T_n}^u \eta (du) \right), \end{aligned}$$

where the third equality comes from the fact that  $\tau$  is an initial time. Since all the processes involved are right continuous and bounded, Lebesgue and Beppo-Levi theorems imply when  $n$  tends to infinity that

$$\begin{aligned} \mathbb{E} (Z_T 1_{\{\tau \leq T-\varepsilon\}}) &= \mathbb{E} \left( Z_T \int_0^{T-\varepsilon} \alpha_T^u \eta (du) \right), \text{ hence} \\ \mathbb{E} (1_{\{\tau \leq T-\varepsilon\}} | \mathcal{F}_T) &= \int_0^{T-\varepsilon} \alpha_T^u \eta (du). \end{aligned}$$

It follows, by application to a null  $\varepsilon$  (the results has been proved for any  $\varepsilon \geq 0$ ) and difference,

$$\mathbb{E} (1_{\{T-\varepsilon < \tau \leq T\}} | \mathcal{F}_T) = \int_{T-\varepsilon}^T \alpha_T^u \eta (du)$$

for any  $\varepsilon$ , and if  $\varepsilon$  tends to 0, Lebesgue's theorem (left member) and Beppo-Levi theorem (right member) imply

$$\mathbb{E} (1_{\{\tau=T\}} | \mathcal{F}_T) = \alpha_T^T \eta (T),$$

which concludes the proof of the lemma. □

We close this section with a brief corollary, dealing with a simple characterization of immersion in the initial time set-up (that can be useful in credit modelling applications):

**Corollary 1** *Under the condition that the initial time  $\tau$  avoids the  $\mathbb{F}$ -stopping times, there is equivalence between  $\mathbb{F}$  immersed in  $\mathbb{G}$  and for any  $u \geq 0$ , the martingale  $\alpha^u$  is constant after  $u$ .*

*Proof.*

• If for any  $u \geq 0$ , the martingale  $\alpha^u$  is constant after  $u$ , then, the martingale part  $M$  of the Doob-Meyer decomposition of  $G$  is constant. Indeed,

$$M_t = \int_0^\infty \alpha_{u \wedge t}^u \eta(du) = \int_0^\infty \alpha_t^u \eta(du) = \mathbb{P}(\tau > 0 | \mathcal{F}_t) = 1,$$

hence for any  $X \in \mathcal{M}(\mathbb{F})$ , one has  $\int_0^{t \wedge \tau} \frac{d\langle X, M \rangle_u}{G_{u-}} = 0$ . Moreover, for any  $F_t \in \mathcal{F}_t$  and bounded Borel function  $h$ , setting  $Z_t = F_t h(t \wedge \tau)$ ,

$$\begin{aligned} \mathbb{E} \left( Z_t \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Big|_{\theta=\tau} \right) &= \mathbb{E} \left( F_t 1_{\tau \leq t} h(\tau) \int_\tau^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Big|_{\theta=\tau} \right) \\ &= \mathbb{E} \left( F_t \int_0^t h(s) \int_s^t \frac{d\langle X, \alpha^s \rangle_u}{\alpha_{u-}^s} \alpha_t^s \eta(ds) \right) = 0 \end{aligned}$$

since  $d\langle X, \alpha^s \rangle_u = 0$  for  $u \geq s$ . It follows from 1 and the fact that, under  $(\mathcal{A})$  condition, the  $B$  part in the decomposition disappears, that  $X \in \mathcal{M}(\mathbb{G})$  and  $\mathbb{F}$  is immersed in  $\mathbb{G}$ .

• Assume now that  $(\mathcal{H})$  hypothesis holds. For any  $X \in \mathcal{M}(\mathbb{F})$ , we have  $X \in \mathcal{M}(\mathbb{G})$  hence with the notation of theorem (1)  $Y - X \in \mathcal{M}(\mathbb{G})$  and is a predictable process with finite variation (formula (1)), hence equal to zero. It follows almost surely that

$$\int_0^{t \wedge \tau} \frac{d\langle X, G \rangle_u}{G_{u-}} + \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\theta \rangle_u}{\alpha_{u-}^\theta} \Big|_{\theta=\tau} = 0,$$

hence each integral is equal to zero almost surely, and in particular the second. It follows that for any  $\mathcal{G}_t$ -measurable random variable  $Z_t$  of the form  $Z_t = F_t h(t \wedge \tau)$ ,

$$\begin{aligned} 0 &= \mathbb{E} \left( Z_t \int_{t \wedge \tau}^t \frac{d\langle X, \alpha^\tau \rangle_u}{\alpha_u^\tau} \right) = \mathbb{E} \left( F_t 1_{\tau \leq t} h(\tau) \int_\tau^t \frac{d\langle X, \alpha^\tau \rangle_u}{\alpha_u^\tau} \right) \\ &= \mathbb{E} \left( F_t \int_0^t h(s) \int_s^t \frac{d\langle X, \alpha^s \rangle_u}{\alpha_u^s} \alpha_t^s \eta(ds) \right) \end{aligned}$$

hence  $\alpha_t^s \int_s^t (d\langle X, \alpha^s \rangle_u / \alpha_u^s) = 0$ , which implies  $d\langle X, \alpha^s \rangle_u = 0$  for any  $u \geq s$  and any  $X \in \mathcal{M}(\mathbb{F})$ . It follows that for any  $u \geq 0$ , the martingale  $\alpha^u$  is constant after  $u$ .  $\square$

As seen in the proof,  $M = cst$  if for any  $u \geq 0$ , the martingale  $\alpha^u$  is constant after  $u$ . As we shall see in the examples presented in the third part, the assumption that the initial time avoids the  $\mathbb{F}$ -stopping time is not too strong in applications such as credit modelling.

The same kind of characterization for immersion under  $(\mathcal{A})$  condition is also easy to establish in a "honest" enlargement set up (there is equivalence between  $G$  is decreasing and predictable and  $\mathbb{F}$  is immersed in  $\mathbb{G}$ ) but the applications of such a result are very restrictive since as proved in [20], if  $G$  is a decreasing process, the time  $\tau$  is a pseudo-stopping time, and it is known that under  $(\mathcal{A})$  condition a honest time can not be a pseudo-stopping<sup>5</sup> (see for example [20]).

<sup>5</sup>We would like to thank A. Nikeghbali for having pointed out this important remark.

## 4 Successive Times

In areas like credit derivatives, a multi-dimensional version of this result is necessary for the modelling. The generalization of the theorem and its complexity depend deeply on the nature of the assumptions made on the random times (the credit events). First the quantities of interest in most of the situations are the ranked defaults - and not the default times themselves - and second, the expansion of the knowledge after the  $k^{\text{th}}$  default must be considered with respect to the filtration enlarged with the  $k - 1$  last credit events, and not to the reference filtration.

The natural framework is therefore the following<sup>6</sup>. Starting from a vector of  $n$  random times  $\theta_1, \dots, \theta_n$ , we define the vecteur of ranked times  $\tau_1, \dots, \tau_n$  ( $\tau_1$  is the smallest  $\theta_i$ , etc.). We say such a vector of increasing times is recursively initial w.r.t.  $\mathbb{F}$ , if for any  $k$  the  $k^{\text{th}}$  time  $\tau_k$  is an initial time w.r.t.  $\mathbb{F}^{k-1} = \mathbb{F} \vee \mathbb{H}^1 \vee \dots \vee \mathbb{H}^{k-1}$  (with  $\mathbb{F}^0 = \mathbb{F}$ ), i.e., there exists a family of  $\mathbb{F}^{k-1}$ -martingales  $\alpha^k(s)$  indexed by  $\mathbb{R}^+$  (for any  $s \geq 0$ ,  $(\alpha_t^k(s))_{t \geq 0} \in \mathcal{M}(\mathbb{F}^{k-1})$ ) and a distribution  $\eta^k$  on  $\mathbb{R}^+$ , such that

$$\mathbb{P}(\tau_k \geq T | \mathcal{F}_T^{k-1}) = \int_T^\infty \alpha_t^k(s) \eta^k(ds).$$

It follows the corollary (recall  $G_t^i = \mathbb{P}(\tau_i \geq t | \mathcal{F}_t^{i-1})$ )

**Corollary 2** *Let  $(\tau_1, \dots, \tau_n)$  be a vector of increasing times recursively initial w.r.t  $\mathbb{F}$ , such that each martingale  $(\alpha_t^i(u))_{t \geq 0}$  is a square integrable  $\mathbb{F}^{i-1}$ -martingale for each  $u \geq 0$  and  $i \leq n$  (resp. in  $BMO$  w.r.t  $\mathbb{F}^{i-1}$ ). For any square integrable  $\mathbb{F}$ -martingale  $X$  (resp.  $X \in \mathcal{M}^{loc}(\mathbb{F})$ ) and any  $k \leq n$ , the process  $Y^k$  defined as*

$$Y_t^k = X_t - \sum_{i=1}^k \int_0^{t \wedge \tau_i} \frac{d\langle X, G^i \rangle_u + dB_u^i}{G_{u-}^i} - \sum_{i=1}^k \int_{t \wedge \tau_i}^t \frac{d\langle X, \alpha^i(\theta) \rangle_u}{\alpha_{u-}^i(\theta)} \Big|_{\theta=\tau_i} \quad (4)$$

is a  $\mathbb{F}^k$ -martingale (resp.  $Y^k \in \mathcal{M}^{loc}(\mathbb{F}^k)$ ).

*Proof.* This result is straightforward to prove by induction. Assume it holds for any vector of size  $k - 1$  (for  $k = 1$  it reduces to formula (1)).

If  $X$  is a square integrable martingale,

$$Y_t^{k-1} = X_t - \sum_{i=1}^{k-1} \int_0^{t \wedge \tau_i} \frac{d\langle X, G^i \rangle_u + dB_u^i}{G_{u-}^i} - \sum_{i=1}^{k-1} \int_{t \wedge \tau_i}^t \frac{d\langle X, \alpha^i(\theta) \rangle_u}{\alpha_{u-}^i(\theta)} \Big|_{\theta=\tau_i} \equiv X_t - A_t^k \in \mathcal{M}(\mathbb{F}^{k-1}),$$

with  $\mathbb{F}^{k-1} = \mathbb{F} \vee \mathbb{H}^1 \vee \dots \vee \mathbb{H}^{k-1}$  generated by the first  $k - 1$  times. The only thing to prove is the existence of  $\langle Y^{k-1}, G^i \rangle = \langle Y^{k-1}, M^i \rangle$  and of  $\langle Y^{k-1}, \alpha^i(\theta) \rangle$ . The first one is obvious since  $M^i$  is  $BMO$  and  $Y_t^{k-1}$  is a martingale. For the second one, remark that  $[Y^{k-1}, \alpha^i(\theta)] = [X, \alpha^i(\theta)] - [A^k, \alpha^i(\theta)]$ . The process  $[X, \alpha^i(\theta)]$  is integrable since the two semi-martingales are integrable, and  $[A^k, \alpha^i(\theta)]$  is a local martingale ( $A^k$  is predictable with finite variation and  $\alpha^i(\theta)$  is a martingale) hence locally integrable. It follows  $[Y^{k-1}, \alpha^i(\theta)]$  is locally integrable and the existence of the sharp bracket. The result follows by application of the proof of the theorem for the expansion of  $\mathbb{F}^{k-1}$  by the  $\mathbb{F}^{k-1}$ -initial time  $\tau^k$ .

If  $X \in \mathcal{M}^{loc}(\mathbb{F})$ ,  $Y_t^{k-1} \in \mathcal{M}^{loc}(\mathbb{F}^{k-1})$ . The result follows from an application of the theorem to  $Y_t^{k-1}$  and  $\tau^k$  which is initial with respect to filtration  $\mathbb{F}^{k-1}$  and in  $BMO(\mathbb{F}^{k-1})$ , since  $\langle Y^{k-1}, G^i \rangle = \langle X, G^i \rangle$  and  $\langle Y^{k-1}, \alpha^i(\theta) \rangle = \langle X, \alpha^i(\theta) \rangle$ .  $\square$

<sup>6</sup>Jeulin [12], p. 86, studies the same problem for honest times

## 5 Cox-like Examples

A random time constructed through the Cox method (see introduction or the following paragraph) is not  $\mathcal{F}_\infty$ -measurable, hence can not be honest. The question of  $(\mathcal{H}')$  hypothesis is therefore relevant. We shall prove that this property is in fact always satisfied, since these times are initial times, and that immersion also holds. We present after that two types of constructions derived from this methodology, still entering the initial times framework but without immersion.

**First example: Cox construction.** D. Lando was the first to propose in [17] the use of the Cox construction in credit modelling, in which a filtration  $\mathbb{F}$  is given as well as a non-negative  $\mathbb{F}$ -adapted process  $\lambda$  and where the default time is defined as:

$$\tau = \inf\{t : \Lambda_t \geq \Theta\}$$

with  $\Lambda_t = \int_0^t \lambda_s ds$ ,  $\Theta$  is a given r.v. independent of  $\mathcal{F}_\infty$  with unit exponential law. For  $T$  and  $t \geq 0$ , under the hypothesis  $\int_0^\infty \lambda_s ds = \infty$ ,

$$G_t^T = \mathbb{P}(\tau > T | \mathcal{F}_t) = \mathbb{P}(\Lambda_T < \Theta | \mathcal{F}_t) = \mathbb{E}(\exp(-\Lambda_T) | \mathcal{F}_t) = \mathbb{E}\left(\int_T^\infty \lambda_s \exp(-\Lambda_s) ds | \mathcal{F}_t\right).$$

Setting  $\psi_s = \lambda_s \exp(-\Lambda_s)$  and  $\gamma(s, t) = \mathbb{E}(\psi_s | \mathcal{F}_t)$ ,

$$G_t^T = \int_T^\infty \mathbb{E}(\psi_s | \mathcal{F}_t) ds = \int_T^\infty \gamma(s, t) ds,$$

Note that  $\gamma(s, t) = \psi_s$  for  $s \leq t$ . Let  $\eta([0, t]) = \eta([0, t]) = \int_0^t \gamma(s, 0) ds = \mathbb{P}(\tau \leq t)$  be the law of the variable  $\tau$ . In any cases, we can write

$$G_t^T = \mathbb{P}(\tau > T | \mathcal{F}_t) = \int_T^\infty \alpha_t^s \eta(ds)$$

with  $\alpha_t^s = \gamma(s, t)/\gamma(s, 0)$  (remark  $\lambda \geq 0 \Rightarrow \psi \geq 0$  but non identically null, hence  $\mathbb{E}(\psi_s) > 0$  for any  $s$ ). This process is by construction for any  $s$  a positive martingale, hence the Cox time  $\tau$  is an initial time. It follows  $(\mathcal{H}')$  hypothesis holds, i.e., any  $\mathbb{F}$ -semi-martingale remains a  $\mathbb{G}$ -semi-martingale. Moreover, this time avoids the  $\mathbb{F}$ -stopping times and it is straightforward that for any  $t \geq s$ ,  $\alpha_t^s = \alpha_s^s$ . It follows from Corollary [?] that  $(\mathcal{H})$  hypothesis also holds. This features of Cox times have already been pointed out in [16].

The main drawback in Cox construction in its application in credit modelling, is the corollary that the survival process  $G = 1 - F$  has no martingale part, which restrict the configurations that can be modelled in such a framework. Inspired by the property that in a progressive expansion of filtration,  $(\mathcal{H})$ -hypothesis is equivalent to the conditionally independence of  $\mathcal{F}_\infty$  and  $\mathcal{H}_t$  given  $\mathcal{F}_t$  (see P. Brémaud and M. Yor in [2]), M. Jeanblanc and Y. Le Cam proposed in [10] alternative constructions, in which the survival process of the credit event is not decreasing. The two main examples are recall in the sequel and we establish the property of initial times gathered by this type of construction.

**Second example: "Cox-like" construction, a finite case.**

Here again, we consider  $\mathbb{F}$  a reference filtration,  $\lambda$  a non-negative  $\mathbb{F}$ -adapted process,  $\Lambda_t = \int_0^t \lambda_s ds$ ,  $\Theta$  a given r.v. independent of  $\mathcal{F}_\infty$  with unit exponential law, and  $V$  be a  $\mathcal{F}_\infty$ -measurable non-negative random variable (such that  $\lambda_u/V$  be integrable for each  $u$ ). We write:

$$\tau = \inf\{t : \Lambda_t \geq \Theta V\}.$$

The variable  $f(\Theta, V) = \Theta V$  is not independent from  $\mathcal{F}_\infty$  anymore. For any  $T$  and  $t$  :

$$\begin{aligned} G_t^T &= \mathbb{E} \left( \mathbb{P} \left( \frac{\Lambda_T}{V} < \Theta \middle| \mathcal{F}_\infty \right) \middle| \mathcal{F}_t \right) = \mathbb{E} \left( \exp - \frac{\Lambda_T}{V} \middle| \mathcal{F}_t \right) \\ &= \int_T^\infty \mathbb{E} (\psi_s | \mathcal{F}_t) ds = \int_T^\infty \gamma(s, t) ds, \end{aligned}$$

with  $\psi_s = \lambda_s/V \exp - \int_0^s \lambda_u/V du$ , and  $\gamma(s, t) = \mathbb{E} (\psi_s | \mathcal{F}_t)$  (remark that for any  $s$ ,  $(\gamma(s, t), t \geq 0)$  is an  $\mathbb{F}$ -martingale). If  $\eta([0, t]) = \int_0^t \gamma(s, 0) ds = \mathbb{P}(\tau \leq t)$  is the law of the variable  $\tau$ , we can write

$$G_t^T = \mathbb{P}(\tau > T | \mathcal{F}_t) = \int_T^\infty \alpha_t^s \eta(ds)$$

with  $\alpha_t^s = \gamma(s, t)/\gamma(s, 0)$ . This process is by construction for any  $s$  a positive martingale, hence the Cox time  $\tau$  is an initial time. It follows  $(\mathcal{H}')$  hypothesis holds, i.e., any  $\mathbb{F}$ -semi-martingale remains a  $\mathbb{G}$ -semi-martingale. Moreover, in this situation, if  $t \geq s$ ,  $\gamma(s, t) \neq \gamma(t, t)$ , hence

$$\alpha_t^s = \frac{\gamma(s, t)}{\gamma(s, 0)} = \frac{\mathbb{E}(\psi_s | \mathcal{F}_t)}{\mathbb{E}(\psi_s)} \neq \alpha_s^s.$$

It follows from corollary 1 that  $(\mathcal{H})$  hypothesis does not hold in this framework (remark this time satisfies  $(\mathcal{A})$  condition since  $\mathbb{P}(\tau = T) = \mathbb{P}(\Lambda_T/V = \Theta) = 0$  for any  $\mathbb{F}$ -stopping time  $T$ , since  $\Theta$  is independent of  $\mathcal{F}_\infty$ ).

Remark that this construction leads to very simple examples in which the initial times can have non square integrable  $\alpha$  (cf. first sectio). Consider for instance a non negative  $\mathbb{F}$ -martingale  $M$  of mean 1/2, not square integrable (i.e., there exist dates  $t$  such that  $\mathbb{E}(M_t^2) = \infty$ ), a date  $T > 0$ , and an integrable  $\mathcal{F}_\infty$ -measurable variable  $K$ , with mean 1/2. We take for random time

$$\tau = \inf\{t : \Lambda_t \geq \Theta V\}.$$

with  $\lambda_s = \lambda$  for any  $s$ , and  $V = 1/(K + M_T) \in \mathcal{F}_\infty$  (which satisfies  $1/V \in L^1$ ). From  $\alpha_t^0 = \gamma(0, t)/\gamma(0, 0)$ ,

$$\alpha_t^0 = \frac{\mathbb{E}(\psi_0 | \mathcal{F}_t)}{\mathbb{E}(\psi_0)} = \frac{\mathbb{E}(\lambda/V | \mathcal{F}_t)}{\mathbb{E}(\lambda/V)} = \frac{\mathbb{E}(K + M_T | \mathcal{F}_t)}{\mathbb{E}(K) + \mathbb{E}(M_T)} = \mathbb{E}(K | \mathcal{F}_t) + M_t \geq M_t$$

hence is non square integrable since  $M$  is not so. It proves that such a property has to be imposed.

This type of construction may present two drawbacks, depending on the field of application: first it is difficult to derive the Doob Meyer decomposition of the process  $F$ , and second the default time in this framework is always finite. A slight modification of the previous example may correct these two points.

**Third example: "Cox-like" construction, a non finite case.** In this part, we force the variable  $V$  to be bigger than 1 and denote by  $U$  a variable uniformly distributed on  $[0, 1]$ , independent of  $\mathcal{F}_\infty$ . We define

$$\tau = \inf\{t : 1 - \exp -\Lambda_t \geq UV\}.$$

The quantity  $1 - \exp -\Lambda_t$  is increasing, starting from 0 and bounded by 1. In some cases, depending on the value of  $V$ , the barrier  $UV$  will be bigger than 1 and never crossed, so that the default time be non finite. We have for any  $T$  and  $t$  :

$$\begin{aligned} G_t^T &= \mathbb{E}(\mathbb{P}(1 - \exp -\Lambda_T < UV | \mathcal{F}_\infty) | \mathcal{F}_t) = \mathbb{E}\left(\mathbb{P}\left(\frac{1 - \exp -\Lambda_T}{V} < U \mid \mathcal{F}_\infty\right) \mid \mathcal{F}_t\right) \\ &= 1 - \mathbb{E}\left(\frac{1 - \exp -\Lambda_T}{V} \mid \mathcal{F}_t\right) = \mathbb{E}\left(\int_T^\infty \psi_s ds \mid \mathcal{F}_t\right) = \int_T^\infty \gamma(s, t) ds, \end{aligned}$$

with  $\psi_s = \lambda_s/V \exp -\int_0^s \lambda_u du$ , and  $\gamma(s, t) = \mathbb{E}(\psi_s | \mathcal{F}_t)$  (remark that for any  $s$ ,  $(\gamma(s, t), t \geq 0)$  is an  $\mathbb{F}$ -martingale). The law of  $\tau$  writes: for any  $t$  (finite)  $\eta([0, t]) = \mathbb{P}(\tau \leq t) = \int_0^t \gamma(s, 0) ds$ ,  $\eta([0, \infty]) = \int_0^\infty \gamma(s, 0) ds = \mathbb{E}(1/V)$  hence  $\eta(\{\infty\}) = 1 - \mathbb{E}(1/V)$ . We can write

$$G_t^T = \mathbb{P}(\tau > T | \mathcal{F}_t) = \int_T^\infty \alpha_t^s \eta(ds)$$

with  $\alpha_t^s = \gamma(s, t)/\gamma(s, 0)$ . This process is by construction for any  $s$  a positive martingale, hence the time  $\tau$  is an initial time. It follows ( $\mathcal{H}'$ ) hypothesis holds, i.e., any  $\mathbb{F}$ -semi-martingale remains a  $\mathbb{G}$ -semi-martingale. In this situation again, if  $t \geq s$ ,

$$\alpha_t^s = \frac{\gamma(s, t)}{\gamma(s, 0)} = \frac{\mathbb{E}(\psi_s | \mathcal{F}_t)}{\mathbb{E}(\psi_s)} \neq \alpha_s^s$$

since  $\psi_s$  is not  $\mathcal{F}_s$  hence  $\mathcal{F}_t$ -measurable because of  $V$ . It follows from corollary 1 that ( $\mathcal{H}$ ) hypothesis does not hold either in this framework (here again, ( $\mathcal{A}$ ) condition holds).

We can compute the Doob Meyer decomposition of  $F = 1 - G$  in this framework, since

$$F_t = \int_0^t \mathbb{E}(\psi_s | \mathcal{F}_t) ds = \mathbb{E}(1/V | \mathcal{F}_t) \int_0^t \left(\lambda_s \exp -\int_0^s \lambda_u du\right) ds = N_t C_t$$

where  $N_t = \mathbb{E}(1/V | \mathcal{F}_t) \in \mathcal{M}(\mathbb{F})$  and  $C_t = \int_0^t (\lambda_s \exp -\int_0^s \lambda_u du) ds$  is an  $\mathbb{F}$ -predictable increasing process. It is therefore by uniqueness the multiplicative decomposition of  $F$ . If  $F_t = M_t + A_t$  is the additive Doob Meyer decomposition of  $F$ , we have

$$dA_t = N_t dC_t = F_t \frac{dC_t}{C_t} \text{ and } dM_t = C_t dN_t = F_t \frac{dN_t}{N_t}.$$

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