Immersion Property and Credit Risk Modelling

Monique Jeanblanc[†], Yann Le Cam[‡],

[†],[‡] Université d'Évry Val d'Essonne
91025 Évry Cedex, France,
[†] Institut Europlace de Finance
[‡] French Treasury

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Abstract

The purpose of this paper is to study the immersion property within a credit risk modelling. The construction of a credit model by enlargement of a reference filtration with the progressive knowledge of a credit event has become a standard for reduced form modelling. It is known that such a construction rises mathematical difficulties, mainly relied to the properties of the random time. Whereas the invariance of the property of semi-martingale in the enlargement is implied by the absence of arbitrage, we address in this paper the question of the invariance of the martingale property.

Introduction

The purpose of this paper is to propose a study of the neutral risk probabilities in the context of credit modelling. Indeed, most of the literature focuses on pricing problems and postulates the existence of a pricing measure, without questioning its features. Within the reduced form approach and particularly under the filtration enlargement framework, such questions may be precisely studied, and lead to interesting properties.

Three steps are developed in the sequel. The first one presents the credit modelling framework and discusses the meaning of the options taken. The second one is a study of the special case where the "reference market" is complete. We shall prove a martingale representation theorem, and establish that under proper conditions the full market is complete as well, and that immersion holds in the filtration enlargement. The last part is dedicated to the incomplete case. Starting from a reference risk-neutral market probability, we construct a unique neutral risk probability that preserves the properties of the reference market and we establish that immersion holds under such a probability.

In this paper, all the processes are constructed on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where the probability $\mathbb{P}$ is referred to as the historical probability.

A financial market is represented in the sequel by a price process $\widetilde{S} = (\widetilde{S}_t, t \leq T)$ (an $\mathbb{R}^{n+2}$ valued process, S^0 denoting the saving accounts, i.e., the risk free asset), and its information by $\mathbb{G}$: The natural (augmented) filtration generated by $\widetilde{S}$. We do not assume that $\mathcal{G}_T = \mathcal{A}$, and we emphasize that $\mathbb{P}$ is a probability defined on $\mathcal{A}$ (even if we shall be interested in the sequel in the restriction of the probabilities on sub-σ-algebras of $\mathcal{A}$). We denote by $\Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})$ the set of $\mathbb{G}$-e.m.m, i.e., the set of probabilities $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that $\widetilde{S} \in \mathcal{M}(\mathbb{G}, \mathbb{Q})$, i.e., the process $(\widetilde{S}_t, t \leq T)$ is a $(\mathbb{G}, \mathbb{Q})$-martingale.

It is well known that there are strong links between no-arbitrage hypothesis and the existence of an equivalent martingale probability (see Kabanov [22], Delbaen and Shachermayer [11]). In this paper we are interested with the condition $\Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})$ is not empty which is equivalent to the No Free Lunch with Vanishing Risk (a condition slightly stronger than absence of arbitrage). The market where the assets $S^i, i = 0, \cdots, n+2$ are traded is complete if any contingent claim is replicable: For any payoff $X_T \in L^2(\mathcal{G}_T)$ there exists a $\mathbb{G}$-adapted self-financed strategy with terminal value X_T. It follows that an arbitrage free market is complete iff under an e.m.m $\mathbb{Q} \in \Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})$, $\widetilde{S}$ admits the PRP (predictable representation theorem). This property is equivalent from Jacod and Yor theorem to the fact that the set composed by the restrictions of the probabilities in $\Theta_{\mathbb{P}}^{\mathbb{G}}(\widetilde{S})$ on $\mathcal{G}_T$ is a singleton, i.e., *there exists a unique martingale probability on* $\mathcal{G}_T$ (this assertion being understood as: The restriction on $\mathcal{G}_T$ of any e.m.m. is unique).

1 Credit modelling framework

We work in this study within a progressive enlargement of filtration set-up, so that to study the pricing of derivatives written on underlyings sensible to a credit event τ. We refer the reader to Elliott et al. [15] or to Jeanblanc and Rutkowski [21] for a detailed presentation of this approach, and to Jeanblanc and Le Cam [20] for the reasons that lead us to adopt it in this context.

In this framework, we shall split the information beared by the market in two components. The first one is generated by what are called in general the default free assets, and the second by the default time (the probability of occurrence of this event depends on factors adapted to the first filtration).

Precisely, we consider the $n+1$-dimensional vector S of the assets $S^0, \cdots, S^n$ and its natural filtration $\mathbb{F}$, referred to as the reference filtration in the sequel[1]. This information flow does not contain the information of the occurrence of the credit event. These assets, that do not bear the

[1] In [3] A. Bélanger et al. refer to $\mathbb{F}$ as the *non firm specific information*. For us, this information flow must be considered as the "market risk" information, and can bear assets linked to the firm, for example its equity or even its directly its spread risk, see later.

direct information of the default, are intended to be modelled by the set of variables $S^i, 0 \leq i \leq n$ (Equity, vanilla options, interest rates, change rates... all information that can be used by a trader that has to manage a position depending on τ, or that can be used by the market to make its idea on the probability of occurrence of the risk, and impact the bid-ask price of instruments written on τ). For example, if τ is the default time of a bond issued by a firm X, it is not a stopping time with respect to the filtration generated by the stock of X and the stochastic interest rates (even it is far from being independent of such variables).

We denote by G the Azéma supermartingale

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$$

Obviously, this process depends on the choice of the probability, nevertheless, for ease of notation, we do not indicate this probability, which will be clear in the context.

We denote by $\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$ the set of $\mathbb{F}$-e.m.m, i.e., the set of probabilities $\mathbb{Q}$ defined on $\mathcal{A}$, equivalent to $\mathbb{P}$ on $\mathcal{A}$, such that $S \in \mathcal{M}(\mathbb{F}, \mathbb{Q})$, i.e., the process $S = (S_t, t \leq T)$ is a $(\mathbb{F}, \mathbb{Q})$-martingale. We assume the hypothesis:

Hypothesis H_1: The reference market is arbitrage free (we assume no interest rate to ease the presentation), i.e., $\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$ is not empty.

We introduce the asset S^{n+1}, that bears direct information on τ, i.e., that satisfies:

$$\mathcal{H}_t \subset \sigma\left(S_s^{n+1}, s \leq t\right) \subset \mathcal{H}_t \vee \mathcal{F}_t \text{ for any } t \geq 0,$$

where the notation $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ stands for the natural augmentation of the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$, modelling the knowledge of the occurrence of the default. This relation means that the default can be read on the value of S^{n+1}, and that S^{n+1} can be priced in terms of τ and $\mathbb{F}$ (think of a risky bond, a defaultable zero coupon or a credit default swap, CDS in the sequel). We denote by $\widetilde{S}$ the vector $\left(S^0, S^1, ..., S^{n+1}\right)$, and by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, the natural augmentation of the filtration generated by $\widetilde{S}$ (the full information of the market). We add the following hypothesis:

Hypothesis H_2: We assume that the full market is arbitrage free, i.e., $\Theta_{\mathbb{P}}^{\mathbb{G}}\left(\widetilde{S}\right)$ is not empty.

In terms of risks analysis, S^{n+1} has in general two types of risks (we consider in this survey only the single default case, hence do not enter a discussion about the correlation risk): a "market risk" - typically a spread risk, i.e., the natural variation of the price of the asset when time goes on - and a jump risk - the specific risk of default, due to the occurrence of τ. *This framework is based on the assumption that the market risk can be hedged with $\mathbb{F}$-adapted instruments, and that the jump risk relies on $\mathbb{H}$-adapted instruments.*

Two points of view can be considered in such a matter. The first one - based on economic analysis - asserts that the spread risk is mainly ruled out by the same noise sources that the assets that generate the reference filtration. For example:

- In the context of firm bonds pricing, Bélanger et al. link in [3] the spread risk of the defaultable zero coupon to the stochastic interest rates. In such a modelling, the credit event is constructed as the hitting time of an independent stochastic barrier by an $\mathbb{F}$-adapted process (the $\mathbb{F}$-intensity), where $\mathbb{F}$ is the filtration bearing the stochastic interest rates movements (basically, the Brownian motion driving the intensity is the same as the noise source of the IR). The parameters of the intensity process depend on the firm (see also [12] where Ehlers and Schönbucher insist on the rôle of the systemic risk implied by the IR on a portfolio of credit risks).

- Moreover in a very close matter, Carr and Wu in [7] or Cremers et al. in [9] show that corporate CDS spreads covary with both the stock option implied volatilies and skewness. It insists on the fact that the factors ruling out the movements of the spread are linked to the variations of the interest rate and of the equity (and its volatility).

- In the context of modelling CDS on debt issued by states (in their example Mexico and Brasil), Carr and Wu study in [8] the correlation between the currency options and the credit spreads. They prove that these quantities are deeply linked and propose a model in which the alea driving the intensity of the default is composed by the sum of a function of the alea of the stochastic volatility of the FX (see Heston [17]), and an independent noise (see also Ehlers and Schönbucher in [13]).

- More generally, this vision is shared by the supporters of structural modelling, in which the default time is triggered by a barrier reached by the equity value (see [25] or [5] for example). In [2], Atlan and Leblanc model the credit time as the reaching time of zero of the Equity of the firm, following a CEV (see also Albanese and Chen in [1] or Linetsky [24]).

The second way is based on the introduction of a new noise source, this alea driving the spread risk, considered as having its own evolution (both approaches can be combined as in [8]). In this construction as well, the "market risk of the defaultable security" does not contain the default occurrence knowledge, and can be sorted in the $\mathbb{F}$-information with the other market risks sensible assets. In reality, it is easy to synthetize an asset that is sensible to this spread risk and not to the jump risk. Take two instruments as S^{n+1} of different maturity for example, namely X^1 and X^2, and assume the market risk is modelled by a (risk neutral) Brownian motion W. If M is the compensated martingale associated with H, we have $dX_t^i = \beta_t^i dM_t + \delta_t^i dW_t$. Set up the self financed portfolio Π that is long at any time of β_t^2 of the asset X^1 and short of β_t^1 of X^2 (and has a position in the savings account to stay self-financed). This portfolio has only sensitivity against the spread risk, and does not jump with τ, since $d\Pi_t = r\Pi_t dt + \beta_t^2 dX_t^1 - \beta_t^1 dX_t^2 = r\Pi_t dt + \left(\beta_t^2\delta_t^1 - \beta_t^1\delta_t^2\right) dW_t$. Remark that with a δ-combination, we can set up a portfolio only sensible of the jump risk (and that has no spread risk).

The two points of view (that need to be combined to achieve a maximum of precision in calibration procedures) converge on the idea that splitting the information of the market in two filtrations is quite natural. Another nomenclature may consist in "market risk filtration"for $\mathbb{F}$, and "default risk filtration"for $\mathbb{H}$.

Hypothesis H_2 implies to work in a mathematical set up where $\mathbb{F}$-semi-martingales remain $\mathbb{G}$-semi-martingales. As developed in Jeanblanc and Le Cam [20], this property does not hold for any random time τ, and we choose to work under the

Hypothesis H_3**:** The credit event is an initial time, that is there exists a family of processes α^u where, for any u, the process $(\alpha_t^u, t \geq 0)$ is an $\mathbb{F}$-martingale such that

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t^u du$$

(in the general definition of initial times, du may be replaced by $\eta(du)$, where η is a finite non-negative measure on $\mathbb{R}^+$). Refer to the thesis of Jiao [18] or to the paper of Jeanblanc and Le Cam [19] for a study of the properties of these times.

In such a context, every $(\mathbb{F}, \mathbb{P})$-martingale X is a $(\mathbb{G}, \mathbb{P})$-semi-martingale and:

$$X_t - \int_0^{t\wedge\tau} \frac{d\left\langle X, Z\right\rangle_u}{G_{u-}} - \int_{t\wedge\tau}^t \left.\frac{d\left\langle X, \alpha^\theta\right\rangle_u}{\alpha_{u-}^\theta}\right|_{\theta=\tau} \in \mathcal{M}(\mathbb{G}, \mathbb{P}).$$

(see [19]). In this paper, we shall implicitly use the

Proposition 1 *If τ is an initial time under $\mathbb{P}$, and $\mathbb{Q}$ is equivalent to $\mathbb{P}$, then τ is a $\mathbb{Q}$-initial time.*

Proof. Let η_∞ be the $\mathcal{G}_\infty$-density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$:

$$d\mathbb{Q}|_{\mathcal{G}_\infty} = \eta_\infty d\mathbb{P}|_{\mathcal{G}_\infty}.$$

For any $T, t > 0$, Bayes rule implies:

$$\mathbb{Q}(\tau > T | \mathcal{F}_t) = \mathbb{E}^{\mathbb{Q}}((1 - H_T)|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}((1 - H_T)\eta_\infty|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_\infty|\mathcal{F}_t)}$$

Assume in a first step that $\eta_\infty = \widetilde{\eta}_\infty h(\tau)$ where $\widetilde{\eta}_\infty$ is an $\mathcal{F}_\infty$-measurable and h is a deterministic function

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}((1 - H_T)\eta_\infty|\mathcal{F}_t) &= \mathbb{E}^{\mathbb{P}}((1 - H_T)\widetilde{\eta}_\infty h(\tau)|\mathcal{F}_t) = \mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_\infty \mathbb{E}^{\mathbb{P}}((1 - H_T)h(\tau)|\mathcal{F}_\infty)|\mathcal{F}_t) \\
&= \mathbb{E}^{\mathbb{P}}\left(\widetilde{\eta}_\infty \int_T^\infty h(u)\alpha_\infty^u du \middle| \mathcal{F}_t\right) = \int_T^\infty \mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_\infty \alpha_\infty^u|\mathcal{F}_t)h(u)du
\end{aligned}$$

It follows that

$$\mathbb{Q}(\tau > T|\mathcal{F}_t) = \int_T^\infty \frac{\mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_\infty \alpha_\infty^u|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_\infty|\mathcal{F}_t)} h(u)du$$

Moreover, if μ_∞ denotes the $\mathcal{F}_\infty$-density of $\mathbb{Q}$ w.r.t. $\mathbb{P}$, i.e., $d\mathbb{Q}|_{\mathcal{F}_\infty} = \mu_\infty d\mathbb{P}|_{\mathcal{F}_\infty}$, μ_∞ writes

$$\mu_\infty = \mathbb{E}^{\mathbb{P}}(\eta_\infty|\mathcal{F}_\infty) = \widetilde{\eta}_\infty \mathbb{E}^{\mathbb{P}}(h(\tau)|\mathcal{F}_\infty) = \widetilde{\eta}_\infty \int_0^\infty h(u)\alpha_\infty^u du := \widetilde{\eta}_\infty h_\infty.$$

It follows that the $\mathbb{F}$-adapted process $\widehat{\alpha}^u_.$ defined for any $u \geq 0$ by:

$$\frac{\widehat{\alpha}^u_t}{h(u)} := \mathbb{E}^{\mathbb{Q}}(\alpha^u_\infty/h_\infty|\mathcal{F}_t) = \frac{\mathbb{E}^{\mathbb{P}}(\alpha^u_\infty\mu_\infty/h_\infty|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\mu_\infty|\mathcal{F}_t)} = \frac{\mathbb{E}^{\mathbb{P}}(\alpha^u_\infty\widetilde{\eta}_\infty|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_\infty|\mathcal{F}_\infty|\mathcal{F}_t)} = \frac{\mathbb{E}^{\mathbb{P}}(\widetilde{\eta}_\infty\alpha^u_\infty|\mathcal{F}_t)}{\mathbb{E}^{\mathbb{P}}(\eta_\infty|\mathcal{F}_t)},$$

is an $(\mathbb{F},\mathbb{Q})$-martingale and that

$$\mathbb{Q}(\tau > T|\mathcal{F}_t) = \int_T^\infty \widehat{\alpha}^u_t du,$$

which means τ is an $\mathbb{F}$-initial time under $\mathbb{Q}$.The general case follows by application of the monotone class theorem. □

It follows that the assumption that the time is initial does not depend on the probability, which will be capital throughout the sequel where we shall study change of equivalent probabilities.

2 Complete reference market

In this section, we make the assumption that the reference market is complete: For any $X_T \in L^2(\mathcal{F}_T)$, there exist n $\mathbb{F}$ -predictable processes φ^i such that $X_T = x + \int_0^T \sum_{1\leq i\leq n}\varphi^i_u dS^i_u$. (We have assumed that the interest rate is null.) Assuming the no-arbitrage hypothesis, this property is equivalent to the fact that the restriction of $\Theta^{\mathbb{F}}_{\mathbb{P}}(S)$ on $\mathcal{F}_T$ is a singleton (Jacod and Yor theorem): It does not imply that there exists a unique probability $\mathbb{Q}$ such that S is an $(\mathbb{F},\mathbb{Q})$-martingale, but that if two probabilities $\mathbb{P}^*$ and $\mathbb{Q}^*$ belong to $\in\Theta^{\mathbb{F}}_{\mathbb{P}}(S)$, then, their restriction to $\mathcal{F}_T$ are equal: $\mathbb{P}^*|_{\mathcal{F}_T} = \mathbb{Q}^*|_{\mathcal{F}_T}$.

We are interested in this section in the properties of the $\mathbb{F}$-adapted assets in the full filtration, and in the completeness of the full market. Under hypothesis H_3, the $\mathbb{F}$-martingales are $\mathbb{F}$-semi-martingales, and the initial time property is stable when changing the filtration. We also assume that τ avoids the $\mathbb{F}$-stopping times.

2.1 The $\mathbb{F}$-adapted assets in the full market

2.1.1 Immersion and $\mathbb{G}$-e.m.m.

For any $\mathbb{Q} \in\Theta^{\mathbb{G}}_{\mathbb{P}}\left(\widetilde{S}\right)$, it follows[2] that $\mathbb{Q} \in\Theta^{\mathbb{F}}_{\mathbb{P}}(S)$. Since the reference market is complete, this implies that the restriction of $\mathbb{Q}$ to the σ-algebra $\mathcal{F}_T$ is unique: All the e.m.ms of the full market have the same restriction on $\mathcal{F}_T$. Such a result will be confirmed by the next representation theorem.

Moreover immersion must hold under every $\mathbb{G}$-e.m.m. Indeed, let $\mathbb{Q} \in\Theta^{\mathbb{G}}_{\mathbb{P}}(S)$ and $X \in \mathcal{M}(\mathbb{F},\mathbb{Q})$, with $X_t = \mathbb{E}^{\mathbb{Q}}(X_T|\mathcal{F}_t)$. As recalled above, the completion of the market implies the existence of $\mathbb{F}$ -predictable processes φ^i such that $X_T = x + \int_0^T \sum_{1\leq i\leq n}\varphi^i_u dS^i_u$. Therefore,

$$\mathbb{E}^{\mathbb{Q}}(X_T|\mathcal{F}_t) = x + \sum_{i\leq n}\int_0^t \varphi^i_u dS^i_u + \mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi^i_u dS^i_u \middle| \mathcal{F}_t\right) = x + \sum_{i\leq n}\int_0^t \varphi^i_u dS^i_u$$

[2] $\Theta^{\mathbb{G}}_{\mathbb{P}}\ \widetilde{S}\ \subset \Theta^{\mathbb{G}}_{\mathbb{P}}(S) \subset \Theta^{\mathbb{F}}_{\mathbb{P}}(S)$, because S is $\mathbb{F}$-adapted

Since $\mathbb{Q} \in \Theta_{\mathbb{P}}^{\mathbb{G}}(S)$ it follows that the process $\int_0^\cdot \varphi_u^i dS_u^i$ is a $(\mathbb{G},\mathbb{Q})$-martingale (we impose for example to X to be a square integrable martingale, so that to avoid cases where the integrals are strict local martingales), hence $\mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi_u^i dS_u^i \middle| \mathcal{G}_t\right) = 0$. Therefore,

$$= x + \sum_{i\leq n}\int_0^t \varphi_u^i dS_u^i = x + \sum_{i\leq n}\int_0^t \varphi_u^i dS_u^i + \mathbb{E}^{\mathbb{Q}}\left(\int_t^T \varphi_u^i dS_u^i \middle| \mathcal{G}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(X_T \middle| \mathcal{G}_t\right),$$

hence $X \in \mathcal{M}(\mathbb{G},\mathbb{Q})$ and immersion holds under $\mathbb{Q}$. Such a result had already been pointed out by Blanchet-Scaillet and Jeanblanc in [6].

This means that if the reference market is complete with neutral risk probability $\mathbb{P}^*$, a construction of the default time in which immersion does not hold imply that $\mathbb{P}^*$ is not a neutral risk measure for the full market. It is then necessary to change the probability, as we shall see in the sequel.

Said differently, if $\mathbb{F}$ is complete and $\mathbb{P}^* \in \Theta_{\mathbb{P}}^{\mathbb{F}}(S)$, *if the* $(\mathbb{F},\mathbb{P}^*)$*-conditional survival process* G^* *has a non constant martingale part,* $\mathbb{P}^*$ *is not a* $\mathbb{G}$*-em.m., i.e.,* $\mathbb{P}^* \notin \Theta_{\mathbb{P}}^{\mathbb{G}}(S)$.

Indeed the following characterization of immersion has been proved in [19]: Under the condition that the default time avoids the $\mathbb{F}$-stopping time, there is equivalence between $\mathbb{F}$ immersed in $\mathbb{G}$ and for any $u \geq 0$, the martingale α^u is constant after u. It follows that under immersion

$$G_t = \int_0^\infty \alpha_{t\wedge u}^u du - \int_0^t \alpha_u^u du = \int_0^\infty \alpha_t^u du - A_t = \mathbb{P}(\tau > 0|\mathcal{F}_t) - A_t = 1 - A_t$$

hence G is decreasing and predictable. By uniqueness of the predictable decomposition of the special $\mathbb{F}$-semi-martingale, if G is decreasing and predictable, $\int_0^\infty \alpha_{t\wedge u}^u du = 1$ for any t hence immersion holds. It follows that immersion is equivalent to the property of G being predictable and decreasing.

2.1.2 A predictable representation theorem in the full market.

For the sake of simplicity, we assume the process S is continuous and one-dimensional. For $\mathbb{P}^* \in \Theta_{\mathbb{P}}^{\mathbb{F}}(S)$, we write

$$G_t^\theta := \mathbb{P}^*(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t^u du \tag{1}$$

and

$$G_t = G_t^t = \int_0^\infty \alpha_{t\wedge u}^u du - \int_0^t \alpha_u^u du \equiv Z_t - A_t, \tag{2}$$

where $Z - A$ stand for the Doob-Meyer decomposition of the supermartingale G and shall both denote $\alpha_t(u)$ or α_t^u in the sequel. Moreover, it is well known (see for example, Bielecki and Rutkowski [4]) that

$$M_t := H_t - \int_0^{t\wedge\tau} \frac{dA_u}{G_{u-}} = H_t - \int_0^t (1 - H_u)\frac{\alpha_u^u}{G_u} du \tag{3}$$

is a $(\mathbb{G},\mathbb{P}^*)$-martingale.

By representation theorem, we denote by a^u the "density" of α^u w.r.t. S (resp. z the density Z), namely the $\mathbb{F}$-predictable such that $d\alpha_t^u = a_t^u dS_t$ (resp. $dZ_t = z_t dS_t$). Then, the process

$$\begin{aligned}
\widehat{S}_t &:= S_t - \int_0^t \frac{(1-H_u)}{G_u} d\langle S, Z\rangle_u + \frac{H_u}{\alpha_u^\theta} d\left\langle S, \alpha^\theta\right\rangle_u \Big|_{\theta=\tau} \\
&= S_t - \int_0^t \left(\frac{(1-H_u)\, z_u}{G_u} + \frac{H_u a_u^\theta}{\alpha_u^\theta} \Big|_{\theta=\tau} \right) d\langle S\rangle_u := S_t - C_t \qquad (4)
\end{aligned}$$

is a $(\mathbb{G}, \mathbb{P}^*)$-local martingale (see [19]).

The next theorem establishes a predictable representation property for $\mathbb{F}$-martingales under a $\mathbb{G}$-e.m.m $\mathbb{P}^*$, as soon as the $\mathbb{F}$-market enjoys this property. Indeed, any $\eta \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ will write as the sum of an integral with respect to $M \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ and an integral with respect to $\widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$. This result extends the representation theorem by Kusuoka [23], to any complete reference market and to the case immersion does not hold.

Theorem 2.1 *For every $\eta \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, there exists two $\mathbb{G}$-predictable process β and γ such that*

$$d\eta_t = \gamma_t d\widehat{S}_t + \beta_t dM_t.$$

Proof. Let $\eta \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$. As we are only interested in finite time horizon, we write $\eta_t = \mathbb{E}(\eta_T | \mathcal{G}_t)$. By a monotone class argument, we reduce ourself to the case where η_T writes $F_T h(\tau \wedge T)$, with $F_T \in \mathcal{F}_T$. We split the problem in three parts:

$$\begin{aligned}
\eta_t &= \mathbb{E}(F_T h(T) 1_{\tau > T} | \mathcal{G}_t) + \mathbb{E}(F_T h(\tau) 1_{\tau \le T} | \mathcal{G}_t) = a_t + \mathbb{E}(F_T h(\tau) 1_{\tau \le T} | \mathcal{G}_t) \\
&= \underbrace{L_t h(T)\, \mathbb{E}(F_T G_T | \mathcal{F}_t)}_{a_t} + \underbrace{L_t \mathbb{E}(F_T h(\tau) 1_{t < \tau \le T} | \mathcal{F}_t)}_{b_t} + \underbrace{H_t \mathbb{E}(F_T h(\tau) 1_{\tau \le t} | \mathcal{F}_t \vee \sigma(\tau))}_{c_t}.
\end{aligned}$$

with $L_t = (1-H_t)/G_t = D_t(1-H_t) \in \mathcal{M}(\mathbb{G}, \mathbb{Q})$, with $D_t = G_t^{-1}$. From the decomposition (2), writing $dG_t = -\alpha_t^t dt + z_t dW_t$,we get: $dD_t = D_t^2 \left(\alpha_t^t dt + z_t^2 D_t d\langle S\rangle_t\right) - D_t^2 z_t dS_t$.

Let us start by developing a : We first remark that a is a $\mathbb{G}$-martingale, so one knows in advance that the predictable bounded variation part will vanish; nevertheless, we keep all these terms in our computation. By representation theorem, we write: $N_t := \mathbb{E}(F_T G_T | \mathcal{F}_t) := n + \int_0^t n_s dS_s$ and $a_t = h(T)(1-H_t) D_t N_t$. It follows, since S is continuous that $[S, H] = 0$ and

$$\begin{aligned}
h^{-1}(T)\, da_t &= -D_t N_t dH_t + (1-H_t) D_t dN_t + (1-H_t) N_t dD_t + (1-H_t)\, d\langle D, N\rangle_t \\
&= -D_t N_t dH_t + (1-H_t) D_t n_t dS_t + (1-H_t) N_t D_t^2 \alpha_t^t dt \\
&\quad + (1-H_t) N_t z_t^2 D_t^3 d\langle S\rangle_t - (1-H_t) N_t D_t^2 z_t dS_t - (1-H_t) D_t^2 n_t z_t d\langle S\rangle_t \\
&= -D_t N_t dM_t - (1-H_t) D_t^2 N_t \alpha_t^t dt + (1-H_t)\left(D_t n_t - N_t D_t^2 z_t\right) dS_t \\
&\quad + (1-H_t) N_t D_t^2 \alpha_t^t dt + (1-H_t)(N_t z_t D_t - n_t) D_t^2 z_t d\langle S\rangle_t
\end{aligned}$$

Using (3), the $\mathbb{G}$-Doob Meyer decomposition of the increasing process H writes $dH_t = dM_t + (1 - H_t) D_t \alpha_t^t dt$ (from $dA_t = \alpha_t^t dt$), with $M \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$. Moreover $S_t = \widehat{S}_t + C_t$ with $\widehat{S}_t \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, and from (4) $(1 - H_t) dC_t = (1 - H_t) z_t D_t d\langle S \rangle_t$. It follows

$$\begin{aligned} h^{-1}(T)\, da_t &= -D_t N_t dM_t + (1 - H_t)\left(D_t n_t - N_t D_t^2 z_t\right) d\widehat{S}_t \\ &\quad + (1 - H_t)\left(\left(D_t n_t - N_t D_t^2 z_t\right) z_t D_t + N_t z_t^2 D_t^3 - n_t D_t^2 z_t\right) d\langle S \rangle_t \\ &= -D_t N_t dM_t + (1 - H_t)\left(D_t n_t - N_t D_t^2 z_t\right) d\widehat{S}_t \end{aligned}$$

To explicit the decomposition of the special $\mathbb{G}$*-semi-martingale* b, we introduce for any u the martingale $N_t^u = \mathbb{E}\left(F_T \alpha_T^u \middle| \mathcal{F}_t\right)$ and its decomposition on $S : N_t^u = y^u + \int_0^t y_s^u dS_s$ provided by the martingale representation theorem on $\mathbb{F}$. By definition of initial times, it follows:

$$b_t = L_t \mathbb{E}\left(F_T \mathbb{E}\left(h(\tau) 1_{t<\tau\le T} \middle| \mathcal{F}_T\right) \middle| \mathcal{F}_t\right) = L_t \mathbb{E}\left(F_T \int_t^T h(u)\, \alpha_T^u du \middle| \mathcal{F}_t\right) = L_t \int_t^T h(u)\, N_t^u du$$

by differentiation:

$$\begin{aligned} db_t &= -D_t \left(\int_t^T h(u)\, N_t^u du\right) dH_t + (1 - H_t)\left(\int_t^T h(u)\, N_t^u du\right) dD_t \\ &\quad - (1 - H_t) D_t h(t)\, N_t^t dt + (1 - H_t) D_t \left(\int_t^T h(u)\, y_t^u du\right) dS_t \\ &\quad - (1 - H_t) D_t^2 z_t \left(\int_t^T h(u)\, y_t^u du\right) d\langle S \rangle_t \end{aligned}$$

and, introducing the $\mathbb{G}$-decomposition of the semi-martingale S and the compensator of H, we obtain finally:

$$\begin{aligned} db_t &= -D_t \left(\int_t^T h(u)\, N_t^u du\right) dH_t + (1 - H_t)\left(\alpha_t^t D_t^2 \int_t^T h(u)\, N_t^u du - D_t h(t)\, N_t^t\right) dt \\ &\quad + (1 - H_t)\left(D_t \int_t^T h(u)\left(y_t^u - D_t N_t^u z_t\right) du\right) dS_t \\ &\quad - (1 - H_t)\left(D_t^2 z_t \int_t^T h(u)\left(y_t^u - N_t^u z_t D_t\right) du\right) d\langle S \rangle_t \\ &= -D_t \left(\int_t^T h(u)\, N_t^u du\right) dM_t + (1 - H_t)\left(D_t \int_t^T h(u)\left(y_t^u - D_t N_t^u z_t\right) du\right) d\widehat{S}_t \\ &\quad - (1 - H_t) D_t h(t)\, N_t^t dt \end{aligned}$$

Decomposition of c. We can write $c_t = H_t \mathbb{E}\left(F_T h(\tau) 1_{\tau\le t} \middle| \mathcal{F}_t \vee \sigma(\tau)\right) = H_t F(t, \tau)$, where for each u the random variable $F(t, u)$ is $\mathcal{F}_t$-measurable and for any t, $u \longmapsto F(t, u)$ is a Borel function. Using the properties of initial times, we compute $F(t, u) = h(u)\, N_t^u / \alpha_t^u$, and for any u, the dynamics write:

$$dF(t, u) = \left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{\left(\alpha_t^u\right)^2}\right) dS_t + \left(N_t^u \frac{\left(a_t^u\right)^2}{\left(\alpha_t^u\right)^3} - \frac{a_t^u y_t^u}{\left(\alpha_t^u\right)^2}\right) d\langle S \rangle_t .$$

It follows that, since

$$\int_0^t F(s,\tau)\,dH_s = F(\tau,\tau)\,\mathbb{1}_{\tau\le t} = \int_0^t F(s,s)\,dH_s,$$

we can write the decomposition of c :

$$\begin{aligned} dc_t &= F(t,\tau)\,dH_t + H_t\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} dS_t + H_t\left(\frac{N_t^u (a_t^u)^2}{(\alpha_t^u)^3} - \frac{a_t^u y_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} d\langle S\rangle_t \\ &= F(t,t)\,dM_t + H_t\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} d\widehat{S}_t + (1-H_t)\,D_t F(t,t)\,\alpha_t^t\,dt \\ &\quad + H_t\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} dC_t + H_t\left(\frac{N_t^u (a_t^u)^2}{(\alpha_t^u)^3} - \frac{a_t^u y_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} d\langle S\rangle_t \\ &= F(t,t)\,dM_t + H_t\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau} d\widehat{S}_t + (1-H_t)\,D_t F(t,t)\,\alpha_t^t\,dt \end{aligned}$$

where the last equality comes from the expression (4) of dC on $\{\tau \le t\}$.

Conclusion. Adding the three parts a, b, and c, we conclude, since $F(t,t)\,\alpha_t^t = h(t)\,N_t^t$, that the $\mathbb{G}$-martingale can be decomposed on the two martingales $\left(M, \widehat{S}\right)$ and writes:

$$\begin{aligned} d\eta_t &= \left(F(t,t) - D_t\left(N_t h(T) + \int_t^T h(u)\,N_t^u du\right)\right) dM_t + ((1-H_t)\,D_t\,(n_t - N_t D_t z_t)\,h(T) \\ &\quad + (1-H_t)\,D_t \int_t^T h(u)\,(y_t^u - N_t^u D_t z_t)\,du + H_t\left(\frac{y_t^u}{\alpha_t^u} - \frac{N_t^u a_t^u}{(\alpha_t^u)^2}\right)\Bigg|_{u=\tau}\Bigg) d\widehat{S}_t, \end{aligned}$$

wich concludes the proof. □

2.1.3 Description of the $\mathbb{G}$-e.m.m.

Applying this theorem to the particular case of a strictly positive martingale - in particular the density of a change of probability - we derive the

Corollary 1 *If $\eta \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$ is strictly positive, then, there exists a pair of predicable processes γ, β*

$$\frac{d\eta_t}{\eta_{t-}} = \gamma_t d\widehat{S}_t + \beta_t dM_t,$$

with $\beta > -1$, i.e., $\eta = \mathcal{E}\left(\gamma \star \widehat{S}\right)\mathcal{E}(\beta \star M)$, with

$$\begin{cases} \mathcal{E}\left(\gamma \star \widehat{S}\right)_t = \exp\left(\int_0^t \gamma_u d\widehat{S}_u - \frac{1}{2}\int_0^t \gamma_u^2 d\langle \widehat{S}\rangle_u\right) \\ \mathcal{E}(\beta \star M)_t = \exp\left(\int_0^t \ln(1+\beta_s)\,dH_s - \int_0^t \beta_s L_s \alpha_s^s ds\right) \end{cases}$$

and $L_s = (1-H_s)/G_s$ (remark that ΔM has size 1).

Let $\mathbb{P}^*$ be an e.m.m. on $\mathcal{F}_T$. If immersion does not hold under $\mathbb{P}^*$ - i.e., if G is not a predictable increasing process - it is sufficient to change the probability from $\mathbb{P}^*$ to any $\mathbb{G}$-e.m.m $\mathbb{Q}$ so that immersion holds, and such a probability change can be expressed in the following way.

Proposition 2 *There exists a probability $\mathbb{Q} \in \Theta_{\mathbb{P}^*}^{\mathbb{G}}(S)$ such that immersion property holds under $\mathbb{Q}$*

Proof. If the $\mathbb{F}$-conditional survival process writes $G_t = \mathbb{P}^*(\tau > t|\mathcal{F}_t) = Z_t - A_t$, the $(\mathbb{G}, \mathbb{P}^*)$-dynamics of S follows the decomposition (4):

$$S_t = \widehat{S}_t + \int_0^t \left(\frac{(1-H_u)\, z_u}{G_u} + \frac{H_u a_u^\theta}{\alpha_u^\theta}\bigg|_{\theta=\tau} \right) d\langle S\rangle_u \text{ with } \widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*),$$

hence $\mathbb{P}^*$ is not a $\mathbb{G}$-e.m.m. From Corollary 1, the set of $\mathbb{G}$-e.m.m can be perfectly described as:

$$\Theta_{\mathbb{P}^*}^{\mathbb{G}}(S) = \left\{ \mathbb{Q} : \frac{d\mathbb{Q}}{d\mathbb{P}^*}\bigg|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(-\frac{(1-H)z}{G} \star \widehat{S} - \frac{Ha^\tau}{\alpha^\tau} \star \widehat{S} \right)_t \mathcal{E}(\beta \star M)_t, \right\}.$$

where $\beta \in \mathcal{Q}$, the set of predictable processes, taking values in $]-1, \infty[$. As a check, under such a probability $\mathbb{Q}$, as $\widehat{S} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, one has

$$\begin{aligned} \widehat{S}_t - \int_0^t \frac{d\langle \widehat{S}, \eta\rangle_u}{\eta_u} &= \widehat{S}_t - \int_0^t d\left\langle \widehat{S}, -\frac{(1-H)z}{G} \star \widehat{S} - \frac{Ha^\theta}{\alpha^\theta}\bigg|_{\theta=\tau} \star \widehat{S} + \beta \star M \right\rangle_u \\ &= \widehat{S}_t + \int_0^t \left(\frac{(1-H_u)\, z_u}{G_u} + \frac{H_u a_u^\tau}{\alpha_u^\tau} \right) d\langle S\rangle_u = S_t \end{aligned}$$

where the second equality comes from the fact that $\langle \widehat{S}\rangle = \langle S\rangle$ and $\left\langle \widehat{S}, M \right\rangle = 0$, since $\widehat{S}$ is continuous and M purely discontinuous. It follows from Girsanov's theorem that S is a $(\mathbb{G}, \mathbb{Q})$-martingale.

The set of $\mathbb{G}$-e.m.m is infinite, parameterized by the predictable processes β. It is straightforward to check that immersion holds under any such a $\mathbb{Q} \in \mathcal{M}(\mathbb{G}, \mathbb{P}^*)$. Indeed for β predictable > -1, if $\mathbb{Q}^\beta$ is the corresponding e.m.m. (denote by $\gamma_u = (1-H_u)\, D_u z_u + H_u a_u^\theta / \alpha_u^\theta|_{\theta=\tau}$):

- $d\mathbb{Q}^\beta|_{\mathcal{F}_\infty} = d\mathbb{P}^*|_{\mathcal{F}_\infty}$. Indeed, for any $F_t \in \mathcal{F}_t$ with $\mathbb{P}^*$-null expectation, $F_t = \int_0^t f_s dS_s$ by PRP and

$$\begin{aligned} \mathbb{E}^\beta(F_t) &= \mathbb{E}^*(F_t \eta_t) = \mathbb{E}^*\left(\int_0^t \eta_s f_s dS_s + \int_0^t F_s\, d\eta_s + \int_0^t f_s d\langle S, \eta\rangle_s \right) \\ &= \mathbb{E}^*\left(\int_0^t \eta_s f_s dS_s + \int_0^t F_s\, d\eta_s + \int_0^t f_s \gamma_s \eta_s d\langle \widehat{S}\rangle_s \right) \\ &= \mathbb{E}^*\left(\int_0^t \eta_s f_s d\widehat{S}_s + \int_0^t F_s\, d\eta_s \right) = 0 = \mathbb{E}^*(F_t), \end{aligned}$$

  where the first line is integration by part formula, the second comes from the dynamics of the density η and the third from the definition of $\widehat{S}$, the expectation being null since $\widehat{S}$ and η belong to $\mathcal{M}(\mathbb{G}, \mathbb{P}^*)$. It follows $d\mathbb{Q}^\beta|_{\mathcal{F}_\infty} = d\mathbb{P}^*|_{\mathcal{F}_\infty}$.

- Let X be a $\left(\mathbb{F},\mathbb{Q}^{\beta}\right)$-martingale. Then, it is a $(\mathbb{F},\mathbb{P}^{*})$ martingale and if $X=x*S$,, from the filtration enlargement by the initial time

$$\widehat{X}_t := \qquad X_t - \int_0^t x_u \left(\frac{(1-H_u)\, z_u}{G_u} + \left.\frac{H_u a_u^{\theta}}{\alpha_u^{\theta}}\right|_{\theta=\tau} \right) d\left\langle S\right\rangle_u \in \mathcal{M}\left(\mathbb{G},\mathbb{P}^{*}\right)$$

Using Girsanov's theorem, the process

$$\widetilde{X}_t = \widehat{X}_t - \int_0^t \frac{d\left\langle X,\eta\right\rangle_u}{\eta_u}$$

is a $\left(\mathbb{F},\mathbb{Q}^{\beta}\right)$-martingale. It remains to note that

$$\widetilde{X}_t = \widehat{X}_t + \int_0^t x_u \left(\frac{(1-H_u)\, z_u}{G_u} + \left.\frac{H_u a_u^{\theta}}{\alpha_u^{\theta}}\right|_{\theta=\tau} \right) d\left\langle S\right\rangle_u = X_t\,.$$

It follows that $X\in\mathcal{M}\left(\mathbb{G},\mathbb{Q}^{\beta}\right)$ hence immersion holds under $\mathbb{Q}^{\beta}$.

2.1.4 Risk premia.

Such a result can be interpreted in the following way. The change from historical $\mathbb{P}$ to neutral risk probability $\mathbb{P}^{*}$ aims at correcting the dynamics from the market risk premium. Indeed, to any financial market can be associated a risk premium, that characterizes the return an investor is expecting over the risk free return (the interest rate), to bear the risk of taking a long position on a derivative written on this market. If N is the martingale modelling the alea (multidimensional, continuous or not) of this market, and if the asset's return writes:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dN_t,$$

the dynamics of any derivatives written on S sold at P, would be $dP_t/P_t=\kappa dt+\alpha dN_t$. A risk free portfolio can be set-up in buying a quantity $S\sigma$ of the derivative, for a total value of $S\sigma.P$ and selling a quantity $P\alpha$ of the asset (completing by ς_t of money market β_t to remain self financed):

$$\begin{aligned} d\Pi_t &= \varsigma_t r\beta_t dt + S_t\sigma dP_t - P_t\alpha dS_t = \varsigma_t r\beta_t dt + S_t\sigma P_t\kappa dt + S_t\sigma P_t\alpha dN_t - P_t\alpha S_t\mu dt - P_t\alpha S_t\sigma dN_t \\ &= \left(\left(\Pi_t - (\sigma-\alpha)\, P_t S_t\right) r + (\sigma\kappa - \alpha\mu)\, P_t S_t\right) dt, \end{aligned}$$

since $\varsigma_t\beta_t = \Pi_t - S_t\sigma P_t + P_t\alpha S_t$ by definition of the portfolio. By absence of arbitrage, its return must be equal to r to preclude arbitrage, so that:

$$(\sigma\kappa-\alpha\mu)\, P_t S_t = r\,(\sigma-\alpha)\, P_t S_t \Longleftrightarrow \frac{\kappa-r}{\alpha} = \frac{\mu-r}{\sigma} = \lambda_S.$$

On the reference market, the neutral risk probability $\mathbb{P}^{*}$ corrects the historical probability $\mathbb{P}$ from the market risk premium . *If immersion does not hold under $\mathbb{P}^{*}$, it means the market risk premium does not take into account the jump risk premium, and it is necessary to change to a $\mathbb{G}$-e.m.m $\mathbb{Q}$ under which S remains a martingale.*

2.2 $\mathbb{G}$-adapted assets in the full market.

When considering as well the $n+2^{th}$ asset S^{n+1}, we shall be able to select a risk neutral probability, in a unique way.

2.2.1 The necessity of the introduction of S^{n+1}.

First, remark that it is necessary to introduce the asset S^{n+1} to the collection S when working on derivatives whose pay-off depends on τ, since it is not possible to hedge the jumping risk with $\mathbb{F}$-adapted assets.

Let us consider for example a credit default swap (contract in which the holder buys a protection in paying a premium at each date of a tenor to the seller until a predefined credit event occurs, and receives if default occurs a recovery fee). To ease the discussion we take a continuous tenor, with a proportional continuous premium κ, and a constant recovery fee δ (β_t denotes the saving account, i.e., the value at t of one unit invested at 0, and $\mathbb{Q}$ a martingale measure associated to this numeraire - that exists by absence of arbitrage). The price of a CDS is the difference between the value of the protection leg and the premium leg: $CDS(t,\delta,\kappa,\mathcal{T}) = \Pr ot_t - \Pr em_t$ with:

$$\begin{aligned}
\Pr em_t &= \beta_t \kappa \mathbb{E}\left(\int_t^T \frac{1-H_u}{\beta_u} du \middle| \mathcal{G}_t\right) = (1-H_t)\frac{\beta_t\kappa}{G_t}\int_t^T \mathbb{E}\left(\frac{1-H_u}{\beta_u}\middle|\mathcal{F}_t\right) du \text{ and} \\
\Pr ot_t &= \beta_t \delta \mathbb{E}\left(\int_t^T \frac{dH_u}{\beta_u} \middle| \mathcal{G}_t\right) = (1-H_t)\frac{\beta_t\delta}{G_t}\int_t^T \mathbb{E}\left(\frac{(1-H_u)\,\alpha_u^u}{\beta_u G_u}\middle|\mathcal{F}_t\right) du.
\end{aligned}$$

Even if every conditional expectation is $\mathbb{F}$-adapted, the price of each leg may be writen as a conditional expectation only if it is possible to replicate the pay-off of the leg with financial assets. Such a writing can not be possible in a model containing only $\mathbb{F}$-adapted assets (S), since the jumping part is not in $\mathcal{F}_T$, hence impossible to replicate.

It is therefore necessary to introduce the asset S^{n+1} - that has a sensibility against the jumps - and prove that a CDS is hedgeable when introducing it in the erplication portfolio, to determine the neutral risk measure and compute the price.

For quoted instruments like CDS, such a formula allows to calibrate the parameters involved in the construction of the default time (and a natural class of assets for S^{n+1} would be the risky bonds associated to τ or a CDS of different maturity).

2.2.2 Completness of the full market.

We introduce the asset S^{n+1} that is is sensible to the jump risk, i.e., $\tau\wedge t$ is $\sigma\left(S_s^{n+1}, s\leq t\right)$-mesurable. To be concrete, we postulate for example that $S_t^{n+1} = \varphi\left(t, S_t, H_t\right)$. Our aim is to prove that if the $\mathbb{F}$-market is complete, the $\mathbb{G}$-market is complete as well. Let $\mathbb{P}$ be the historical probability.

We know that the compensator of G writes $dA_t = \alpha_t^t dt$, hence from

$$dM_t = dH_t - \frac{1-H_t}{G_t}\alpha_t^t\, dt$$

it follows, since each process of the right-hand member of the equality is of finite variation and the second is continuous, that

$$[M]_t = [H]_t = \sum_{s\leq t} \Delta H_s^2 = \sum_{s\leq t} \Delta H_s = H_t, \text{ hence}$$

$$[M]_t - \int_0^t \frac{1-H_s}{G_s} \alpha_s^s ds = M_t \in \mathcal{M}(\mathbb{G},\mathbb{P}).$$

It follows $d\langle M\rangle_t = (1-H_t)\,\alpha_t^t/G_t dt$. Assume also that $d\langle S\rangle \ll dt$.

We assume that the asset S writes

$$dS_t = dS^*{}_t + b_t dt = d\widehat{S}_t + (c_t + b_t)\,dt,$$

with $S^* \in \mathcal{M}(\mathbb{F},\mathbb{P})$, $\widehat{S} \in \mathcal{M}(\mathbb{G},\mathbb{P})$ and $c_u = (1-H_u)\,z_u/G_u + H_u a_u^\theta/\alpha_u^\theta\big|_{\theta=\tau}$, and the asset S^{n+1} writes:

$$dS_t^{n+1} = \mu_t dt + \varepsilon_t d\widehat{S}_t + \zeta_t dM_t,$$

where M is the compensated martingale of H ($M \in \mathcal{M}(\mathbb{G},\mathbb{P})$) and $\mu_t dt$ is the drift term[3] (assume the three processes μ, ε and ζ are predictable). The set of $\mathbb{G}$-neutral risk probabilities writes, by Corollary 1:

$$\Theta_{\mathbb{P}}^{\mathbb{G}}(S) = \left\{ \mathbb{Q} \sim \mathbb{P}\ \exists \alpha, \beta \in \mathcal{Q}, \left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(-\alpha \star \widehat{S}\right)_t \mathcal{E}\left(-\beta \star M\right)_t \right\}.$$

By Theorem 1, the $\mathbb{G}$-neutral risk probability is uniquely defined by:

$$\alpha_t = c_t + b_t, \text{ and } \beta_t = G_t \frac{\mu_t - \varepsilon_t (c_t + b_t)}{\alpha_t^t \zeta_t}.$$

It follows the

Proposition 3 *If the reference market $\mathbb{F}$ defined by the assets $S = \left(S^0, S^1, ..., S^n\right)$ is complete, then the full market $\mathbb{G}$ composed of the default free assets $S = \left(S^0, S^1, ..., S^n\right)$ and of the default sensitive asset S^{n+1} is complete.*

Once this probability has been defined, it is possible to price and hedge the τ-sensitive claims with $\widetilde{S}$, like for example CDS on τ (of different maturities if S^{n+1} is a CDS) or derivatives written on S^{n+1}.

3 Incomplete markets.

Assumption of absence of arbitrage is central in analysis and is systematic in the context of derivatives modelling. It is a natural assumption since as soon as an arbitrage is identified, it is exercised. In the opposite, assumption of completeness is not always satisfied, and is often violated when writing a

[3] Each coefficient can be easily derived from the function F by application of Ito's formula and compensation.

model to price exotic derivatives. When a structured product has a high sensibility to many complex risks (more than the delta and gamma risks - against the small movements of the underlying or its erratic short term large movements) it is necessary to choose a model that incorporates the fair price of the vanilla options of the needed maturities and strikes. Indeed, the hedge against the vega risk for example is provided by purchasing vanilla options, and their prices must be taken into account by the model when computing the selling price of the product, to avoid a negative arbitrage when setting up the strategy. The trader may price the product with an incomplete model so that to have a good flexibility in the calibration on the class of products useful for the hedge.

When dealing with an incomplete market - whose information is supported by the filtration $\mathbb{F}$ - it is by definition impossible to replicate every pay-off, hence there is no unicity of the $\mathbb{F}$-e.m.m. The definition of the price of a derivative may be tricky in an incomplete framework, since there is no existence of a replication portfolio so that to prove that the conditional expectation of the pay-off correponds to a fair price. However, as the set $\Theta_{\mathbb{P}}^{\mathbb{F}}(S)$ is convex and not reduced to a point (by absence of arbitrage), there exists for each claim $X_T \in \mathcal{F}_T$ an interval of prices, from

$$\left]\inf_{\mathbb{Q}\in\Theta_{\mathbb{P}}^{\mathbb{F}}(S)} \mathbb{E}^{\mathbb{Q}}(X_T); \sup_{\mathbb{Q}\in\Theta_{\mathbb{P}}^{\mathbb{F}}(S)} \mathbb{E}^{\mathbb{Q}}(X_T)\right[. \tag{5}$$

N. El Karoui and M.C. Quenez proved in [14] that if the derivative of payoff X_T is sold at a price inside this interval, it does not lead to an arbitrage. Indeed, they establish the relation:

$$\sup_{\mathbb{Q}\in\Theta_S(\mathbb{F}^S)} \mathbb{E}^{\mathbb{Q}}(X_T) = \inf_{\varphi\in\Sigma(\mathbb{F}^S)} V_0(\varphi)$$

where $\Sigma\left(\mathbb{F}^S\right)$ is the set of sur-replicating strategies, i.e., the non empty set of strategies such that $V_T(\varphi) \geq X_T$.

It follows that in such a context, definition of a unique price and martingale property of its dynamics are not canonical.

3.1 A market neutral risk probability

Let us consider two classical examples of incomplete markets. We deal here with the only reference market, who can be an equity or fixed income market:

- Heston's model: The asset's dynamics are modelled by a stochastic volatility diffusion. Under the historical probability $\mathbb{P}$, the price of the (traded) underlying asset S is given in terms of a stochastic volatility (non traded):

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sigma_t dW_t, \\ d\sigma_t &= \eta_t dt + \alpha_t\left(\rho dW_t + \sqrt{1-\rho^2}dB_t\right), \end{aligned} \tag{6}$$

  with W and B are independent Brownian motions. The market is not complete since it is not possible to replicate a B-sensitive claim with only the asset S, i.e., with W. In other words, it is impossible to vega hedge a portfolio with the underlying. The set of $\mathbb{F}$-e.m.m is parameterized by the $\mathbb{F}$-predictable processes κ and writes

$$\Theta_{\mathbb{P}}(S)=\left\{\mathbb{Q}\sim\mathbb{P},\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\mathcal{E}\left(-\frac{\mu}{\sigma}\star W\right)_t\mathcal{E}(\kappa\star B)_t\right\}.$$

Indeed, under such a probability $\mathbb{Q}^{\kappa}$ the dynamics of the process write:

$$\begin{aligned}\frac{dS_t}{S_t}&=\sigma_t d\widetilde{W}_t,\\ d\sigma_t&=\left(\eta_t-\rho\frac{\alpha_t\mu_t}{\sigma_t}+\sqrt{1-\rho^2}\alpha_t\kappa_t\right)dt+\alpha_t\left(\rho d\widetilde{W}_t+\sqrt{1-\rho^2}d\widetilde{B}_t\right),\end{aligned}$$

and $S\in\mathcal{M}(\mathbb{F},\mathbb{Q}^{\kappa})$.

It is classical that if the calls are liquid and their price are computed through this model (i.e., the true dynamics of the asset is the model dynamics (6)), it is possible to use such instruments in the hedging portfolio and to work under a complete market framework. See Romano and Touzi [26].

- The asset's dynamics are modelled by a jumping diffusion. For example, assume that the price of the underlying follows, under the historical probability $\mathbb{P}$:

$$\frac{dS_t}{S_{t-}}=\mu_t dt+\sigma_t dW_t+\varphi_t dM_t$$

where W is a Brownian motion and $dM_t=dN_t-\lambda_t dt$, with N an inhomogeneous Poisson process with deterministic intensity λ. Then, the market is incomplete and the set of $\mathbb{F}$-e.m.m is parametrized by two $\mathbb{F}$-predictable processes α and β, with $\beta>-1$ as follows:

$$\Theta_{\mathbb{P}}(S)=\left\{\mathbb{Q}\sim\mathbb{P},\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\mathcal{E}(\alpha\star W)_t\,\mathcal{E}(\beta\star M)_t\,,\ \text{with } \mu_t+\sigma_t\alpha_t+\lambda_t\beta_t\varphi_t=0\right\}.$$

Indeed, under such a probability $\mathbb{Q}^{\alpha,\beta}$ the processes $\widetilde{W},\widetilde{M}$ defined as

$$\begin{aligned}d\widetilde{W}_t&=dW_t-\alpha_t dt\\ d\widetilde{M}_t&=dM_t-\beta_t d\langle M\rangle_t=dM_t-\lambda_t\beta_t dt\end{aligned}$$

are martingales by Girsanov's theorem, hence the dynamics of the process write:

$$\frac{dS_t}{S_{t^-}}=\sigma_t d\widetilde{W}_t+\varphi_t d\widetilde{M}_t.$$

Under each new $\mathbb{F}$-e.m.m., the asset is a martingale, but in general, the law of the process changes with any change of e.m.m. It follows contingent claims may have different prices - where price is here a short cut for discounted $\mathbb{F}$-conditional expectation (cf. 5). For example, changing the drift of the stochastic volatility by changing the probability will change the price of the call options, as well as changing the intensity of the jump process in the second example[4].

In practice, once a class of model is chosen by the trader for pricing a derivative, the procedure of calibration aims at choosing the parameters of this class that make the pricing of well chosen hedging instruments (for example vanilla options) the closest to the market cotations. This operation can be assimilated to the selection of the neutral risk probability within the sub-set of $\mathbb{F}$-e.m.m. that preserves the class of the model, so that to stick to the true market probability. For example, if the target dynamics of the volatility in a stoch vol framework are a piecewise constant CIR process - i.e., the so-called Heston class (see [17]) writing under the neutral risk probability:

$$\begin{aligned} \frac{dS_t}{S_t} &= \mu_t dt + \sqrt{V_t} dW_t, \\ dV_t &= \lambda (a - V_t) dt + \alpha \sqrt{V_t} \left(\rho dW_t + \sqrt{1-\rho^2} dB_t \right), \end{aligned} \tag{7}$$

- the set of $\mathbb{F}$-e.m.m. is parametrized by the couples of real numbers (β, δ) such that $\kappa_t = \beta\sqrt{V_t} + \delta/\sqrt{V_t}$ (with the above notations). This condtion of remaining in the Heston class is not due to absence of arbitrage consideration but is a constraint imposed by the teams. Each change of probability (i.e., of (β, δ)) implies a change in call prices.

The calibration procedure can be interpreted as the selection of the appropriate (β, δ), namely the appropriate $\mathbb{F}$-e.m.m. In some more complicated examples, matching the set of calls can be not sufficient for determining the parameters. The definition of a series of constraints allows to solve this problem and make the pricing consistent with the market. This contrained procedure can become quite sophisticated in case of degenerated model, as in [10] or in [16].

When an incomplete model is chosen - in general for its hability to well reproduce a given class of calibration instruments and for its nice features regarding to the products to price - the selection of a probability is systematically performed by the calibration procedure. Under this condition, the law of the price process is uniquely determined, to be the closest to the observed prices and to a set of well chosen constraints (historical data etc.). A change of probability within the set of $\mathbb{F}$-e.m.m. will change the price of the selected options or break the imposed constraints.

3.2 Filtration enlargement

As emphasized in the first section, even if the set of $\mathbb{F}$-e.m.m. is not reduced to a singleton in the incomplete situation, we are often conduced to focus on one particular probability. Equivalently, we

[4] In the opposite, series of examples are known where a change of probability does not imply a change in the price of call options (jumping processes with free parameters on the intensity, the mean and the variance of jumps, or a mix between stoch vol and jumps). These models are in general called degenerated. In such cases, the absence of change in call options prices is due to the fact that the change of probability changes the law of the underlying, but preserves its marginals (recall $f(x,t) = \partial_x^2 C(x,t)$, with $C(x,t)$ the price of the call of maturity t stroken at x, and $f(x,t)$ the density of S_t). However, the increments laws may be completely different (think of the important differences in the value of a forward start option between a pricing within a stoch vol model and a pricing within a local vol model, the latter being calibrated on the former).

assume that the market has chosen an $\mathbb{F}$-e.m.m. for pricing the default free derivatives.

This point justifies that our attention is fixed from now on, on a given $\mathbb{F}$-neutral risk probability, i.e., a probability $\mathbb{P}^*$ defined on $\mathcal{A}$ equivalent to the historical probability $\mathbb{P}$, such that $S \in \mathcal{M}(\mathbb{F}, \mathbb{P}^*)$.

Our goal is to prove that there exists a unique probability $\mathbb{Q}$ equivalent to $\mathbb{P}^*$ such that $\widetilde{S}$ is a $(\mathbb{G}, \mathbb{Q})$-martingale and $\mathcal{F}_T$ is preserved, i.e.,

$$\mathbb{E}^{\mathbb{Q}}(X_T) = \mathbb{E}^*(X_T), \text{ for any } X_T \in L^2(\mathcal{F}_T),$$

this constraint being naturally imposed regarding to the discussion of the previous section.

3.2.1 The $\mathbb{G}$-e.m.m.

If $Y_T \in L^2(\mathcal{F}_T)$ with null expectation, the martingale $(Y_t = \mathbb{E}^*(Y_T|\mathcal{F}_t); 0 \le t \le T)$ belongs to the class $\mathcal{H}^2(\mathbb{F}, \mathbb{P}^*)$ the set of square integrable $(\mathbb{F}, \mathbb{P})$-martingales. Since $S \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}^*)$ and due to the Hilbert property of this functional space, there exists a unique decomposition $Y = x \star S + N$ where $N \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}^*)$ and the martingales (S, N) are orthogonal.

Using this result, we now decompose the martingales Z and α^θ:

$$G = Z - A = z \star S + N^G - A, \qquad \alpha^\theta = a^\theta \star S + N^\theta,$$

with $\left(N^G, N^\theta\right) \in \mathcal{H}^2(\mathbb{F}, \mathbb{P}^*)$ and are orthogonal to S. As $S = \widehat{S} + s$ is the decomposition of the $(\mathbb{G}, \mathbb{P}^*)$-semi-martingale, with:

$$ds_t = \frac{1 - H_t}{G_t} d\langle S, G\rangle_t + \frac{H_t}{\alpha_t^\theta} d\langle S, \alpha^\theta\rangle_t\Big|_{\theta=\tau} = \left(\frac{1 - H_t}{G_t} z_t + \frac{H_t}{\alpha_t^\theta} a^\theta\Big|_{\theta=\tau}\right) d\langle S\rangle_t,$$

the set $\Theta_{\mathbb{P}^*}^{\mathbb{G}}(S)$ writes (if we only focus on $\mathcal{H}^2$ change of probability measures, and apply the same martingale decomposition in the Hilbert space $\mathcal{H}^2(\mathbb{G}, \mathbb{P}^*)$):

$$\Theta_{\mathbb{P}^*}^{\mathbb{G}}(S) = \left\{ \mathbb{Q}, \frac{d\mathbb{Q}}{d\mathbb{P}^*}\Big|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(\varphi \star \widehat{S}\right)_t \mathcal{E}(\psi \star M)_t \mathcal{E}\left(N^\perp\right)_t \right\}$$

with $\varphi = -ds/d\left\langle \widehat{S} \right\rangle$, and where $N^\perp$ is a martingale in the set $\mathcal{M}(\mathbb{G}, \mathbb{P}^*)$, orthogonal to the pair $\widehat{S}, M$ and where ψ is a $\mathbb{G}$-predictable process, such that $\psi > -1$. We see that the set of e.m.m. $\Theta_{\mathbb{P}^*}^{\mathbb{G}}(S)$ is parameterized by the pair $(\psi, N^\perp)$.

Moreover, assuming that $dS_t^{n+1} = \mu_t dt + \varepsilon_t d\widehat{S}_t + \zeta_t dM_t + d\Sigma_t$ under $\mathbb{P}^*$ where Σ is a $(\mathbb{G}, \mathbb{P}^*)$ martingale, orthogonal to the pair $\left(\widehat{S}, M\right)$, the set $\Theta_{\mathbb{P}^*}^{\mathbb{G}}\left(\widetilde{S}\right)$ writes:

$$\Theta_{\mathbb{P}^*}^{\mathbb{G}}\left(\widetilde{S}\right) = \left\{ \mathbb{Q}, \frac{d\mathbb{Q}}{d\mathbb{P}^*}\Big|_{\mathcal{G}_t} = \eta_t = \mathcal{E}\left(\varphi \star \widehat{S}\right)_t \mathcal{E}(\psi \star M)_t \mathcal{E}\left(N^\perp\right)_t \right\}$$

with $\varphi = -ds/d\langle\widehat{S}\rangle$ and where $N^\perp$ is a $(\mathbb{G}, \mathbb{P}^*)$-martingale orthogonal to $(\widehat{S}, M)$ and ψ is the $\mathbb{G}$-predictable process defined as

$$\mu_t dt = \varepsilon_t ds_t - d\left\langle \Sigma, N^\perp \right\rangle - \zeta_t \psi_t d\langle M\rangle_t$$

(here again we assume that all quadratic variations are a.c.w.r.t. dt, otherwise some continuous relations are to be imposed). We see that the set of e.m.m. $\Theta^{\mathbb{G}}_{\mathbb{P}^*}\left(\widetilde{S}\right)$ is parameterized by $N^{\perp}$, a $(\mathbb{G},\mathbb{P}^*)$-martingale orthogonal to the pair $\left(\widehat{S},M\right)$.

Let $X_T\in L^2\left(\mathcal{F}_T\right)$, such that $\mathbb{E}^*\left(X_T\right)=0$. The $(\mathbb{F},\mathbb{P}^*)$-martingale X writes $X=x\star S+N$, with $(S,N)\in\mathcal{M}(\mathbb{F},\mathbb{P}^*)$. The decomposition of this special $(\mathbb{G},\mathbb{P}^*)$-semi-martingale is $X=x\star\widehat{S}+x\star s+\widehat{N}+n$, with $\left(\widehat{S},\widehat{N}\right)\in\mathcal{M}(\mathbb{G},\mathbb{P}^*)$ and:

$$dn_t=\frac{1-H_t}{G_t}d\left\langle N,G\right\rangle_t+\frac{H_t}{\alpha^\theta_t}d\left\langle N,\alpha^\theta\right\rangle_t\Big|_{\theta=\tau}=\frac{1-H_t}{G_t}d\left\langle N,N^G\right\rangle_t+\frac{H_t}{\alpha^\theta_t}d\left\langle N,N^\theta\right\rangle_t\Big|_{\theta=\tau}$$

(recall $ds_t=\left(\left(1-H_t\right)z_t/G_t+H_ta^\theta_t/\alpha^\theta_t\big|_{\theta=\tau}\right)d\left\langle S\right\rangle_t$). It follows that

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left(X_T\right) &= \mathbb{E}^*\left(X_T\eta_T\right)=\mathbb{E}^*\left(\int_0^T\eta_tdX_t+[X,\eta]_T\right)+\mathbb{E}^*\left(\int_0^T\eta_t\left(x_tds_t+dn_t\right)+\left\langle X,\eta\right\rangle_T\right)\\
&= \mathbb{E}^*\left(\int_0^T\eta_t\left(x_tds_t+dn_t\right)+\int_0^T\eta_td\left\langle x\star\widehat{S}+\widehat{N},\varphi\star\widehat{S}+\psi\star M+N^\perp\right\rangle_t\right)\\
&= \mathbb{E}^*\left(\int_0^T\eta_t\left(x_tds_t+dn_t\right)+\int_0^T\eta_tx_t\varphi_td\left\langle\widehat{S}\right\rangle_t+\int_0^T\eta_td\left\langle\widehat{N},N^\perp\right\rangle_t\right)\\
&= \mathbb{E}^*\int_0^T\eta_td\left(n_t+\left\langle\widehat{N},N^\perp\right\rangle_t\right).
\end{aligned}$$

Introduce the $(\mathbb{F},\mathbb{P}^*)$-martingale

$$N^{\mathbb{F},\perp}_t=\int_0^t-\frac{1-H_u}{G_u}dN^G_u-\frac{H_u}{\alpha^\theta_u}dN^\theta_u\Big|_{\theta=\tau}$$

and choose for $N^\perp$ in the definition of $\mathbb{Q}$ the $(\mathbb{G},\mathbb{P}^*)$-martingale

$$N^\perp_t=N^{\mathbb{F},\perp}_t+\int_0^t\frac{1-H_u}{G_u}d\left\langle N^{\mathbb{F},\perp},G\right\rangle_u+\frac{H_u}{\alpha^\theta_u}d\left\langle N^{\mathbb{F},\perp},\alpha^\theta\right\rangle_u\Big|_{\theta=\tau}\in\mathcal{M}(\mathbb{G},\mathbb{P}^*).$$

With this definition of $N^\perp$, we have:

$$d\left\langle\widehat{N},N^\perp\right\rangle=d\left\langle N,N^{\mathbb{F},\perp}_t\right\rangle=-\frac{1-H_t}{G_t}d\left\langle N,N^G\right\rangle_t-\frac{H_t}{\alpha^\theta_t}d\left\langle N,N^\theta\right\rangle_t\Big|_{\theta=\tau}=-dn_t.$$

Moreover, under each $\mathbb{Q}$ defined with another orthogonal martingale $\widetilde{N}^\perp$, it is possible to find $N\in\mathcal{M}(\mathbb{F},\mathbb{P}^*)$ such that

$$\mathbb{E}^*\int_0^T\eta_td\left(n_t+\left\langle\widehat{N},N^\perp\right\rangle_t\right)\neq0,$$

since therefore $n_t\neq\left\langle\widehat{N},N^\perp\right\rangle_t$ and η_t does not depend on N.

It follows there exists a unique pair $N^\perp,\psi)$ where the $(\mathbb{G},\mathbb{P}^*)$-martingale is orthogonal to the pair $\left(\widehat{S},M\right)$ and $\psi_t=\left(\varepsilon_tds_t-\mu_tdt-d\left\langle\Sigma,N^\perp\right\rangle\right)/\zeta_td\left\langle M\right\rangle_t$ is a $\mathbb{G}$-predictable process such that $\mathbb{E}^{\mathbb{Q}}\left(X_T\right)=0$ for any $X_T\in L^2\left(\mathcal{F}_T\right)$, and we have the

Proposition 4 *There exists a unique $\mathbb{G}$-e.m.m. $\mathbb{Q}\in\Theta^{\mathbb{G}}_{\mathbb{P}^*}\left(\widetilde{S}\right)$, that preserves $\mathcal{F}_\infty$, i.e.,*

$$\mathbb{E}^{\mathbb{Q}}\left(X_T\right)=\mathbb{E}^*\left(X_T\right),\ \textit{for any } X_T\in L^2\left(\mathcal{F}_T\right).$$

3.2.2 Immersion under $\mathbb{Q}$

Let $X \in \mathcal{H}^2(\mathbb{F},\mathbb{Q})$, $X_t = \mathbb{E}^{\mathbb{Q}}\left(X_T \middle| \mathcal{F}_t\right)$. As $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}^*|_{\mathcal{F}_\infty}$, $X_t = \mathbb{E}^*\left(X_T \middle| \mathcal{F}_t\right)$. Indeed, for $F_t \in \mathcal{F}_t$,

$$\mathbb{E}^*\left(X_T F_t\right) = \mathbb{E}^{\mathbb{Q}}\left(X_T F_t\right) = \mathbb{E}^{\mathbb{Q}}\left(X_t F_t\right) = \mathbb{E}^*\left(X_t F_t\right).$$

It follows that $X = x \star S + N$, where the $(\mathbb{F},\mathbb{P}^*)$-martingales (S,N) are orthogonal. The $(\mathbb{G},\mathbb{P}^*)$-decomposition writes:

$$X = x \star S + N = x \star \widehat{S} + x \star s + \widehat{N} + n$$

for $\left(\widehat{S},\widehat{N}\right) \in \mathcal{M}(\mathbb{G},\mathbb{P}^*)$, with

$$ds_t = \left(\frac{1-H_t}{G_t} z_t + \frac{H_t}{\alpha_t^\theta} a^\theta \bigg|_{\theta=\tau} \right) d\left\langle S\right\rangle, \text{ and } dn_t = \frac{1-H_t}{G_t} d\left\langle N, N^G\right\rangle_t + \frac{H_t}{\alpha_t^\theta} d\left\langle N, N^\theta\right\rangle_t \bigg|_{\theta=\tau}.$$

Under $\mathbb{Q}$:

$$X = x \star \widetilde{S} + x \star \left\langle \widehat{S}, \log\eta \right\rangle + x \star s + \widetilde{N} + \left\langle \widehat{N}, \log\eta \right\rangle + n,$$

and by definition of $\mathbb{Q}$, $\left\langle \widehat{S}, \log\eta \right\rangle = -s$ and $\left\langle \widehat{N}, \log\eta \right\rangle = -n$ (see above), hence

$$X = x \star \widetilde{S} + \widetilde{N} \in \mathcal{M}(\mathbb{G},\mathbb{Q}),$$

and it follows the

Proposition 5 *Under the $\mathbb{G}$-e.m.m. $\mathbb{Q} \in \Theta_{\mathbb{P}^*}^{\mathbb{G}}\left(\widetilde{S}\right)$, that preserves $\mathcal{F}_\infty$, immersion holds.*

4 Conclusion

In this paper, we have given some arguments that show that it is natural to assume that immersion hypothesis holds for a study of a single default. However, it is well known that it is usually impossible to assume this hypothesis in case of (non-ordered) multi-defaults,and that the martingale parts of the survival probabilities reflects the correlation between the different default times.

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