Density results in Sobolev spaces where the functions vanish on a part of the boundary

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Abstract

In this article, we consider the subspace of $W^{m,r}$ of functions that vanish on a part γ_0 of the boundary. The convolution-translation method, a variant of the standard mollifier technique, allows to prove the density of smooth functions, which vanish in a neighborhood of γ_0, in this subspace. The result is first proved for $m = 1$, then generalized to the case where $m \geq 1$, in any dimension, in the framework of Lipschitz-continuous domain. We don't need, as may be expected, to make additionnal assumptions on the boundary of γ_0. An application of the same type of technique is given in the space of functions L^2 with divergence in L^2. Contrary to the previous case, assumptions are needed on the boundary of γ_0.

Résumé

Cet article considère le sous-espace de $W^{m,r}$ des fonctions qui s'annulent sur une partie γ_0 de la frontière. On montre que la méthode de convolution-translation, variante de la technique standard de convolution par une fonction C^∞ à support compact, permet de démontrer la densité des fonctions régulières, nulles sur un voisinage de γ_0, dans ce sous-espace. Le résultat est démontré, d'abord pour $m = 1$, puis généralisé à $m \geq 1$, en dimension quelconque, dans le cadre des domaines Lipschitziens, et il n'est pas néssaire, contrairement à ce qu'il était attendu, de faire des hypothèses supplémentaires sur la frontière de γ_0. On donne une application du même type de technique, dans l'espace des fonctions L^2 à divergence dans L^2, qui, elle, nécessite des hypothèses sur la frontière de γ_0.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary is Lipschitz-continuous. This article mainly deals with functions of $W^{m,r}(\Omega, \gamma_0)$, where $r > 1$ and $m \geq 1$, which are functions of $W^{m,r}(\Omega)$ that vanish on an open part γ_0 of the boundary $\partial\Omega$. More precisely, we study the density of smooth functions that vanish on a neighborhood of γ_0 in the space $W^{m,r}(\Omega, \gamma_0)$. This density is well known in particular cases and used in [1]. It is proven in two dimensions for $m = 1$ in [2] by introducing a convolution-translation operator. However, as far as I know, there is no proof in the general case. The reason why is, perhaps, that these results of density appeared true, without no doubt, but the general proof seemed difficult or, at least, not particularly interesting. Thus, it seemed to me useful to give a detailed proof of these significant results.

Let γ_1 denotes the complementary set of $\overline{\gamma_0}$ in the boundary $\partial\Omega$. In two dimensions, it is generally assumed, as in [2], that $\overline{\gamma_0}\cap\overline{\gamma_1}$ is composed of a finite numbers of points. But, in the general case, do we have to assume conditions on this intersection, as may be expected, to prove the density theorem ? In this article, we assume that the intersection $\overline{\gamma_0} \cap \overline{\gamma_1}$ has a finite number of connected components, but no additionnal assumption is necessary to prove the density theorem. We use a modified mollification technique, initiated by [7] and rediscovered simultaneously in [2] and [5], which consists to combine a convolution and a translation. First, we localize and we establish a partition of unity, which allows us to distinguish three parts in the boundary. On a neighborhood of a point of γ_0, we make a translation outside the domain, in a neighborhood of a point of γ_1, we make a translation inside the domain and, next, we apply, in theses two cases, the mollification technique. On the third part of the boundary, which is composed of neightborhhoods of the connected components of $\overline{\gamma_0} \cap \overline{\gamma_1}$, because of Poincaré's inequality, we approximate the function by 0.

In dimension $d \geq 3$, the neightborhood of the connected components of $\overline{\gamma_0} \cap \overline{\gamma_1}$ are no longer simple balls, which complicates the previous approximation by 0: we consider an optimal covering by balls and a special technique of permutation and partition to deal with the intersections of balls in the estimates.

In this paper, the main result is Theorem 3.1, which establishes the density in $W^{1,r}(\Omega, \gamma_0)$, that is to say the result of density for $m = 1$. The generalization to the case $m \geq 1$, which is Theorem 4.1, is straightforward. Finally, the partition of unity on $\overline{\Omega}$, that we used to prove the previous theorems of density, allows us to prove a result in the space $H(\mathrm{div}, \Omega)$, which, contrary to what one might expect, is not obvious: a function of $H(\mathrm{div}, \Omega)$, that vanishes on γ_0 and γ_1, where γ_0 and γ_1 are defined previously, vanishes on $\partial\Omega$. However, to prove this result, contrary to the proofs of the density theorems, we shall need to make additionnal assumptions on the set $\overline{\gamma_0} \cap \overline{\gamma_1}$.

This article is organized as follows. In Section 2, we define the adequate covering of $\overline{\Omega}$ and the partition of unity subordinated to this covering. In Section 3, we prove our main result of density in $W^{1,r}(\Omega, \gamma_0)$. Section 4 is devoted to the generalization of this result to the space $W^{m,r}(\Omega, \gamma_0)$,with $m \geq 1$. Finally, in Section 5, we give an application of the previous partition of unity in the space $H(\mathrm{div}, \Omega)$.

We end this introduction with some notation that we shall use further on. We recall that Ω is a bounded domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary is Lipschitz-continuous. Let

γ_0 and γ_1 be two non empty open parts of $\partial\Omega$ such that they have a finite number of connected components and such that they verify

$$\partial\Omega = \overline{\gamma_0} \cup \overline{\gamma_1}, \qquad \gamma_0 \cap \gamma_1 = \emptyset, \qquad \overline{\gamma_0} \cap \overline{\gamma_1} = \bigcup_{k=1}^{q} K_k, \tag{1.1}$$

where K_k, $1 \le k \le q$, denotes the connected components of $\overline{\gamma_0} \cap \overline{\gamma_1}$ and, for $1 \le k \le q$, let us set

$$\forall \alpha > 0,\ G_{k,\alpha} = \{\mathbf{x} \in \mathbb{R}^d,\ d(\mathbf{x}, K_k) < \alpha\}, \tag{1.2}$$

where $d(.,.)$ is the euclidian distance in $\mathbb{R}^d$. Afterwards, we choose α such that

$$0 < \alpha < \frac{1}{2} \min_{1 \le i,j \le q,\ i \ne j} d(K_i, K_j) \qquad \text{and} \qquad \alpha \le 1. \tag{1.3}$$

.

We define : for each real $r > 1$ and each integer $m \ge 1$,

$$W^{m,r}(\Omega, \gamma_0) = \{v \in W^{m,r}(\Omega),\ (\frac{\partial^j v}{\partial n^j})_{|\gamma_0} = 0,\ j = 0, \ldots, m-1\}, \tag{1.4}$$

$$\mathcal{D}(\overline{\Omega}, \gamma_0) = \{v \in \mathcal{D}(\overline{\Omega}),\ v \text{ is equal to 0 in a neighborhood of } \gamma_0\}. \tag{1.5}$$

Finally, we shall use the space

$$H(\text{div}, \Omega) = \{\mathbf{v} \in L^2(\Omega)^d,\ \text{div}\,\mathbf{v} \in L^2(\Omega)\}.$$

2 Partition of unity

2.1 First covering of $\overline{\Omega}$

Since the boundary of Ω is Lipschitz-continuous, for every $\mathbf{x} \in \partial\Omega$, there exist an open hypercube $C_{\mathbf{x}}$, neighborhood of $\mathbf{x}$ in $\mathbb{R}^d$, and new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, where $\mathbf{y}' = (y_1, \ldots, y_{d-1})$, such that:

i) $C_{\mathbf{x}} = \prod_{j=1}^{d}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$.

ii) There exists a Lipschitz-continuous function $\Phi^{\mathbf{x}}$ defined in $\prod_{j=1}^{d-1}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$ of constant $L_{\mathbf{x}}$ such that $\forall \mathbf{y}' \in \prod_{j=1}^{d-1}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[,\ |\Phi^{\mathbf{x}}(\mathbf{y}')| \le \frac{a_{\mathbf{x},d}}{2}$ and

$$\Omega \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d < \Phi^{\mathbf{x}}(\mathbf{y}')\}, \qquad \partial\Omega \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\}. \tag{2.1}$$

Moreover, $\forall \mathbf{x} \in \gamma_0 \cup \gamma_1$, $\forall j = 1, \ldots, d$, we choose the reals $a_{\mathbf{x},j}$ such that $C_{\mathbf{x}} \cap \overline{\gamma_0} \cap \overline{\gamma_1} = \emptyset$. Since $\forall x \in \gamma_0$, $C_{\mathbf{x}} \cap \overline{\gamma_1} = \emptyset$ and $\forall \mathbf{x} \in \gamma_1$, $C_{\mathbf{x}} \cap \overline{\gamma_0} = \emptyset$, we have

$$\forall \mathbf{x} \in \gamma_0,\ \gamma_0 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\},\ \forall \mathbf{x} \in \gamma_1,\ \gamma_1 \cap C_{\mathbf{x}} = \{\mathbf{y} \in C_{\mathbf{x}},\ y_d = \Phi^{\mathbf{x}}(\mathbf{y}')\}.$$

For every strictly positive real number α verifying (1.3), let us define a finite open covering of $\overline{\Omega}$ as it follows : First, we have

$$\partial\Omega \subset (\bigcup_{\mathbf{x}\in\gamma_0\cup\gamma_1} C_{\mathbf{x}}) \cup (\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}})).$$

Note that, owing to (1.3), $G_{i,\alpha} \cap G_{j,\alpha} = \emptyset$, $1 \le i,j \le q$, $i \ne j$. Second, the compactness implies that there exists a finite open covering of $\partial\Omega$:

$$\partial\Omega \subset (\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}}) \cup (\bigcup_{k=q+1}^{r_\alpha} C_{\mathbf{m}_{k,\alpha}}), \tag{2.2}$$

where the open sets $C_{\mathbf{x}}$ are defined by (2.1) and $G_{k,\alpha}$ is defined by (1.2). Moreover, there exists an open set $C_{0,\alpha}$ such that

$$\overline{C_{0,\alpha}} \subset \Omega \quad \text{and} \quad \overline{\Omega} \subset C_{0,\alpha} \cup (\bigcup_{k=1}^{q} G_{k,\frac{\alpha}{2}}) \cup (\bigcup_{k=q+1}^{r_\alpha} C_{\mathbf{m}_{k,\alpha}}), \tag{2.3}$$

which is an open covering of $\overline{\Omega}$ denoted $\mathcal{R}_\alpha$.

2.2 Second covering of $\overline{\Omega}$ and associated partition of unity

Let ρ be a standard mollifier, which means that ρ is a positive C^∞ function in $\mathrm{I\!R}^d$ supported in the unit ball and such that $\int_{\mathrm{I\!R}^d} \rho(\mathbf{x})\, d\mathbf{x} = 1$. For every $p \in \mathrm{I\!N}^*$, we define

$$\forall \mathbf{x} \in \mathrm{I\!R}^d,\ \rho_p(\mathbf{x}) = p^d \rho(p\mathbf{x}). \tag{2.4}$$

Let φ belong to $C^1(\mathrm{I\!R}_+)$ such that

$$\forall t \in [0, \frac{9}{16}],\ \varphi(t) = 1,\ \forall t \ge \frac{11}{16},\ \varphi(t) = 0 \quad \text{and} \quad \forall t \in \mathrm{I\!R}_+,\ |\varphi'(t)| \le A.$$

For example, we can choose φ defined on $[\frac{9}{16}, \frac{11}{16}]$ by $\varphi(t) = \dfrac{1+\cos(8\pi t - \frac{9\pi}{2})}{2}$, with $A = 4\pi$. Let us recall (see [5]) that, for $k = 1,\ldots,q$ and $i = 1\ldots,d$, $\mathbf{x} \mapsto \partial_i d(\mathbf{x}, K_k)$ belongs to $L^\infty(\mathrm{I\!R}^d)$ and verifies

$$\forall i = 1,\ldots,d,\ \forall \mathbf{x} \in \mathrm{I\!R}^d,\ |\partial_i d(\mathbf{x}, K_k)| \le 1. \tag{2.5}$$

Then, we set

$$\forall k,\ 1 \le k \le q,\ \theta_{\alpha,k} = \varphi(\frac{1}{\alpha} d(\,.\,, K_k)) * \rho_{p_\alpha}, \tag{2.6}$$

with $p_\alpha = [\frac{16}{\alpha}] + 1$, where $[x]$ denotes the integral part of the real number x, and ρ_p defined by (2.4). This function belongs to $\mathcal{D}(G_{k,\alpha})$ and verifies, for $i = 1,\ldots,d$,

$$\forall \mathbf{x} \in G_{k,\frac{\alpha}{2}},\ \theta_{\alpha,k}(\mathbf{x}) = 1,\ \forall \mathbf{x} \notin G_{k,\frac{3\alpha}{4}},\ \theta_{\alpha,k}(\mathbf{x}) = 0 \text{ and } \forall \mathbf{x} \in \mathrm{I\!R}^d,\ |\partial_i \theta_{\alpha,k}(\mathbf{x})| \le \frac{A}{\alpha}. \tag{2.7}$$

Considering successively that: $\theta_{\alpha,j} + (1 - \theta_{\alpha,j}) = 1$, for $j = 1, \dots, q$, we obtain

$$\theta_{\alpha,1} + (1-\theta_{\alpha,1})\theta_{\alpha,2} + \ldots + (\prod_{j=1}^{q-1}(1-\theta_{\alpha,j}))\theta_{\alpha,q} + \prod_{j=1}^{q}(1-\theta_{\alpha,j}) = 1.$$

But, since the sets $G_{j,\alpha}$ are disconnected and since $\theta_{\alpha,j}$ belongs to $\mathcal{D}(G_{j,\alpha})$, for $1 \le j \le q$, we have $(\prod_{j=1}^{k-1}(1-\theta_{\alpha,j}))\theta_{\alpha,k} = \theta_{\alpha,k}$. Thus, we obtain

$$\theta_{\alpha,1} + \theta_{\alpha,2} + \ldots + \theta_{\alpha,q} + \prod_{j=1}^{q}(1-\theta_{\alpha,j}) = 1.$$

Hence, we derive, for every $u \in W^{1,r}(\Omega, \gamma_0)$

$$u = \theta_{\alpha,1}u + \theta_{\alpha,2}u + \ldots + \theta_{\alpha,q}u + (\prod_{j=1}^{q}(1-\theta_{\alpha,j}))u. \tag{2.8}$$

Let $\{\beta_{\alpha,j}\}_{j=0}^{r_\alpha}$ be a partition of unity on $\overline{\Omega}$ (see [3] or [4]), subordinated to the covering $\mathcal{R}_\alpha$ defined by (2.3). Substituting the functions $\beta_{\alpha,j}$ in (2.8) yields

$$u = \theta_{\alpha,1}u + \theta_{\alpha,2}u + \ldots + \theta_{\alpha,q}u + \sum_{k=0}^{r_\alpha}(\prod_{j=1}^{q}(1-\theta_{\alpha,j}))\beta_{\alpha,k}u.$$

Considering that, for every $1 \le k \le q$, $\prod_{j=1}^{q}(1-\theta_{\alpha,j})\beta_{\alpha,k} = 0$, since, if $\mathbf{x} \in G_{k,\frac{\alpha}{2}}$, $\theta_{\alpha,k}(x) = 1$, we obtain

$$u = \sum_{k=0}^{r_\alpha} \varphi_{\alpha,k}u, \tag{2.9}$$

where $\varphi_{\alpha,k} = (\prod_{j=1}^{q}(1-\theta_{\alpha,j}))\beta_{\alpha,k}$, $k = 0$ or $q+1 \le k \le r_\alpha$ and $\varphi_{\alpha,k} = \theta_{\alpha,k}$, $1 \le k \le q$. Thus, for α verifying (1.3), $\mathcal{P}_\alpha = \{\varphi_{\alpha,k}\}_{k=0}^{r_\alpha}$ is a partition of unity on $\overline{\Omega}$, subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, with

$$\mathcal{O}_{0,\alpha} = C_{0,\alpha},\ \mathcal{O}_{k,\alpha} = G_{k,\alpha} \text{ for } 1 \le k \le q \quad \text{and} \quad \mathcal{O}_{k,\alpha} = C_{\mathbf{m}_{k,\alpha}} \text{ for } q+1 \le k \le r_\alpha, \tag{2.10}$$

where the sets $C_{0,\alpha}$, $G_{k,\alpha}$ and $C_{\mathbf{x}}$ are respectively defined by (2.3), (1.2) and (2.1).

3 Density result in $W^{1,r}(\Omega, \gamma_0)$

Theorem 3.1 *Let $r > 1$ be a real number. Let Ω a bounded domain in $I\!R^d$ whose boundary is Lipschitz-continuous and let γ_0 be an open part of $\partial\Omega$ verifying (1.1). Let the spaces $W^{1,r}(\Omega, \gamma_0)$ and $\mathcal{D}(\overline{\Omega}, \gamma_0)$ be defined respectively by (1.4) and (1.5). Then the space $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is dense in $W^{1,r}(\Omega, \gamma_0)$.*

Proof. From now on, we suppose that α verifies (1.3), so we can consider the partition $\mathcal{P}_\alpha$ defined by (2.10). For every real number $\varepsilon > 0$, let us define a real $\alpha_\varepsilon > 0$ such that, for $0 < \alpha \leq \alpha_\varepsilon$, the partition of unity $\mathcal{P}_\alpha$ subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$ allows us to construct an approximation $u_\varepsilon \in \mathcal{D}(\overline{\Omega},\ \gamma_0)$ of $u \in W^{1,r}(\Omega,\ \gamma_0)$ in $W^{1,r}$ norm.

Lemma 3.2 *For every real number $\varepsilon > 0$, there exists a real number α_ε verifying (1.3) such that, for every $0 < \alpha \leq \alpha_\varepsilon$,*

$$\forall k = 1, \ldots, q, \qquad \|\theta_{\alpha,k} u\|_{W^{1,r}(G_{k,\alpha}\cap\Omega)} \leq \frac{\varepsilon}{4q}. \tag{3.1}$$

Proof. For $k = 1, \ldots, q$, let $\{B(\mathbf{x}_i, \alpha)\}_{i=1}^{p}$ be an open optimal covering of $\overline{G_{k,\alpha}}$, where $B(\mathbf{x}_i, \alpha)$ denotes the open ball with center $\mathbf{x}_i$ and radius α. This means that there is no covering of $\overline{G_{k,\alpha}}$ with less than p balls of radius α. Let $i \in \mathbb{N}^*$ such that $1 \leq i \leq p$. Note that $B(\mathbf{x}_i, \alpha) \cap \overline{G_{k,\alpha}} \neq \emptyset$ and let $\mathbf{z}_i$ belongs to $B(\mathbf{x}_i, \alpha) \cap \overline{G_{k,\alpha}}$. Then, there exists $\mathbf{y}_i \in K_k$, such that $d(\mathbf{z}_i, \mathbf{y}_i) \leq \alpha$, which implies $d(\mathbf{x}_i, \mathbf{y}_i) < 2\alpha$. Hence, we derive

$$G_{k,\alpha} \subset \bigcup_{i=1}^{p} B(\mathbf{x}_i, \alpha) \subset \bigcup_{i=1}^{p} B(\mathbf{x}_i, 2\alpha), \tag{3.2}$$

such that the covering $\{B(\mathbf{x}_i, \alpha)\}_{i=1}^{p}$ is maximal and the covering $\{B(\mathbf{x}_i, 2\alpha)\}_{i=1}^{p}$ verifies, $\forall i = 1, \ldots, p$,

$$B(\mathbf{x}_i, 2\alpha) \cap K_k \neq \emptyset. \tag{3.3}$$

Note that, $\forall \mathbf{x} \in \mathbb{R}^d$, $\forall n \in \mathbb{N}^*$ and $\forall \alpha > 0$, there exists a covering $\{B(\mathbf{x}'_i, \alpha)\}_{i=1}^{p_{n,d}}$ of the ball $B(\mathbf{x}, n\alpha)$ with $p_{n,d} = ([n\sqrt{d}] + 1)^d$, where $[x]$ denotes the integral part of the real number x. Indeed, the ball of radius $n\alpha$ is inscribed in a hypercube of edge $2n\alpha$ and the hypercube of edge $\dfrac{2\alpha}{\sqrt{d}}$ is inscribed in a ball of radius α. Let $i \in \mathbb{N}^*$ such that $1 \leq i \leq p$ and let us set

$$N_i = \{j \in \mathbb{N}^*, 1 \leq j \leq p, B(\mathbf{x}_j, 2\alpha) \cap B(\mathbf{x}_i, 2\alpha) \neq \emptyset\}. \tag{3.4}$$

On the one hand, we have

$$\bigcup_{j\in N_i} B(\mathbf{x}_j, \alpha) \subset \bigcup_{j\in N_i} B(\mathbf{x}_j, 2\alpha) \subset B(\mathbf{x}_i, 6\alpha).$$

On the other hand, the previous note implies

$$B(\mathbf{x}_i, 6\alpha) \subset \bigcup_{j=1}^{p_{6,d}} B(\mathbf{x}'_j, \alpha).$$

Since the covering $\{B(\mathbf{x}_i, \alpha)\}_{i=1}^{p}$ of $\overline{G_{k,\alpha}}$ is maximal, we derive

$$\forall i \in \mathbb{N}^*,\ 1 \leq i \leq p,\ \operatorname{card} N_i \leq p_{6,d} = ([6\sqrt{d}] + 1)^d = M_d, \tag{3.5}$$

where N_i is defined by (3.4).

Let $\tilde{u} \in W^{1,r}(\mathbb{R}^d)$ be an extension of $u \in W^{1,r}(\Omega, \gamma_0)$ outside Ω. For every $i = 1, \ldots, p$, setting $\mathbf{x} = \mathbf{x}_i + \alpha\mathbf{z}$ yields

$$\|\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \leq \alpha^d \int_{B(\mathbf{0},2)} |\tilde{u}(\mathbf{x}_i + \alpha\mathbf{z})|^r \, d\mathbf{z}. \tag{3.6}$$

Since u vanishes on $B(\mathbf{x}_i, 2\alpha) \cap \gamma_0$, which has a strictly positive measure because of (3.3), $\tilde{u}$ also vanishes at least on the same set and we can apply Poincaré inequality to deduce that there exists a constant $C_1 > 0$ such that

$$\int_{B(\mathbf{0},2)} |\tilde{u}(\mathbf{x}_i + \alpha\mathbf{z})|^r \, d\mathbf{z} \leq C_1 \int_{B(\mathbf{0},2)} |\nabla_{\mathbf{z}} \tilde{u}(\mathbf{x}_i + \alpha\mathbf{z})|^r \, d\mathbf{z}. \tag{3.7}$$

Next, using again $\mathbf{x} = \mathbf{x}_i + \alpha\mathbf{z}$, we have

$$\int_{B(\mathbf{0},2)} |\nabla_{\mathbf{z}} \tilde{u}(\mathbf{x}_i + \alpha\mathbf{z})|^r \, d\mathbf{z} = \alpha^r \int_{B(\mathbf{0},2)} |\nabla \tilde{u}(\mathbf{x}_i + \alpha\mathbf{z})|^r \, d\mathbf{z} = \alpha^{r-d} \int_{B(\mathbf{x_i},2\alpha)} |\nabla \tilde{u}(\mathbf{x})|^r \, d\mathbf{x},$$

which gives, owing to (3.6) and (3.7),

$$\|\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \leq C_1 \alpha^r \|\nabla \tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))}. \tag{3.8}$$

Then, from (3.2), we derive

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq \|\tilde{u}\|^r_{L^r(G_{k,\alpha})} \leq \sum_{i=1}^{p} \|\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))},$$

and, in view of (3.8), we obtain

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq C_1 \alpha^r \sum_{i=1}^{p} \|\nabla \tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))}. \tag{3.9}$$

Now, we can assume that the integrals $\|\nabla \tilde{u}\|_{L^r(B(\mathbf{x}_{\psi(i)},2\alpha))}$ are in decreasing order with respect to i where ψ is a permutation of the set $\{1, \ldots, p\}$. To simplify the notation, we still denote the index i instead of $\psi(i)$. Thus, we assume that, for $i = 1, \ldots, p-1$,

$$\|\nabla \tilde{u}\|_{L^r(B(\mathbf{x}_i,2\alpha))} \geq \|\nabla \tilde{u}\|_{L^r(B(\mathbf{x}_{i+1},2\alpha))}. \tag{3.10}$$

Next, we construct by finite induction a partition of $I = \{i \in \mathbb{N}^*,\ 1 \leq i \leq p\}$ in the following way: we define $I_0 = I$, $i_1 = 1$ and for $k \geq 1$

$$J_k = \{j \in I_{k-1},\ B(\mathbf{x}_j, 2\alpha) \cap B(\mathbf{x}_{i_k}, 2\alpha) \neq \emptyset\}, \quad I_k = \{j \in I_{k-1},\ B(\mathbf{x}_j, 2\alpha) \cap B(\mathbf{x}_{i_k}, 2\alpha) = \emptyset\}$$

and $i_{k+1} = \min I_k$ if $I_k \neq \emptyset$. Note that $i_{k+1} > i_k$, because, by construction, $i_{k+1} \geq i_k$ and $i_k \notin I_k$. Let $l \geq 1$ such that $I_l = \emptyset$ and $I_{l-1} \neq \emptyset$. Considering that $I_{k-1} = J_k \cup I_k$ for $k = 1, \ldots, l$, we obtain the following partition of I

$$I = \bigcup_{k=1}^{l} J_k. \tag{3.11}$$

Moreover, by construction, the balls $B(x_{i_k}, 2\alpha)$, $k = 1, \ldots l$, are disconnected two by two. Hence, on the one hand, we derive

$$\|\nabla \tilde{u}\|^r_{L^r(\bigcup\limits_{i=1}^{p} B(\mathbf{x}_i,2\alpha))} \geq \sum_{k=1}^{l} \|\nabla \tilde{u}\|^r_{L^r(B(\mathbf{x}_{i_k},2\alpha))}. \tag{3.12}$$

On the other hand, we have

$$\sum_{i=1}^{p} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} = \sum_{k=1}^{l}\Big(\sum_{j\in J_k} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_j,2\alpha))}\Big).$$

But, in view of (3.5) and (3.10), we can write

$$\sum_{j\in J_k} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_j,2\alpha))} \leq M_d\|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_{i_k},2\alpha))}.$$

Thus, we derive

$$\sum_{i=1}^{p} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \leq M_d \sum_{k=1}^{l} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_{i_k},2\alpha))}.$$

Finally, owing to (3.12), we obtain the crucial estimate

$$\sum_{i=1}^{p} \|\nabla\tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \leq M_d\|\nabla\tilde{u}\|^r_{L^r(\bigcup_{i=1}^{p} B(\mathbf{x}_i,2\alpha))} \leq M_d\|\nabla\tilde{u}\|^r_{L^r(G_{k,4\alpha})},$$

which gives, in view of (3.9),

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq C_1 M_d \alpha^r \|\nabla\tilde{u}\|^r_{L^r(G_{k,4\alpha})}. \tag{3.13}$$

Finally, for $i = 1,\ldots,d$, $\partial_i(\theta_{\alpha,k}u) = \partial_i(\theta_{\alpha,k})u + \theta_{\alpha,k}\partial_i u$, where $\theta_{\alpha,k}$ is defined by (2.6). From (2.7) and (3.13), we derive

$$\|\partial_i(\theta_{\alpha,k})u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq \frac{A^r}{\alpha^r}\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq C_1 M_d A^r \|\nabla\tilde{u}\|^r_{L^r(G_{k,4\alpha})}.$$

Considering (2.7) again and

$$\|\partial_i(\theta_{\alpha,k}u)\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq 2^{r-1}\big(\|\partial_i(\theta_{\alpha,k})u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} + \|\theta_{\alpha,k}\partial_i u\|^r_{L^r(G_{k,\alpha}\cap\Omega)}\big),$$

we obtain, for $k = 1,\ldots,q$

$$\|\theta_{\alpha,k}u\|^r_{W^{1,r}(G_{k,\alpha}\cap\Omega)} \leq \|\tilde{u}\|^r_{L^r(G_{k,4\alpha})} + 2^{r-1}\,(C_1\,M_d\,A^r + 1)d\|\nabla\tilde{u}\|^r_{L^r(G_{k,4\alpha})}.$$

Note that

$$\bigcap_{\alpha>0} G_{k,4\alpha} = K_k$$

and the measure of K_k is 0 in $\mathbb{R}^d$. Since $\tilde{u}$ belongs to $W^{1,r}(\mathbb{R}^d)$, for $k = 1,\ldots,q$, we have

$$\lim_{\alpha\to 0} \|\theta_{\alpha,k}u\|_{W^{1,r}(G_{k,\alpha}\cap\Omega)} = 0.$$

Thus, there exists a real $\alpha_\varepsilon > 0$ such that the inequalities (3.1) and (1.3) are verified. $\Diamond$

Let us note that, considering the partition of unity $\mathcal{P}_\alpha$ defined by (2.10), such that $0 < \alpha \leq \alpha_\varepsilon$, and in view of $\theta_{\alpha,k} \in \mathcal{D}(G_{k,\alpha})$, (3.1) can be written

$$\forall k = 1,\ldots,q,\ \|\varphi_{\alpha,k}u\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4q}, \tag{3.14}$$

so that, for every $k = 1, \ldots, q$, we can approximate $\boldsymbol{\varphi}_{\alpha,k} u$ by 0 in $\mathcal{O}_{k,\alpha} = G_{k,\alpha}$.

We now deal with the case $k = 0$, that is to say, we want approximate $\varphi_{\alpha,0} u$ in $\mathcal{O}_{0,\alpha}$. Let us recall that $\varphi_{\alpha,0} u$ has a compact support in $\mathcal{O}_{0,\alpha}$ with $\overline{\mathcal{O}_{0,\alpha}} \subset \Omega$. Therefore, we have

$$d(\text{supp}\,(\varphi_{\alpha,0} u), \partial\mathcal{O}_{0,\alpha}) = \mu_0 > 0 \tag{3.15}$$

and we can note that $\widetilde{\varphi_{\alpha,0}} u$ belongs to $W^{1,r}(\mathbb{R}^d)$, where the latter denotes the extension by zero. Then, for every $p \in \mathbb{N}^*$, we define u_p by

$$\forall \mathbf{x} \in \mathbb{R}^d, \;\; u_p(\mathbf{x}) = ((\widetilde{\varphi_{\alpha,0}} u) * \rho_p)(\mathbf{x}) = \int_{B(\mathbf{0},1/p)} \widetilde{\varphi_{\alpha,0}} u(\mathbf{x}-\mathbf{y}) \rho_p(\mathbf{y})\, d\mathbf{y},$$

where ρ_p is defined by (2.4). In a standard way, we obtain that

$$\lim_{p \to +\infty} u_p = \widetilde{\varphi_{\alpha,0}} u \qquad \text{in} \qquad W^{1,r}(\mathbb{R}^d),$$

which implies that there exists $P_\varepsilon \in N^*$ such that, $\forall p \geq P_\varepsilon$,

$$\|\varphi_{\alpha,0} u - u_p\|_{W^{1,r}(\mathcal{O}_{0,\alpha})} \leq \frac{\varepsilon}{4}. \tag{3.16}$$

Next, taking care of the support of u_p, we choose $p \geq \dfrac{3}{\mu_0}$ and we define the set $E = \{\mathbf{x} \in \overline{\mathcal{O}_{0,\alpha}}, \; d(\mathbf{x}, \partial\mathcal{O}_{0,\alpha}) \leq \dfrac{\mu_0}{3}\}$. This implies that $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$ and $\forall \mathbf{x} \in E$,

$$d(\mathbf{x}-\mathbf{y}, \text{supp}\,(\varphi_{\alpha,0} u)) \geq d(\partial\mathcal{O}_{0,\alpha}, \text{supp}\,(\varphi_{\alpha,0} u)) - d(\mathbf{x}-\mathbf{y}, \mathbf{x}) - d(\mathbf{x}, \partial\mathcal{O}_{0,\alpha}) \geq \frac{\mu_0}{3} > 0.$$

In the same way, we have $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$ and $\forall \mathbf{x} \in \overline{\Omega} \setminus \mathcal{O}_{0,\alpha}$,

$$d(\mathbf{x}-\mathbf{y}, \text{supp}\,(\varphi_{\alpha,0} u)) \geq \frac{2\mu_0}{3} > 0.$$

Hence, we derive that u_p vanishes on $E \cup (\overline{\Omega} \setminus \mathcal{O}_{0,\alpha})$. Setting $u_{\varepsilon,0} = u_{m_\varepsilon}$, where m_ε is defined by $m_\varepsilon = \max([\dfrac{3}{\mu_0}], P_\varepsilon)$ ($[r]$ is the integral part of r), and considering the supports of $\varphi_{\alpha,0} u$ and $u_{\varepsilon,0}$, yield

$$\|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\mathcal{O}_{0,\alpha} \cap \Omega)} = \|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4} \qquad \text{with} \qquad u_{\varepsilon,0} \in \mathcal{D}(\mathcal{O}_{0,\alpha}), \tag{3.17}$$

where $\overline{\mathcal{O}_{0,\alpha}} \subset \Omega$.

The next lemma gives an approximation of $\varphi_{\alpha,k} u$ in $\mathcal{O}_{k,\alpha}$, for $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, that is to say an approximation of u localized around γ_1.

Lemma 3.3 *Let α be a real number verifying (1.3). For every real number $\varepsilon > 0$ and for every $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, there exists a function $u_{\varepsilon,k} \in \mathcal{D}(\overline{\Omega})$ with compact support in $\mathcal{O}_{k,\alpha} \cap \overline{\Omega}$ such that*

$$\|\varphi_{\alpha,k} u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}, \tag{3.18}$$

where r_α is defined by (2.2).

Proof. For $k = q+1, \dots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_1$, we want to approximate $\varphi_{\alpha,k} u$. To simplify the notations, we drop the indexes, replacing $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and we set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \mu > 0. \tag{3.19}$$

Considering (2.1) and (2.10), we may assume that $\mathcal{O}$ is an open hypercube, neighborhood of a point of γ_1, such that, in new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, we have

$$\mathcal{O} \cap \Omega = \{\mathbf{y} \in \mathcal{O},\ y_d < \Phi(\mathbf{y}')\} \qquad \text{and} \qquad \gamma_1 \cap \mathcal{O} = \{\mathbf{y} \in \mathcal{O},\ y_d = \Phi(\mathbf{y}')\}, \tag{3.20}$$

where Φ is a Lipschitz-continuous function, defined in $\prod_{j=1}^{d-1}]-a_j, a_j[$, of constant L.

Let $n \in \mathbb{N}^*$. We set

$$u_n(\mathbf{y}) = u(\mathbf{y}', y_d - 1/n), \tag{3.21}$$

which is a function defined on

$$\Omega_n = \{\mathbf{y} \in \mathbb{R}^d\ (\mathbf{y}', y_d - 1/n) \in \mathcal{O} \cap \Omega\}.$$

The set Ω_n is obtained by translating $\mathcal{O} \cap \Omega$ to the direction of positive y_d. We denote by $\tilde{u}_n$ the extension of u_n by zero. Considering the support of u, we can see that the restriction of $\tilde{u}_n$ to $\mathcal{O} \cap \Omega$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$.

Next, since the translation is continuous on $L^r(\mathbb{R}^d)$, we derive

$$\lim_{n \to +\infty} \tilde{u}_{n|\mathcal{O}\cap\Omega} = u \qquad \text{in} \qquad L^r(\mathcal{O} \cap \Omega).$$

Moreover, as $\partial_i(\tilde{u}_{n|\mathcal{O}\cap\Omega}) = (\widetilde{\partial_i u})_{n|\mathcal{O}\cap\Omega}$, where the wide latter denotes the extension by zero of $(\partial_i u)_n$ in $\mathcal{O} \cap \Omega \setminus \Omega_n$, as we can verify by deriving in the sense of distribution, we have the same convergence for the partial derivatives. Thus, we obtain

$$\lim_{n \to +\infty} \tilde{u}_{n|\mathcal{O}\cap\Omega} = u \qquad \text{in} \qquad W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.22}$$

For every $n \in \mathbb{N}^*$ and $p \in \mathbb{N}^*$, we define

$$u_{n,p} = \tilde{u}_n * \rho_p. \tag{3.23}$$

The standard properties of the convolution imply

$$\lim_{p \to +\infty} u_{n,p} = \tilde{u}_n \qquad \text{in} \qquad L^r(\mathbb{R}^d). \tag{3.24}$$

Next

$$\partial_i u_{n,p} = \partial_i \tilde{u}_n * \rho_p.$$

We cannot pass to the limit in $L^r(\mathbb{R}^d)$ because, usually, $\partial_i \tilde{u}_n$ is not in $L^r(\mathbb{R}^d)$. First, let us show that, for p large enough, $\tilde{u}_{n|\mathcal{O}_p}$ belongs to $W^{1,r}(\mathcal{O}_p)$, where $\mathcal{O}_p$ is defined by

$$\mathcal{O}_p = \{\mathbf{y} \in \mathbb{R}^d,\ d(\mathbf{y}, \mathcal{O} \cap \Omega) < 1/p\}. \tag{3.25}$$

We set

$$\Gamma_n = \{\mathbf{y} \in \mathbb{R}^d, (\mathbf{y}', y_d - 1/n) \in \partial\Omega \cap \mathcal{O}\}, \tag{3.26}$$

and, thus, we can write

$$\partial\Omega_n = \overline{\Gamma_n} \cup \overline{\Gamma'_n}, \qquad \text{with } \Gamma_n \cap \Gamma'_n = \emptyset.$$

We can note that, since, $\forall \mathbf{y} \in \Gamma'_n$, $(\mathbf{y}', y_d - 1/n) \in \Omega \cap (\partial\mathcal{O})$,

$$\forall \mathbf{y} \in \Gamma'_n, \qquad u_n(\mathbf{y}) = 0. \tag{3.27}$$

Let us estimate, for every $\mathbf{z} \in \Gamma_n$, the distance $d(\mathbf{z}, \overline{\mathcal{O} \cap \Omega}) = d(\mathbf{z}, \overline{\mathcal{O}} \cap \partial\Omega)$. Indeed, $\forall \mathbf{y} \in \overline{\mathcal{O} \cap \Omega}$, $[\mathbf{z}, \mathbf{y}] \cap (\overline{\mathcal{O}} \cap \partial\Omega) \neq \emptyset$.

$$\forall \mathbf{z} \in \Gamma_n, \ \forall \mathbf{y} \in (\overline{\mathcal{O}} \cap \partial\Omega), \ \|\mathbf{z} - \mathbf{y}\|^2 = \|\mathbf{z}' - \mathbf{y}'\|^2 + (1/n + \Phi(\mathbf{z}') - \Phi(\mathbf{y}'))^2.$$

The properties of Φ yield

$$1/n + \Phi(\mathbf{z}') - \Phi(\mathbf{y}') \geq 1/n - L\|\mathbf{z}' - \mathbf{y}'\|.$$

Then, if $\|\mathbf{z}' - \mathbf{y}'\| \leq 1/(2nL)$, we have $\|\mathbf{z} - \mathbf{y}\| \geq 1/(2n)$, and, if $\|\mathbf{z}' - \mathbf{y}'\| \geq 1/(2nL)$, we have $\|\mathbf{z} - \mathbf{y}\| \geq 1/(2nL)$. Therefore, we obtain

$$d(\Gamma_n, \overline{\mathcal{O} \cap \Omega}) \geq \min(1/(2n), 1/(2nL)). \tag{3.28}$$

Next, we have by definition

$$\forall \psi \in \mathcal{D}(\mathcal{O}_p), \qquad < \partial_i \tilde{u}_n, \psi >_{\mathcal{D}(\mathcal{O}_p)} = -\int_{\mathcal{O}_p} \tilde{u}_n(\mathbf{x}) \partial_i \psi(\mathbf{x})\, d\mathbf{x} = -\int_{\mathcal{O}_p \cap \Omega_n} u_n(\mathbf{x}) \partial_i \psi(\mathbf{x})\, d\mathbf{x}.$$

Since u_n belongs to $W^{1,r}(\Omega_n)$, the Green's formula yields

$$< \partial_i \tilde{u}_n, \psi >_{\mathcal{D}(\mathcal{O}_p)} = \int_{\mathcal{O}_p \cap \Omega_n} \partial_i u_n(\mathbf{x}) \psi(\mathbf{x})\, d\mathbf{x} - \int_{\partial(\mathcal{O}_p \cap \Omega_n)} u_n(\mathbf{s}) \psi(\mathbf{s})\, n_i\, d\mathbf{s}.$$

Let us choose

$$1/p < \min(1/(2n), 1/(2nL)). \tag{3.29}$$

Then, owing to (3.28), we have for every $\mathbf{y} \in \overline{\mathcal{O}_p}$

$$d(\mathbf{y}, \overline{\mathcal{O} \cap \Omega}) \leq 1/p < \min(1/(2n), 1/(2nL)) \leq d(\Gamma_n, \overline{\mathcal{O} \cap \Omega}),$$

which implies

$$\Gamma_n \cap \overline{\mathcal{O}_p} = \emptyset.$$

Hence, we obtain

$$\partial(\mathcal{O}_p \cap \Omega_n) \subset (\partial(\mathcal{O}_p) \cup \partial(\Omega_n)) \cap \overline{\mathcal{O}_p} \subset (\partial(\mathcal{O}_p) \cup \overline{\Gamma'_n}).$$

Therefore, with (3.27) in addition, $u_n \psi$ vanishes on $\partial(\mathcal{O}_p \cap \Omega_n)$ and we derive

$$< \partial_i \tilde{u}_n, \psi >_{\mathcal{D}(\mathcal{O}_p)} = \int_{\mathcal{O}_p \cap \Omega_n} \partial_i u_n(\mathbf{x}) \psi(\mathbf{x})\, d\mathbf{x},$$

that is to say

$$\tilde{u}_{n|\mathcal{O}_p} \text{ belongs to } W^{1,r}(\mathcal{O}_p) \quad \text{and} \quad \partial_i \tilde{u}_{n|\mathcal{O}_p} = \widetilde{\partial_i u_n}, \tag{3.30}$$

where the wide latter is the extension by zero of $\partial_i u_n \in L^r(\Omega_n \cap \mathcal{O}_p)$ in $\mathcal{O}_p$.

Second, let us show, that, for p large enough, $\partial_i u_{n,p} = \widetilde{\partial_i u_n} * \rho_p$. In view of the Fubini Theorem,

$$\begin{aligned}\forall \psi \in \mathcal{D}(\mathcal{O} \cap \Omega), \qquad < \partial_i u_{n,p}, \psi >_{\mathcal{D}(\mathcal{O}\cap\Omega)} &= -\int_{\mathcal{O}\cap\Omega} (\int_{B(\mathbf{0},1/p)} \tilde{u}_n(\mathbf{x}-\mathbf{y})\rho_p(\mathbf{y})\, d\mathbf{y})\partial_i\psi(\mathbf{x})\, d\mathbf{x} \\ &= -\int_{B(\mathbf{0},1/p)} \rho_p(\mathbf{y})(\int_{\mathcal{O}\cap\Omega} \tilde{u}_n(\mathbf{x}-\mathbf{y})\partial_i\psi(\mathbf{x})\, d\mathbf{x})d\mathbf{y}.\end{aligned}$$

In view of (3.30), for every $\mathbf{y} \in B(\mathbf{0}, 1/p)$, $\mathbf{x} \mapsto \tilde{u}_n(\mathbf{x}-\mathbf{y})$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$ and

$$\forall \mathbf{x} \in \mathcal{O} \cap \Omega,\ ,\forall \mathbf{y} \in B(\mathbf{0}, 1/p), \qquad \partial_i \tilde{u}_n(\mathbf{x}-\mathbf{y}) = \widetilde{\partial_i u_n}(\mathbf{x}-\mathbf{y}).$$

Then, the Green's formula and the Fubini Theorem yield

$$< \partial_i u_{n,p}, \psi >_{\mathcal{D}(\mathcal{O}\cap\Omega)} = \int_{\mathcal{O}\cap\Omega} (\int_{B(\mathbf{0},1/p)} \widetilde{\partial_i u_n}(\mathbf{x}-\mathbf{y})\rho_p(\mathbf{y})\, d\mathbf{y})\psi(\mathbf{x})\, d\mathbf{x},$$

which implies, for every p verifying (3.29),

$$\partial_i u_{n,p} = \widetilde{\partial_i u_n} * \rho_p.$$

From the standard properties of the convolution, we derive

$$\lim_{p\to+\infty} (\partial_i u_{n,p})_{|\mathcal{O}\cap\Omega} = \widetilde{\partial_i u_n} \quad \text{and} \quad L^r(\mathcal{O} \cap \Omega)$$

and, in view of (3.24),

$$\lim_{p\to+\infty} (u_{n,p})_{|\mathcal{O}\cap\Omega} = \tilde{u}_n \quad \text{in} \quad W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.31}$$

Then, (3.22) and (3.31) yield that there exists $N_\varepsilon \in \mathbb{N}^*$ such that, for $\min(n, p) \geq N_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{1,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{3.32}$$

Finally, we set $\mathbf{z}_n = (\mathbf{0}, 1/n)$ and, for $\min(n, p) \geq \dfrac{6}{\mu}$ where μ is defined by (3.19), we consider the set

$$E = \{\mathbf{x} \in \overline{\mathcal{O}} \cap \overline{\Omega},\ d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) \leq \frac{\mu}{3}\}.$$

$\forall \mathbf{x} \in E$, $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$, we have

$$d(\mathbf{x}-\mathbf{y}-\mathbf{z}_n, \text{supp } u) \geq d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) - d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) - d(\mathbf{x}, \mathbf{x}-\mathbf{y}-\mathbf{z}_n) \geq \frac{\mu}{3} > 0.$$

In the same way, $\forall \mathbf{x} \in \overline{\Omega} \setminus \mathcal{O}$, $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$, we have

$$d(\mathbf{x}-\mathbf{y}-\mathbf{z}_n, \text{supp } u) \geq d(\mathbf{x}, \text{supp } u) - d(\mathbf{x}, \mathbf{x}-\mathbf{y}-\mathbf{z}_n) \geq \frac{2\mu}{3} > 0.$$

Hence, we derive that, for every $\mathbf{x} \in E \cup (\overline{\Omega} \setminus \mathcal{O})$ and $\mathbf{y} \in B(\mathbf{0}, 1/p)$, $\mathbf{x} - \mathbf{y} - \mathbf{z}_n$ does not belong to $\operatorname{supp} u$, which implies $u_{n,p}(\mathbf{x}) = 0$. Thus, the function $u_\varepsilon = u_{m_\varepsilon, m_\varepsilon}$, where $u_{n,p}$ is defined by (3.23) and m_ε by $m_\varepsilon = \max([\frac{6}{\mu}] + 1, N_\varepsilon)$, belongs to $\mathcal{D}(\overline{\Omega})$ with a compact support in $\mathcal{O} \cap \overline{\Omega}$ and verifies

$$\|u - u_\varepsilon\|_{W^{1,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{1,r}(\Omega)} \le \frac{\varepsilon}{4\, r_\alpha},$$

which ends the proof of the lemma. ◊

The next lemma deals with an approximation of $\varphi_{\alpha,k} u$ in $\mathcal{O}_{k,\alpha}$, for $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, that is to say an approximation of u localized around γ_0, that is the part of the boundary where u vanishes.

Lemma 3.4 *Let α be a real number verifying (1.3). For every real number $\varepsilon > 0$ and for every $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, there exists a function $u_{\varepsilon,k} \in \mathcal{D}(\Omega \cap \mathcal{O}_{k,\alpha})$, such that*

$$\|\varphi_{\alpha,k} u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} \le \frac{\varepsilon}{4\, r_\alpha}, \tag{3.33}$$

where r_α is defined by (2.2).

Proof. As in the previous lemma, to simplify the notations, we drop the indexes, replacing, for $k = q+1, \ldots, r_\alpha$, $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and we set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \nu > 0. \tag{3.34}$$

Considering (2.1) and (2.10), we may assume that $\mathcal{O}$ is an open hypercube, neighborhood of a point of γ_0, such that, in new orthogonal coordinates $\mathbf{y} = (\mathbf{y}', y_d)$, we have

$$\mathcal{O} \cap \Omega = \{\mathbf{y} \in \mathcal{O},\ y_d < \Phi(\mathbf{y}')\} \qquad \text{and} \qquad \gamma_0 \cap \mathcal{O} = \{\mathbf{y} \in \mathcal{O},\ y_d = \Phi(\mathbf{y}')\}, \tag{3.35}$$

where Φ is a Lipschitz-continuous function, defined in $\prod_{j=1}^{d-1}]-a_j, a_j[$, of constant L.

Let $n \in \mathbb{N}^*$. We set

$$u_n(\mathbf{y}) = u(\mathbf{y}', y_d + 1/n), \tag{3.36}$$

which is a function defined on

$$\Omega_n = \{\mathbf{y} \in \mathbb{R}^d,\ (\mathbf{y}', y_d + 1/n) \in \mathcal{O} \cap \Omega\}.$$

The set Ω_n is obtained by translating $\mathcal{O} \cap \Omega$ to the direction of negative y_d, that is to say, contrary to previously, inside the domain Ω. We denote by $\tilde{u}_n$ the extension of u_n by zero outside Ω_n. Considering the support of u and since u vanishes on γ_0, we can see that the restriction of $\tilde{u}_n$ to $\mathcal{O} \cap \Omega$ belongs to $W^{1,r}(\mathcal{O} \cap \Omega)$ and, as in the previous lemma, we have

$$\lim_{n \to +\infty} \tilde{u}_{n|\mathcal{O}\cap\Omega} = u \qquad \text{in} \qquad W^{1,r}(\mathcal{O} \cap \Omega). \tag{3.37}$$

Note that, if $\frac{1}{n} \leq \nu$, where ν is defined by (3.34), then u_n has a compact support in $\mathcal{O} \cap \Omega$ and, therefore, $\tilde{u}_n$ belongs to $W^{1,r}(\mathbb{R}^d)$. Hence, setting

$$u_{n,p} = \tilde{u}_n * \rho_p,$$

we derive

$$\lim_{p \to +\infty} (u_{n,p})_{|\mathcal{O}\cap\Omega} = (\tilde{u}_n)_{|\mathcal{O}\cap\Omega} \qquad \text{in} \qquad W^{1,r}(\mathcal{O} \cap \Omega),$$

which implies, in view of (3.37), that there exists $N'_\varepsilon \in \mathbb{N}^*$, such that, for $\min(n,p) \geq N'_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{1,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{3.38}$$

We set

$$\Gamma_n^* = \{\mathbf{y} \in \mathbb{R}^d, (\mathbf{y}', y_d + 1/n) \in \partial\Omega \cap \mathcal{O}\}, \tag{3.39}$$

Note that $d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) = d(\partial\Omega \cap \overline{\mathcal{O}}, \Gamma_n^*)$ because $\forall \mathbf{z} \in \partial\Omega \cap \overline{\mathcal{O}}$ and $\forall \mathbf{y} \in \Omega_n$, $[\mathbf{z}, \mathbf{y}] \cap \Gamma_n^* \neq \emptyset$. Moreover, in the same way as for Γ_n, we obtain the analogous of (3.28):

$$d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) = d(\partial\Omega \cap \overline{\mathcal{O}}, \Gamma_n^*) \geq \min(1/(2n), 1/(2nL)) = \delta_n. \tag{3.40}$$

We recall that

$$u_{n,p}(\mathbf{x}) = \int_{B(\mathbf{0},1/p)} \tilde{u}_n(\mathbf{x} - \mathbf{y}) \rho_p(\mathbf{y})\, d\mathbf{y}.$$

Let us define the two following sets:

$$E = \{\mathbf{x} \in \overline{\Omega \cap \mathcal{O}},\ d(\mathbf{x}, \partial\Omega \cap \overline{\mathcal{O}}) \leq \delta_n/3\} \qquad \text{and} \qquad F = \{\mathbf{x} \in \overline{\Omega \cap \mathcal{O}},\ d(\mathbf{x}, \partial\mathcal{O} \cap \overline{\Omega}) \leq \nu/3\}.$$

On the one hand, choosing $p \geq \frac{3}{\delta_n}$, $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$ and $\forall \mathbf{x} \in E$, we have

$$d(\mathbf{x} - \mathbf{y}, \partial\Omega_n) \geq d(\partial\Omega \cap \overline{\mathcal{O}}, \partial\Omega_n) - d(\mathbf{x}, \partial\Omega \cap \overline{\mathcal{O}}) - d(\mathbf{x}, \mathbf{x} - \mathbf{y}) \geq \frac{\delta_n}{3} > 0,$$

which implies $\tilde{u}_n(\mathbf{x} - \mathbf{y}) = 0$. Thus, we obtain

$$\forall \mathbf{x} \in E,\ u_{n,p}(\mathbf{x}) = 0. \tag{3.41}$$

On the other hand, setting $\mathbf{z}_n = (\mathbf{0}, 1/n)$ and choosing n and p large enough such that $1/n + 1/p \leq \frac{\nu}{3}$, $\forall \mathbf{y} \in B(\mathbf{0}, 1/p)$ and $\forall \mathbf{x} \in E$, we have

$$d(\mathbf{x} - \mathbf{y} + \mathbf{z}_n, \operatorname{supp} u) \geq d(\partial\Omega \cap \overline{\Omega}, \operatorname{supp} u) - d(\mathbf{x}, \partial\Omega \cap \overline{\Omega}) - d(\mathbf{x}, \mathbf{x} - \mathbf{y} + \mathbf{z}_n) \geq \frac{\nu}{3} > 0,$$

which implies $\tilde{u}_n(\mathbf{x} - \mathbf{y}) = 0$ and, therefore,

$$\forall \mathbf{x} \in F,\ u_{n,p}(\mathbf{x}) = 0. \tag{3.42}$$

Thus, since $\partial(\Omega \cap \mathcal{O}) = (\partial\mathcal{O} \cap \overline{\Omega}) \cup (\partial\Omega \cap \overline{\mathcal{O}})$, owing to (3.41) and (3.42), for $n \geq \frac{6}{\nu}$ and $p \geq \max(\frac{6}{\nu}, \frac{3}{\delta_n})$, with δ_n defined in (3.40), $u_{n,p}$ belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$. Finally, in view of (3.38), the function $u_\varepsilon = u_{n_\varepsilon,p_\varepsilon}$, where

$$n_\varepsilon = \max([\frac{6}{\nu}] + 1, N'_\varepsilon) \qquad \text{and} \qquad p_\varepsilon = \max([\frac{6}{\nu}] + 1, [\frac{3}{\min(1/(2n_\varepsilon), 1/(2n_\varepsilon L))}] + 1, N'_\varepsilon),$$

belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$ and verifies

$$\|u - u_\varepsilon\|_{W^{1,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{1,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha},$$

hence, the lemma follows. ♢

We can now complete the proof of Theorem 3.1. Let $\varepsilon > 0$ be a given real number. Lemma 3.2 leads us to define a partition of unity $\mathcal{P}_\alpha$, with $\alpha \leq \alpha_\varepsilon$, where $\mathcal{P}_\alpha$ is defined by (2.10). Next, (3.17), Lemma 3.3 and Lemma 3.4 allow us to construct a function u_ε of $\mathcal{D}(\overline{\Omega})$ defined by :

$$u_\varepsilon = u_{\varepsilon,0} + \sum_{q+1 \leq k \leq r_\alpha} u_{\varepsilon,k}. \tag{3.43}$$

Then, we have

$$\|u - u_\varepsilon\|_{W^{1,r}(\Omega)} \leq \|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{1,r}(\Omega)} + \sum_{q+1 \leq k \leq r_\alpha} \|\varphi_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{1,r}(\Omega)} + \sum_{k=1}^{q} \|\boldsymbol{\varphi}_{\alpha,k}\, u\|_{W^{1,r}(\Omega)},$$

which implies, in view of (3.14), (3.17), (3.18) and (3.33)

$$\|u - u_\varepsilon\|_{W^{1,r}(\Omega)} \leq \varepsilon. \tag{3.44}$$

Moreover, owing to Lemma 3.3, we obtain that, for every $k = q+1, \ldots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_1$ (note that by construction $\mathcal{O}_{k,\alpha} \cap \overline{\gamma_0} = C_{\mathbf{m}_{k,\alpha}} \cap \overline{\gamma_0} = \emptyset$), $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$ and, consequently, u_ε belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$, where $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is defined by (1.5). Thus, Theorem 3.1 is proven. ♢

4 Density result in $W^{m,r}(\Omega, \gamma_0)$

Let $k \geq 1$ be an integer and let us suppose that the boundary $\partial\Omega$ is of class $C^{k,1}$, which means that, for every $\mathbf{x} \in \partial\Omega$, the functions $\Phi^{\mathbf{x}}$, defined by (2.1), are of class $C^{k,1}$. The following theorem generalizes Theorem 3.1.

Theorem 4.1 *Let $r > 1$ be a real and $m \geq 1$ be an integer. Let Ω a bounded domain in $I\!R^d$ whose boundary is of class $C^{k,1}$, where k is an integer such that $k + 1 \geq m$, and let γ_0 be an open part of $\partial\Omega$ verifying (1.1). Let the spaces $W^{m,r}(\Omega, \gamma_0)$ and $\mathcal{D}(\overline{\Omega}, \gamma_0)$ be defined respectively by (1.4) and (1.5). Then the space $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is dense in $W^{m,r}(\Omega, \gamma_0)$.*

Proof. Let us prove the result for $m = 2$, the extension to the general case is straightforward. We suppose that u belongs to $W^{2,r}(\Omega, \gamma_0)$. The proof of this theorem is analogous to that of Theorem 3.1. Indeed, we use the same covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, defined by (2.10), and an associated partition of unity $\widetilde{\mathcal{P}}_\alpha$, analogous to $\mathcal{P}_\alpha$, defined as follows :

First, we define the functions $\tilde{\theta}_{\alpha,k}$, for $k = 1, \dots, q$, by

$$\forall k,\ 1 \le k \le q,\ \tilde{\theta}_{\alpha,k} = \tilde{\varphi}(\frac{1}{\alpha} d(\,.\,, K_k)) * \rho_{p_\alpha}, \tag{4.1}$$

with $p_\alpha = [\frac{16}{\alpha}] + 1$ and ρ_p defined by (2.4), where the function $\tilde{\varphi}$ belongs to $C^2(\mathbb{R}^+)$ and verifies

$$\forall t \in [0, \frac{9}{16}],\ \tilde{\varphi}(t) = 1,\ \forall t \ge \frac{11}{16},\ \tilde{\varphi}(t) = 0 \quad \text{and} \quad \forall t \in \mathbb{R}_+,\ |\tilde{\varphi}'(t)| \le A,\ |\tilde{\varphi}''(t)| \le B.$$

For example, we can choose $\tilde{\varphi}$ defined on $[\frac{9}{16}, \frac{11}{16}]$ by

$$\tilde{\varphi}(t) = 15(16^4) \int_t^{\frac{11}{16}} (x - \frac{9}{16})^2 (x - \frac{11}{16})^2\, dx.$$

Since the boundary is at least of class $C^{1,1}$ the first and second order partial derivatives of the function $\mathbf{x} \mapsto d(\mathbf{x}, K_k)$ belongs to $L^\infty(\mathbb{R}^d)$ (see [6]). Setting $M = \|\partial^2 d(., K_k)\|_{L^\infty(\mathbb{R}^d)}$, we derive the following estimations for the functions $\tilde{\theta}_{\alpha,k} \in \mathcal{D}(G_{k,\alpha})$ and its derivatives, for $k = 1, \dots, q$, for $i, j = 1, \dots, d$,

$$\begin{aligned} &\forall \mathbf{x} \in G_{k,\frac{\alpha}{2}},\ \tilde{\theta}_{\alpha,k}(\mathbf{x}) = 1,\ \forall \mathbf{x} \notin G_{k,\frac{3\alpha}{4}},\ \tilde{\theta}_{\alpha,k}(\mathbf{x}) = 0 \\ \text{and } &\forall \mathbf{x} \in \mathbb{R}^d,\ |\partial_i \tilde{\theta}_{\alpha,k}(\mathbf{x})| \le \frac{A}{\alpha}, \qquad |\partial_i \partial_j \tilde{\theta}_{\alpha,k}(\mathbf{x})| \le \frac{C}{\alpha^2}, \end{aligned} \tag{4.2}$$

where $G_{k,\alpha}$ is defined by (1.2) and $C = B + AM$.

Second, we set $\widetilde{\mathcal{P}}_\alpha = \{\tilde{\varphi}_{\alpha,k}\}_{k=0}^{r_\alpha}$ with

$$\tilde{\varphi}_{\alpha,k} = (\prod_{j=1}^{q} (1 - \tilde{\theta}_{\alpha,j}))\beta_{\alpha,k},\ k = 0 \text{ or } q + 1 \le k \le r_\alpha \text{ and } \tilde{\varphi}_{\alpha,k} = \tilde{\theta}_{\alpha,k},\ 1 \le k \le q. \tag{4.3}$$

As previously, for every real ε, we must compute a parameter α'_ε, allowing us to construct an adequate partition of unity $\widetilde{\mathcal{P}}_\alpha$ with $\alpha \le \alpha'_\varepsilon$. Thus, we prove an analogous lemma to Lemma 3.2.

Lemma 4.2 *For every real number $\varepsilon > 0$, there exists a real number α'_ε verifying (1.3) such that, for every $0 < \alpha \le \alpha'_\varepsilon$,*

$$\forall k = 1, \dots, q, \qquad \|\tilde{\theta}_{\alpha,k} u\|_{W^{2,r}(G_{k,\alpha} \cap \Omega)} \le \frac{\varepsilon}{4q}. \tag{4.4}$$

Proof. By the same way as in the Lemma 3.2, using an extension $\tilde{u} \in W^{2,r}(\mathbb{R}^d)$ of $u \in W^{2,r}(\Omega, \gamma_0)$ we prove

$$\lim_{\alpha \to 0} \|\tilde{\theta}_{\alpha,k} u\|_{W^{1,r}(G_{k,\alpha} \cap \Omega)} = 0. \tag{4.5}$$

On the one hand, for $j = 1, \ldots, d$ and $i = 1, \ldots, p$, $\partial_j \tilde{u}$ vanishes on $B(\mathbf{x}_i, 2\alpha) \cap \gamma_0$, which has a strictly positive measure, and we can apply Poincaré inequality to deduce

$$\|\nabla \tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))} \leq C_1 \alpha^r \|\partial^2 \tilde{u}\|^r_{L^r(B(\mathbf{x}_i,2\alpha))},$$

where C_1 is the constant defined in (3.7). As in the proof of Lemma 3.2, setting the integrals $\|\partial^2 \tilde{u}\|_{L^r(B(\mathbf{x}_i,2\alpha))}$ in decreasing order, by analogous method we obtain

$$\|\nabla \tilde{u}\|^r_{L^r(G_{k,\alpha})} \leq C_1 \alpha^r M_d \|\partial^2 \tilde{u}\|^r_{L^r(G_{k,4\alpha})}, \tag{4.6}$$

where M_d is defined by (3.5). Moreover, owing to (3.13), we derive

$$\|u\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq 4^r\, C_1^2 \alpha^{2r} M_d^2 \|\partial^2 \tilde{u}\|^r_{L^r(G_{k,16\alpha})}. \tag{4.7}$$

On the other hand, we can write

$$\partial_i \partial_j (\tilde{\theta}_{\alpha,k}\, u) = \partial_i \partial_j (\tilde{\theta}_{\alpha,k})\, u + \partial_i (\tilde{\theta}_{\alpha,k}) \partial_j u + \partial_j (\tilde{\theta}_{\alpha,k})\, \partial_i u + (\tilde{\theta}_{\alpha,k}) \partial_i \partial_j u.$$

Then, in view of (4.2), (4.6) and (4.7), we obtain

$$\|\partial^2 (\tilde{\theta}_{\alpha,k}\, u)\|^r_{L^r(G_{k,\alpha}\cap\Omega)} \leq 4^{r-1}((4\,C)^r\,(d\,C_1\,M_d)^2 + 2d\,C_1\,M_d\,A^r + 1)\|\partial^2 \tilde{u}\|^r_{L^r(G_{k,16\alpha})}.$$

Hence, since

$$\lim_{\alpha\to 0} \|\partial^2 \tilde{u}\|_{L^r(G_{k,16\alpha})} = 0,$$

we derive

$$\lim_{\alpha\to 0} \|\partial^2 (\tilde{\theta}_{\alpha,k}\, u)\|_{L^r(G_{k,\alpha}\cap\Omega)} = 0,$$

which implies, owing to (4.5),

$$\lim_{\alpha\to 0} \|\tilde{\theta}_{\alpha,k} u\|_{W^{2,r}(G_{k,\alpha}\cap\Omega)} = 0$$

and the result of the lemma follows. $\diamondsuit$

We consider a partition of unity $\widetilde{\mathcal{P}}_\alpha$ defined by (4.1), with $0 < \alpha \leq \alpha'_\varepsilon$, subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, defined by (2.10), where α'_ε is defined in the Lemma 4.2. Since $\tilde{\theta}_{k,\alpha}$ belongs to $\mathcal{D}(G_{k,\alpha})$, (4.4) can be written, with the notation of the partition $\widetilde{\mathcal{P}}_\alpha$,

$$\forall k = 1, \ldots, q,\ \|\tilde{\varphi}_{\alpha,k} u\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4q}, \tag{4.8}$$

so that, for every $k = 1, \ldots, q$, we can approximate $\tilde{\varphi}_{\alpha,k} u$ by 0 in $\mathcal{O}_{k,\alpha} = G_{k,\alpha}$.

We now deal with the case $k = 0$, that is to say, we want approximate $\tilde{\varphi}_{\alpha,0} u$ in $\mathcal{O}_{0,\alpha}$. In the same way as in the proof of Theorem 3.1, we set $u_p = (\widetilde{\boldsymbol{\varphi}_{\alpha,0} u}) * \rho_p$, where the wide latter denotes the extension by zero. In a standard way, considering that $\widetilde{\tilde{\varphi}_{\alpha,0} u} \in W^{2,r}(\mathbb{R}^d)$, we obtain that

$$\lim_{p\to+\infty} u_p = \widetilde{\tilde{\varphi}_{\alpha,0} u} \quad \text{in} \quad W^{2,r}(\mathbb{R}^d),$$

which implies that there exists $P'_\varepsilon \in N^*$ such that, $\forall p \geq P'_\varepsilon$,

$$\|\tilde{\varphi}_{\alpha,0} u - u_p\|_{W^{2,r}(\mathcal{O}_{0,\alpha})} \leq \frac{\varepsilon}{4}. \tag{4.9}$$

Then considering $\mu'_0 = d(\text{supp}(\tilde{\varphi}_{\alpha,0}\, u), \partial\mathcal{O}_{0,\alpha}) > 0$ and setting $u_{\varepsilon,0} = u_{m'_\varepsilon}$, where m'_ε is defined by $m'_\varepsilon = \max([\frac{3}{\mu'_0}], P'_\varepsilon)$ ($[r]$ is the integral part of r), yield

$$\|\tilde{\varphi}_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{2,r}(\mathcal{O}_{0,\alpha}\cap\Omega)} = \|\tilde{\varphi}_{\alpha,0} u - u_{\varepsilon,0}\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4} \qquad \text{with} \qquad u_{\varepsilon,0} \in \mathcal{D}(\mathcal{O}_{0,\alpha}), \tag{4.10}$$

where $\overline{\mathcal{O}_{0,\alpha}} \subset \Omega$.

Next, we take care of an approximation of $\tilde{\varphi}_{\alpha,k}\, u$ in $\mathcal{O}_{k,\alpha}$, for $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_1$, that is to say an approximation of u localized around γ_1. As in Lemma 3.3, to simplify the notations, we drop the indexes, replacing $\tilde{\varphi}_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and we set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \mu' > 0. \tag{4.11}$$

We define u_n, Ω_n by (3.21) and we denote by $\tilde{u}_n$ the extension of u_n by zero. We can verify, by deriving in the sense of distribution, that

$$\partial_i(\tilde{u}_{n|\mathcal{O}\cap\Omega}) = (\widetilde{\partial_i u})_{n|\mathcal{O}\cap\Omega}, \qquad \partial_j\partial_i(\tilde{u}_{n|\mathcal{O}\cap\Omega}) = (\widetilde{\partial_j\partial_i u})_{n|\mathcal{O}\cap\Omega},$$

where the wide latter denotes the extension by zero in $\mathcal{O} \cap \Omega \setminus \Omega_n$, which implies that the restriction of $\tilde{u}_n$ to $\mathcal{O} \cap \Omega$ belongs to $W^{2,r}(\mathcal{O} \cap \Omega)$ and the following convergence

$$\lim_{n\to+\infty} \tilde{u}_{n|\mathcal{O}\cap\Omega} = u \qquad \text{in} \qquad W^{2,r}(\mathcal{O}\cap\Omega). \tag{4.12}$$

Next, we define $u_{n,p}$ by (3.23) and in the same way as in Lemma 3.3, we prove that $\tilde{u}_{n|\mathcal{O}_p}$ belongs to $W^{2,r}(\mathcal{O}_p)$, where $\mathcal{O}_p$ is defined by (3.25), and $\partial_j\partial_i\tilde{u}_{n|\mathcal{O}_p} = \widetilde{\partial_j\partial_i u_n}$, where the wide latter is the extension by zero of $\partial_j\partial_i u_n \in L^r(\Omega_n \cap \mathcal{O}_p)$ in $\mathcal{O}_p$. Moreover, as in the proof of (3.31), we can show that, for p verifying (3.29),

$$\partial_j\partial_i u_{n,p} = \widetilde{\partial_j\partial_i u_n} * \rho_p \qquad \text{almost everywhere in} \qquad \mathcal{O}\cap\Omega$$

and we obtain

$$\lim_{p\to+\infty} (u_{n,p})_{|\mathcal{O}\cap\Omega} = \tilde{u}_n \qquad \text{in} \qquad W^{2,r}(\mathcal{O}\cap\Omega).$$

Hence, with (4.12), we derive that there exists $N'_\varepsilon \in \mathbb{N}^*$ such that, for $\min(n,p) \geq N'_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{4.13}$$

Thus, the function $u_\varepsilon = u_{m'_\varepsilon, m'_\varepsilon}$, where m'_ε is defined by

$$m'_\varepsilon = \max([\frac{6}{\mu'}] + 1, N'_\varepsilon),$$

with μ' defined by (4.11), belongs to $\mathcal{D}(\overline{\Omega})$ with a compact support in $\mathcal{O} \cap \overline{\Omega}$ and verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}.$$

Then, with the initial notation, we obtain, for every $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_1$,

$$\|\tilde{\varphi}_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}, \tag{4.14}$$

where the function $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\overline{\Omega})$ with compact support in $\mathcal{O}_{k,\alpha} \cap \overline{\Omega}$, which ends the problem of the approximation of u localized around γ_1.

Finally, we still have an approximation of $\tilde{\varphi}_{\alpha,k}\, u$ in $\mathcal{O}_{k,\alpha}$ to do, for $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_0$, that is to say an approximation of u localized around γ_0, that is the part of the boundary where u vanishes.

As previously, to simplify the notations, we replace, for $k = q+1, \ldots, r_\alpha$, $\varphi_{\alpha,k}\, u$ by u and $\mathcal{O}_{k,\alpha}$ by $\mathcal{O}$, so that we may assume that u has compact support in $\mathcal{O} \cap \overline{\Omega}$ and we set

$$d(\partial\mathcal{O} \cap \overline{\Omega}, \text{supp } u) = \nu' > 0. \tag{4.15}$$

We again define u_n by (3.36) and $\tilde{u_n}$ again denotes the extension of u_n by zero. In the same way as in the proof of Lemma 3.4, we have

$$\lim_{n\to+\infty} \tilde{u_n}_{|\mathcal{O}\cap\Omega} = u \qquad \text{in} \qquad W^{2,r}(\mathcal{O}\cap\Omega) \tag{4.16}$$

and, for $\dfrac{1}{n} \leq \nu'$, u_n has a compact support in $\mathcal{O} \cap \Omega$. Moreover, setting again $u_{n,p} = \tilde{u_n} * \rho_p$ yields that there exists $N''_\varepsilon \in \mathbb{N}^*$ such that, for $\min(n,p) \geq N''_\varepsilon$,

$$\|u - u_{n,p}\|_{W^{2,r}(\mathcal{O}\cap\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}. \tag{4.17}$$

Then, the function $u_\varepsilon = u_{n'_\varepsilon, p'_\varepsilon}$, where

$$n'_\varepsilon = \max([\frac{6}{\nu'}] + 1, N''_\varepsilon) \qquad \text{and} \qquad p'_\varepsilon = \max([\frac{6}{\nu'}] + 1, [\frac{3}{\min(1/(2n'_\varepsilon), 1/(2n'_\varepsilon L))}] + 1, N''_\varepsilon),$$

belongs to $\mathcal{D}(\Omega \cap \mathcal{O})$ and verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\mathcal{O}\cap\Omega)} = \|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}.$$

With the initial notation, we obtain, for every $k = q+1, \ldots, r_\alpha$ such that $\mathbf{m}_{k,\alpha} \in \gamma_0$,

$$\|\tilde{\varphi}_{\alpha,k}\, u - u_{\varepsilon,k}\|_{W^{2,r}(\Omega)} \leq \frac{\varepsilon}{4\, r_\alpha}, \tag{4.18}$$

where the function $u_{\varepsilon,k}$ belongs to $\mathcal{D}(\Omega \cap \mathcal{O}_{k,\alpha})$, which ends the problem of the approximation of u localized around γ_0.

We can complete the proof of Theorem 4.1. Let $\varepsilon > 0$ be a given real number. Lemma 4.2 leads us to define an adequate partition of unity $\widetilde{\mathcal{P}}_\alpha$, with $0 < \alpha \leq \alpha_\varepsilon$. Next (4.9), (4.14) and (4.18) allows us to construct a function u_ε of $\mathcal{D}(\overline{\Omega})$ defined by :

$$u_\varepsilon = u_{\varepsilon,0} + \sum_{q+1 \leq k \leq r_\alpha} u_{\varepsilon,k},$$

that verifies

$$\|u - u_\varepsilon\|_{W^{2,r}(\Omega)} \leq \varepsilon.$$

With the same argument as in the end of the proof of Theorem 3.1, we prove that u_ε belongs to $\mathcal{D}(\overline{\Omega}, \gamma_0)$, where $\mathcal{D}(\overline{\Omega}, \gamma_0)$ is defined by (1.5). Thus, Theorem 4.1 is proven. $\diamondsuit$

5 Another application of the partition of unity $\mathcal{P}_\alpha$

Let Ω a bounded domain in $\mathbb{R}^d$, $d \geq 2$, whose boundary $\partial\Omega$ and the parts of boundary γ_0 and γ_1 verify (1.1). Let us suppose, in addition, that, if $d \geq 3$, for $k = 1, \ldots, q$ and for every $\mathbf{x} \in \overline{\gamma_0} \cap \overline{\gamma_1}$, there exist an open hypercube $C_\mathbf{x}$, neighborhood of $\mathbf{x}$ in $\mathbb{R}^d$, and new orthogonal coordinates $\mathbf{y} = (\mathbf{y}'', y_{d-1}, y_d)$, where $\mathbf{y}'' = (y_1, \ldots, y_{d-2})$, such that:

i) $C_\mathbf{x} = \prod_{j=1}^{d}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$.

ii) There exist Lipschitz-continuous functions $\Phi_1^\mathbf{x}$ and $\Phi_2^\mathbf{x}$ defined in $\prod_{j=1}^{d-2}]-a_{\mathbf{x},j}, a_{\mathbf{x},j}[$ of constants, respectively, $L_{1,\mathbf{x}}$ and $L_{2,\mathbf{x}}$ such that

$$K_k \cap C_\mathbf{x} = \{\mathbf{y} \in C_\mathbf{x},\ y_{d-1} = \Phi_1^\mathbf{x}(\mathbf{y}''),\ y_d = \Phi_2^\mathbf{x}(\mathbf{y}'')\}. \tag{5.1}$$

Let us consider the partition of unity $\mathcal{P}_\alpha = \{\varphi_{\alpha,k}\}_{k=0}^{r_\alpha}$ on $\overline{\Omega}$, subordinated to the covering $\{\mathcal{O}_{k,\alpha}\}_{k=0}^{r_\alpha}$, defined by (2.10). The approximation constructed in the proof of the density Theorem 3.1 allows us to prove the following theorem.

Theorem 5.1 *Let Ω a bounded domain in $\mathbb{R}^d$ whose boundary is Lipschitz-continuous and let γ_0 and γ_1 be open parts of $\partial\Omega$ verifying 1.1 and such that, in addition, the connected components of $\overline{\gamma_0} \cap \overline{\gamma_1}$ verify (5.1). Let $\mathbf{v}$ belong to $H(\mathrm{div}, \Omega)$ such that $\mathbf{v} \,.\, \mathbf{n}_{|\gamma_0} = 0$ and $\mathbf{v} \,.\, \mathbf{n}_{|\gamma_1} = 0$. Then, we have $\mathbf{v} \,.\, \mathbf{n}_{|\partial\Omega} = 0$.*

Proof. Note that this theorem is not trivial, because, in the general case, these normal components are not defined as functions defined almost everywhere. Let us recall that, if γ_0 and γ_1 verify (1.1), $H^{-1/2}(\gamma_0) = (H_{00}^{1/2}(\gamma_0))'$, where

$$H_{00}^{1/2}(\gamma_0) = \{u_{|\gamma_0},\ u \in H^1(\Omega),\ u_{|\gamma_1} = 0\},$$

and we have the following Green's formula : $\forall u \in H^1(\Omega),$ with $u_{|\gamma_1} = 0,\ \forall \mathbf{v} \in H(\mathrm{div}, \Omega),$

$$\int_\Omega (\mathrm{div}\, \mathbf{v})\, u \, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u \, d\mathbf{x} = < \mathbf{v} \,.\, \mathbf{n}, u >_{H^{-1/2}(\gamma_0)} . \tag{5.2}$$

Let $\mathbf{v} \in H(\mathrm{div}, \Omega)$ verifying the assumptions of Theorem 5.1 and let $\mu \in H^{1/2}(\partial\Omega)$. There exists $u \in H^1(\Omega)$ such that

$$< \mathbf{v} \,.\, \mathbf{n}, \mu >_{H^{-1/2}(\partial\Omega)} = < \mathbf{v} \,.\, \mathbf{n}, u >_{H^{-1/2}(\partial\Omega)} = \int_\Omega (\mathrm{div}\, \mathbf{v})\, u \, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u \, d\mathbf{x}. \qquad (5.3)$$

Since $\mathcal{D}(\overline{\Omega})$ is dense in $H^1(\Omega)$, we can assume that u belongs to $\mathcal{D}(\overline{\Omega})$. We begin to prove a lemma that deals with the sets $G_{k,\alpha}$, $k = 1, \ldots, q$, defined by (1.2).

Lemma 5.2 *Let the sets $G_{k,\alpha}$, $k = 1, \ldots, q$, be defined by (1.2). There exist real numbers $\alpha_0 > 0$ and $M > 0$ such that, for every $0 < \alpha \le \alpha_0$ verifying (1.3) and $k = 1, \ldots, q$,*

$$|G_{k,\alpha}| \le M \, \alpha^2. \qquad (5.4)$$

Proof. When $d = 2$, for $k = 1, \ldots, q$, the set $G_{k,\alpha}$ is the open ball with center $\mathbf{x}_k$ and radius α and the estimate (5.4) is trivial with $M = \pi$. Let us assume that $d \ge 3$. Since, for $k = 1, \ldots, q$, the set K_k is compact, we derive

$$K_k \subset \bigcup_{j=1}^{s_k} C_{\mathbf{m}_j},$$

where $\mathbf{m}_j \in K_k$, for $j = 1, \ldots, s_k$, and where $C_{\mathbf{x}}$ is defined at the beginning of the section. There exists a real α'_0 such that, for $\alpha \le \alpha'_0$, we have

$$G_{k,\alpha} \subset \bigcup_{j=1}^{s_k} C_{\mathbf{m}_j}. \qquad (5.5)$$

On the one hand, let M_k be a real number such that, in local coordinates, for every $j = 1, \ldots, s_k$, $\forall \mathbf{y} \in G_{k,\alpha} \bigcap C_{\mathbf{m}_j}$,

$$\|\mathbf{y}''\| \le M_k, \qquad (5.6)$$

where $\mathbf{y} = (\mathbf{y}'', y_{d-1}, y_d)$ and $\| \,.\, \| = d(\,.\,, \mathbf{0})$ is the euclidian norm in $\mathbb{R}^d$. Moreover, we define the following notations: for every subset E of $C_{\mathbf{m}_j}$, in local coordinates :

$$(E)_{\mathbf{z}''} = \{\mathbf{y} \in E, \ \mathbf{y}'' = \mathbf{z}''\}$$

and for every $j = 1, \ldots, q$,

$$C'_j = C_{\mathbf{m}_j} \setminus (\bigcup_{1 \le l \le s_k, \ l \ne j} C_{\mathbf{m}_l}).$$

On the other hand, we can assume that, for $k = 1, \ldots, q$, for $j, l = 1, \ldots s_k$, with $l \ne j$,

$$\partial C_{\mathbf{m}_j} \cap \partial C_{\mathbf{m}_l} \cap K_k = \emptyset. \qquad (5.7)$$

Indeed, if not, we can cover K_k by a finite number of hypercubes $C'_{\mathbf{x}}$ smaller than the hypercubes $C_{\mathbf{x}}$ and next, if necessary, increase a little the edges of the hypercubes $C'_{\mathbf{x}}$ such that the previous result is true. From (5.7), we derive, for $k = 1, \ldots, q$, for $j, l = 1, \ldots s_k$, with $l \ne j$,

$$d(\overline{C_{\mathbf{m}_j}} \setminus C_{\mathbf{m}_l}, K_k \cap (\overline{C_{\mathbf{m}_l}} \setminus C_{\mathbf{m}_j})) > 0.$$

Then, we set, for $k = 1, \ldots, q$,

$$\alpha'_k = \min_{\substack{j,l=1,\ldots,s_k \\ j \neq l}} d(\overline{C_{\mathbf{m}_j}} \setminus C_{\mathbf{m}_l}), K_k \cap (\overline{C_{\mathbf{m}_l}} \setminus C_{\mathbf{m}_j}))$$

and we chooze

$$0 < \alpha \leq \alpha_0 = \min_{k=0,\ldots,q} (\alpha'_k), \tag{5.8}$$

where α'_0 is defined by (5.5). Then, we obtain, for $k = 1, \ldots, q$, owing to (5.6),

$$|G_{k,\alpha}| = \sum_{j=1}^{s_k} \int_{G_{k,\alpha} \cap C'_j} d\,\mathbf{y} \leq \sum_{j=1}^{s_k} \int_{\|\mathbf{y}''\| \leq M_k} d\mathbf{y}''\Big(\int_{(G_{k,\alpha} \cap C'_j)_{\mathbf{y}''}} dy_{d-1} dy_d\Big). \tag{5.9}$$

Let $\mathbf{y}$ belong to $(G_{k,\alpha} \cap C'_j)_{\mathbf{y}''}$. In view of (5.8), for $k = 1, \ldots, q$, we have

$$d(\mathbf{y}, K_k) = d(\mathbf{y}, \tilde{\mathbf{y}}),$$

with $\tilde{\mathbf{y}} \in K_k \cap C_{\mathbf{m}_j}$. But

$$(d(\mathbf{y}, K_k))^2 = \|\mathbf{y}'' - \tilde{\mathbf{y}''}\|^2 + (y_{d-1} - \Phi_1^{\mathbf{m}_j}(\tilde{\mathbf{y}''}))^2 + (y_d - \Phi_2^{\mathbf{m}_j}(\tilde{\mathbf{y}''}))^2.$$

Since $d(\mathbf{y}, K_k) < \alpha$, we derive

$$\|\mathbf{y}'' - \tilde{\mathbf{y}''}\| \leq \alpha, \ |y_{d-1} - \Phi_1^{\mathbf{m}_j}(\tilde{\mathbf{y}''})| \leq \alpha \quad \text{and} \quad |y_d - \Phi_2^{\mathbf{m}_j}(\tilde{\mathbf{y}''})| \leq \alpha.$$

Then (5.1) yields

$$|y_{d-1} - \Phi_1^{\mathbf{m}_j}(\mathbf{y}'')| \leq (1 + L_{1,\mathbf{m}_j})\alpha, \qquad |y_d - \Phi_2^{\mathbf{m}_j}(\mathbf{y}'')| \leq (1 + L_{2,\mathbf{m}_j})\alpha.$$

Hence, considering (5.9), we obtain

$$|G_{k,\alpha}| \leq 4|B_{d-2}(\mathbf{0}, M_k)|\Big(\sum_{j=1}^{s_k}((1 + L_{1,\mathbf{m}_j})(1 + L_{2,\mathbf{m}_j}))\Big)\,\alpha^2,$$

where $B_{d-2}(\mathbf{0}, M_k)$ is the open ball with center $\mathbf{0}$ and radius M_k in $\mathbb{R}^{d-2}$ and setting

$$M = \max_{k=1,\ldots,q} \Big(4|B_{d-2}(\mathbf{0}, M_k)|\Big(\sum_{j=1}^{s_k}((1 + L_{1,\mathbf{m}_j})(1 + L_{2,\mathbf{m}_j}))\Big)\Big),$$

the result of the lemma follows. $\diamondsuit$

First, for $k = 1, \ldots, q$, we have

$$\|\theta_{\alpha,k}\, u\|^2_{L^2(G_{k,\alpha} \cap \Omega)} = \int_\Omega \theta^2_{\alpha,k}(\mathbf{x}) u^2(\mathbf{x})\, d\mathbf{x}.$$

Let $\mathbf{x} \in \Omega$ and let $\alpha \leq \frac{4}{3}\, d(x, \partial\Omega)$. Since $\mathbf{x} \notin G_{k, \frac{3\alpha}{4}}$, owing to (2.7), we derive $\theta_{\alpha,k}(\mathbf{x}) = 0$, which implies

$$\lim_{\alpha \to 0} \theta^2_{\alpha,k}(\mathbf{x})\, u^2(\mathbf{x}) = 0.$$

But, we have

$$\theta_{\alpha,k}^2\, u^2 \leq u^2 \qquad \text{in} \qquad \Omega,$$

and $u^2 \in L^1(\Omega)$. Therefore, we obtain

$$\lim_{\alpha\to 0} \|\theta_{\alpha,k}\, u\|_{L^2(G_{k,\alpha}\cap\Omega)} = 0$$

and there exists a real $\tilde{\alpha}_\varepsilon$ such that, for every $0 < \alpha \leq \tilde{\alpha}_\varepsilon$ verifying (1.3),

$$\forall k = 1, \ldots, q, \qquad \|\theta_{\alpha,k}\, u\|_{L^2(G_{k,\alpha}\cap\Omega)} \leq \frac{\varepsilon}{4q}. \tag{5.10}$$

Next, let us construct an approximation $u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^1$ of $u \in \mathcal{D}(\overline{\Omega})$ in $H^1(\Omega)$, such that $(u_\varepsilon^0)_{|\gamma_1} = 0$ and $(u_\varepsilon^1)_{|\gamma_0} = 0$. Applying Lemma 3.3 to the function u, with $r = 2$, but with γ_0 instead of γ_1, yields functions $u_{\varepsilon,k}$, $k = q+1, \ldots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_0$, veryfying (3.18). Indeed, we can apply Lemma 3.3 with every function $u \in \mathcal{D}(\overline{\Omega})$ in place of functions of $H^1(\Omega,\ \gamma_0)$. Then, with, in addition, $u_{\varepsilon,0}$ defined by (3.17), we set

$$u_\varepsilon^0 = u_{\varepsilon,0} + \sum_{\substack{q+1\leq k\leq r_\alpha \\ \mathbf{m}_{k,\alpha}\in\gamma_0}} u_{\varepsilon,k}. \tag{5.11}$$

Note that the functions $u_{\varepsilon,k}$ have compact support in $\mathcal{O}_{k,\alpha} \cap \overline{\Omega}$ with $\mathbf{m}_{k,\alpha} \in \gamma_0$ and $u_{\varepsilon,0} \in \mathcal{D}(\Omega)$. Thus, owing to (2.1) and (2.10), these functions $u_{\varepsilon,k}$ vanish on γ_1 and, therefore, $(u_\varepsilon^0)_{|\gamma_1} = 0$.

We apply again Lemma 3.3 to the function u, which gives functions $u_{\varepsilon,k}$, for $k = q+1, \ldots, r_\alpha$ with $\mathbf{m}_{k,\alpha} \in \gamma_1$, verifying (3.18). Then, we set

$$u_\varepsilon^1 = \sum_{\substack{q+1\leq k\leq r_\alpha \\ \mathbf{m}_{k,\alpha}\in\gamma_1}} u_{\varepsilon,k}. \tag{5.12}$$

In the same way, considering the support of the functions $u_{\varepsilon,k}$ where $\mathbf{m}_{k,\alpha} \in \gamma_1$, we obtain $(u_\varepsilon^1)_{|\gamma_0} = 0$. Next, we have, considering $u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^1$,

$$\|u - u_\varepsilon\|_{L^2(\Omega)} \leq \|\varphi_{\alpha,0}\, u - u_{\varepsilon,0}\|_{L^2(\Omega)} + \sum_{\substack{q+1\leq k\leq r_\alpha \\ \mathbf{m}_{k,\alpha}\in\gamma_0}} \|\varphi_{\alpha,k}\, u - u_{\varepsilon,k}\|_{L^2(\Omega)}$$

$$+ \sum_{\substack{q+1\leq k\leq r_\alpha \\ \mathbf{m}_{k,\alpha}\in\gamma_1}} \|\varphi_{\alpha,k}\, u - u_{\varepsilon,k}\|_{L^2(\Omega)} + \sum_{k=1}^{q} \|\theta_{\alpha,k}\, u\|_{L^2(\Omega)}.$$

Then, (5.10), (3.17 and (3.18) imply, for $0 < \alpha \leq \tilde{\alpha}_\varepsilon$ verifying (1.3),

$$\|u - u_\varepsilon\|_{L^2(\Omega)} \leq \varepsilon.$$

Thus, we have

$$\lim_{\varepsilon\to 0} u_\varepsilon = u \qquad \text{in} \qquad L^2(\Omega), \tag{5.13}$$

where $u_\varepsilon = u_\varepsilon^0 + u_\varepsilon^1$ with $(u_\varepsilon^0)_{|\gamma_1} = 0$ and $(u_\varepsilon^1)_{|\gamma_0} = 0$.

Let us show the weak convergence of u_ε towards u in $H^1(\Omega)$. We choose $\varepsilon \leq 1$. On the one hand, in view of (3.17) and (3.18) applied with $r = 2$ and with u localized on γ_0 instead of γ_1 and, next, with u localized on γ_1, as we made previously for the convergence in L^2, we have

$$\|u - u_\varepsilon\|_{H^1(\Omega)} \leq \|\varphi_{\alpha,0} u - u_{\varepsilon,0}\|_{H^1(\Omega)} + \sum_{q+1 \leq k \leq r_\alpha} \|\varphi_{\alpha,k}\, u - u_{\varepsilon,k}\|_{H^1(\Omega)} + \sum_{k=1}^{q} \|\theta_{\alpha,k}\, u\|_{H^1(\Omega)},$$

which implies

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} + \frac{3}{4} + \sum_{k=1}^{q} \|\theta_{\alpha,k}\, u\|_{H^1(\Omega)}.$$

On the other hand, considering (2.7) yields

$$\|\theta_{\alpha,k}\, u\|^2_{H^1(\Omega)} \leq \|\theta_{\alpha,k}\, u\|^2_{L^2(\Omega)} + 2\|\theta_{\alpha,k}\, \nabla u\|^2_{L^2(\Omega)} + 2 \sum_{j=1}^{d} \int_{\Omega \cap G_{k,\alpha}} (\partial_j\, \theta_{\alpha,k}(\mathbf{x}))^2\, u(\mathbf{x})^2\, d\mathbf{x}$$
$$\leq \|u\|^2_{L^2(\Omega)} + 2\|\nabla u\|^2_{L^2(\Omega)} + 2d \frac{A^2}{\alpha^2} \|u\|^2_{L^\infty(\Omega)} |G_{k,\alpha}|.$$

Hence, for $\alpha \leq \alpha_0$, owing to (5.4), we derive

$$\|\theta_{\alpha,k}\, u\|_{H^1(\Omega)} \leq \sqrt{2}\|u\|_{H^1(\Omega)} + \sqrt{2d\, M}\, A \|u\|_{L^\infty(\Omega)},$$

which implies

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq (1 + q\sqrt{2})\|u\|_{H^1(\Omega)} + \frac{3}{4} + q\sqrt{2d\, M}\, A \|u\|_{L^\infty(\Omega)}.$$

Finally, in view of (5.13), we have

$$\lim_{\varepsilon \to 0} u_\varepsilon = u \qquad \text{weakly} \qquad \text{in} \qquad H^1(\Omega). \tag{5.14}$$

We can now complete the proof of the Theorem 5.1. First, we can write

$$\int_\Omega (\operatorname{div} \mathbf{v})\, u_\varepsilon\, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u_\varepsilon\, d\mathbf{x} = \int_\Omega (\operatorname{div} \mathbf{v})\, u^0_\varepsilon\, d\mathbf{x}$$
$$+ \int_\Omega \mathbf{v} \,.\, \nabla u^0_\varepsilon\, d\mathbf{x} + \int_\Omega (\operatorname{div} \mathbf{v})\, u^1_\varepsilon\, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u^1_\varepsilon\, d\mathbf{x}.$$

Applying twice the Green's formula (5.2) and owing to the assumptions on $\mathbf{v}$ yields

$$\int_\Omega (\operatorname{div} \mathbf{v})\, u_\varepsilon\, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u_\varepsilon\, d\mathbf{x} = < \mathbf{v} \,.\, \mathbf{n}, u^0_\varepsilon >_{H^{-1/2}(\gamma_0)} + < \mathbf{v} \,.\, \mathbf{n}, u^1_\varepsilon >_{H^{-1/2}(\gamma_1)} = 0.$$

Then, (5.3) and (5.14) imply

$$< \mathbf{v} \,.\, \mathbf{n}, \mu >_{H^{-1/2}(\partial\Omega)} = \int_\Omega (\operatorname{div} \mathbf{v})\, u\, d\mathbf{x} + \int_\Omega \mathbf{v} \,.\, \nabla u\, d\mathbf{x} = 0,$$

which ends the proof of the theorem. ♢

References

[1] C. Bernardi and Y. Maday, Spectral element discretizations of the Laplace equation with mixed boundary conditions, preprint Laboratoire d'Analyse Numérique R01015 (2001).

[2] A. Blouza and H. Le Dret, An up-to-the boundary vertion of Friedrichs's lemma and applications to the linear Koiter shell model, SIAM J. Math. ANAl. 33 (2001), pp. 877-895.

[3] H. Brezis, *Analyse fonctionnelle, théorie et applcations*, Masson, Paris, 1992.

[4] V. Girault and P.A. Raviart, *Finite Element Approximation for Navier-Stokes Equations. Theory and Algorithms,* SMC 5, Springer-Verlag, Berlin, 1986.

[5] V. Girault and L.R. Scott, Analysis of two-dimensional grade-two fluid model with a tangential boundary condition, J. Math. Pures Appl., 78 (1999), pp. 981-1011.

[6] V. Girault and L.R. Scott, Analysis of two-dimensional grade-two fluid model with a tangential boundary condition, Research Report UH/MD - 246, Department of Mathema tics University of Houston (1998).

[7] J.P. Puel and M.C. Roptin, Lemme de Friedrichs. Theéorème de densité résultant du lemme de Friedrichs, Rapport de stage dirigé par C. Goulaouic, Diplôme d'Etudes Approfondies, Université de Rennes, 1967.