

Convolution Semigroup for Distortion Functions and Spectral Risk Measures, with Numerical Applications

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Abstract

We focus our work here on some very recent results obtained by Cherny and Madan (see [2], [3], [4], [5], [6], [7]). They developed a rigorous mathematical framework for the study of coherent risk measures. The first sections mainly review the existing literature. We present it here for sake of completeness as well as to point out possible extensions. Our main contribution is to provide some numerical and empirical facts concerning Spectral Risk Measures, and in particular study Coherent Acceptability Indices proposed in [4]. One result, for instance, is that this Index is infinite for distributions that are symmetric around 0.

Keywords : risk measures, functional analysis, incomplete markets, VaR.

1 Notations and introduction to Spectral Risk Measures on L^∞

In the following, unless otherwise stated, we will consider a fixed probability space (Ω, \mathcal{F}, P) . For a random variable X with continuous distribution we denote by $q_\lambda(X) = \inf \{x : P(X \leq x) > \lambda\}$ for $\lambda \in [0, 1]$ the right λ -quantile of X .

1.1 Convex and Coherent Measure of Risk

We might sometimes speak of risk measures, sometimes of utility functions. If ρ is a risk measure, then the corresponding utility function u reads $u(X) = -\rho(X)$.

Definition 1 *Convex Risk Measure (Concave utility function)*

A mapping ρ from L^∞ to \mathbb{R} is called a convex risk measure if and only if the corresponding utility satisfies the following axioms :

- *Monotonicity:* If $X \leq Y$ then $u(X) \leq u(Y)$
- *Translation Invariance:* $\forall m \in \mathbb{R}, u(X + m) = u(X) + m$
- *Concavity:* $\forall \lambda \in [0, 1], u(\lambda X + (1 - \lambda)Y) \geq \lambda u(X) + (1 - \lambda)u(Y)$

Definition 2 *Coherent Risk Measure*

A coherent utility function on L^∞ is a concave utility function with the additional Positive Homogeneity Property :

$$\forall \lambda \geq 0, u(\lambda X) = \lambda u(X)$$

One can easily see that, for coherent risk measures, the convexity property reduces to a *Superadditivity Property*. Furthermore, any Coherent risk measure is normalized, i.e. $\rho(0) = 0$.

Dealing with utility functions is more convenient than with coherent risk measures as it allows us to get rid of many minus signs. In the following, we denote \mathcal{P} the set of probability measures on \mathcal{F} that are absolutely continuous with respect to the reference probability P . We have the following representation theorem (see [14], Proposition 4.14 p. 161):

Proposition 3 *A function u is a coherent utility function on L^∞ if and only if there exists a nonempty set $\mathcal{D} \subseteq \mathcal{P}$ such that*

$$u(X) = \inf_{Q \in \mathcal{D}} \mathbb{E}_Q(X)$$

We note that the set \mathcal{D} may obviously not be unique. We therefore define the determining set as the largest L^1 -closed convex subset of \mathcal{P} satisfying the above theorem (see [5]). We give some elements of proofs in the Section 2. Note that we work here on L^∞ . This representation theorem can be extended to $L^p, p \geq 1$ with some additional conditions on the risk measure. We now provide a few examples.

1.2 Tail VaR

Let $\lambda \in [0, 1]$. The Tail VaR ([14], p.180) of order λ is defined as

$$\rho_\lambda(X) = - \inf_{Q \in \mathcal{D}_\lambda} \mathbb{E}_Q(X)$$

where $\mathcal{D}_\lambda = \left\{ Q \in \mathcal{P} : \frac{dQ}{dP} \leq \frac{1}{\lambda} \right\}$. The corresponding utility function is obviously defined as $u_\lambda(X) = \inf_{Q \in \mathcal{D}_\lambda} \mathbb{E}_Q(X)$.

1.3 Weighted VaR

Let u_λ be a Tail VaR utility function for $\lambda \in [0, 1]$ and μ a probability measure on $[0, 1]$. We define the Weighted VaR utility function as

$$u_\mu(X) = \int_{[0,1]} u_\lambda(X) \mu(d\lambda)$$

and, from **Proposition 3**, we have the representation

$$u_\mu(X) = \inf_{Q \in \mathcal{D}_\mu} \mathbb{E}_Q(X)$$

where \mathcal{D}_μ is the determining set of u_μ . We refer to [5] for the representation of this set. The following representation will be of fundamental importance for us

Theorem 4 *Let u_μ be a Weighted VaR utility function, then there exists an increasing, concave function Ψ_μ on $[0, 1]$ with $\Psi(0) = 0$, $\Psi_\mu(0+) = \mu(\{0\})$ and $\Psi_\mu(1) = 1$ such that*

$$u_\mu(X) = \int_{\mathbb{R}} x d(\Psi_\mu(F_X(x)))$$

where F_X is the distribution function of X . The function Ψ is called a Distortion function.

To prove it, we need the following lemma :

Lemma 5 *Let μ be a probability measure on $[0, 1]$ and define a function Ψ as $\Psi(0) = \mu(\{0\})$ and*

$$\forall 0 < x \leq 1, \Psi(x) = \mu(\{0\}) + \int_0^x \int_{(t,1]} s^{-1} \mu(ds) dt$$

Then Ψ is increasing, concave and $\Psi(1) = 1$.

Proof The fact that Ψ is increasing is obvious. Now, let $\lambda \in [0, 1]$ and $(x, y) \in (0, 1]^2$. We have

$$\begin{aligned}\Psi(\lambda x + (1 - \lambda)y) &= \mu(\{0\}) + \int_0^{\lambda x + (1 - \lambda)y} \int_{(t, 1]} s^{-1} \mu(ds) dt \\ &= \mu(\{0\}) + \int_0^{\lambda x} \int_{(t, 1]} s^{-1} \mu(ds) dt + \int_{\lambda x}^{\lambda x + (1 - \lambda)y} \int_{(t, 1]} s^{-1} \mu(ds) dt\end{aligned}$$

For the first integral, we have

$$\int_0^{\lambda x} \int_{(t, 1]} s^{-1} \mu(ds) dt = \lambda \int_0^x \int_{(\lambda t, 1]} s^{-1} \mu(ds) dt$$

And hence

$$\lambda \int_0^x \int_{(\lambda t, 1]} s^{-1} \mu(ds) dt + \lambda \mu(\{0\}) \geq \lambda \Psi(x) \quad (1)$$

Now,

$$\int_{\lambda x}^{\lambda x + (1 - \lambda)y} \int_{(t, 1]} s^{-1} \mu(ds) dt = (1 - \lambda) \int_0^y \int_{(\lambda x + (1 - \lambda)t, 1]} s^{-1} \mu(ds) dt$$

And

$$(1 - \lambda) \int_0^y \int_{(\lambda x + (1 - \lambda)t, 1]} s^{-1} \mu(ds) dt + (1 - \lambda) \mu(\{0\}) \geq (1 - \lambda) \Psi(y) \quad (2)$$

Combining (1) and (2) proves the concavity of Ψ . Moreover

$$\begin{aligned}\Psi(1) &= \mu(\{0\}) + \int_0^1 \int_{(0, 1]} s^{-1} \mu(ds) dt \\ &\stackrel{Fubini}{=} \mu(\{0\}) + \int_{(0, 1]} s^{-1} \int_0^1 \mathbb{I}_{\{t < s \leq 1\}} dt \mu(ds) \\ &= \mu(\{0\}) + \mu((0, 1]) = 1\end{aligned}$$

■

Remark 6 We also note that $\Psi'(x) = \int_x^1 \frac{\mu(ds)}{s}$

We now prove the following theorem :

Theorem 7 For $X \in L^\infty$, we have

$$\begin{aligned}\rho_\mu(-X) &= \Psi(0+) \operatorname{ess\,sup}(X) + \int_{(0,1]} q_X(t) \Psi'(1-t) dt \\ &= \int_{-\infty}^0 [\Psi(P(X > x)) - 1] dx + \int_0^{+\infty} \Psi(P(X > x)) dx\end{aligned}$$

Where q_X is the quantile function of X .

Proof We recall the following facts, for $\lambda \in [0, 1]$:

$$V@R_\lambda(-X) = \inf\{\alpha : P(-X \leq -\alpha) \leq \lambda\} = \inf\{\alpha : P(X \leq \alpha) \geq 1 - \lambda\} = q_X(1 - \lambda)$$

and the Average $V@R$ is defined as

$$AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(X) d\gamma$$

Now,

$$\begin{aligned}\rho_\mu(-X) &= \int_{(0,1]} AV@R_\lambda(-X) \mu(d\lambda) \\ &= \int_{(0,1]} \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(-X) d\gamma \mu(d\lambda) \\ &= \int_{(0,1]} \frac{1}{\lambda} \int_0^\lambda q_X(1 - \gamma) d\gamma \mu(d\lambda) \\ &\stackrel{Fubini}{=} \int_0^1 \int_\gamma^1 \lambda^{-1} q_X(1 - \gamma) d\gamma \mu(d\lambda) \\ &= \int_{(0,1]} q_X(1 - \gamma) d\gamma \int_\gamma^1 \frac{\mu(d\lambda)}{\lambda} = \Psi(0+) \operatorname{ess\,sup}(X) + \int_{(0,1]} q_X(t) \Psi'(1-t) dt\end{aligned}$$

This proves the first equality of the theorem. Now, for the second one, first assume that X only takes positive values. Then

$$\begin{aligned}\int_0^1 q_X(t) \Psi'(1-t) dt &= \int_0^1 \Psi'(1-t) \int_0^{+\infty} \mathbb{I}_{\{F_X(x) < t\}} dx dt \\ &\stackrel{Fubini}{=} \int_0^{+\infty} dx \int_0^1 \Psi'(1-t) \mathbb{I}_{\{F_X(x) < t\}} dt \\ &= \int_0^{+\infty} dx \int_0^{1-F_X(x)} \Psi'(t) \mathbb{I}_{\{t < 1-F_X(x)\}} dt \\ &= \int_0^{+\infty} dx \int_0^{1-F_X(x)} \Psi'(t) dt = \int_0^{+\infty} [\Psi(1-F_X(x)) - \Psi(0+)] dx \\ &= \int_0^{+\infty} \Psi(1-F_X(x)) dx - \Psi(0+) \operatorname{ess\,sup}(X)\end{aligned}$$

Where we used the fact that

$$q_X(t) = \sup \{ \alpha : P(X \leq \alpha) < t \} = \int_0^{+\infty} \mathbb{I}_{\{F_X(x) < t\}} dx$$

In this case ($X \geq 0$), we remark that the first integral in the second equality of the theorem is null, indeed : $\forall x \leq 0, P(X > x) = 1 \Rightarrow \Psi(P(X > x) - 1) = 0$. For a general $X \in L^\infty$, we let $C = -\text{ess inf}(X)$, i.e, $X + C \geq 0$. By translation invariance,

$$\rho_\mu(-(X + C)) = \rho_\mu(-X) + C$$

And we get the required result. ■

If we now integrate by part the second equality of the above theorem, we have, for $X \in L^\infty$:

$$\begin{aligned} \rho_\mu(-X) &= \int_{-\infty}^0 [\Psi(P(X > x)) - 1] dx + \int_0^{+\infty} \Psi(P(X > x)) dx \\ &= [x\Psi(-F_X(x))]_{-\infty}^0 + [x\Psi(P(X > x))]_0^{+\infty} \\ &\quad - \int_{-\infty}^0 x d(\Psi(-F_X(x))) - \int_0^{+\infty} x d(\Psi(P(X > x))) \\ &= - \int_{-\infty}^0 x d(\Psi(-F_X(x))) - \int_0^{+\infty} x d(\Psi(-F_X(x))) \\ &= - \int_{-\infty}^{+\infty} x d(\Psi(-F_X(x))) \\ &= \int_{\mathbb{R}} x d(\Psi \circ F_X)(x) \end{aligned}$$

Where we assume

$$\begin{cases} \lim_{x \rightarrow -\infty} x\Psi(-F_X(x)) = 0 \\ \lim_{x \rightarrow +\infty} x\Psi(P(X > x)) = 0 \end{cases}$$

so that the two brackets are null.

1.4 Extreme Measure

Let us consider a Coherent Risk Measure $\rho(\cdot)$, then, we know it has a representation of the form $\rho(X) = -\inf_{Q \in \mathcal{D}} \mathbb{E}_Q(X)$. The set \mathcal{D} might not be unique; therefore, we consider the largest of these sets (see the Remark just after **Theorem 4**). For instance, for the AV@R, the determining set was denoted D_λ . Let us now denote by $\chi_\rho(X)$ the class of extreme measures for a given random variable X and a risk measure ρ by

$$\chi_\rho(X) = \left\{ Z = \frac{dQ}{dP}, Q \in \mathcal{D}_\rho : \mathbb{E}_Q(X) = \inf_{\tilde{Q} \in \mathcal{D}_\rho} \mathbb{E}_{\tilde{Q}}(X) \right\}$$

This means that if a density Z belongs to $\chi_\rho(X)$ then $\mathbb{E}_Q(X) = -\rho(X)$ where $\frac{dQ}{dP} = Z$. From [5], we know that if X has a continuous distribution and the risk measure is finite, then $\chi_\rho(X)$ is a singleton. We can now consider this extreme measure in the context of spectral risk measures (Weighted Var) :

Theorem 8 (See [9])

Let X be a random variable with a continuous distribution and μ a utility function of the Weighted VaR class. Then

$$\chi_\mu(X) = \{Z_\mu(\cdot)\} = \left\{ \int_{[F_X(\cdot), 1]} \frac{\mu(d\lambda)}{\lambda} \right\}$$

And we have

$$Q_\mu^* = (\Psi_\mu \circ F_X) X$$

Where Q_μ^* is the probability such that $\frac{dQ_\mu^*}{dP} = Z_\mu$.

2 Representation Theorems

In this section, we give some formal proofs of the Representation Theorems mentioned above on L^∞ , and we see how they can be extended to L^p spaces. The main references for the L^∞ theorems is [14]. Concerning L^p ($1 \leq p < \infty$) spaces, we refer the reader to [13] and [16].

2.1 Convex Risk Measures on L^∞

Theorem 9 Any Convex Risk Measure ρ on L^∞ can be written as

$$\rho(X) = \max_{Q \in \mathcal{M}_{1,f}(P)} [\mathbb{E}_Q(-X) - \underline{\alpha}(Q)]$$

Where $\mathcal{M}_{1,f}(P)$ represents the set of all finitely additive functions $Q : \mathcal{F} \rightarrow [0, 1]$ such that $Q(\Omega) = 1$ and

$$\forall Q \in \mathcal{M}_{1,f}(P), \underline{\alpha}(Q) = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_Q(-X)$$

Where $\mathcal{A}_\rho = \{X \in \mathcal{X} : \rho(X) \leq 0\}$ is the acceptance set of ρ .

We give here some elements of proof and refer to ([14], Theorem 4.31 p.172) for a detailed proof.

Proof For any $X \in L^\infty$, let $Y = \rho(X) + X$. By cash invariance, $Y \in \mathcal{A}_\rho$. So

$$\forall Q \in \mathcal{M}_{1,f}(P), \underline{\alpha}(Q) = \sup_{\tilde{Y} \in \mathcal{A}_\rho} \mathbb{E}_Q(-\tilde{Y}) \geq \mathbb{E}_Q(-Y) = \mathbb{E}_Q(-X) - \rho(X)$$

And hence

$$\forall Q \in \mathcal{M}_{1,f}(P), \underline{\alpha}(Q) = \sup_{X \in L^\infty} [\mathbb{E}_Q(-X) - \rho(X)]$$

Which means that $\underline{\alpha}$ corresponds to the Fenchel-Legendre Transform of the function ρ on L^∞ , i.e. $\underline{\alpha}(Q) = \rho^*(l_Q)$ where $l_Q(X) = \mathbb{E}_Q(-X)$ and

$$\begin{aligned} \rho^* : (L^\infty)' &\rightarrow \mathbb{R} \\ l &\mapsto \sup_{X \in L^\infty} [l(X) - \rho(X)] \end{aligned}$$

Now, because ρ is convex and Lipschitz continuous wrt to the sup norm (property of monetary risk measures), then its acceptance set \mathcal{A}_ρ is also convex, hence, all the level sets are convex and strongly closed. Thus, using **Theorem 20** (see Appendix), they are also weakly closed. The Fenchel-Moreau theorem therefore applies and $\rho^{**} = \rho$, where

$$\rho^{**} = \sup_{l \in (L^\infty)'} [l(X) - \rho^*(l)]$$

So

$$\begin{aligned} \forall X \in L^\infty, \rho(X) = \rho^{**}(X) &= \sup_{l_Q \in (L^\infty)'} [l_Q(X) - \rho^*(l_Q)] \\ &= \sup_{Q \in \text{ba}(P)} [\mathbb{E}_Q(-X) - \underline{\alpha}(Q)] \end{aligned}$$

Eventually, the function $\Phi : Q \mapsto \mathbb{E}_Q(-X) - \underline{\alpha}(Q)$ is upper semicontinuous, and because $\mathcal{M}_{1,f}(P)$ is weak*-compact in $(L^\infty)'$, then, by the Banach-Alaoglu Theorem (see Appendix), the function attains its maximum on $\mathcal{M}_{1,f}(P)$. ■

2.2 Coherent Risk Measures on L^∞

As the class of Coherent Risk Measures is a subset of all the Convex Risk Measures, the Representation Theorem for Convex Risk Measures (**Theorem 9**) obviously holds. But it can be precised in the following sense :

Theorem 10 Any Coherent Risk Measure ρ on L^∞ can be represented as

$$\forall X \in L^\infty, \rho(X) = \max_{Q \in \mathcal{Q}} \mathbb{E}(-X)$$

Where $\mathcal{Q} = \{Q \in \mathcal{M}_{1,f}(P) : \underline{\alpha}(Q) = 0\}$

The proof is trivial due to the positive homogeneity property of Coherent Risk Measures.

2.3 Risk Measures on $L^p, 1 \leq p < \infty$

This subsection is of fundamental importance for us. Indeed, we would like to work, not on L^∞ but rather on L^p , for $1 \leq p < \infty$ (in the following, L^p will denote this very space), which is more consistent with the real world and which will provide some useful restrictions as we will see later. The main point here is to be able to extend a few results :

- Does the **Representation Theorem 10** still hold for Coherent Risk Measures ? (and also for Convex Risk Measures ?)
- Can we still define the Average V@R in the same way ?
- Does the construction of the Spectral Risk Measure still hold or do we have to add some conditions on the probability measure μ ?

We first start by the Representation Theorem for Convex Risk Measures on L^p . The conditions needed in the $p = \infty$ case are not sufficient anymore, and we now need some continuity conditions on the measure for a representation to hold. We just recall here some very recent results and refer to the corresponding papers ([13] and [16]) for more details. We have

Theorem 11 Let ρ be a convex risk measure on L^p , then the following are equivalent :

- ρ is continuous.
- $\text{Int}(\mathcal{A}_\rho) \neq \emptyset$
- ρ is continuous at one point in its domain.
- ρ is lower semi-continuous and $\text{Dom}(\rho) = L^p$.

Where \mathcal{A}_ρ represents the acceptance set of ρ , i.e. $\mathcal{A}_\rho = \{X \in L^p : \rho(X) \leq 0\}$. In any case, we have

$$\forall X \in L^p, \rho(X) = \max_{Z \in (L^p)^*} \left[\mathbb{E}(XZ) - \rho^*(Z) \right]$$

Proof See [13].

■

We can now state the equivalent for Coherent Risk Measures on L^p :

Theorem 12 Let ρ be a coherent risk measure on L^p . Then the following points are equivalent :

- ρ is continuous.
- $\text{Dom}(\rho^*)$ is $\sigma((L^p)^*, L^p)$ -compact.

In either case, ρ only takes finite values and the following representation holds

$$\forall X \in L^p, \rho(X) = \max_{Y \in \text{Dom}(\rho^*)} \mathbb{E}(XY)$$

Proof Suppose that $\overline{\text{Dom}(\rho^*)}$ is compact. Then, for any $(X, Y) \in L^p \times (L^p)^*$, $g_X : Y \mapsto \mathbb{E}(XY) - \rho^*(Y)$ is upper semi-continuous, therefore, using the James' Weak Compactness theorem (see Appendix), its maximum is attained on $\overline{\text{Dom}(\rho^*)}$. Hence we can set

$$\rho(X) = \max_{Y \in \text{Dom}(\rho^*)} \left(\mathbb{E}(XY) - \rho^*(Y) \right)$$

Hence, from **Theorem (11)** for convex risk measures on L^p we deduce that ρ must be continuous. We also note, via this representation, that ρ only takes finite values.

Now, suppose that ρ is continuous. From the Representation Theorem for convex risk measures on L^p and due to the positive homogeneity of ρ , we have

$$\begin{aligned} \forall (\lambda, Y) \in \mathbb{R}_+^* \times \text{Dom}(\rho^*), \rho^*(Y) &= \sup_{X \in L^p} \left(\mathbb{E}(XY) - \rho(X) \right) \\ &= \sup_{X \in L^p} \left(\mathbb{E}(\lambda XY) - \rho(\lambda X) \right) = \lambda \rho^*(Y) \end{aligned}$$

Hence $\rho^*(Y) \in \{0, \infty\}$. So that we eventually have

$$\forall X \in L^p, \rho(X) = \max_{Y \in \text{Dom}(\rho^*)} \mathbb{E}(XY)$$

This in particular means that $\forall X \in L^p, \sup_{Y \in \text{Dom}(\rho^*)} |\mathbb{E}(XY)| < \infty$. From the Banach-Steinhaus theorem, $\text{Dom}(\rho^*)$ is bounded (and closed) and the Banach-Alaoglu theorem states that $\sigma((L^p)^*, L^p)$ is compact. ■

2.4 Application : $AV@R_\lambda$ in L^p

We here review the very recent result by Filipovic and Svindland (see [13]). Let $\lambda \in (0, 1]$ and define the following set :

$$\mathcal{D}_\lambda := \left\{ Z \in (L^\infty)^* : Z \geq -\frac{1}{\lambda} \right\}$$

From the Banach-Alaoglu theorem, \mathcal{D}_λ is compact in $\sigma((L^p)^*, L^p)$ and we can therefore define the Average V@R on L^p as the continuous risk measure

$$\forall X \in L^p, AV@R_\lambda(X) = \max_{Z \in \mathcal{D}_\lambda} \mathbb{E}(XZ)$$

2.5 Application : Spectral Risk Measure (WV@R) on L^p

This subsection is of fundamental importance for us, for our main tool is this spectral risk measure. We have just seen in the previous subsection that $AV@R$ could be defined on L^p . Now, the question is : does the following object exists, for $X \in L^p$?

$$\rho(X) = \int_0^{+\infty} AV@R_\lambda(X) \mu(d\lambda)$$

Where $\mu(\cdot)$ is a risk measure defined on $[0, +\infty)$. We first recall the representation of the Weighted V@R on L^∞ as in **Theorem 7** :

$$\rho_\mu(-X) = \Psi(0+) \text{ess sup}(X) + \int_0^1 q_X(\gamma) \Psi'(1-\gamma) d\gamma$$

Where $\Psi'(x) = \int_x^1 \frac{\mu(d\lambda)}{\lambda}$, for $0 \leq x \leq 1$. As X is no more bounded on L^p , we shall first require that $\Psi(0+) = \mu(\{0\}) = 0$. Now, we have:

Open Question: Let $X \in L^p, p \geq 1$. Under what conditions (On either the probability measure μ of the Distortion function Ψ) does the following representation holds?

$$\rho_\mu(-X) = \int_0^1 q_X(\gamma) \Psi'(1-\gamma) d\gamma$$

Where $\Psi'(x) = \int_x^1 \frac{\mu(d\lambda)}{\lambda}$, for $0 \leq x \leq 1$.

3 Extension to Convolution Semigroup

3.1 Coherent Acceptability Indices

Definition 13 *Acceptability index*¹ (from [2])

A map $\alpha : L^\infty \rightarrow \mathbb{R}_+$ is a coherent acceptability index if and only if there exists a collection $(\mathcal{D}_x)_{x \in \mathbb{R}_+}$ of subsets of \mathcal{P} such that for $x \leq y$, $\mathcal{D}_x \subseteq \mathcal{D}_y$ and

$$\alpha(X) = \inf \left\{ x \in \mathbb{R}_+ : \inf_{Q \in \mathcal{D}_x} \mathbb{E}_Q(X) < 0 \right\} \text{ with } \inf \emptyset = \infty$$

Hence, the definition of the coherent acceptability index associated to a utility function μ of the Weighted VaR class also reads

$$\alpha(X) = \inf \left\{ x \in \mathbb{R}_+ : \mathbb{E}_{(\Psi_\mu^x \circ F_X)_P}(X) < 0 \right\} \text{ with } \inf \emptyset = \infty$$

Where the family $(\Psi_\mu^x)_{x \in \mathbb{R}_+}$ is defined relatively to the family of determining sets $(\mathcal{D}_x)_{x \in \mathbb{R}_+}$. This holds true because we saw before that for a given WV@R, the infimum was attained on the Determining Set of the risk measure. Furthermore, Cherny and Filipovic ([9]) recently narrowed the class of Distortion functions and introduced a so-called convolution semigroup :

Theorem 14 *Let μ be a probability measure on $[0, 1]$. A family $(\Psi_\mu^t)_{t \geq 0}$ of Distortions is a concave distortion semigroup is and only if there exists a concave function $G : [0, 1] \rightarrow \mathbb{R}_+^*$ such that*

$$\forall t \geq 0, \forall x \in (0, 1], \Psi_\mu^t(x) = \inf \left\{ y \in [x, 1] : \int_x^y \frac{ds}{G(s)} = t \right\}$$

Where $\inf \emptyset = 1$. Furthermore we have the inverse relation

$$\forall x \in (0, 1), G(x) = \lim_{t \searrow 0} \frac{\Psi_\mu^t(x) - x}{t}$$

3.2 Statement of the problem

To summarize the above notations, there is a one-to-one correspondance between a spectral risk measure and a convolution semigroup of increasing concave functions. We would like to study the class of spectral risk measures more deeply. We had

$$\forall x \in \mathbb{R}_+, \frac{dQ_\mu^x}{dP} = \partial \Psi_\mu^x \circ F_X \tag{3}$$

Which is directly linked with the pricing with utility function that, in the traditional framework of complete markets reads $\frac{dQ}{dP} = cU'(X)$ where P is the physical measure, Q the risk-neutral one, U the utility function of an agent and

¹For clarity and brevity reasons, Cherny and Madan (see their revised version of [8]) now prefer to use the term Acceptability Index, dropping the Coherent term.

c a normalizing constant. In an incomplete market framework, there might be an infinity of equivalent martingale measures such that (3) is satisfied. Our problem can be decomposed into the following subproblems :

- We might not be able to observe the whole semigroup, but just one element of the family (think, for instance, about rating transition matrices : we just observe the 1-year matrix, but in order to price a 6-month CDS, we need the 6-month matrix). So, which conditions must we impose on a function Ψ so that the family is indeed a distortion convolution semigroup ? How can we reconstruct the whole semigroup from one element of the family? Is there unicity ? For the rating transition matrix analogy, some results have already been obtained by [15].
- What are the consequences on the extreme measure Q_μ^x and its Radon-Nikodym derivative wrt P ?
- Can the relation between Spectral Risk Measures and Distortion functions be extended to L^p spaces, with some additional conditions on the risk measure ?

4 Application and Numerical Results

In this section, we wish to provide some numerical results. Basically, what we do is the following: consider some distributions (preferably fat-tailed distributions, though we will also provide the corresponding results for the Gaussian), and numerically and graphically study the risk associated (with respect to the different families of risk measures); this will also provide us with the corresponding coherent acceptability index (whenever it exists). Let X be a random cashflow over a given period of time and $(\Psi_t)_{t \geq 0}$ a Distortion function. Then the risk measure is given by

$$\rho_{\Psi_t}(X) = \int_{\mathbb{R}} q_{-X}(u) \Psi'_t(1-u) du$$

In the following, whenever there is no ambiguity on the Distortion function we are using, we will write ρ_t instead of ρ_{Ψ_t} . Consider a portfolio with 0 value at the beginning of the period, so that the total loss of the portfolio is exactly $L = -X$. As in [7], we defined an acceptability index $\alpha(X)$ as follows: we are interested here in finding the optimal index t^* such that

$$t^* = \alpha(X) = \sup \{t \geq 0 : \rho_t(X) < 0\}$$

With $\sup \{\emptyset\} = \infty$ We take here a strict inequality in the definition of the index for practical reasons we will explain later.

4.1 Results: Overview and Methodology

The methodology is as follows: We discretise the integral corresponding to the spectral risk measure (50 steps); we numerically (using the *Quantile* function in

MAPLE) invert the probability distribution of the simulated Loss process. For consistency reasons, for a fixed ν , we keep the same simulated Loss Distribution for all t and all x . When the product $q(\cdot)\Psi(\cdot)$ is converging sufficiently fast to 0 in 0 and 1, then MAPLE is able to obtain some results in a reasonable period of time. For more rigorous results, in particular for the Gaussian or the Gumbel Distributions, it is much more efficient to implement the numerical integration in a fast-computing environment, such as C++, or Python. We used Python here.

We refer the reader to the Appendix concerning the Distortion we use here. In particular, $\Psi^{(1)}$ refers to the AIW Index, $\Psi^{(2)}$ to the AIMIN Index, $\Psi^{(3)}$ to the AIMAX Index, $\Psi^{(4)}$ to the Exponential Utility function.

Numerically, if we want to compute such a risk measure, we bump into several problems. The most immediate one is to determine the Quantile function of the Distribution, which we very scarcely have in closed-form. Suppose that we use a software providing the Quantile function at each point (such as MAPLE for instance), if we want to discretise the integral, we have a further problem around 0 : the quantile function goes to $-\infty$ whereas $\Psi'(\cdot)$ tends to $+\infty$. So we need a discretisation step too large to allow stable results. Another approach, which we take here, is to consider an approximate analytical formula for the Quantile function and derive an analytical result for the integral. We have

$$\forall u \in \mathbb{R}, \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du, F_{X_{0,1}}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du = \frac{1}{2} + \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)$$

Where $X_{0,\sigma^2} \hookrightarrow \mathcal{N}(0, \sigma^2)$. And hence

$$\forall x \in [0, 1], F_{X_{0,1}}^{\leftarrow}(x) = \sqrt{2} \operatorname{erf}^{-1}(2x - 1)$$

And obviously

$$\forall x \in [0, 1], F_{X_{0,1}}^{\leftarrow}(x) = \frac{F_{X_{\mu,\sigma^2}}^{\leftarrow}(x) - \mu}{\sigma}$$

So that

$$q_{-X_{0,1}} = \frac{q_{-X_{\mu,\sigma}} + \mu}{\sigma}$$

And we deduce the Spectral Risk Measure associated to the Gaussian Distribution:

$$\begin{aligned} \forall t \geq 0, \rho_{\Psi_t}(X_{\mu,\sigma}) &= \int_0^1 (\sigma q_{-X_{0,1}}(u) - \mu) \Psi_t'^{(i)}(1-u) du \\ &= -\mu + \sigma\sqrt{2} \int_0^1 \operatorname{erf}^{-1}(2u - 1) \Psi_t'(1-u) du \end{aligned}$$

Numerically speaking, we could either use a brute Taylor series expansion (see Appendix) for the erf^{-1} function, but some more efficient methods have been proposed, in particular, the Acklam algorithm to compute the inverse CDF of the Gaussian. The following theorem is very general:

Theorem 15 Let X be a Distribution symmetric around 0 and $(\Psi_t(\cdot))_{t \geq 0}$ a Concave Distortion Semigroup such that, $\forall x \in [0, 1]$, $\partial_x \Psi_0(x)$ is a constant (which is the case for the AIMIN and AIMA X Distortions), then

$$\forall t > 0, \rho_{\Psi_t}(X) > 0$$

And

$$\rho_{\Psi_0}(X) = 0$$

Therefore

$$\alpha(X) = \infty$$

Proof

$$\begin{aligned} \forall t > 0, \rho_{\Psi_t}(X) &= \int_0^1 q_{-X}(u) \Psi'_t(1-u) du \\ &= \int_0^{1/2} q_{-X}(u) \Psi'_t(1-u) du + \int_{1/2}^1 q_{-X}(u) \Psi'_t(1-u) du \\ &= \int_0^{1/2} q_{-X}(u) \Psi'_t(1-u) du + \int_0^{1/2} q_{-X}\left(u + \frac{1}{2}\right) \Psi'_t\left(\frac{1}{2} - u\right) du \\ &= \int_0^{1/2} q_{-X}(u) \left[\Psi'_t(1-u) - \Psi'_t\left(\frac{1}{2} - u\right) \right] du \end{aligned}$$

Where we used the fact that $q_{-X}\left(u + \frac{1}{2}\right) = -q_{-X}(u)$ for $u \in [0, \frac{1}{2}]$. Now, for $u \in [0, \frac{1}{2}]$, we have $(1-u) \in [\frac{1}{2}, 1]$, $(\frac{1}{2} - u) \in [0, \frac{1}{2}]$, and because, for each $t > 0$, $\Psi_t(\cdot)$ is a concave function, $\Psi'_t(\frac{1}{2} - u) > \Psi'_t(1-u)$, for $u \in [0, \frac{1}{2}]$. We also have that, for $u \in [0, \frac{1}{2}]$, $q_{-X}(u) < 0$, and hence the result. The second statement of the theorem is immediate form the last line of the formula and the last statement is an immediate consequence of the first statement. ■

4.2 Analytical results for the Gaussian Distribution

We saw above that, for the $\mathcal{N}(0, \sigma^2)$ distribution, we always have $\rho_{\Psi_0^{(2)}}(X) = \rho_{\Psi_0^{(3)}}(X) = 0$ because the distribution is symmetric around 0. Suppose now that $X_\mu \hookrightarrow \mathcal{N}(\mu, \sigma^2)$, with $\mu \neq 0$, then we have the trivial identities:

$$\begin{aligned} F_{-X_{\mu, \sigma}}(x) &= \mathbb{P}(-X_{\mu, \sigma} \leq x) = F_{-X_{0, 1}}\left(\frac{x + \mu}{\sigma}\right) \\ q_{-X_{\mu, \sigma}}(x) &= \inf \left\{ \alpha : F_{-X_{\mu, \sigma}}(\alpha) \geq x \right\} = \sigma q_{-X_{0, 1}}(x) - \mu \end{aligned}$$

And hence:

$$\begin{aligned}
\rho_{\Psi_t}(X_{\mu,\sigma}) &= \int_0^1 q_{-X_{\mu,\sigma}}(u) \Psi_t'(1-u) du \\
&= \int_0^1 [\sigma q_{X_{0,1}}(u) - \mu] \Psi_t'(1-u) du \\
&= \sigma \rho_t(X_{0,1}) - \mu [\Psi_t^{(i)}(1) - \Psi_t(0)] \\
&= \sigma \rho_t(X_{0,1}) - \mu
\end{aligned}$$

Because, by construction, $\Psi_t(1) = 1$ and $\Psi_t(0) = 0$. So we just need to focus on the centered Gaussian Distribution. It is convenient to write it in the following form:

$$\forall \mu, \tilde{\mu}, \sigma, \tilde{\sigma}, \forall t \geq 0, \quad \boxed{\rho_{\Psi_t}(X_{\tilde{\mu},\tilde{\sigma}}) = \frac{\tilde{\sigma}}{\sigma} \rho_{\Psi_t}(X_{\mu,\sigma}) + \mu \frac{\tilde{\sigma}}{\sigma} - \tilde{\mu}}$$

4.2.1 AIMIN Distortion for Gaussian Distribution

Lemma 16 *Let $X \hookrightarrow \mathcal{N}(0,1)$ and Ψ_t be an AIMIN Concave Distortion Semigroup. Then $t \mapsto \rho_{\Psi_t}(X)$ is an increasing function.*

Proof *The proof is similar to the proof above. For ease of notation, let $\theta = e^t$:*

$$\begin{aligned}
\forall \theta \geq 1, \partial_\theta \rho_{\Psi_\theta^{(2)}}(X) &= \int_0^1 q_{-X}(u) u^{\theta-1} [1 + \log(u)] du \\
&= \int_0^{1/2} q_{-X}(u) u^{\theta-1} [1 + \log(u)] du \\
&\quad - \int_0^{1/2} q_{-X}(v) (1-v)^{\theta-1} [1 + \log(1-v)] dv \\
&= \int_0^{1/2} q_{-X}(u) [g_\theta(u) - g_\theta(1-u)] du
\end{aligned}$$

Where $g_\theta : u \mapsto u^\theta [1 + \log(u)]$. It is easy to see that, $\forall \theta \geq 1, \forall u \in [0, \frac{1}{2}), g_\theta(u) - g_\theta(1-u) < 0$. And hence the lemma is proved. ■

Comment *Let $X \hookrightarrow \mathcal{N}(\mu,\sigma)$ and Ψ_t be a Concave Distortion Semigroup. Then, if $\mu > 0$ then $\alpha(X) \in \mathbb{R}_+^*$, otherwise $\alpha(X) = \infty$. ■*

Conjecture 17 *Should still hold for any Concave Distortion Semigroup.*

We consider a centered Gaussian Distribution $\mathcal{N}(0, \sigma^2)$ and we have (we expand the erf^{-1} function up to order $n = 2p + 1$):

$$\begin{aligned}
\forall t \geq 0, \rho_{\Psi_t^{(2)}}(X_{\sigma^2}) &= \sigma\sqrt{2} \int_0^1 \text{erf}^{-1}(2u-1) \Psi_t^{(i)}(1-u) du \\
&= \sigma e^t \sqrt{2} \int_0^1 \text{erf}^{-1}(2u-1) u^{e^t-1} du \\
&= \sigma e^t \sqrt{2} \int_0^1 \left[\sum_{i=0}^p \beta_{2i+1} (2u-1)^{2i+1} + \mathcal{O}(u^{2p+1}) \right] u^{e^t-1} du \\
&\approx \sigma e^t \sqrt{2} \sum_{i=0}^p \beta_{2i+1} \int_0^1 (2u-1)^{2i+1} u^{e^t-1} du \\
&\approx \sigma e^t \sqrt{2} \sum_{i=0}^p \beta_{2i+1} \sum_{k=0}^{2i+1} C_{2i+1}^k (-1)^{2i+1-k} 2^k \int_0^1 u^{k+e^t-1} du \\
&\approx -\sigma e^t \sqrt{2} \sum_{i=0}^p \beta_{2i+1} \sum_{k=0}^{2i+1} \frac{C_{2i+1}^k (-2)^k}{k+e^t}
\end{aligned}$$

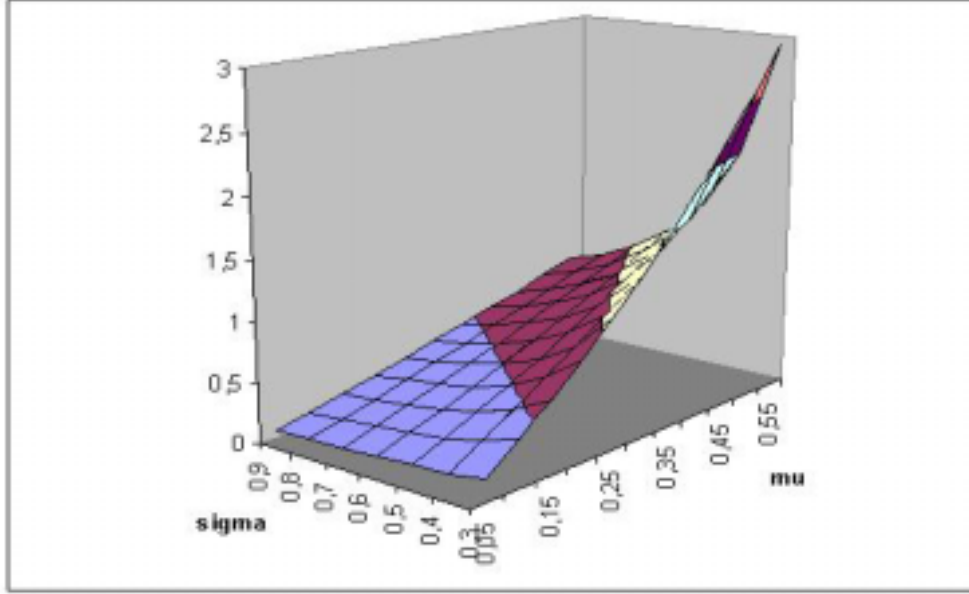


Figure 1: Acceptability Index $\alpha(X_{\mu, \sigma})$ for the AIMAX Distortion

4.2.2 Numerical Implementation and issues

From what we have seen above, we know, that, for any $\mu > 0, \sigma$, $\alpha(X_{\mu,\sigma})$ exists and is unique. The graphs below present some numerical results for this Acceptability Index. Some comments arise from the numerical implementation:

- The implementation has been carried out in Python (MAPLE being way too slow for numerical integration and minimisation).
- The number of steps needed for a stable integration heavily depends on the region defined by the couple (μ, σ) . We show hereafter some numerical caveats.

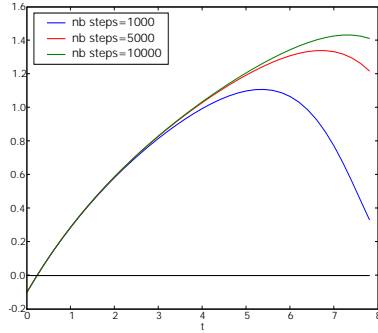


Figure 2: AIMIN Risk Measure with for $\mathcal{N}(\mu = 0.1, \sigma = 0.5)$

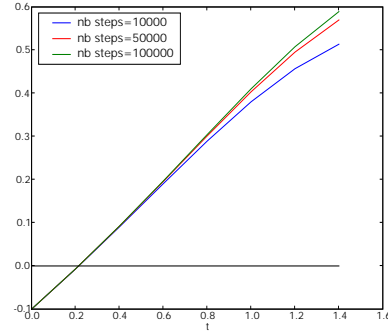


Figure 3: AIMAX Risk Measure with for $\mathcal{N}(\mu = 0.1, \sigma = 0.5)$

From these two figures, we observe that, for some range of (μ, σ) , we do not need too many steps for the integration, as we are only interested by the point $t^* : \rho_{\Psi_{t^*}}(X) = 0$. A higher value of μ means a downward translation of the graphs. So, for large values of μ , the integration requires a very large number of steps, and hence the minimisation procedure will be much longer.

Comment

We provide an intuitive reason for the shape of these graphs, in particular when t gets large. The graphs correspond to the function $t \mapsto \int_{[0,1]} q_{-X}(u) \Psi'_t(1-u) du$.

$\Psi'_t(\cdot)$ is a strictly decreasing function from $+\infty$ to 0 (Ψ_t is strictly concave). The higher the t , the more concave the function Ψ_t , and hence, in the integral, the higher the weight in the upper side of the integral. Hence, if the number of steps in the integration is not large enough, we omit many terms, in particular many large terms. For more accurate results, we should adopt an adaptive numerical scheme, specifying a number of steps proportional to the concavity of Ψ_t at each point.

However, for our purposes, and in particular, with the values we chose for the

parameters, this is not of fundamental importance, as we are just interested by the point where the function vanishes. But for some values of parameters, this integration might be computationally really intensive. ■

4.3 Some analytical results for the Student Distribution

In some cases, the inverse CDF of the Student is available in closed-form, see [17] for the details. In particular, we have

- For $n = 1$, this is the Cauchy distribution and $F^{\leftarrow}(u) = \tan\left(\pi\left(u - \frac{1}{2}\right)\right)$.
- For $n = 2$, $F^{\leftarrow}(u) = \frac{2u-1}{\sqrt{2u(1-u)}}$. Practical speaking, though, this case is less interesting as it has infinite variance.
- For $n = 4$, $F^{\leftarrow}(u) = \operatorname{sgn}\left(u - \frac{1}{2}\right) \sqrt{\frac{2}{\sqrt{u(1-u)}} \cos\left(\frac{1}{3} \arccos\left(2\sqrt{u(1-u)}\right)\right)} - 4$. This case is particularly interesting as it has finite variance and infinite Kurtosis.

We note that, as Student distributions are symmetric around 0, we have, for all ν , $q_{X_\nu} = q_{-X_\nu}$. We study here the case $\nu = 2$ (infinite variance). For a Concave Distortion Semigroup $(\Psi_t(\cdot))_{t \geq 0}$, we then have

$$\forall t \geq 0, \rho_{\Psi_t}(X_2) = \int_0^1 q_{X_2}(u) \Psi_t'(1-u) du$$

Let us consider first the AIMIN Distortion:

$$\begin{aligned} \forall t \geq 0, \rho_{\Psi_t^{(2)}}(X_2) &= e^t \int_0^1 \frac{2u-1}{\sqrt{2u(1-u)}} u^{e^t-1} du \\ &= \frac{\sqrt{2\pi}\Gamma\left(\frac{1}{2} + e^t\right)}{\Gamma(1 + e^t)} - \frac{\sqrt{\frac{\pi}{2}}\Gamma\left(e^t - \frac{1}{2}\right)}{\Gamma(e^t)} \end{aligned}$$

(We refer the interested reader to the Appendix for the proof of the result). We plot below the graph of the Spectral Risk Measure as a function of t . We observe that, contrary to the Normal Distribution, it is not strictly increasing. As we proved above, $\forall t > 0$, $\rho_{\Psi_t^{(2)}}(X_2) > 0$, we also have $\alpha(X_2) = \infty$.

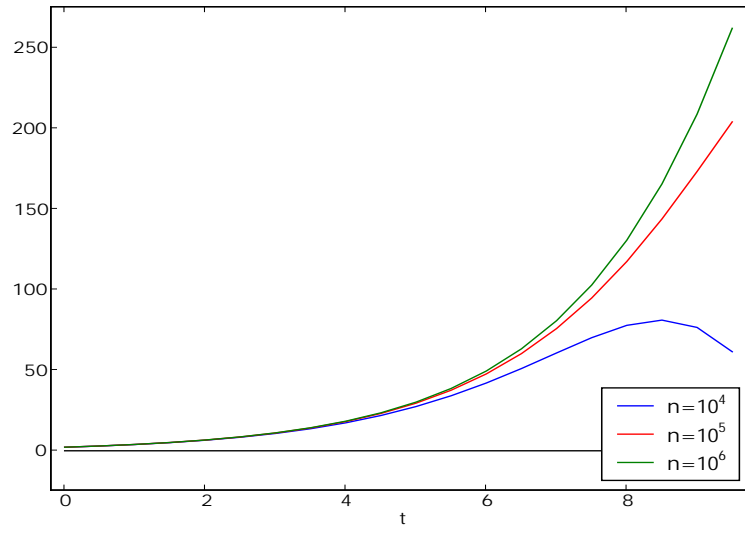


Figure 4: AIMAX Risk Measure for a Student Distribution with $\nu = 2$, and n number of steps for the integration

4.4 An asymmetric Distribution: Gumbel

We now consider an asymmetric distribution. The reason for choosing the Gumbel Distribution is that the inverse CDF is known in closed form. Indeed, let $X \hookrightarrow \mathcal{G}(a, b)$, with $a \in \mathbb{R}$, $b > 0$, we have

$$\forall u \in (0, 1), q_X(u) = a - b \log(-\log(u))$$

And

$$\forall u \in (0, 1), q_{-X}(u) = -a + b \log(-\log(1-u))$$

We now consider the AIMIN Spectral Risk Measure:

$$\begin{aligned} \forall t \geq 0, \rho_{\Psi_t}(X) &= e^t \int_0^1 [-a + b \log(-\log(1-u))] u^{e^t-1} du \\ &= -a + b e^t \int_0^1 \log(-\log(1-u)) u^{e^t-1} du \end{aligned}$$

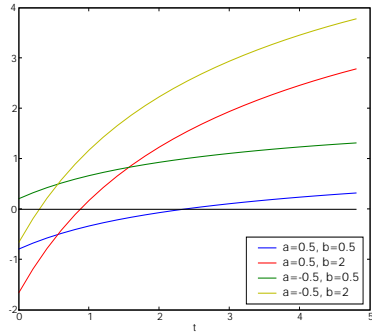


Figure 5: Some examples of AIMIN Risk Measure for Gumbel Distribution (10^3 steps)

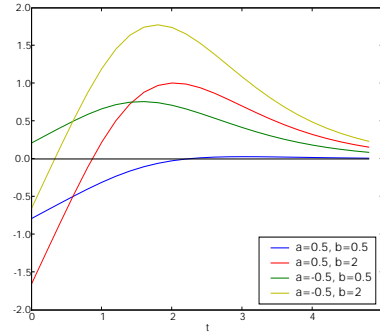


Figure 6: Some examples of AIMAX Risk Measure for Gumbel Distribution (10^3 steps)

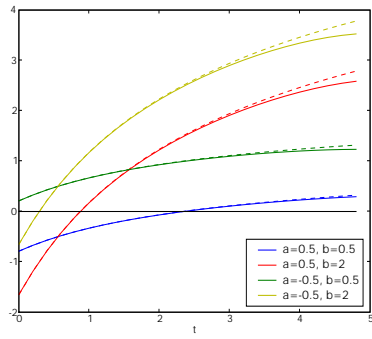


Figure 7: Numerical issues with the AIMIN Risk Measure for the Gumbel Distribution (solid line: 10^3 steps, dashed line: 10^4 steps)

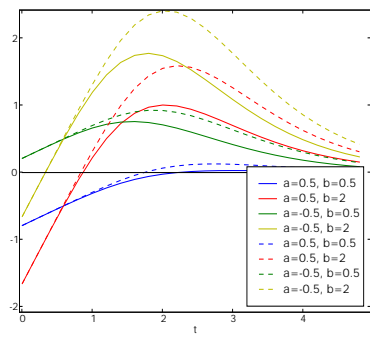


Figure 8: Numerical issues with the AIMAX Risk Measure for the Gumbel Distribution (solid line: 10^4 steps, dashed line: 10^6 steps)

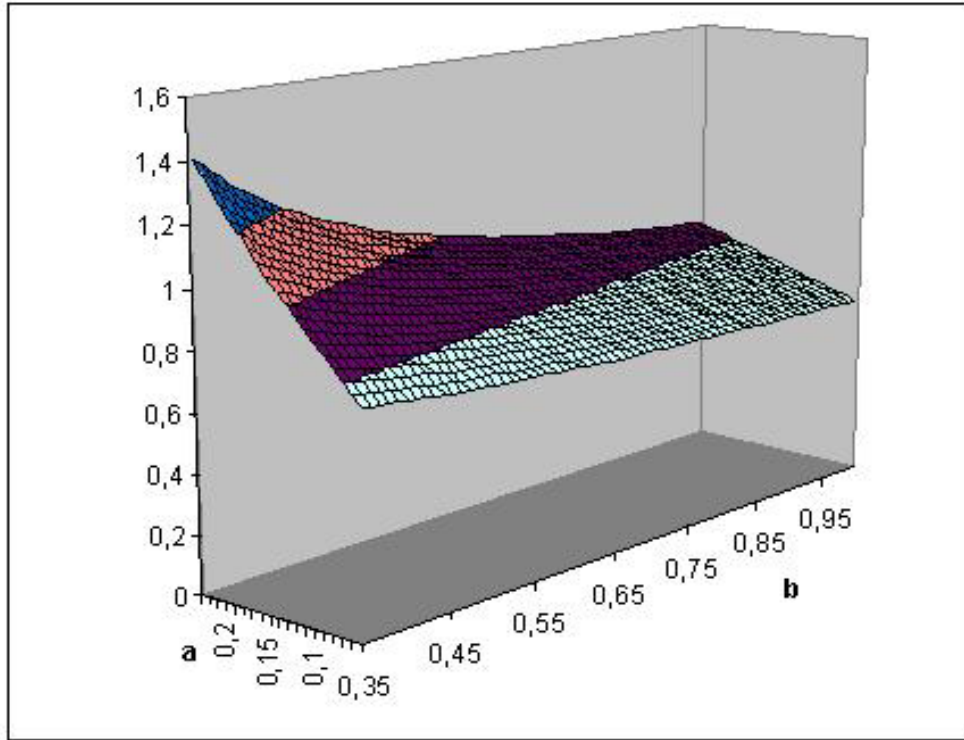


Figure 9: Optimal $\alpha(X)$ when $X \hookrightarrow \mathcal{G}(a, b)$, with 10^4 steps

A Reminders from Functional Analysis

Definition 18 A topological vector space E is said to be locally convex if its topology has a base consisting of convex sets.

Example : If E is a Banach space with norm $\|\cdot\|$, then the open balls $B_\delta = \{y \in E, \|y - x\| \leq \delta\}$, for $\delta > 0$ form a base for the topology of E . As all these balls are convex, then E is locally convex.

Counter-Example : $L^0(\Omega)$ is not locally convex if Ω is atomless.

Theorem 19 Banach-Alaoglu Theorem

If E is a normed linear space, then the closed unit ball of its dual space is compact in the weak*-topology. Thus, each net of points of this ball has a convergent subnet.

Theorem 20 Dunford-Schwartz, V.3.13, in [11] A convex subset of a locally convex linear topological space is weakly*-closed iff it is closed.

Theorem 21 Fenchel-Moreau

Let (E, τ) be a locally convex topological vector space with topological dual E' . Let $f : E \rightarrow \mathbb{R} \cup \{\infty\}$ be a proper, convex and lower semicontinuous function. Then f is identical to its doubly conjugate f^{**} :

$$\forall x \in E, f(x) = \sup_{x^* \in E'} (\langle x, x^* \rangle - f^*(x^*))$$

Where

$$\forall x^* \in E', f^*(x^*) = \sup_{x \in E} (\langle x^*, x \rangle - f(x))$$

Theorem 22 Dunford & Schwartz IV.8.16 (in [11])

Let (S, Σ, μ) be a positive measure space. There is an isometric isomorphism between $(L^\infty(S, \Sigma, \mu))'$ and $ba(S, \Sigma, \mu)$ determined by

$$\forall f \in L^\infty(S, \Sigma, \mu), x^* f = \int_S f(s) \lambda(ds)$$

Where $ba(S, \Sigma, \mu)$ represents the space of finitely additive set functions with finite total variation that are absolutely continuous with respect to μ .

Theorem 23 Krein-Smulian

Let X be a Banach space. A convex set in X^* is X -closed if and only if its intersection with every positive multiple of the closed unit sphere of X^* is X -closed.

Theorem 24 James' Weak Compactness

Let A be a nonempty weakly closed subset of a Banach space X . Then the following are equivalent :

- A is weakly compact.

- Whenever x^* is a bounded linear functional on X , the supremum of $|x^*|$ on A is attained.
- Whenever u^* is a bounded real-linear function on X , the supremum of $|u^*|$ on A is attained, so is the supremum of u^* on A .

Theorem 25 *Extended Namioka*

Let (E, τ) be a Fréchet Lattice and $f : E \rightarrow \mathbb{R} \cup \{\infty\}$ a proper, convex, increasing function. Then f is continuous on $I_f = \text{Int}(\text{Dom}f)$, where $\text{Dom}f = \{x \in E, f(x) < \infty\}$.

Lemma 26 *Dini's lemma*

Let (X, d) be a compact nonempty metric space, and $(f_n)_{n \in \mathbb{N}}$ an increasing sequence in $C^{\mathbb{R}}(X)$. If the sequence converges pointwise to a function $f \in C^{\mathbb{R}}(X)$, it also converges uniformly to f . The notation $C^{\mathbb{R}}(X)$ refers to the space of continuous functions from X to \mathbb{R} .

Theorem 27 *Banach-Steinhaus Theorem (Uniform Boundedness Principle)*

Let \mathcal{F} be a nonempty family of bounded linear operators from a Banach space X into a normed space Y . If $\sup\{\|Tx\|, T \in \mathcal{F}\}$ is finite for each $x \in X$, then $\sup\{\|T\|, T \in \mathcal{F}\}$ is finite.

Corollary 28 Let (T_n) be a sequence of bounded linear operators from a Banach space X into a normed space Y such that $\lim_n T_n x$ exists for each $x \in X$. Let $T : X \rightarrow Y$ by $Tx = \lim_n T_n x$. Then T is a bounded linear operator from X to Y .

B Overview of the distributions used

B.1 Gaussian

$X \hookrightarrow \mathcal{N}(\mu, \sigma)$ for $\mu \geq 0, \sigma > 0$. The density reads

$$\forall x \in \mathbb{R}, f(x) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}, \Phi(x) = \int_{-\infty}^x f(u) du$$

$$\mathbb{E}(X) = \mu, \mathbb{V}(X) = \sigma^2$$

B.2 Student

$X \hookrightarrow \mathcal{S}(\nu)$ for $\nu \in \mathbb{N}$.

$$\forall x \in \mathbb{R}, f(x) = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}\nu)}{\sqrt{\pi\nu}\Gamma(\frac{1}{2}\nu)} \left(1 + \frac{x^2}{\nu}\right)^{-\frac{1}{2} + \frac{1}{2}\nu}$$

$$F(x) = \begin{cases} \frac{1}{2} + \frac{\arctan\left(\frac{x}{\sqrt{\nu}}\right)}{\pi} + \frac{x\sqrt{\nu}}{\pi(\nu+x^2)} \left[\frac{\arcsin\left(\sqrt{\frac{\nu}{\nu+x^2}}\right)}{\sqrt{\frac{\nu}{\nu+x^2}}\sqrt{1-\frac{\nu}{\nu+x^2}}} - \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\sqrt{\pi}H\left(\left[1, \frac{1}{2}+\frac{\nu}{2}\right], \left[1+\frac{\nu}{2}\right], \frac{\nu}{\nu+x^2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{\nu/2-1/2}} \right], & \text{if } \nu = 1 \\ \frac{1}{2} + \frac{x}{2\sqrt{\nu+x^2}} \left[\sqrt{\frac{\nu+x^2}{x^2}} - \frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)H\left(\left[1, \frac{1}{2}+\frac{\nu}{2}\right], \left[1+\frac{\nu}{2}\right], \frac{\nu}{\nu+x^2}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)\left(1+\frac{x^2}{\nu}\right)^{\nu/2}} \right], & \text{otherwise} \end{cases}$$

Where $H([u], [l], x)$ represents the Generalised Hypergeometric Function with upper parameters u and lower parameters l , evaluated at x .

B.3 Gumbel

$X \hookrightarrow \mathcal{G}(a, b)$ for $a \in \mathbb{R}$, $b > 0$. a is a location parameter (where the density attains its maximum), and b a scale parameter. We have:

$$\forall x \in \mathbb{R}, f(x) = \frac{1}{b} e^{-\frac{x-a}{b}} e^{-e^{-\frac{x-a}{b}}}, F(x) = e^{-e^{-\frac{x-a}{b}}}$$

$$\mathbb{E}(X) = a + \gamma b, \mathbb{E}(X) = \frac{1}{6} b^2 \pi^2$$

Where γ is the Euler constant.

C Families of Distortion Functions

In the following, we might consider the following families of Distortion functions (for $x \in [0, 1]$, the families are indexed by t):

- $\psi_t^{(1)}(x) = xe^t \wedge t$: AIW Index
- $\psi_t^{(2)}(x) = 1 - (1-x)e^t$: AIMIN Index
- $\psi_t^{(3)}(x) = xe^{-t}$: AIMAX Index
- $\psi_t^{(4)}(x) = \frac{e^t}{1-e^t} (e^{-tx} - 1)$: Corresponds to the Exponential Utility function

D Taylor Series Expansion for the erf^{-1} function

We just indicate the first orders (we used the order $n = 10$).

$$\begin{aligned}
 \text{erf}(x) &= \frac{1}{2}\sqrt{\pi}x + \mathcal{O}(x^3), \quad n = 2 \\
 &= \frac{1}{2}\sqrt{\pi}x + \frac{1}{24}\pi^{\frac{3}{2}}x^3 + \mathcal{O}(x^5), \quad n = 4 \\
 &= \frac{1}{2}\sqrt{\pi}x + \frac{1}{24}\pi^{\frac{3}{2}}x^3 + \frac{7}{960}\pi^{\frac{5}{2}}x^5 + \mathcal{O}(x^7), \quad n = 6 \\
 &= \frac{1}{2}\sqrt{\pi}x + \frac{1}{24}\pi^{\frac{3}{2}}x^3 + \frac{7}{960}\pi^{\frac{5}{2}}x^5 + \frac{127}{80640}\pi^{\frac{7}{2}}x^7 + \mathcal{O}(x^9), \quad n = 8 \\
 &= \frac{1}{2}\sqrt{\pi}x + \frac{1}{24}\pi^{\frac{3}{2}}x^3 + \frac{7}{960}\pi^{\frac{5}{2}}x^5 + \frac{127}{80640}\pi^{\frac{7}{2}}x^7 + \frac{4369}{11612160}\pi^{\frac{9}{2}}x^9 + \mathcal{O}(x^{11}), \quad n = 10
 \end{aligned}$$

For higher orders, one can use the algorithm by Steinbrecher in [18] :

$$\text{erf}^{-1}(x) = \sum_{n \geq 0} \frac{\gamma_n}{2n+1} \left(\frac{\sqrt{\pi}}{2}x \right)^{2n+1}$$

Where

$$\gamma_0 = 1, \quad \forall n \geq 1, \quad \gamma_n = \sum_{k=0}^{n-1} \frac{\gamma_k \gamma_{n-k-1}}{(k+1)(2k+1)}$$

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