# Fractional intertwinings between two Markov semigroups


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**Abstract** We define the notion of $\alpha$-intertwining between two Markov Feller semigroups on $\mathbb{R}_+$ and we give some examples. The 1-intertwining, in particular, is merely the intertwining via the first derivative operator. It can be used in the study of the existence of pseudo-inverses, a notion recently introduced by Madan-Roynette-Yor [12] and Roynette-Yor [15].



**Key words** Markov semigroup, intertwining, fractional derivative, fractional integration, pseudo-inverse


# 1 Introduction

During the last two decades, a number of examples of intertwinings between two Markov semigroups $(P_t)$ and $(Q_t)$ defined respectively on $(E, \mathcal{E})$ and $(F, \mathcal{F})$, via a Markovian kernel $\Lambda : (E, \mathcal{E}) \longrightarrow (F, \mathcal{F})$, have been discovered and exploited. Precisely, $(P_t)$ and $(Q_t)$ are intertwined via $\Lambda$ if

$$Q_t \, \Lambda = \Lambda \, P_t \tag{1}$$

where, for two kernels $M$ and $N$, $M\,N$ denotes the composition of these kernels.

To illustrate, let us present an important example. Let us denote, for $\alpha > 0$ and $\beta > 0$ , by $\Lambda_{\alpha,\beta}$ the "multiplication kernel" by a beta$(\alpha, \beta)$ variable, that is:

$$\forall x \in \mathbb{R}_+ \quad \Lambda_{\alpha,\beta} f(x) = E[f(x\, Z_{\alpha,\beta})], \;\; f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+, \text{ Borel,}$$

$$\text{and} \quad P(Z_{\alpha,\beta} \in \mathrm{d}z) = \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha,\beta)}\, \mathrm{d}z \quad (0 < z < 1).$$

We also denote, for $\delta > 0$, by $(Q_t^\delta)$ the semigroup of the squared Bessel process with dimension $\delta$. One then has, for $\alpha > 0$ and $\beta > 0$, the following intertwining relation (Yor [17]):

$$Q_t^{2(\alpha+\beta)}\, \Lambda_{\alpha,\beta} = \Lambda_{\alpha,\beta}\, Q_t^{2\alpha}. \tag{2}$$

This result is an extension (at the semigroup level) of the so-called beta-gamma algebra. In the case $\alpha = 1/2$ and $\beta = 1$, the intertwining (2) is closely related to Pitman's theorem asserting that $R_t := 2\,S_t - B_t$, $t \geq 0$, is a 3-dimensional Bessel process, when $(B_t)$ is the Brownian motion starting from 0, and $S_t = \sup_{s \leq t} B_s$.

We refer to the paper Carmona-Petit-Yor [4] and to the list of references therein, for various examples of intertwinings. These are obtained very often, as explained in that paper, via a filtering type framework. We also mention the papers of Biane [2] and of Matsumoto-Yor [13, 14].

In the present paper, we shall discuss a kind of intertwining which is different from that induced by a Markovian kernel $\Lambda$ via (1). Our motivation comes from the following relation:

$$\forall f \in E_1 \quad Q_t^{\delta+2}\, D\, f = D\, Q_t^\delta\, f \tag{3}$$

where $E_1$ denotes the space of $C^1$-functions on $\mathbb{R}_+$ which tend to 0 at infinity as well as their first derivative, $D$ denotes the first derivative and, as previously, for $\delta > 0$, $(Q_t^\delta)$ denotes the semigroup of the squared Bessel process with dimension $\delta$. This formula (3) is found in an equivalent but different form, in Hirsch-Song [7], and has been helpful in the discussion of pseudo-inverses of squared Bessel processes developed recently by Roynette-Yor [15] (see also Section 5 below).

In this paper, we are more generally interested in the intertwining via the fractional derivative operator $D^\alpha$, that we call the $\alpha$-intertwining. So the organization of the present paper is as follows:

- in Section 2, we define precisely the operators $D^\alpha$ and $V^\alpha$, respectively of differentiation and of integration of order $\alpha$, for $\alpha \in (0,1]$,
- in Section 3, we prove some equivalent forms of the $\alpha$-intertwining,
- in Section 4, we discuss two classes of examples. The first one is in the framework of branching processes with immigration (Kawazu-Watanabe [8]), and the second one is in the framework of processes obtained in Yor [17] by intertwining (in the sense of (1)) from squared Bessel processes,
- in Section 5, we present as an application of the 1-intertwining, an approach to the problem of the existence of some pseudo-inverses.

# 2 Operators $D^\alpha$ and $V^\alpha$

## 2.1

Let $E$ be the space of continuous functions on $\mathbb{R}_+ = [0, +\infty)$, tending to 0 at infinity, equipped with the norm:

$$\| f \|_E = \sup_{x \in \mathbb{R}_+} |f(x)|.$$

We denote by $E_1$ the space of $C^1$-functions $f$ on $\mathbb{R}_+$ such that $f$ and its derivative $f'$ belong to $E$, equipped with the norm:

$$\| f \|_{E_1} = \| f \|_E + \| f' \|_E.$$

We denote by $D$ the closed operator on $E$ with domain $\mathrm{dom}D = E_1$ and defined by

$$\forall f \in E_1 \quad Df = f'.$$

We define $E_{-1}$ as the set of functions $f \in E$ such that $\lim_{x\to\infty} \int_0^x f(t)\,\mathrm{d}t$ exists (then denoted by $\int_0^\infty f(t)\,\mathrm{d}t$).

We denote by $V$ the closed operator on $E$ with domain $\mathrm{dom}V = E_{-1}$ and defined by

$$\forall f \in E_{-1} \quad Vf(x) = \int_x^\infty f(t)\,\mathrm{d}t.$$

The space $E_{-1}$ is equipped with the norm

$$\| f \|_{E_{-1}} = \| f \|_E + \| Vf \|_E.$$

## 2.2

If $F$ is a space of functions on $\mathbb{R}_+$, $F^+$ (resp. $F^c$) will denote the subspace of $F$ consisting of functions which are nonnegative (resp. with compact support in $\mathbb{R}_+$). Obviously, $E_1$ is dense in $E$, $E_1^c$ is dense in $E_1$, and $E^c$ is dense in $E_{-1}$.

## 2.3

If we consider $D$ as an operator from $E_1$ onto $E_{-1}$ and $V$ as an operator from $E_{-1}$ onto $E_1$, then these operators are isometries satisfying

$$V\,D = -I_{E_1} \quad \text{and} \quad D\,V = -I_{E_{-1}}$$

where $I_{E_1}$ (resp. $I_{E_{-1}}$) denotes the identity operator on $E_1$ (resp. $E_{-1}$).

One also has $D = -V^{-1}$ and $V = -D^{-1}$ as closed (non-everywhere defined) operators on $E$.

## 2.4

Let $0 < \alpha < 1$. We define $D^\alpha$ as the closure in $E$ of the operator defined on $E_1$ by

$$\forall f \in E_1, \forall x \in \mathbb{R}_+, \quad D^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_0^\infty [f(x+t) - f(x)]\, t^{-1-\alpha}\, \mathrm{d}t.$$

In other words, $D^\alpha$ is defined as $-(-D)^\alpha$ in the sense of fractional powers of closed operators (Balakrishnan [1]).

We remark that $D^\alpha(E_1^c) \subset E^c$.

**Lemma 2.1** *For $f \in E_1$,*

$$D^\alpha f(x) = \lim_{a\to\infty} \frac{1}{\Gamma(1-\alpha)} \int_0^a f'(x+t)\, t^{-\alpha}\, \mathrm{d}t$$

*uniformly with respect to $x \in \mathbb{R}_+$. Moreover, $D^\alpha$ is continuous from $E_1$ into $E$.*

**Proof** If $f \in E_1$ and $a > 0$,

$$D^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \left[\alpha \int_a^\infty [f(x+t) - f(x+a)]\, t^{-1-\alpha}\, \mathrm{d}t + \int_0^a f'(x+t)\, t^{-\alpha}\, \mathrm{d}t\right].$$

Hence the uniform limit holds and

$$\| D^\alpha f \|_E \le \frac{1}{\Gamma(1-\alpha)} (2 \| f \|_E + (1-\alpha)^{-1} \| f' \|_E).$$

□

## 2.5

According to general results on fractional powers (Komatsu [9, Theorem 4.4]), one has a precise description of the domain of $D^\alpha$ as the set of $f \in E$ such that

$$\lim_{\epsilon \to 0} \int_\epsilon^\infty [f(x+t) - f(x)]\, t^{-1-\alpha}\, \mathrm{d}t$$

exists uniformly with respect to $x$. Moreover,

$$\forall f \in \mathrm{dom} D^\alpha \quad D^\alpha f(x) = \frac{\alpha}{\Gamma(1-\alpha)} \lim_{\epsilon \to 0} \int_\epsilon^\infty [f(x+t) - f(x)]\, t^{-1-\alpha}\, \mathrm{d}t.$$

## 2.6

Let $0 < \alpha < 1$. We define $V^\alpha$ as the closure in $E$ of the operator defined on $E_{-1}$ by

$$\forall f \in E_{-1}, \forall x \in \mathbb{R}_+, \quad V^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \lim_{a \to \infty} \int_0^a f(x+t)\, t^{-1+\alpha}\, \mathrm{d}t.$$

Therefore, by Lemma 2.1, we have

$$\forall f \in E_{-1} \quad V^\alpha f = -D^{1-\alpha} V f.$$

In other words, $V^\alpha$ is defined as the $\alpha$-power of $V$ in the sense of fractional powers of closed operators (Balakrishnan [1]).

We remark that $V^\alpha(E^c) \subset E^c$. Moreover, by Lemma 2.1, $V^\alpha$ is continuous from $E_{-1}$ into $E$.

## 2.7

One has a precise description of the domain of $V^\alpha$ as the set of $f \in E$ such that

$$\lim_{a \to \infty} \int_0^a f(x+t)\, t^{-1+\alpha}\, \mathrm{d}t$$

exists uniformly with respect to $x$. Moreover,

$$\forall f \in \mathrm{dom} V^\alpha \quad V^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \lim_{a \to \infty} \int_0^a f(x+t)\, t^{-1+\alpha}\, \mathrm{d}t.$$

This result, in a much more general framework, can be found in Hirsch [5, Corollaire du Théorème 4]. We also refer to Hirsch [6] and to the references therein.
We notice that we could use, in what follows, another precise description of the domain, that given in Komatsu [9, Theorem 2.10].

### 2.8

By the general theory of fractional powers (see, for example, Komatsu [10, Theorem 3.2]), for $0 < \alpha < 1$, $D^\alpha = -(V^\alpha)^{-1}$ and $V^\alpha = -(D^\alpha)^{-1}$ as closed operators on $E$. This result plays an important role in what follows. As seen above, this also holds for $\alpha = 1$, setting $D^1 = D$ and $V^1 = V$.

### 2.9

Let $0 < \alpha \leq 1$. We denote by $\widetilde{V^\alpha}$ the kernel defined, for a generic nonnegative Borel function on $\mathbb{R}_+$, $f$, by

$$\forall x \in \mathbb{R}_+ \quad \widetilde{V^\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty f(x+t)\, t^{-1+\alpha}\, \mathrm{d}t \ \leq +\infty.$$

We have the following relation between $V^\alpha$ and $\widetilde{V^\alpha}$.

**Lemma 2.2** *If $f \in E^+$, then $f \in \mathrm{dom} V^\alpha$ if and only if $\widetilde{V^\alpha} f \in E$. In this case, $V^\alpha f = \widetilde{V^\alpha} f$.*

**Proof** This follows easily from the precise description of $\mathrm{dom} V^\alpha$ given in Subsection 2.7, by using Dini's lemma. □

# 3 Intertwining of order $\alpha$

In this section, we fix $0 < \alpha \leq 1$ and we consider two Feller semigroups: $P = (P_t)$ and $Q = (Q_t)$, on $\mathbb{R}_+$. We denote by $\widetilde{P_t}$ and $\widetilde{Q_t}$ the kernels associated with $P_t$ and $Q_t$.
If $A$ and $B$ are two (non-everywhere defined) operators on $E$, we denote by $A \subset B$ the relation:

$$\mathrm{dom}A \subset \mathrm{dom}B \quad \text{and} \quad \forall f \in \mathrm{dom}A,\ Af = Bf,$$

and by $A\,B$ the composition of operators $A$ and $B$, whose domain is:

$$\mathrm{dom}(A\,B) = \{f \in \mathrm{dom}B\ ;\ Bf \in \mathrm{dom}A\}.$$

**Definition 3.1** The pair $(P, Q)$ is said *$\alpha$-intertwined* if, for any $t > 0$,

$$Q_t\, D^\alpha \subset D^\alpha\, P_t,$$

which means that, if $f \in \mathrm{dom}D^\alpha$, then $P_t f \in \mathrm{dom}D^\alpha$ and $D^\alpha\, P_t f = Q_t\, D^\alpha f$.

Property 1, in the following theorem, gives a useful characterization of the $\alpha$-intertwining.

**Theorem 3.2** *The following properties are equivalent:*

1. *For all* $t > 0$, $\widetilde{P_t}\,\widetilde{V^\alpha} = \widetilde{V^\alpha}\,\widetilde{Q_t}$.
2. *For all* $t > 0$, $P_t\,V^\alpha \subset V^\alpha\,Q_t$.
3. $(P, Q)$ *is* $\alpha$*-intertwined.*

**Proof**

**1. ⇒ 2.** Suppose that property 1 holds and let $f \in (E^c)^+$. Then $\widetilde{V^\alpha} f = V^\alpha f$ and

$$\widetilde{V^\alpha}\,Q_t f = P_t\,V^\alpha f \in E.$$

Therefore, by Lemma 2.2,

$$Q_t f \in \mathrm{dom} V^\alpha \quad \text{and} \quad V^\alpha\,Q_t f = P_t\,V^\alpha f. \tag{4}$$

As $E^c$ is dense in $E_{-1}$ and $V^\alpha$ is continuous from $E_{-1}$ into $E$, we see, using the fact that $V^\alpha$ is closed, that (4) is true for $f \in E_{-1}$.
Now, if $f \in \mathrm{dom} V^\alpha$, by the definition of $V^\alpha$, there exists a sequence $(f_n)$ in $E_{-1}$ which converges to $f$ in $E$ and such that $(V^\alpha f_n)$ converges to $V^\alpha f$ in $E$. As $V^\alpha$ is closed, $f$ satisfies (4), and hence, property 2 holds.

**2. ⇒ 1.** Suppose that property 2 holds. Then, by Lemma 2.2,

$$\forall f \in (E^c)^+ \quad \widetilde{P_t}\,\widetilde{V^\alpha} f = \widetilde{V^\alpha}\,\widetilde{Q_t} f$$

and therefore property 1 holds.

**2. ⇒ 3.** Suppose that property 2 holds and let $f \in \mathrm{dom} D^\alpha$. We set $g = D^\alpha f$. As $V^\alpha = -(D^\alpha)^{-1}$, then $g \in \mathrm{dom} V^\alpha$ and $V^\alpha g = -f$. By property 2, $Q_t g \in \mathrm{dom} V^\alpha$ and

$$V^\alpha\,Q_t\,D^\alpha f = V^\alpha\,Q_t g = P_t\,V^\alpha g = -P_t f.$$

Using again $V^\alpha = -(D^\alpha)^{-1}$, we have:

$$P_t f \in \mathrm{dom} D^\alpha \quad \text{and} \quad D^\alpha\,P_t f = Q_t D^\alpha f,$$

and therefore, property 3 holds.

**3. ⇒ 2.** The proof is analogous to the previous one.

□

# 4 Examples

## 4.1 Branching processes with immigration

Recall that a *Bernstein function* is a nonnegative continuous function $H$ on $\mathbb{R}_+$, which is $C^1$ on $(0,+\infty)$, and such that $H'$ is completely monotone.

We consider a Bernstein function $F$ such that $F(0)=0$. In other words, $F$ is the Laplace exponent of a subordinator. We have (Bernstein's theorem):

$$F(x) = a\,x + \int (1-\mathrm{e}^{-tx})\; n(\mathrm{d}t) \tag{5}$$

with $a \geq 0$ and $n$ a $\sigma$- finite, positive measure on $(0,+\infty)$ such that $\displaystyle\int \frac{t}{1+t}\, n(\mathrm{d}t) < +\infty$.

We consider another Bernstein function, $G$, defined by:

$$G(x) = b\,x + \int_0^\infty (1-\mathrm{e}^{-tx})\,\varphi(t)\;\mathrm{d}t \tag{6}$$

with $b \geq 0$ and $\varphi$ a nonnegative decreasing function on $(0,+\infty)$ such that $\displaystyle\int_0^\infty \frac{t}{1+t}\,\varphi(t)\;\mathrm{d}t < +\infty$.

We now set $R(x) = -x\,G(x)$. We then easily have:

$$\begin{aligned} R(x) &= -b\,x^2 + \left(\int_0^\infty \frac{t^3}{1+t^2}\;\mathrm{d}\varphi(t)\right)\,x \\ &\quad + \int_0^\infty \left(\mathrm{e}^{-tx} - 1 + \frac{x\,t}{1+t^2}\right)\;\mathrm{d}\varphi(t) \end{aligned} \tag{7}$$

and $\displaystyle\int_0^\infty \frac{t^2}{1+t}\;(-\mathrm{d}\varphi(t)) < +\infty$.

We then define $\psi_G(t,\lambda)$ and $\varphi_{F,G}(t,\lambda)$ $(t \geq 0, \lambda \geq 0)$ by

$$\frac{\partial}{\partial t}\psi_G = R(\psi_G) \quad \text{and} \quad \psi_G(0,\lambda) = \lambda \tag{8}$$

$$\varphi_{F,G}(t,\lambda) = \exp\left\{-\int_0^t F\left(\psi_G(s,\lambda)\right)\;\mathrm{d}s\right\}. \tag{9}$$

By Theorem 1.1 in Kawazu-Watanabe [8], it follows from (5), (7), (8) and (9), that there exists a Markovian Feller semigroup: $Q^{F,G} = (Q_t^{F,G})$, whose Laplace transform is given by:

$$\forall \lambda > 0,\ \forall x \geq 0, \quad Q_t^{F,G}(e_\lambda)(x) = \varphi_{F,G}(t,\lambda)\;\exp[-x\,\psi_G(t,\lambda)]$$

where $e_\lambda(s) = \mathrm{e}^{-\lambda s}$.

**Theorem 4.1** *For $\alpha \in (0,1]$, the pair $(Q^{F,G}, Q^{F+\alpha G,G})$ is $\alpha$-intertwined.*

**Proof** For every $\lambda > 0$, $V^\alpha(e_\lambda) = \lambda^{-\alpha} e_\lambda$. Therefore,

$$\widetilde{Q_t^{F,G}}\, \widetilde{V^\alpha} e_\lambda = \lambda^{-\alpha}\, \varphi_{F,G}(t,\lambda)\, e_{\psi_G(t,\lambda)}$$

and

$$\widetilde{V^\alpha}\, \widetilde{Q_t^{F+\alpha G,G}} e_\lambda = \varphi_{F+\alpha G,G}(t,\lambda)\, \psi_G^{-\alpha}(t,\lambda)\, e_{\psi_G(t,\lambda)}.$$

But, by (8) and (9),

$$\psi_G(t,\lambda) = \lambda\, \varphi_{G,G}(t,\lambda) \quad \text{and} \quad \varphi_{F+\alpha G,G} = \varphi_{F,G}\, (\varphi_{G,G})^\alpha.$$

Consequently, for every $\lambda > 0$,

$$\widetilde{Q_t^{F,G}}\, \widetilde{V^\alpha} e_\lambda = \widetilde{V^\alpha}\, \widetilde{Q_t^{F+\alpha G,G}} e_\lambda.$$

Thus, by injectivity of the Laplace transform, the property 1 in Theorem 3.2 is satisfied. □

We now can extend the previous theorem to any $\alpha > 0$.

It is not difficult to see, using the explicit definition of $D^\alpha$ (Subsection 2.5) and of $V^\alpha$ (Subsection 2.7), that, if $0 < \alpha \le 1$,

$$D\, D^\alpha = D^\alpha\, D \quad \text{and} \quad V\, V^\alpha = V^\alpha\, V.$$

Therefrom, for $\alpha > 0$, one defines

$$D^\alpha := D^n\, D^{\alpha'} = D^{\alpha'}\, D^n \quad \text{and} \quad V^\alpha := V^n\, V^{\alpha'} = V^{\alpha'}\, V^n$$

with $\alpha = n + \alpha'$, $n \in \mathbb{N}$ and $\alpha' \in (0,1]$. We still define, for any $\alpha > 0$, the $\alpha$-intertwining by Definition 3.1. Then, reasoning by induction, we obtain:

**Corollary 4.2** *For any $\alpha > 0$, the pair $(Q^{F,G}, Q^{F+\alpha G,G})$ is $\alpha$-intertwined, and, for $t > 0$, we also have*

$$Q_t^{F,G}\, V^\alpha \subset V^\alpha\, Q_t^{F+\alpha G,G}.$$

#### 4.1.1 Example

Let $\delta > 0$, $F(x) = \delta\, x$, $G(x) = 2\, x$ (which corresponds, in (5) and (6), to $a = \delta$, $n = 0$, $b = 2$, $\varphi = 0$).

Then

$$\psi_G(t, \lambda) = \frac{\lambda}{1 + 2t\lambda} \quad \varphi_{F,G}(t, \lambda) = (1 + 2t\lambda)^{-\delta/2}$$

and $Q^{F,G} = Q^\delta$, the semigroup of the squared Bessel process of dimension $\delta$. By Corollary 4.2, we have:

$$\forall \delta > 0,\ \forall \alpha > 0, \quad (Q^\delta, Q^{\delta+2\alpha}) \text{ is } \alpha\text{-intertwined.}$$

In particular, $Q_t^{\delta+2}\, D \subset D\, Q_t^\delta$, which is the property (3) in the introduction.

#### 4.1.2 Example (Kawazu-Watanabe [8, Example 1.1])

Let $\delta > 0$, $r > 0$, $0 < \beta < 1$. We set $F(x) = \delta\, x^\beta$ and $G(x) = r\, x^\beta$ (which corresponds, in (5) and (6), to $a = 0$, $b = 0$, $\varphi(t) = \dfrac{r\beta}{\Gamma(1-\beta)}\, t^{-\beta-1}$, $n(\mathrm{d}t) = \dfrac{\delta}{r}\, \varphi(t)\, \mathrm{d}t$).

Then

$$\psi_G(t, \lambda) = \frac{\lambda}{(1 + r\beta\lambda^\beta t)^{1/\beta}} \quad \varphi_{F,G}(t, \lambda) = (1 + r\beta\lambda^\beta t)^{-\frac{\delta}{r\beta}}$$

We denote by $Q^{\delta,r,\beta}$ the associated semigroup. By Corollary 4.2, we have:

$$\forall r, \delta > 0,\ \forall \beta \in (0,1),\ \forall \alpha > 0, \quad (Q^{\delta,r,\beta}, Q^{\delta+r\alpha,r,\beta}) \text{ is } \alpha\text{-intertwined.}$$

We remark that the process associated with $Q^{\delta,r,\beta}$ is a *semi-stable Markov process of order* $\beta^{-1}$ in the sense of Lamperti [11], which means:

$$\forall a > 0,\ \forall t \geq 0, \forall f \in E, \quad Q_{at}^{\delta,r,\beta} f(x) = Q_t^{\delta,r,\beta}(\tau_{a^{\beta^{-1}}} f)(a^{-\beta^{-1}} x)$$

where $\tau_{a^{\beta^{-1}}} f(x) = f(a^{\beta^{-1}} x)$.

## 4.2 Processes obtained by intertwining from squared Bessel processes

#### 4.2.1 Kernel $M_a$

For $a > 0$, we denote by $M_a$ the multiplication kernel by $2\,\gamma_a$, where $\gamma_a$ is an exponential variable of index $a$, that is:

$$\forall x \geq 0 \quad M_a f(x) = \frac{1}{\Gamma(a)} \int_0^\infty f(2xs)\, s^{a-1}\, \mathrm{e}^{-s}\, \mathrm{d}s.$$

We have the following density lemma.

**Lemma 4.3** *If $a > 1$, then the space spanned by $\{M_a e_\lambda \ ; \ \lambda > 0\}$ is dense in $E_{-1}$.*

**Proof** Let $L$ be a continuous linear form on $E_{-1}$. There exist two signed bounded measures on $\mathbb{R}_+$, $\mu$ and $\nu$, such that

$$\forall f \in E_{-1} \quad Lf = \int f \, \mathrm{d}\mu + \int V f \, \mathrm{d}\nu.$$

For any $\lambda > 0$, $M_a e_\lambda(x) = (1 + 2\lambda x)^{-a}$, then, since $a > 1$, $M_a e_\lambda \in E_{-1}$. Denoting by $\mathcal{L}$ the Laplace transform, we have

$$\begin{aligned} L(M_a e_\lambda) &= \frac{1}{\Gamma(a)} \left[ \int_0^\infty \mathcal{L}\mu(2\lambda s)\, s^{a-1}\, \mathrm{e}^{-s} \, \mathrm{d}s + \int_0^\infty \frac{1}{2\lambda s} \mathcal{L}\nu(2\lambda s)\, s^{a-1}\, \mathrm{e}^{-s} \, \mathrm{d}s \right] \\ &= \frac{(2\lambda)^{-a}}{\Gamma(a)} \int_0^\infty [u\, \mathcal{L}\mu(u) + \mathcal{L}\nu(u)]\, u^{a-2}\, \mathrm{e}^{-u/2\lambda} \, \mathrm{d}u. \end{aligned}$$

Suppose now that $L(M_a e_\lambda) = 0$ for every $\lambda > 0$. Then, by the injectivity of the Laplace transform,

$$\forall u > 0 \quad u\, \mathcal{L}\mu(u) + \mathcal{L}\nu(u) = 0.$$

Let $N(x) = \nu([0, x])$. Then, using again the injectivity of the Laplace transform, we obtain

$$\mu(\mathrm{d}x) = -N(x)\, \mathrm{d}x.$$

Consequently, for $f \in E^c$,

$$Lf = -\int_0^\infty f(x)\, N(x) \, \mathrm{d}x + \int_0^\infty V f(x)\, \nu(\mathrm{d}x) = 0.$$

As $E^c$ is dense in $E_{-1}$, we get $L = 0$, and the result follows by the Hahn-Banach theorem. □

In what follows, for $x > 0$, we set:

$$c(x) = 2^x\, \Gamma(x).$$

**Lemma 4.4** *For any $a > 0$ and $\alpha \in (0, 1]$,*

$$c(a)\ M_a\, \widetilde{V^\alpha} = c(a + \alpha)\, \widetilde{V^\alpha}\, M_{a+\alpha}.$$

**Proof** An easy calculation yields, for $\lambda > 0$ and $x \geq 0$,

$$c(a)\ M_a\, \widetilde{V^\alpha} e_\lambda(x) = c(a)\, \lambda^{-\alpha}\, (1 + 2\lambda x)^{-a} = c(a + \alpha)\, \widetilde{V^\alpha}\, M_{a+\alpha} e_\lambda(x),$$

and the result follows by the injectivity of the Laplace transform. □

### 4.2.2 Semigroup $Q^{\delta',\delta}$

Let $\delta \geq 2$ and $0 < \delta' < \delta$. We still denote by $Q^{\delta'} = (Q_t^{\delta'})$ the semigroup of the squared Bessel process of dimension $\delta'$. According to Yor [17] (see also Carmona-Petit-Yor [3]), there exists a Feller semigroup $Q^{\delta',\delta}$ satisfying, for all $t \geq 0$,

$$Q_t^{\delta',\delta} \, M_{\delta/2} = M_{\delta/2} \, Q_t^{\delta'}. \tag{10}$$

**Theorem 4.5** *For every $\alpha > 0$, the pair $(Q^{\delta',\delta}, Q^{\delta'+2\alpha,\delta+2\alpha})$ is $\alpha$-intertwined.*

**Proof** Suppose first $0 < \alpha \leq 1$. By Lemma 4.4, (10) and the example 4.1.1, we get, for any $\lambda > 0$,

$$Q_t^{\delta',\delta} \, V^\alpha \, M_{(\delta/2)+\alpha} e_\lambda = V^\alpha \, Q_t^{\delta'+2\alpha,\delta+2\alpha} \, M_{(\delta/2)+\alpha} e_\lambda.$$

As $(\delta/2)+\alpha$ is $> 1$, by Lemma 4.3, the continuity of $V^\alpha$ on $E_{-1}$ and the fact that $V^\alpha$ is a closed operator on $E$, we obtain

$$\forall f \in E_{-1} \quad Q_t^{\delta',\delta} \, V^\alpha f = V^\alpha \, Q_t^{\delta'+2\alpha,\delta+2\alpha} f,$$

and then, by definition of $V^\alpha$,

$$Q_t^{\delta',\delta} \, V^\alpha \subset V^\alpha \, Q_t^{\delta'+2\alpha,\delta+2\alpha}.$$

Property 2 in Theorem 3.2 is therefore satisfied.

The case $\alpha > 1$ can be obtained by induction as explained at the end of Subsection 4.1. □

### 4.2.3 Kernel $\widehat{M}_a$

For $a > 0$, we denote by $\widehat{M}_a$ the multiplication kernel by $(2\,\gamma_a)^{-1}$, where $\gamma_a$ is as before an exponential variable of index $a$, that is:

$$\forall x \geq 0 \quad \widehat{M}_a f(x) = \frac{1}{\Gamma(a)} \int_0^\infty f\left(\frac{x}{2s}\right) s^{a-1} \, \mathrm{e}^{-s} \, \mathrm{d}s.$$

We have the following uniqueness lemma.

**Lemma 4.6** *Let $a > 0$ and let $f$ and $g$ be nonnegative Borel functions on $\mathbb{R}_+$. Suppose*

$$\forall x \geq 0 \quad \widehat{M}_a f(x) = \widehat{M}_a g(x) < +\infty.$$

*Then $f = g$ almost everywhere on $\mathbb{R}_+$.*

**Proof** We have

$$\widehat{M}_a f(x) = \frac{x^a}{\Gamma(a)} \int_0^\infty f\left(\frac{1}{2u}\right) u^{a-1}\, \mathrm{e}^{-ux}\, \mathrm{d}u.$$

Then the result follows from the injectivity of the Laplace transform. □

We also have the analogue of Lemma 4.4.

**Lemma 4.7** *For any $a > 0$ and $\alpha \in (0, 1]$,*

$$c(a)\, \widetilde{V^\alpha}\, \widehat{M_a} = c(a+\alpha)\, \widehat{M}_{a+\alpha}\, \widetilde{V^\alpha}.$$

**Proof** An easy calculation yields, for $\lambda > 0$ and $x \geq 0$,

$$c(a)\, \widetilde{V^\alpha}\, \widehat{M}_a e_\lambda(x) = 2^{a+\alpha}\lambda^{-\alpha} \int_0^\infty \mathrm{e}^{-\frac{\lambda x}{2t}}\, \mathrm{e}^{-t}\, t^{a+\alpha-1}\, \mathrm{d}t = c(a+\alpha)\, \widehat{M}_{a+\alpha} \widetilde{V^\alpha} e_\lambda(x),$$

and the result follows again from the injectivity of the Laplace transform. □

#### 4.2.4 Semigroup $\widehat{Q}^{\delta',\delta}$

Let $\delta > 2$ and $0 < \delta' < \delta$. We still denote by $Q^{\delta'} = (Q_t^{\delta'})$ the semigroup of the squared Bessel process of dimension $\delta'$. According to Yor [17] (see also Carmona-Petit-Yor [3]), there exists a Feller semigroup $\widehat{Q}^{\delta',\delta}$ satisfying, for all $t \geq 0$,

$$Q_t^{\delta'}\, \widehat{M}_{(\delta-\delta')/2} = \widehat{M}_{(\delta-\delta')/2}\, \widehat{Q}_t^{\delta',\delta}. \tag{11}$$

**Theorem 4.8** *For $\alpha > 0$ such that $\delta' + 2\alpha < \delta$, the pair $(\widehat{Q}^{\delta',\delta}, \widehat{Q}^{\delta'+2\alpha,\delta})$ is $\alpha$-intertwined.*

**Proof** Suppose first $0 < \alpha \leq 1$ and $\delta' + 2\alpha < \delta$. By Lemma 4.7, (11) and the example 4.1.1, we get, for any $f \in (E^c)^+$,

$$\widehat{M}_{(\delta-\delta')/2}\, \widehat{Q}_t^{\delta',\delta}\, \widetilde{V^\alpha} f = \widehat{M}_{(\delta-\delta')/2}\, \widetilde{V^\alpha}\, \widehat{Q}^{\delta'+2\alpha,\delta} f.$$

Using Lemma 4.6, we see that Property 1 in Theorem 3.2 is therefore satisfied.

The case $\alpha > 1$, $\delta' + 2\alpha < \delta$, can be obtained by induction as explained at the end of Subsection 4.1. □

## 5 An application of the 1-intertwining

Let $\beta > 0$. We consider a Markovian Feller semigroup: $P = (P_t)$, on $\mathbb{R}_+$, that we assume to be semi-stable of order $\beta$ (Lamperti [11]):

$$\forall a > 0,\, \forall t \geq 0, \forall f \in E, \quad P_{at} f(x) = P_t(\tau_{a^\beta} f)(a^{-\beta} x)$$

where $\tau_{a^\beta} f(x) = f(a^\beta x)$. We denote by $((X_t), (\mathbb{P}_x))$ the process associated with $P$. Moreover, we assume that there exists a Markovian Feller semigroup: $Q = (Q_t)$, on $\mathbb{R}_+$, which is also semi-stable of order $\beta$, and such that the pair $(P, Q)$ is 1-intertwined.

We set, for $0 \leq x < y$ and $t \geq 0$,

$$F_x^y(t) = \mathbb{P}_x(X_t \geq y).$$

**Theorem 5.1** *Let $0 \leq x < y$. We have:*

*i)* $\lim_{t\to 0} F_x^y(t) = 0$.

*ii)* *If* $\mathbb{P}_0(X_1 = 0) = 0$, *then* $\lim_{t\to\infty} F_x^y(t) = 1$.

*iii)* *If, for any $t > 0$, $P_t$ and $Q_t$ admit densities $p_t$ and $q_t$ which are continuous on $\mathbb{R}_+ \times \mathbb{R}_+^*$, then*

$$\forall t > 0 \quad \frac{\mathrm{d}}{\mathrm{d}t} F_x^y(t) = \frac{\beta}{t}\,[y\, p_t(x,y) - x\, q_t(x,y)].$$

**Proof**

**i)** As $F_x^y$ is a nonnegative u.s.c. function on $\mathbb{R}_+$ and $F_x^y(0) = 0$, clearly property $i)$ holds.

**ii)** Let, for $\varepsilon > 0$, $g_\varepsilon$ be a nonnegative decreasing continuous function which vanishes on $[y + \varepsilon, +\infty)$ and is equal to 1 on $[0, y]$. We have:

$$F_x^y(t) \geq 1 - P_t g_\varepsilon(x) = 1 - P_1[\tau_{t^\beta} g_\varepsilon](t^{-\beta} x).$$

As $g_\varepsilon$ is decreasing, for any $s > 0$,

$$\liminf_{t\to\infty} F_x^y(t) \geq 1 - P_1[\tau_{s^\beta} g_\varepsilon](0).$$

Taking the limit for $s$ tending to infinity, we get

$$\liminf_{t\to\infty} F_x^y(t) \geq 1 - \mathbb{P}_0(X_1 = 0) = 1.$$

**iii)** Let $0 < \varepsilon < y - x$ and $f_\varepsilon$ be a nonnegative decreasing $C^1$-function which vanishes on $[y, +\infty)$ and is equal to 1 on $[0, y - \varepsilon]$. We set

$$F_\varepsilon(t) := 1 - P_t f_\varepsilon(x) = 1 - P_1[\tau_{t^\beta} f_\varepsilon](t^{-\beta} x).$$

Using the 1-intertwining, we get:

$$\begin{aligned} F'_\varepsilon(t) &= \beta\, t^{-1}\, x\, Q_1[\tau_{t^\beta} f'_\varepsilon](t^{-\beta} x) - \beta\, t^{-1}\, P_1[\tau_{t^\beta}(Y f'_\varepsilon)](t^{-\beta} x) \\ &= \frac{\beta}{t}\, [x\, Q_t(f'_\varepsilon)(x) - P_t(Y\, f'_\varepsilon)(x)] \end{aligned}$$

where $Y$ denotes the identity function: $Y(x) = x$. Property *iii)* follows, taking the limit for $\varepsilon$ tending to 0.

□

The following corollary is an approach to the problem of existence of pseudo-inverses.

**Corollary 5.2** *Suppose that the hypotheses in iii) of Theorem 5.1 hold and, moreover,*

$$\exists m \in [0, 1],\ \forall (u, v) \in \mathbb{R}_+ \times \mathbb{R}_+^*, \quad v^m\, p_t(u, v) - u^m\, q_t(u, v) \geq 0. \tag{12}$$

*Then, for $0 \leq x < y$, $F_x^y$ is the distribution function of a random variable $Y_x^y$ on $\mathbb{R}_+$ whose density is:*

$$t \in \mathbb{R}_+ \longrightarrow \frac{\beta}{t}\, [y\, p_t(x, y) - x\, q_t(x, y)].$$

[ $(Y_x^y)_{y>x}$ is a pseudo-inverse of the process $X$ in the sense of Madan-Roynette-Yor[12] and Roynette-Yor [15].]

**Proof** Under the condition (12), we have, for $0 \leq x < y$,

$$y\, p_t(x, y) \geq y^{1-m}\, x^m\, q_t(x, y) \geq x\, q_t(x, y),$$

which proves, by Theorem 5.1, that $F_x^y$ is an increasing function and, therefore, a distribution function. □

**Example** Suppose $(P, Q) = (Q^\delta, Q^{\delta+2})$ as in Example 4.1.1. We also assume $\delta \geq 1$. Then the hypotheses stated at the beginning of this section are fulfilled with $\beta = 1$. The hypotheses of Corollary 5.2 are also satisfied with $m = 1/2$. This is equivalent to the inequality $I_{\nu+1} \leq I_\nu$ on $\mathbb{R}_+$, with $\nu = (\delta/2) - 1 \geq -1/2$ and $I_{\nu+1}$ (resp.$I_\nu$ ) denoting as usual the modified Bessel function of index $\nu + 1$ (resp. $\nu$). This inequality is well-known and can be proved by different methods. We have for example, for $\delta > 1$,

$$\frac{I_{\nu+1}}{I_\nu}\left(\frac{\sqrt{xy}}{t}\right) = \mathbb{E}_x\left(\exp\left[-\frac{\delta-1}{2}\int_0^t \frac{1}{X_s}\,\mathrm{d}s\right] \mid X_t = y\right)$$

where $X$ is the squared Bessel process of dimension $\delta$ (see Yor [16], and also Hirsch-Song [7]).

The above example is studied in detail in Roynette-Yor [15].

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