

Pricing of contingent claims in a two-dimensional model with random dividends*

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We study a model of a financial market, in which two risky assets are paying dividends with rates, changing from one fixed value to another when some credit event occurs. The credit events are associated with the first times at which the asset values fall below some given constant levels. The behavior of the asset values is described by exponential diffusion processes with random drift rates and independent driving Brownian motions. We obtain closed form expressions for the rational prices of certain European and barrier-type contingent claims whose structure is similar to the first- and the second-to-default options in credit risk theory.

1 Introduction

In the present paper, we study a first passage time model for two dividend paying assets, whose dividend rates are random and change from one fixed value to another, during the allowed infinite time horizon. The times of change of the dividend rates are assumed to be the

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first times at which the asset values reach some given lower constant barriers. Such a model corresponds to a situation in which the fall of one of the asset values leads to the changes of dividend rates not of the same asset only, but of the other ones as well. For instance, this may happen in a model with one firm, having several branches, in which a change in the dividend rate of one of the branches makes an influence on the dividend policy not of the same branch only, but of the other ones as well. The obtained structure of dependent credit events can be also considered as a contribution to the wide range of the first passage time models with dependent defaults (see, e.g. Zhou [14], Giesecke [4], Overbeck and Schmidt [9], Valužis [13] and also Bielecki and Rutkowski [1; Chapter X] or Schönbucher [12; Chapter X] for further references). Note that some other models with random dividends were earlier considered in the literature (see, e.g. Geske [3]).

The purpose of the present paper is to derive the rational prices of certain European and barrier-type contingent claims whose structure is similar to the so-called first- and second-to-default options. The models are considered in which the information is generated by both asset values and by the value of one of the assets only. The risk-neutral dynamics of the asset values is modeled by geometric Brownian motions with random drift rates, changing their values from one fixed to another at the first times when the value processes fall to some constant levels, on the infinite time interval. For simplicity of exposition, we restrict our consideration to a two-dimensional case and assume that the driving Brownian motions are independent. The rational prices of the claims are expressed through the transition density of the joint marginal distribution of a linearly drifted Brownian motion and its running minimum, and the density of its first passage time on a constant level. The consideration of dependent driving Brownian motions would lead to more complicated and less explicit formulas (see, e.g. Iyengar [6], He et al. [5] or Patras [10]). The results of the paper can be naturally extended to the case of a model with several underlying risky assets whose value processes are driven by independent Brownian motions with random drift rates.

The paper is organized as follows. In Section 2, we introduce the two-dimensional structural model, described above. We also recall the expressions for the joint transition density of a linearly drifted Brownian motion and its running minimum, and for the density of the first time at which it hits a constant level. These explicit expressions are used for the derivation of subsequent formulas. In Section 3, we derive expressions for the rational prices of contingent claims, having the structure similar to the first- and the second-to-default options, with respect to the filtration, generated by both asset value processes (full information). In Section 4, we

present expressions for the same contingent claims with respect to the filtration, generated by one of the processes only (partial information). In Section 5, we consider some particular cases of the first-to-default options under full information, in which the expressions of rational prices can be simplified, and thus, they become more appropriate for Monte Carlo simulations. The main results of the paper are stated in Theorems 3.1, 3.2, 4.1 and 4.2.

2 The model

In this section, we introduce a model with two underlying risky asset processes, paying random dividends. We solve the problem of pricing of derivatives such as European claims, or products, having a structure similar to the first- and the second-to-default options, under full and partial information.

2.1 The dynamics of asset prices

For a precise formulation of the problem, let us consider a probability space (Ω, \mathcal{G}, P) with two *independent* standard Brownian motions $W^i = (W_t^i)_{t \geq 0}$, $i = 1, 2$. Suppose that there exist two processes $X^i = (X_t^i)_{t \geq 0}$, $i = 1, 2$, given by:

$$X_t^i = x_i \exp \left(\left(r - \frac{\sigma_i^2}{2} - \delta_{i,0} \right) t - (\delta_{i,1} - \delta_{i,0}) (t - \tau_1)^+ - (\delta_{i,2} - \delta_{i,0}) (t - \tau_2)^+ + \sigma_i W_t^i \right) \quad (2.1)$$

where $(t - \tau_i)^+ = \max\{t - \tau_i, 0\}$, $r \geq 0$, and σ_i , $\delta_{i,\ell}$, x_i are some given strictly positive constants for every $i = 1, 2$ and $\ell = 0, 1, 2$. The processes X^i , $i = 1, 2$, describe the risk-neutral dynamics of the values of some assets paying dividends, and τ_i , $i = 1, 2$, are random times to be specified below, at which some credit events occur that lead to the changes of dividend rates. In more details, for every $i = 1, 2$ fixed, the asset number i pays dividends at the rate $\delta_{i,0}$ until the time $\tau_1 \wedge \tau_2$ at which the first credit event occurs and the dividend rate is changed to $\delta_{i,\ell}$, where $\ell = 1$ if $\tau_1 \wedge \tau_2 = \tau_1$, and $\ell = 2$ if $\tau_1 \wedge \tau_2 = \tau_2$. Then, the asset i pays dividends with the rate $\delta_{i,\ell}$ until the time $\tau_1 \vee \tau_2$ at which the second credit event occurs and the dividend rate is changed to $\delta_{i,3} = \delta_{i,1} + \delta_{i,2} - \delta_{i,0}$. After both credit events occur, the asset i pays dividends with the rate $\delta_{i,3}$. Here r is the interest rate of a riskless banking account, and σ_i is the volatility coefficient.

Following the structural approach, let us define the random time τ_i by:

$$\tau_i = \inf\{t \geq 0 \mid X_t^i \leq b_i\} \quad (2.2)$$

where $b_i > 0$ is a given constant. By construction, τ_i is a stopping time with respect to the natural filtration $\mathcal{G}_t = \sigma(X_s^1, X_s^2 | 0 \leq s \leq t)$, generated by X^i , $i = 1, 2$. Then, the existence of such a pair of processes (X^1, X^2) can be easily deduced from the classical diffusion model with constant dividend rates, by means of standard change-of-measure arguments.

2.2 The payoffs of contingent claims

The purpose of the present paper is to determine the rational (ex-dividend) prices of two contingent claims, having the following payoff structure. Note that we have $3 - i = 2$ for $i = 1$, and we have $3 - i = 1$ for $i = 2$, obviously. In the first claim, the amount $C(X_T^1, X_T^2)$ is paid at the maturity T if and only if no credit event occurs before the maturity. Additionally, the amount $D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i})$ is paid at the time τ_i if the time of the first credit event is τ_i that occurs before the maturity, for every $i = 1, 2$. In the second claim, the amount $C_i(X_T^i, X_T^{3-i})$ is paid at the maturity T if and only if two credit events occur before the maturity, and the time of the second credit event is τ_i . Additionally, the amount $D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i})$ is paid at the time τ_i if the time of the second credit event is τ_i that occurs before the maturity, for every $i = 1, 2$. Without loss of generality, we further assume that the payoffs are already discounted by the banking account, which is equivalent to letting r equal to zero. We shall also extend our study to the case of European contingent claims with payoffs $C(X_T^1, X_T^2)$.

Note that the contingent claims described above have the structure similar to the *first-* and the *second-to-default* options in credit risk theory. The rational price processes V_t^i , $i = 1, 2$, of such claims are given by:

$$V_t^1 = E \left[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) + \sum_{i=1}^2 D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) \middle| \mathcal{G}_t \right] \quad (2.3)$$

and

$$V_t^2 = E \left[\sum_{i=1}^2 (D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq T) \middle| \mathcal{G}_t \right] \quad (2.4)$$

for any $0 \leq t \leq T$, respectively (see, e.g. [1; Chapter X] or [12; Chapter X]). Here, the expectations are taken with respect to the martingale measure, and $I(\cdot)$ denotes the indicator function. We further assume that $C(x_1, x_2)$, $C_i(x_i, x_{3-i})$ and $D_{k,i}(t, x_i, x_{3-i})$, $k, i = 1, 2$, are nonnegative functions such that the integrals appearing below are well defined. For example, for a *barrier-type basket call* or *put* option with the strike price $K > 0$, we may set $C(x_1, x_2)$ and $C_i(x_i, x_{3-i})$ being equal to $(\alpha_i x_i + \alpha_{3-i} x_{3-i} - K)^+$ or $(K - \alpha_i x_i - \alpha_{3-i} x_{3-i})^+$ with some

$\alpha_i \geq 0$ for every $i = 1, 2$, respectively. We may also assume the recovery to be linear, by setting $D_{k,i}(t, x_i, x_{3-i}) = \gamma + \beta_i x_i + \beta_{3-i} x_{3-i}$ with some $\gamma \geq 0$ and $\beta_i \geq 0$, for every $k, i = 1, 2$.

Moreover, we shall also determine the rational prices of the contingent claims under the assumption that the information available on the market is generated by one of the assets only. In that case, the rational price processes $V_t^{i,j}$, $i = 1, 2$, of the claims are given by:

$$V_t^{1,j} = E \left[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) + \sum_{i=1}^2 D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) \middle| \mathcal{G}_t^j \right] \quad (2.5)$$

and

$$V_t^{2,j} = E \left[\sum_{i=1}^2 (D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq T) \middle| \mathcal{G}_t^j \right] \quad (2.6)$$

for any $0 \leq t \leq T$, respectively. Here $\mathcal{G}_t^j = \sigma(X_s^j | 0 \leq s \leq t)$ is the natural filtration of the process X^j for every $j = 1, 2$.

2.3 The minimum process

For the process X^i , let us introduce the corresponding *running minimum* process $M^i = (M_t^i)_{t \geq 0}$ given by:

$$M_t^i = \min_{0 \leq s \leq t} X_s^i \wedge m_i \quad (2.7)$$

for any $x_i \geq m_i > b_i > 0$ fixed. Then, it is seen from the structure of (2.2) that the default time τ_i takes the form:

$$\tau_i = \inf\{t \geq 0 \mid M_t^i \leq b_i\} \quad (2.8)$$

and, in order to obtain the initial values of the expressions in (2.3)-(2.4) and (2.5)-(2.6), we shall put $x_i = m_i$ for every $i = 1, 2$.

It thus follows from (2.8) that the event $\{\tau_i > t\}$ can be expressed as $\{M_t^i > b_i\}$ for any $t \geq 0$, so that, the process X^i admits the representation:

$$dX_t^i = -X_t^i (\delta_{i,0} + (\delta_{i,1} - \delta_{i,0}) I(M_t^1 \leq b_1) + (\delta_{i,2} - \delta_{i,0}) I(M_t^2 \leq b_2)) dt + X_t^i \sigma_i dW_t^i \quad (2.9)$$

with $X_0^i = x_i$. Therefore, we may conclude that $(X^1, M^1, X^2, M^2) = (X_t^1, M_t^1, X_t^2, M_t^2)_{t \geq 0}$ is a (time-homogeneous) strong Markov process with respect to the filtration $(\mathcal{G}_t)_{t \geq 0}$. In the sequel, we also use the notation $\tau_i = \tau_i(x_1, m_1, x_2, m_2)$ for $x_i \geq m_i > b_i > 0$ and every $i = 1, 2$.

Note that, by means of standard arguments of filtering theory (see, e.g. [8; Chapter IX]), it is shown that the process X^i admits the following representation in its own filtration:

$$dX_t^i = -X_t^i (\delta_{i,0} + (\delta_{i,i} - \delta_{i,0}) I(M_t^i \leq b_i) + (\delta_{i,3-i} - \delta_{i,0}) P[M_t^{3-i} \leq b_{3-i} | \mathcal{G}_t^i]) dt + X_t^i \sigma_i d\bar{W}_t^i \quad (2.10)$$

with $X_0^i = x_i$, where the innovation process $\bar{W}^i = (\bar{W}_t^i)_{t \geq 0}$ defined by:

$$\bar{W}_t^i = W_t^i - \frac{\delta_{i,3-i} - \delta_{i,0}}{\sigma_i} \int_0^t (I(M_s^{3-i} \leq b_{3-i}) - P[M_s^{3-i} \leq b_{3-i} | \mathcal{G}_s^i]) ds \quad (2.11)$$

is a standard Brownian motion with respect to the filtration $(\mathcal{G}_t^i)_{t \geq 0}$, according to P. Lévy's characterization theorem (see, e.g. [11; Chapter IV, Theorem 3.6]).

2.4 Some formulas

Let us introduce the process $X^{i,j} = (X_t^{i,j})_{t \geq 0}$ defined by:

$$X_t^{i,j} = x_i \exp \left(- \left(\delta_{i,j} + \frac{\sigma_i^2}{2} \right) t + \sigma_i W_t^i \right) \quad (2.12)$$

and its running minimum process $M^{i,j} = (M_t^{i,j})_{t \geq 0}$ given by:

$$M_t^{i,j} = \min_{0 \leq s \leq t} X_s^{i,j} \wedge m_i \quad (2.13)$$

for any $x_i \geq m_i > b_i > 0$ and every $i = 1, 2$ and $j = 0, 1, 2, 3$. Observe that it is seen from (2.1) and (2.12) that $X_t^i = X_t^{i,0}$ holds for all $0 \leq t \leq \tau_1 \wedge \tau_2$, since we have put $r = 0$. It is known (see, e.g. [11; Chapter III, Section 3], [7; Appendix E] or [2; Part II, Section 2]) that the transition density $g_{i,j}$ of the Markov process $(X^{i,j}, M^{i,j})$ defined by:

$$P_{x_i, m_i}[X_t^{i,j} \in dz, M_t^{i,j} \in dy] = g_{i,j}(x_i, m_i; t, z, y) dz dy \quad (2.14)$$

admits the representation:

$$g_{i,j}(x_i, m_i; t, z, y) = \frac{2}{\sigma_i^3 \sqrt{2\pi t^3}} \frac{\ln(y^2/(x_i z))}{zy} \exp \left(- \frac{\ln^2(y^2/(x_i z))}{2\sigma_i^2 t} + \frac{\rho_{i,j}}{\sigma_i} \ln(z/x_i) - \frac{\rho_{i,j}^2 t}{2} \right) \quad (2.15)$$

for all $t > 0$ and $z \geq y$ with $m_i \geq y > 0$, and equals zero otherwise. Here, P_{x_i, m_i} denotes the probability under the assumption that $(X^{i,j}, M^{i,j})$ starts at (x_i, m_i) , and we set $\rho_{i,j} = -\delta_{i,j}/\sigma_i - \sigma_i/2$.

Let us also define the corresponding hitting time $\tau_{i,j}$ of the form:

$$\tau_{i,j} = \inf\{t \geq 0 \mid X_t^{i,j} \leq b_i\} \quad (2.16)$$

for every $i = 1, 2$ and $j = 0, 1, 2$. It is known that the density $h_{i,j}$ of $\tau_{i,j}$ defined by:

$$P_{x_i, m_i}[\tau_{i,j} \in dt] = h_{i,j}(x_i; t) dt \quad (2.17)$$

admits the representation:

$$h_{i,j}(x_i; t) = \frac{\ln(x_i/b_i)}{\sigma_i \sqrt{2\pi t^3}} \exp \left(- \frac{(\ln(x_i/b_i) + \rho_{i,j} \sigma_i t)^2}{2\sigma_i^2 t} \right) \quad (2.18)$$

for all $t > 0$ and $x_i \geq m_i > b_i > 0$.

3 The case of full information

In this section, we compute the conditional expectations in (2.3) and (2.4).

3.1 The first-to-default

3.1.1. Let us begin by computing the terms for the *first-to-default* in (2.3). For this, applying the Markov property of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) E_{x_1, m_1, x_2, m_2} [C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [C(X_{T'}^1, X_{T'}^2) I(T' < \tau'_1 \wedge \tau'_2)]
\end{aligned} \tag{3.1}$$

where we set $T' = T - t$ and $\tau'_i = \tau_i(X_t^1, M_t^1, X_t^2, M_t^2) = \tau_i(x_1, m_1, x_2, m_2) - t$ for each $0 \leq t \leq T$. Here, E_{x_1, m_1, x_2, m_2} denotes the expectation under the assumption that the process (X^1, M^1, X^2, M^2) starts at (x_1, m_1, x_2, m_2) with some $x_i \geq m_i > b_i > 0$ for every $i = 1, 2$. Then, using the fact that the event $\{\tau_i > t\}$ can be represented in the form $\{M_t^i > b_i\}$, we have:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C(X_{T'}^1, X_{T'}^2) I(T' < \tau'_1 \wedge \tau'_2)] \\
&= E_{x_1, m_1, x_2, m_2} [C(X_{T'}^1, X_{T'}^2) I(M_{T'}^1 > b_1, M_{T'}^2 > b_2)] \\
&= E_{x_1, m_1, x_2, m_2} [C(X_{T'}^{1,0}, X_{T'}^{2,0}) I(M_{T'}^{1,0} > b_1, M_{T'}^{2,0} > b_2)]
\end{aligned} \tag{3.2}$$

where $\tau'_i = \tau_i(x_1, m_1, x_2, m_2)$ and the processes $(X^{i,0}, M^{i,0})$, $i = 1, 2$, are defined in (2.12)-(2.13) above. Hence, we obtain from (3.1) and (3.2) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) \int_{b_1}^{\infty} \int_{b_1}^{\infty} \int_{b_2}^{\infty} \int_{b_2}^{\infty} C(x'_1, x'_2) \prod_{\ell=1}^2 g_{\ell,0}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell
\end{aligned} \tag{3.3}$$

where the functions $g_{i,0}$, $i = 1, 2$, are given in (2.15) above.

3.1.2. In a similar way, using again the Markov property of the process (X^1, M^1, X^2, M^2) ,

we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_i \leq t, \tau_i < \tau_{3-i}) \\
&\quad + E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= D_{1,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_i \leq t, \tau_i < \tau_{3-i}) \\
&\quad + I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T', \tau'_i < \tau'_{3-i})]
\end{aligned} \tag{3.4}$$

for $0 \leq t \leq T$ and

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T', \tau'_i < \tau'_{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T', M_{\tau'_i}^{3-i} > b_{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [D_{1,i}(t + \tau_{i,0}, b_i, X_{\tau_{i,0}}^{3-i,0}) I(\tau_{i,0} \leq T', M_{\tau_{i,0}}^{3-i,0} > b_{3-i})]
\end{aligned} \tag{3.5}$$

for $x_i \geq m_i > b_i > 0$, where the processes $(X^{3-i,0}, M^{3-i,0})$ as well as the hitting times $\tau_{i,0}$, $i = 1, 2$, are defined in (2.12)-(2.13) and (2.16) above. Therefore, taking into account the independence of $\tau_{i,0}$ and $(X^{3-i,0}, M^{3-i,0})$, we conclude from (3.4) and (3.5) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) \int_0^{T-t} \int_{b_{3-i}}^\infty \int_{b_{3-i}}^\infty D_{1,i}(t + u, b_i, x'_{3-i}) h_{i,0}(X_t^i; u) \\
&\quad \times g_{3-i,0}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.6}$$

where the functions $g_{i,0}$ and $h_{i,0}$, $i = 1, 2$, are given in (2.15) and (2.18).

Summarizing the facts proved above, let us now formulate the following assertion.

Theorem 3.1. *The rational price of the first-to-default option in (2.3) under full information is given by the sum of the terms in (3.3) and (3.6).*

3.2 The second-to-default

3.2.1. Let us continue with computing the terms for the *second-to-default* in (2.4). For this, applying the Markov property of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, X_{\tau_i}^i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \\
&= E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} < \tau_i \leq t) | \mathcal{G}_t] \\
&\quad + E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(\tau_{3-i} \leq t < \tau_i \leq T) | \mathcal{G}_t] \\
&\quad + E_{x_1, m_1, x_2, m_2} [(D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + C_i(X_T^i, X_T^{3-i})) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \\
&= I(\tau_{3-i} < \tau_i \leq t) (D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) + E_{X_t^1, M_t^1, X_t^2, M_t^2} [C_i(X_{T'}^i, X_{T'}^{3-i})]) \\
&\quad + I(\tau_{3-i} \leq t < \tau_i) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) + C_i(X_{T'}^i, X_{T'}^{3-i})) I(\tau'_i \leq T')] \\
&\quad + I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) + C_i(X_{T'}^i, X_{T'}^{3-i})) I(\tau'_{3-i} < \tau'_i \leq T')]
\end{aligned} \tag{3.7}$$

for all $0 \leq t \leq T$, and we then continue with computing each of the terms separately.

3.2.2. Firstly, we see that:

$$E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i})] = E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^{i,3}, X_{T'}^{3-i,3})] \tag{3.8}$$

for $x_i \geq m_i$ with $b_i \geq m_i > 0$, where the processes $(X^{i,3}, M^{i,3})$, $i = 1, 2$, are defined in (2.12)-(2.13) above. Then, using the independence of $(X^{i,3}, M^{i,3})$ and $(X^{3-i,3}, M^{3-i,3})$, we have:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_T^i, X_T^{3-i}) I(\tau_{3-i} < \tau_i \leq t) | \mathcal{G}_t] \\
&= I(\tau_{3-i} < \tau_i \leq t) \int_0^\infty \int_0^{b_i} \int_0^\infty \int_0^{b_{3-i}} C_i(x'_i, x'_{3-i}) \prod_{\ell=i}^{3-i} g_{\ell,3}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell
\end{aligned} \tag{3.9}$$

where the functions $g_{i,3}$, $i = 1, 2$, are given in (2.15) above.

3.2.3. Secondly, we observe that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau_{i,3-i}, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}) I(\tau_{i,3-i} \leq T')]
\end{aligned} \tag{3.10}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, where the processes $(X^{3-i,3-i}, M^{3-i,3-i})$ as well as the hitting times $\tau_{i,3-i}$, $i = 1, 2$, are defined in (2.12)-(2.13) and (2.16) above. Then, taking into account the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$,

we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau_i, b_i, X_{\tau_i}^{3-i}) I(\tau_{3-i} \leq t < \tau_i \leq T) | \mathcal{G}_t] \\
&= I(\tau_{3-i} \leq t < \tau_i) \int_0^{T-t} \int_0^\infty \int_0^{b_{3-i}} D_{2,i}(t + u, b_i, x'_{3-i}) h_{i,3-i}(X_t^i; u) \\
&\quad \times g_{3-i,3-i}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.11}$$

where the functions $g_{3-i,3-i}$ and $h_{i,3-i}$, $i = 1, 2$, are given in (2.15) and (2.18) above.

3.2.4. Thirdly, applying the strong Markov property of (X^1, M^1, X^2, M^2) , we have:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{C}_i(X_{\tau'_i}^i, M_{\tau'_i}^i, X_{\tau'_i}^{3-i}, M_{\tau'_i}^{3-i}; T' - \tau'_i) I(\tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{C}_i(b_i, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}, M_{\tau_{i,3-i}}^{3-i,3-i}; T' - \tau_{i,3-i}) I(\tau_{i,3-i} \leq T')]
\end{aligned} \tag{3.12}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, where the functions \widehat{C}_i , $i = 1, 2$, are defined by:

$$\begin{aligned}
\widehat{C}_i(x_i, m_i, x_{3-i}, m_{3-i}; T' - u) &= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'-u}^i, X_{T'-u}^{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'-u}^{i,3}, X_{T'-u}^{3-i,3})]
\end{aligned} \tag{3.13}$$

for $x_\ell \geq m_\ell$ with $b_\ell \geq m_\ell > 0$, $\ell = 1, 2$, and any $0 \leq u \leq T'$ fixed. Thus, using the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, we obtain from (3.12) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_T^i, X_T^{3-i}) I(\tau_{3-i} \leq t < \tau_i \leq T)] \\
&= I(\tau_{3-i} \leq t < \tau_i) \int_0^{T-t} \int_0^\infty \int_0^{b_{3-i}} \widehat{C}_i(b_i, b_i, x'_{3-i}, m'_{3-i}; T - t - u) h_{i,3-i}(X_t^i; u) \\
&\quad \times g_{3-i,3-i}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i}
\end{aligned} \tag{3.14}$$

where, by virtue of the independence of $(X^{i,3}, M^{i,3})$ and $(X^{3-i,3}, M^{3-i,3})$, it follows from (3.13) that:

$$\begin{aligned}
& \widehat{C}_i(x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - u) \\
&= \int_0^\infty \int_0^{b_i} \int_0^\infty \int_0^{b_{3-i}} C_i(x''_i, x''_{3-i}) \prod_{\ell=i}^{3-i} g_{\ell,3}(x'_\ell, m'_\ell; T - t - u, x''_\ell, m''_\ell) dx''_\ell dm''_\ell
\end{aligned} \tag{3.15}$$

and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 1, 2, 3$, are given in (2.15) and (2.18) above.

3.2.5. Now, applying the strong Markov property of (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_{3-i} < \tau'_i \leq T')] \tag{3.16} \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(M_{\tau'_{3-i}}^i > b_i, \tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{D}_{2,i}(t + \tau'_{3-i}, X_{\tau'_{3-i}}^i, M_{\tau'_{3-i}}^i, X_{\tau'_{3-i}}^{3-i}, M_{\tau'_{3-i}}^{3-i}; T' - \tau'_{3-i}) I(M_{\tau'_{3-i}}^i > b_i, \tau'_{3-i} \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widehat{D}_{2,i}(t + \tau_{3-i,0}, X_{\tau_{3-i,0}}^{i,0}, M_{\tau_{3-i,0}}^{i,0}, b_{3-i}, b_{3-i}; T' - \tau_{3-i,0}) I(M_{\tau_{3-i,0}}^{i,0} > b_i, \tau_{3-i,0} \leq T')]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$, where the functions $\widehat{D}_{2,i}$, $i = 1, 2$, are defined by:

$$\begin{aligned}
& \widehat{D}_{2,i}(t + v, x_i, m_i, x_{3-i}, m_{3-i}; T' - v) \tag{3.17} \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + v + \tau'_i, b_i, X_{\tau'_i}^{3-i}) I(\tau'_i \leq T' - v)] \\
&= E_{x_1, m_1, x_2, m_2} [D_{2,i}(t + v + \tau_{i,3-i}, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}) I(\tau_{i,3-i} \leq T' - v)]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, and any $0 \leq v \leq T'$ fixed. Hence, using the independence of $\tau_{3-i,0}$ and $(X^{i,0}, M^{i,0})$, we obtain from (3.16) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{2,i}(\tau_i, b_i, X_{\tau_i}^{3-i}) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \tag{3.18} \\
&= I(t < \tau_1 \wedge \tau_2) \int_0^{T-t} \int_0^\infty \int_{b_i}^\infty \widehat{D}_{2,i}(t + v, x'_i, m'_i, b_{3-i}, b_{3-i}; T - t - v) h_{3-i,0}(X_t^{3-i}; v) \\
&\quad \times g_{i,0}(X_t^i, M_t^i; v, x'_i, m'_i) dv dx'_i dm'_i
\end{aligned}$$

where, by virtue of the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, it follows from (3.17) that:

$$\begin{aligned}
& \widehat{D}_{2,i}(t + v, x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - v) \tag{3.19} \\
&= \int_0^{T-t-v} \int_0^\infty \int_0^{b_{3-i}} D_{2,i}(t + v + u, b_i, x''_{3-i}) h_{i,3-i}(x'_i; u) \\
&\quad \times g_{3-i,3-i}(x'_{3-i}, m'_{3-i}; u, x''_{3-i}, m''_{3-i}) du dx''_{3-i} dm''_{3-i}
\end{aligned}$$

and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 0, 1, 2$, are given in (2.15) and (2.18) above.

3.2.6. Finally, we see that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(\tau'_{3-i} < \tau'_i \leq T')] \tag{3.20} \\
&= E_{x_1, m_1, x_2, m_2} [C_i(X_{T'}^i, X_{T'}^{3-i}) I(M_{\tau'_{3-i}}^i > b_i, \tau'_i \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widetilde{C}_i(X_{\tau'_{3-i}}^i, M_{\tau'_{3-i}}^i, X_{\tau'_{3-i}}^{3-i}, M_{\tau'_{3-i}}^{3-i}; T' - \tau'_{3-i}) I(M_{\tau'_{3-i}}^i > b_i, \tau'_{3-i} \leq T')] \\
&= E_{x_1, m_1, x_2, m_2} [\widetilde{C}_i(X_{\tau_{3-i,0}}^i, M_{\tau_{3-i,0}}^i, b_{3-i}, b_{3-i}; T' - \tau_{3-i,0}) I(M_{\tau_{3-i,0}}^i > b_i, \tau_{3-i,0} \leq T')]
\end{aligned}$$

for $x_i \geq m_i > b_i > 0$, where the functions \tilde{C}_i , $i = 1, 2$, are defined by:

$$\begin{aligned} & \tilde{C}_i(x_i, m_i, x_{3-i}, m_{3-i}; T' - v) \\ &= E_{x_1, m_1, x_2, m_2}[\hat{C}_i(X_{\tau'_i}^i, M_{\tau'_i}^i, X_{\tau'_i}^{3-i}, M_{\tau'_i}^{3-i}; T' - v - \tau'_i) I(\tau'_i \leq T' - v)] \\ &= E_{x_1, m_1, x_2, m_2}[\hat{C}_i(b_i, b_i, X_{\tau_{i,3-i}}^{3-i,3-i}, M_{\tau_{i,3-i}}^{3-i,3-i}; T' - v - \tau_{i,3-i}) I(\tau_{i,3-i} \leq T' - v)] \end{aligned} \quad (3.21)$$

for $x_i \geq m_i > b_i > 0$ and $x_{3-i} \geq m_{3-i}$ with $b_{3-i} \geq m_{3-i} > 0$, and any $0 \leq v \leq T'$ fixed, where the functions \hat{C}_i , $i = 1, 2$, are defined in (3.13) above. Hence, using again the independence of $\tau_{3-i,0}$ and $(X^{i,0}, M^{i,0})$, we obtain from (3.20) that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2}[C_i(X_T^i, X_T^{3-i}) I(t < \tau_{3-i} < \tau_i \leq T) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) \int_0^{T-t} \int_{b_i}^\infty \int_{b_i}^\infty \tilde{C}_i(x'_i, m'_i, b_{3-i}, b_{3-i}; T - t - v) h_{3-i,0}(X_t^{3-i}; v) \\ & \quad \times g_{i,0}(X_t^i, M_t^i; v, x'_i, m'_i) dv dx'_i dm'_i \end{aligned} \quad (3.22)$$

where, by virtue of the independence of $\tau_{i,3-i}$ and $(X^{3-i,3-i}, M^{3-i,3-i})$, it follows from (3.21) that:

$$\begin{aligned} & \tilde{C}_i(x'_i, m'_i, x'_{3-i}, m'_{3-i}; T - t - v) \\ &= \int_0^{T-t-v} \int_0^\infty \int_0^{b_{3-i}} \hat{C}_i(b_i, b_i, x''_{3-i}, m''_{3-i}; T - t - v - u) h_{i,3-i}(x'_i; u) \\ & \quad \times g_{3-i,3-i}(x'_{3-i}, m'_{3-i}; u, x''_{3-i}, m''_{3-i}) du dx''_{3-i} dm''_{3-i} \end{aligned} \quad (3.23)$$

the functions \hat{C}_i admit the representation (3.15), and the functions $g_{i,\ell}$ and $h_{i,\ell}$, $i = 1, 2$, $\ell = 0, 1, 2$, are given in (2.15) and (2.18).

Therefore, summarizing the facts proved above, we are now ready to formulate the following assertion.

Theorem 3.2. *The rational price of the second-to-default option in (2.4) under full information is given by the sum of the terms in (3.9), (3.11), (3.14), (3.18) and (3.22).*

4 The case of partial information

In this section, we compute the conditional expectations in (2.5) and (2.6).

4.1 The first-to-default

4.1.1. Let us proceed by computing the terms for the *first-to-default* in (2.5). For this, let $H(x_j, m_j, x_{3-j}, m_{3-j})$ be a nonnegative continuous function for any $j = 1, 2$ fixed. By virtue of the independence of the processes $(X^{j,0}, M^{j,0})$ and $(X^{3-j,0}, M^{3-j,0})$, defined in (2.12)-(2.13), we get:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(t < \tau_j \wedge \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(t < \tau_j) E_{x_1, m_1, x_2, m_2} [H(X_t^{j,0}, M_t^{j,0}, X_t^{3-j,0}, M_t^{3-j,0}) I(M_t^{3-j,0} > b_{3-j}) | \mathcal{G}_t^j] \\ &= I(t < \tau_j) \int_{b_{3-j}}^{\infty} \int_{b_{3-j}}^{\infty} H(X_t^{j,0}, M_t^{j,0}, x'_{3-j}, m'_{3-j}) g_{3-j,0}(x_{3-j}, m_{3-j}; t, x'_{3-j}, m'_{3-j}) dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.1)$$

for all $0 \leq t \leq T$, where the functions $g_{3-j,0}$, $j = 1, 2$, are given in (2.15) above.

4.1.2. Now, we see that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(M_{\tau_j}^{3-j} > b_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(M_{\tau_{j,0}}^{3-j,0} > b_{3-j}) | \mathcal{G}_t^j] \end{aligned} \quad (4.2)$$

for $0 \leq t \leq T$, where the hitting times $\tau_{j,0}$, $j = 1, 2$, are defined in (2.16). Thus, using the independence of $\tau_{j,0}$ and $(X^{3-j,0}, M^{3-j,0})$, we have:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_{b_{3-j}}^{\infty} \int_{b_{3-j}}^{\infty} D_{1,j}(\tau_j, b_j, x'_{3-j}) g_{3-j,0}(x_{3-j}, m_{3-j}; \tau_j, x'_{3-j}, m'_{3-j}) dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.3)$$

where the functions $g_{3-j,0}$, $j = 1, 2$, are given in (2.15) above.

4.1.3. Then, we observe that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, X_{\tau_j}^j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j < \tau_{3-j} \leq t) | \mathcal{G}_t^{3-j}] \\ &\quad + E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \end{aligned} \quad (4.4)$$

for $0 \leq t \leq T$ and $j = 1, 2$. Hence, by virtue of the independence of $\tau_{j,0}$ and $(X^{3-j,0}, M^{3-j,0})$, we obtain:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j < \tau_{3-j} \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(\tau_{j,0} < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \end{aligned} \quad (4.5)$$

and

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\
&= I(t < \tau_{3-j}) E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t) | \mathcal{G}_t^{3-j}] \\
&= I(t < \tau_{3-j}) E_{x_1, m_1, x_2, m_2} [D_{j,1}(\tau_{j,0}, b_j, X_{\tau_{j,0}}^{3-j,0}) I(\tau_{j,0} \leq t) | \mathcal{G}_t^{3-j}]
\end{aligned} \tag{4.6}$$

and thus, we conclude from (4.4) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [D_{1,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_j \leq t, \tau_j < \tau_{3-j}) | \mathcal{G}_t^{3-j}] \\
&= \int_0^{\tau_{3-j} \wedge t} D_{1,j}(u, b_j, X_u^{3-j}) h_{j,0}(x_j; u) du
\end{aligned} \tag{4.7}$$

where the functions $h_{j,0}$, $j = 1, 2$, are given in (2.18).

Summarizing the facts proved above, let us formulate the following assertion.

Theorem 4.1. *The rational price of the first-to-default option in (2.5) under partial information is given by the sum of the terms in (4.1) and (4.3) or (4.7), where the function H is given appropriately by the corresponding value in (3.3) or (3.6), respectively.*

4.2 The second-to-default

4.2.1. Let us conclude with computing the terms for the *second-to-default* in (2.6). Firstly, taking into account the Markovian structure of the process (X^1, M^1, X^2, M^2) , we get:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} \leq t < \tau_j) | \mathcal{G}_t^j] \\
&= I(t < \tau_j) \int_0^t \int_0^\infty \int_0^{b_{3-j}} H(X_t^j, M_t^j, x'_{3-j}, m'_{3-j}) h_{3-j,0}(x_{3-j}; u) \\
&\quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; t - u, x'_{3-j}, m'_{3-j}) du dx'_{3-j} dm'_{3-j}
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} \leq t < \tau_j) | \mathcal{G}_t^{3-j}] \\
&= I(\tau_{3-j} \leq t) \int_{b_j}^\infty \int_{b_j}^\infty \widehat{H}(x'_j, m'_j, X_t^{3-j}, M_t^{3-j}; t - \tau_{3-j}) g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) dx'_j dm'_j
\end{aligned} \tag{4.9}$$

where the function \widehat{H} is defined by:

$$\begin{aligned}
& \widehat{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t - v) \\
&= \int_{b_j}^\infty \int_{b_j}^\infty H(x''_j, m''_j, x'_{3-j}, m'_{3-j}) g_{j,3-j}(x'_j, m'_j; t - v, x''_j, m''_j) dx''_j dm''_j
\end{aligned} \tag{4.10}$$

for $0 \leq t \leq T$, and the functions $g_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2$, are given in (2.15) above.

4.2.2. Now, by virtue of Markovian structure of the process (X^1, M^1, X^2, M^2) , we see that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{2,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_0^{\tau_j} \int_0^\infty \int_0^{b_{3-j}} D_{2,j}(\tau_j, b_j, x'_{3-j}) h_{3-j,0}(x_{3-j}; u) \\ & \quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; \tau_j - u, x'_{3-j}, m'_{3-j}) du dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.11)$$

and

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [D_{2,j}(\tau_j, b_j, X_{\tau_j}^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) \int_{\tau_{3-j}}^t \int_{b_j}^\infty \int_{b_j}^\infty D_{2,j}(u, b_j, X_u^{3-j}) h_{j,3-j}(x'_j; u - \tau_{3-j}) \\ & \quad \times g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) du dx'_j dm'_j \end{aligned} \quad (4.12)$$

for $0 \leq t \leq T$, where the functions $g_{j,\ell}$ and $h_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2$, are given in (2.15) and (2.18) above.

4.2.3. Finally, using again the Markovian structure of the process (X^1, M^1, X^2, M^2) , we obtain:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^j] \\ &= I(\tau_j \leq t) \int_0^{\tau_j} \int_0^\infty \int_0^{b_{3-j}} \tilde{H}(X_t^j, M_t^j, x'_{3-j}, m'_{3-j}; t - \tau_j) h_{3-j,0}(x_{3-j}; v) \\ & \quad \times g_{3-j,3-j}(b_{3-j}, b_{3-j}; \tau_j - v, x'_{3-j}, m'_{3-j}) dv dx'_{3-j} dm'_{3-j} \end{aligned} \quad (4.13)$$

where the function \tilde{H} is defined by:

$$\begin{aligned} & \tilde{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t - v) \\ &= \int_0^\infty \int_0^{b_{3-j}} H(x'_j, m'_j, x''_{3-j}, m''_{3-j}) g_{3-j,3}(x'_j, m'_j; t - v, x''_{3-j}, m''_{3-j}) dx''_{3-j} dm''_{3-j} \end{aligned} \quad (4.14)$$

and

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [H(X_t^j, M_t^j, X_t^{3-j}, M_t^{3-j}) I(\tau_{3-j} < \tau_j \leq t) | \mathcal{G}_t^{3-j}] \\ &= I(\tau_{3-j} \leq t) \int_{\tau_{3-j}}^t \int_{b_j}^\infty \int_{b_j}^\infty \bar{H}(b_j, b_j, X_t^{3-j}, M_t^{3-j}; t - u) h_{j,3-j}(x'_j; u - \tau_{3-j}) \\ & \quad \times g_{j,0}(x_j, m_j; \tau_{3-j}, x'_j, m'_j) du dx'_j dm'_j \end{aligned} \quad (4.15)$$

where the function \bar{H} is defined by:

$$\begin{aligned} \bar{H}(x'_j, m'_j, x'_{3-j}, m'_{3-j}; t - u) \\ = \int_0^\infty \int_0^{b_j} H(x''_j, m''_j, x'_{3-j}, m'_{3-j}) g_{j,3}(x'_j, m'_j; t - u, x''_j, m''_j) dx''_j dm''_j \end{aligned} \quad (4.16)$$

for $0 \leq t \leq T$, and the functions $g_{j,\ell}$ and $h_{j,\ell}$, $j = 1, 2$, $\ell = 0, 1, 2, 3$, are given in (2.15) and (2.18).

Therefore, summarizing the facts proved above, we are now ready to formulate the following assertion.

Theorem 4.2. *The rational price of the second-to-default option in (2.6) under partial information is given by the sum of the terms in (4.1), (4.8), (4.9), (4.11), (4.12), (4.13) and (4.15), where the function H is given appropriately by the corresponding value in (3.9), (3.14) or (3.22), respectively.*

5 Some remarks and examples

In this section, we derive some expressions for the rational prices of the European and the barrier-type first-to-default basket call options with linear recovery, under full information.

5.1 The European claims

The rational price process V_t of a standard (non-defaultable) European contingent claim with the payoff $C(X_T^1, X_T^2)$ is given by:

$$V_t = E[C(X_T^1, X_T^2) | \mathcal{G}_t] \quad (5.1)$$

for any $0 \leq t \leq T$. Observe that the value in (5.1) can be represented as:

$$\begin{aligned} E[C(X_T^1, X_T^2) | \mathcal{G}_t] &= E[C(X_T^1, X_T^2) I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\ &\quad + E[C(X_T^1, X_T^2) I(\tau_1 \vee \tau_2 \leq T) | \mathcal{G}_t] \\ &\quad + \sum_{i=1}^2 E[C(X_T^1, X_T^2) I(\tau_i \leq T < \tau_{3-i}) | \mathcal{G}_t] \end{aligned} \quad (5.2)$$

for $0 \leq t \leq T$. The first term in the right-hand side of expression (5.2) was computed in (3.1)-(3.3), while the other terms can be computed similarly to the formulas in (3.8)-(3.9), (3.12)-(3.15) and (3.20)-(3.23).

5.2 The first-to-default basket call

According to the example mentioned above, for the rest of the paper, let us set $C(x_1, x_2) = (\alpha_1 x_1 + \alpha_2 x_2 - K)^+$ with some $K > 0$ and $\alpha_i \geq 0$, for every $i = 1, 2$. In this case, we get from (3.1) that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_T^1 + \alpha_2 X_T^2 - K)^+ I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_T^1 + \alpha_2 X_T^2 - K) I(T < \tau_1 \wedge \tau_2, \alpha_1 X_T^1 + \alpha_2 X_T^2 > K) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(\alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 - K) I(T' < \tau'_1 \wedge \tau'_2, \alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 > K)] \end{aligned} \quad (5.3)$$

for all $0 \leq t \leq T$. Hence, we have from (3.2) that:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 - K) I(T' < \tau'_1 \wedge \tau'_2, \alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 > K)] \\ &= E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 - K) I(M_{T'}^1 > b_1, M_{T'}^2 > b_2, \alpha_1 X_{T'}^1 + \alpha_2 X_{T'}^2 > K)] \\ &= E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_{T'}^{1,0} + \alpha_2 X_{T'}^{2,0} - K) I(M_{T'}^{1,0} > b_1, M_{T'}^{2,0} > b_2, \alpha_1 X_{T'}^{1,0} + \alpha_2 X_{T'}^{2,0} > K)] \end{aligned} \quad (5.4)$$

for $x_i \geq m_i > b_i > 0$, where the processes $(X^{i,0}, M^{i,0})$, $i = 1, 2$, are defined in (2.12)-(2.13) above.

Let us now observe that, following the line of the arguments from [11; Theorem A.6.1], it is shown that there exists a probability measure \tilde{P} such that it is locally equivalent to P on the filtration $(\mathcal{G}_t)_{t \geq 0}$ and its density process is given by:

$$\frac{d\tilde{P}^i}{dP} \Big|_{\mathcal{G}_t} = \exp \left(\sigma_i W_t^i - \frac{\sigma_i^2}{2} t \right) \quad (5.5)$$

for all $t \geq 0$ and every $i = 1, 2$. Then, by Girsanov's theorem (see, e.g. [8; Theorem 6.3]), we may conclude that the process $\tilde{W}^i = (\tilde{W}_t^i)_{t \geq 0}$, defined by $\tilde{W}_t^i = W_t^i - \sigma_i t$, is a standard Brownian motion under the measure \tilde{P}^i . Thus, it is seen from (2.1) that the process X^i has the expression:

$$X_t^i = x_i \exp \left(\left(\frac{\sigma_i^2}{2} - \delta_{i,0} \right) t - (\delta_{i,1} - \delta_{i,0}) (t - \tau_1)^+ - (\delta_{i,2} - \delta_{i,0}) (t - \tau_2)^+ + \sigma_i \tilde{W}_t^i \right) \quad (5.6)$$

for every $i = 1, 2$. We also note that, using explicit expression (2.12), we obtain from (5.5) that:

$$\frac{d\tilde{P}^i}{dP} \Big|_{\mathcal{G}_t} = e^{\delta_{i,0} t} \frac{X_t^{i,0}}{x_i} \quad (5.7)$$

for all $t \geq 0$.

Therefore, we conclude from (5.3)-(5.4) and (5.7) that the expression in (3.3) takes the form:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [(\alpha_1 X_T^1 + \alpha_2 X_T^2 - K)^+ I(T < \tau_1 \wedge \tau_2) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) \left(-K P_{X_t^1, M_t^1, X_t^2, M_t^2} [M_{T-t}^{1,0} > b_1, M_{T-t}^{2,0} > b_2, \alpha_1 X_{T-t}^{1,0} + \alpha_2 X_{T-t}^{2,0} > K] \right. \\
&\quad \left. + \sum_{i=1}^2 e^{-\delta_i, 0(T-t)} \alpha_i x_i \tilde{P}_{X_t^1, M_t^1, X_t^2, M_t^2}^i [M_{T-t}^{1,0} > b_1, M_{T-t}^{2,0} > b_2, \alpha_1 X_{T-t}^{1,0} + \alpha_2 X_{T-t}^{2,0} > K] \right)
\end{aligned} \tag{5.8}$$

where

$$\begin{aligned}
& P_{X_t^1, M_t^1, X_t^2, M_t^2} [M_{T-t}^{1,0} > b_1, M_{T-t}^{2,0} > b_2, \alpha_1 X_{T-t}^{1,0} + \alpha_2 X_{T-t}^{2,0} > K] \\
&= \int_{b_1}^{\infty} \int_{b_1}^{\infty} \int_{b_2}^{\infty} \int_{b_2}^{\infty} I(\alpha_1 x'_1 + \alpha_2 x'_2 > K) \prod_{\ell=1}^2 g_{\ell,0}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell
\end{aligned} \tag{5.9}$$

and

$$\begin{aligned}
& \tilde{P}_{X_t^1, M_t^1, X_t^2, M_t^2}^i [M_{T-t}^{1,0} > b_1, M_{T-t}^{2,0} > b_2, \alpha_1 X_{T-t}^{1,0} + \alpha_2 X_{T-t}^{2,0} > K] \\
&= \int_{b_1}^{\infty} \int_{b_1}^{\infty} \int_{b_2}^{\infty} \int_{b_2}^{\infty} I(\alpha_1 x'_1 + \alpha_2 x'_2 > K) \prod_{\ell=1}^2 \tilde{g}_{\ell,0}(X_t^\ell, M_t^\ell; T-t, x'_\ell, m'_\ell) dx'_\ell dm'_\ell
\end{aligned} \tag{5.10}$$

with the functions $\tilde{g}_{i,0}$, $i = 1, 2$, defined as $g_{i,0}$ in (2.15) above with $\tilde{\rho}_{i,j} = -\delta_{i,j}/\sigma_i + \sigma_i/2$ in place of $\rho_{i,j}$.

5.3 The linear recovery

Let us now set $D_{1,i}(t, x_i, x_{3-i}) = \gamma + \beta_i x_i + \beta_{3-i} x_{3-i}$ with some $\gamma \geq 0$ and $\beta_i \geq 0$ for every $i = 1, 2$. In a similar way, we get from (3.4) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) E_{x_1, m_1, x_2, m_2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(\tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\
&= I(t < \tau_1 \wedge \tau_2) E_{X_t^1, M_t^1, X_t^2, M_t^2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(\tau_i' \leq T', \tau_i' < \tau_{3-i}')]
\end{aligned} \tag{5.11}$$

for all $0 \leq t \leq T$. Hence, we have from (3.5) that:

$$\begin{aligned}
& E_{x_1, m_1, x_2, m_2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(\tau_i' \leq T', \tau_i' < \tau_{3-i}')] \\
&= E_{x_1, m_1, x_2, m_2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(\tau_i' \leq T', M_{\tau_i}^{3-i} > b_{3-i})] \\
&= E_{x_1, m_1, x_2, m_2} [(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_{i,0}}^{3-i,0}) I(\tau_{i,0} \leq T', M_{\tau_{i,0}}^{3-i,0} > b_{3-i})]
\end{aligned} \tag{5.12}$$

for $x_i \geq m_i > b_i > 0$, where the processes $(X^{3-i,0}, M^{3-i,0})$ as well as the hitting times $\tau_{i,0}$, $i = 1, 2$, are defined in (2.12)-(2.13) and (2.16) above.

Therefore, we conclude from (5.11)-(5.12) and (5.7) that the expression in (3.6) takes the form:

$$\begin{aligned} & E_{x_1, m_1, x_2, m_2}[(\gamma + \beta_i b_i + \beta_{3-i} X_{\tau_i}^{3-i}) I(t < \tau_i \leq T, \tau_i < \tau_{3-i}) | \mathcal{G}_t] \\ &= I(t < \tau_1 \wedge \tau_2) \times \left((\gamma + \beta_i b_i) P_{X_t^1, M_t^1, X_t^2, M_t^2} [M_{\tau_{i,0}}^{3-i,0} > b_{3-i}, \tau_{i,0} \leq T - t] \right. \\ & \quad \left. + \beta_{3-i} x_{3-i} \tilde{E}_{X_t^1, M_t^1, X_t^2, M_t^2}^{3-i} [e^{-\delta_{3-i,0}(t+\tau_{i,0})} I(M_{\tau_{i,0}}^{3-i,0} > b_{3-i}, \tau_{i,0} \leq T - t)] \right) \end{aligned} \quad (5.13)$$

where

$$\begin{aligned} & P_{X_t^1, M_t^1, X_t^2, M_t^2} [M_{\tau_{i,0}}^{3-i,0} > b_{3-i}, \tau_{i,0} \leq T - t] \\ &= \int_0^{T-t} \int_{b_{3-i}}^\infty \int_{b_{3-i}}^\infty h_{i,0}(X_t^i; u) g_{3-i,0}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i} \end{aligned} \quad (5.14)$$

and

$$\begin{aligned} & \tilde{E}_{X_t^1, M_t^1, X_t^2, M_t^2}^{3-i} [e^{-\delta_{3-i,0}(t+\tau_{i,0})} I(M_{\tau_{i,0}}^{3-i,0} > b_{3-i}, \tau_{i,0} \leq T - t)] \\ &= \int_0^{T-t} \int_{b_{3-i}}^\infty \int_{b_{3-i}}^\infty e^{-\delta_{3-i,0}(t+u)} \tilde{h}_{i,0}(X_t^i; u) \tilde{g}_{3-i,0}(X_t^{3-i}, M_t^{3-i}; u, x'_{3-i}, m'_{3-i}) du dx'_{3-i} dm'_{3-i} \end{aligned} \quad (5.15)$$

with the functions $\tilde{g}_{i,0}$ and $\tilde{h}_{i,0}$, $i = 1, 2$, defined as $g_{i,0}$ and $h_{i,0}$ in (2.15) and (2.18) above with $\tilde{\rho}_{i,j} = -\delta_{i,j}/\sigma_i + \sigma_i/2$ in place of $\rho_{i,j}$.

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