

A construction of processes with one dimensional martingale marginals, based upon path-space Ornstein-Uhlenbeck processes and the Brownian sheet

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Abstract Using a variation from the construction of the Ornstein-Uhlenbeck process on canonical path-space $C([0, 1]; \mathbb{R})$ in terms of the Brownian sheet, we obtain a large class of processes, adapted to the Brownian filtration, which admit the one dimensional marginals of a martingale.

Key words 1-martingale; convex order; Ornstein-Uhlenbeck semigroup; Ornstein-Uhlenbeck process; canonical path-space; Wiener space; Brownian chaos; Brownian sheet;

1 Introduction

1.1

Recently, Carr et al [5] showed that the arithmetic average A of geometric Brownian motion $\mathcal{E}^{(\lambda)}$:

$$A_t = \frac{1}{t} \int_0^t \mathcal{E}_s^{(\lambda)} ds, \quad t \geq 0$$

where

$$\mathcal{E}_s^{(\lambda)} = \exp\left(\lambda B_s - \frac{\lambda^2}{2} s\right), \quad s \geq 0$$

with (B_s) a standard Brownian motion and $\lambda \in \mathbb{R}$, is increasing in the convex order, meaning that:

for every convex function $f : \mathbb{R} \rightarrow \mathbb{R}$, the function

$$t \rightarrow \mathbb{E}[f(A_t)] \quad (\in (-\infty, +\infty])$$

increases.

1.2

Later, Baker-Yor [1] showed that $(A_t, t \geq 0)$ is a 1-martingale, meaning that it has the one dimensional marginals of a martingale $(M_t, t \geq 0)$. Thus, the previous result in 1.1 follows from Jensen's inequality.

Precisely, in [1], it is shown that one can take

$$M_t = \int_0^1 \exp\left(\lambda W_{u,t} - \frac{\lambda^2}{2} ut\right) du, \quad t \geq 0$$

which is a $(\mathcal{W}_t := \sigma\{W_{u,s}; u \leq 1, s \leq t\})$ -martingale, where $\{W_{u,s}\}$ denotes the standard Brownian sheet (see, e.g., Cairoli-Walsh [4] for a deep study of the Brownian sheet).

1.3

Our aim in this paper is to develop systematically the above approach in 1.2, by exhibiting many processes $(\Phi_t^\sharp, t \leq 1)$, which are adapted to the Brownian filtration $(\mathcal{B}_t := \sigma\{B_s; s \leq t\}, t \leq 1)$ and are 1-martingales. Basically, our findings rest on the simple, but powerful, observation that, for any fixed t ,

$$(B_{tu}, u \geq 0) \stackrel{(\text{law})}{=} (W_{u,t}, u \geq 0)$$

1.4

Our paper is organised as follows:

- In Section 2, we define, via Brownian chaos expansions, Markov operators $(R_t, t \leq 1)$ on $L^2(\mathcal{B}_1)$. These operators can also be described from the Ornstein-Uhlenbeck semigroup (T_h) and scaling operators on canonical path-space.
- In Section 3, we show that for any $\Phi \in L^2(\mathcal{B}_1)$, the process Φ^\sharp defined by

$$\Phi_t^\sharp(B) = R_t \Phi(B) = T_{-\log t} \Phi(t^{-1/2} B_{t\bullet}), \quad t \leq 1$$

has the same one dimensional marginals as the process Φ^m defined by

$$\Phi_t^m(W) = T_{-\log t} \Phi(t^{-1/2} W_{\bullet,t}), \quad t \leq 1$$

which is a (\mathcal{W}_t) -martingale. We illustrate this result with many examples, notably the example presented in 1.2 above. Moreover, we show that there exist Φ 's such that the processes $(\Phi_t^\sharp(B), t \leq 1)$ and $(\Phi_t^m(W), t \leq 1)$ are identical in law; some such examples are closely connected with the construction of non-canonical Brownian motions (see, Jeulin-Yor [8], Hitsuda [6], Chiu [3], Hibino-Hitsuda-Muraoka [7], ...). However, this identity in law between the two processes is the exception, rather than the rule (see notably Proposition 5.5 below).

- In Section 4, we exhibit a Markov process $(Y^h, h \geq 0)$ with semigroup $Q_h = R_{e^{-h}}, h \geq 0$, and we study the infinitesimal generator of (Q_h) .
- In Section 5, we study the vector spaces $\mathcal{M}, \mathcal{V}, \mathcal{S}$, which consist of Φ 's in $L^2(\mathcal{B}_1)$ such that Φ^\sharp is respectively a (\mathcal{B}_t) -martingale, a continuous process of bounded variation on $[0, 1]$, a (\mathcal{B}_t) -semimartingale.

2 The operators $(R_t, 0 \leq t \leq 1)$

2.1 Notation

We first introduce some basic notation.

- We denote by E the standard Wiener space $C([0, 1]; \mathbb{R})$ equipped with the sup-norm:

$$\|B\| = \sup_{0 \leq t \leq 1} |B_t|$$

The generic element of E shall often be denoted by B or B_\bullet , the Wiener measure on E by $\mathbb{P}_{(B)}$, and the corresponding expectation by $\mathbb{E}_{(B)}$. If no confusion is possible, we omit (B) in the notation.

We will use the notation L^2 to denote the L^2 -space with respect to $\mathbb{P}_{(B)}$. The corresponding norm will be denoted by $\|\cdot\|_{L^2}$.

- We denote by $(\mathcal{B}_t)_{0 \leq t \leq 1}$ the usual Brownian filtration on E .
- If $0 < t \leq 1$, we denote by σ_t the scaling operator on E defined by:

$$\sigma_t(B) = \frac{1}{\sqrt{t}} B_\bullet$$

We also denote $\sigma_t(B)$ by $B^{(t)}$, that is:

$$\forall u \in [0, 1] \quad B_u^{(t)} = \frac{1}{\sqrt{t}} B_{tu}$$

- If $0 \leq t \leq 1$ and $n \geq 1$, we set

$$\Delta_n(t) = \{u \in \mathbb{R}^n ; t \geq u_1 \cdots \geq u_n \geq 0\}$$

and $\Delta_n = \Delta_n(1)$.

If φ_n is defined on $\Delta_n(t)$, $\|\varphi_n\|_{\Delta_n(t)}$ denotes the L^2 -norm of φ_n with respect to the Lebesgue measure on $\Delta_n(t)$.

If φ_n is defined on Δ_n and $0 < t \leq 1$, we set

$$\forall u \in \Delta_n(t) \quad \varphi_n^t(u) = \varphi_n\left(\frac{1}{t}u\right)$$

Thus,

$$\|\varphi_n^t\|_{\Delta_n(t)} = t^{n/2} \|\varphi_n\|_{\Delta_n}$$

- Let $(B_t)_{0 \leq t \leq 1}$ be a standard linear Brownian motion on the time interval $[0, 1]$. We set, for $0 \leq t \leq 1$, $n \geq 1$ and $\varphi_n \in L^2(\Delta_n(t))$,

$$\begin{aligned} I_n^t(\varphi_n)(B) &= \int_{\Delta_n(t)} \varphi_n(u) d^{(n)}B_u \\ &= \int_0^t dB_{u_1} \int_0^{u_1} dB_{u_2} \cdots \int_0^{u_{n-1}} dB_{u_n} \varphi_n(u_1, \dots, u_n) \end{aligned}$$

We omit t in the notation if $t = 1$; thus, $I_n(\varphi_n) = I_n^1(\varphi_n)$.

2.2 Definition of R_t

Let $\Phi \in L^2$ given by its chaos expansion:

$$\Phi = \sum_{n \geq 0} I_n(\varphi_n) \quad (1)$$

where, by convention, $I_0(\varphi_0)$ is set for $\mathbb{E}(\Phi)$. We define $R_t\Phi$ by

$$\begin{aligned} R_0\Phi &= \mathbb{E}(\Phi) \quad \text{and, for } 0 < t \leq 1, \\ R_t\Phi(B) &= \mathbb{E}(\Phi) + \sum_{n \geq 1} \int_{\Delta_n} \varphi_n(u) d^{(n)}B_{tu} \end{aligned} \quad (2)$$

Equivalently, for $0 < t \leq 1$,

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n \geq 1} t^{n/2} I_n(\varphi_n)(B^{(t)}) \quad (3)$$

We also have, for $0 < t \leq 1$,

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n \geq 1} \int_{\Delta_n(t)} \varphi_n\left(\frac{1}{t}u\right) d^{(n)}B_u \quad (4)$$

which can be written:

$$R_t\Phi(B) = \mathbb{E}(\Phi) + \sum_{n \geq 1} I_n^t(\varphi_n^t)(B) \quad (5)$$

2.3 Some properties of $(R_t, t \leq 1)$

Proposition 2.1 *i) $(R_t)_{0 \leq t \leq 1}$ is a family of linear contractions of L^2 .*

More precisely, if Φ is given by its chaos expansion (1),

$$\|R_t\Phi\|_{L^2}^2 = (\mathbb{E}(\Phi))^2 + \sum_{n \geq 1} t^n \|I_n(\varphi_n)\|_{\Delta_n}^2 \leq \|\Phi\|_{L^2}^2 \quad (6)$$

In particular, if $\mathbb{E}(\Phi) = 0$, $\|R_t\Phi\|_{L^2} \leq \sqrt{t} \|\Phi\|_{L^2}$.

ii) R_1 is the identity operator on L^2 and, for any $t, s \in [0, 1]$,

$$R_t R_s = R_{ts}$$

iii) For any $\Phi \in L^2$, the map

$$t \in [0, 1] \longrightarrow R_t \Phi \in L^2$$

is continuous.

Proof

Property i) is a direct consequence of formula (3) and of the fact that σ_t preserves the Wiener measure. The expression (5) could also be used.

Property ii) is clear.

The continuity of the map $t \in [0, 1] \longrightarrow R_t \Phi \in L^2$ at 0 follows from i). By the expression (5), the continuity on $(0, 1]$ is easy if, for any $n \geq 1$, $\varphi_n \in C(\Delta_n)$. The general case follows by density, according to i). □

2.4 Relation with the Ornstein-Uhlenbeck semigroup

We first recall the definition of the Ornstein-Uhlenbeck semigroup: $T = (T_h)_{h \geq 0}$, on L^2 (see, e.g., Bouleau-Hirsch [2, Chapter II, Section 2] or Nualart [9, p. 49, Definition 1.4.1]). If $\Phi \in L^2$, $\Phi = \sum_{n \geq 0} I_n(\varphi_n)$, and $h \geq 0$,

$$T_h \Phi = \sum_{n \geq 0} e^{-nh/2} I_n(\varphi_n) \tag{7}$$

Then, the so-called Mehler's formula holds:

$$T_h \Phi(B) = \mathbb{E}_{(\tilde{B})} [\Phi(e^{-h/2} B + \sqrt{1 - e^{-h}} \tilde{B})] \tag{8}$$

where \tilde{B} denotes a Brownian motion independent of B .

We also set, for $0 < t \leq 1$,

$$\Sigma_t \Phi(B) = \Phi(B^{(t)}) = \Phi \circ \sigma_t(B)$$

As σ_t preserves the Wiener measure, Σ_t is an isometry of L^2 .

Lemma 2.2 For $h \geq 0$ and $0 < t \leq 1$

$$T_h \Sigma_t = \Sigma_t T_h$$

Proof

By Mehler's formula (8),

$$T_h \Sigma_t \Phi(B) = \mathbb{E}_{(\tilde{B})}[\Phi(e^{-h/2} B^{(t)} + \sqrt{1 - e^{-h}} \tilde{B}^{(t)})]$$

and

$$\Sigma_t T_h \Phi(B) = \mathbb{E}_{(\tilde{B})}[\Phi(e^{-h/2} B^{(t)} + \sqrt{1 - e^{-h}} \tilde{B})]$$

Therefore the equality follows from the fact that $\tilde{B}^{(t)}$ and \tilde{B} have the same law. □

We now note the following useful expression of R_t .

Proposition 2.3 *One has, for any $t \in (0, 1]$,*

$$R_t = T_{-\log t} \Sigma_t = \Sigma_t T_{-\log t}$$

Proof

By formulas (3) and (7), we have $R_t = \Sigma_t T_{-\log t}$, and we may then apply the previous lemma. □

Corollary 2.3.1 *For $0 \leq t \leq 1$, R_t is a Markovian operator, and*

$$\forall \Phi \in L^2 \quad R_t \Phi(B) = \mathbb{E}_{(\tilde{B})}[\Phi(B_{t\bullet} + \sqrt{1 - t} \tilde{B})] \quad (9)$$

2.5 Extension

We can more generally define, for $\alpha > 0$ and $\beta \geq 0$, a family of Markovian operators $(R_t^{\alpha, \beta})_{0 \leq t \leq 1}$ by

$$R_t^{\alpha, \beta} = T_{-\alpha \log t} \Sigma_{t^\beta} = \Sigma_{t^\beta} T_{-\alpha \log t}$$

In particular, we have, for $\Phi \in L^2$, $\Phi = \sum_{n \geq 0} I_n(\varphi_n)$,

$$R_t^{\alpha, \beta} \Phi(B) = \mathbb{E}(\Phi) + \sum_{n \geq 1} t^{\alpha n/2} I_n(\varphi_n)(B^{(t^\beta)})$$

or

$$R_t^{\alpha, \beta} \Phi(B) = \mathbb{E}_{(\tilde{B})}[\Phi(t^{(\alpha-\beta)/2} B_{t^\beta \bullet} + \sqrt{1 - t^\alpha} \tilde{B})]$$

The previous results given for R_t are easily extended to $R_t^{\alpha, \beta}$.

3 Definitions and some properties of the processes Φ^\sharp and Φ^m

3.1 Notation

We denote by E_2 the space $C([0, 1]^2, \mathbb{R})$ equipped with the law $P_{(W)}$ of a Brownian sheet $(W_{s,t})_{0 \leq s, t \leq 1}$. The generic element of E_2 will be denoted by W .

We also define the filtration $(\mathcal{W}_t)_{0 \leq t \leq 1}$ on E_2 by

$$\mathcal{W}_t = \sigma\{W_{u,v}; 0 \leq u \leq 1, 0 \leq v \leq t\}$$

3.2 Definitions of Φ^\sharp and Φ^m

Let $\Phi \in L^2$. We associate with Φ two processes denoted by Φ^\sharp and Φ^m , and defined as follows:

Φ^\sharp is the process defined on the filtered probability space $(E, P_{(B)}, (\mathcal{B}_t)_{0 \leq t \leq 1})$ by

$$\Phi_t^\sharp(B) = R_t \Phi(B) = \mathbb{E}_{(\tilde{B})}[\Phi(B_{t\bullet} + \sqrt{1-t}\tilde{B})] \quad (10)$$

Φ^m is the process defined on the filtered probability space $(E_2, P_{(W)}, (\mathcal{W}_t)_{0 \leq t \leq 1})$ by

$$\Phi_t^m(W) = T_{-\log t} \Phi \left(\frac{1}{\sqrt{t}} W_{\bullet, t} \right) = \mathbb{E}_{(\tilde{B})}[\Phi(W_{\bullet, t} + \sqrt{1-t}\tilde{B})] \quad (11)$$

In formula (10), \tilde{B} denotes a Brownian motion independent of B , and in formula (11), \tilde{B} denotes a Brownian motion independent of W .

3.3 Main properties of Φ^\sharp and Φ^m

The following theorem summarizes our main objective in this paper.

Theorem 3.1 *Let $\Phi \in L^2$.*

- i) The process Φ^\sharp is (\mathcal{B}_t) -adapted and L^2 -continuous.*
- ii) The process Φ^m is a (\mathcal{W}_t) -martingale.*
- iii) For any $t \in [0, 1]$, Φ_t^\sharp and Φ_t^m have the same law.*

Proof

The property i) is clear by the definition and Proposition 2.1, iii).
 Let $0 \leq s \leq t$. We have by (11):

$$\Phi_t^m(W) = \mathbb{E}_{(\tilde{B})}[\Phi(W_{\bullet,s} + (W_{\bullet,t} - W_{\bullet,s}) + \sqrt{1-t}\tilde{B})]$$

Therefore, from the properties of the Brownian sheet,

$$\mathbb{E}(\Phi_t^m | \mathcal{W}_s)(W) = \mathbb{E}_{(\hat{B}, \tilde{B})}[\Phi(W_{\bullet,s} + \sqrt{t-s}\hat{B} + \sqrt{1-t}\tilde{B})]$$

where (\hat{B}, \tilde{B}) denotes a two-dimensional Brownian motion, independent of W .

We can thus write:

$$\mathbb{E}(\Phi_t^m | \mathcal{W}_s)(W) = \mathbb{E}_{(\tilde{B})}[\Phi(W_{\bullet,s} + \sqrt{1-s}\tilde{B})] = \Phi_s^m(W)$$

The property iii) follows from the formulas (10) and (11), since the processes $B_{t\bullet}$ and $W_{\bullet,t}$ have the same law. □

In particular, the processes Φ^\sharp are 1-*martingales*, meaning that they have the same one dimensional marginals as a martingale. This result is remarkable because, as we will see in the next subsection, there exist many such processes which are continuous and of finite variation. Now, continuous processes with square-integrable variation cannot be 2-*martingales*, that is they cannot have the same two dimensional marginals as a martingale, unless they are constant. A more general result is shown below.

Lemma 3.2 *Let $V = (V_t)_{0 \leq t \leq 1}$ be a continuous process of finite variation such that*

$$i) \mathbb{E} \left[\left(\int_0^1 |dV_s| \right)^2 \right] < \infty$$

ii) V has orthogonal increments

Then, $V_t = V_0$ for $0 \leq t \leq 1$.

Proof

Let (σ_n) be a sequence of subdivisions of $[0, t]$ whose meshes tend to 0 when n tends to infinity. By hypothesis ii),

$$\mathbb{E} \left[\sum_{\sigma_n} (V_{t_{i+1}} - V_{t_i})^2 \right] = \mathbb{E} [(V_t - V_0)^2]$$

On the other hand,

$$\left[\sum_{\sigma_n} (V_{t_{i+1}} - V_{t_i})^2 \right] \leq \left(\int_0^1 |dV_s| \right)^2$$

and

$$\left[\sum_{\sigma_n} (V_{t_{i+1}} - V_{t_i})^2 \right] \leq \sup_{\sigma_n} |V_{t_{i+1}} - V_{t_i}| \left(\int_0^1 |dV_s| \right)$$

Therefore the result follows from i), by using the dominated convergence theorem. □

An interesting consequence of Theorem 3.1 is the following result, which actually is a general result valid for any 1-martingale.

Proposition 3.3 *The process Φ^\sharp is increasing for the convex order, which means: For any convex function f on \mathbb{R} , the map*

$$t \in [0, 1] \longrightarrow \mathbb{E}[f(\Phi_t^\sharp)] \in (-\infty, +\infty]$$

is increasing.

Proof

This follows directly from the fact that Φ^\sharp is a 1-martingale, by Jensen's inequality for conditional expectations. □

Concerning the continuity of the process Φ^\sharp , there is the following partial result.

Proposition 3.4 *Suppose that Φ is continuous on E and that there exist $\lambda \geq 0$ and $c < 1/2$ such that*

$$\forall B \in E \quad |\Phi(B)| \leq \lambda \exp(c \|B\|^2)$$

Then Φ^\sharp admits a continuous version which is given by

$$\forall B \in E, \forall t \in [0, 1], \quad \Phi_t^\sharp(B) = \mathbb{E}_{(\tilde{B})}[\Phi(B_{t\bullet} + \sqrt{1-t}\tilde{B})]$$

Proof The result follows from the dominated convergence theorem thanks to the following lemma.

Lemma 3.4.1 *One has, for $0 \leq c < 1/2$,*

$$(1 - 2c)^{-1/2} \leq \mathbb{E}[\exp(c \| B \|^2)] \leq 2(1 - 2c)^{-1/2}$$

Proof

The first inequality is obvious since $\| B \|^2 \geq (B_1)^2$.

For the second inequality, set $S(B) = \sup_{0 \leq s \leq 1} B_s$. We have $\| B \| = \sup(S(B), S(-B))$, and, $S(B)$ and $S(-B)$ have the same law as $|B_1|$. Therefore,

$$\mathbb{E}[\exp(c \| B \|^2)] = \mathbb{E}[\sup(\exp(c S(B)^2), \exp(c S(-B)^2))] \leq 2 \mathbb{E}[\exp(c (B_1)^2)]$$

which yields the result. □

□

Concerning the continuity of the process Φ^m , we have:

Proposition 3.5 *The process Φ^m admits a continuous version on $[0, 1]$.*

Proof

Let, for $0 \leq s, t \leq 1$,

$$\mathcal{W}_{s,t} = \sigma\{W_{u,v} ; 0 \leq u \leq s, 0 \leq v \leq t\}$$

and

$$\Phi_{s,t} = \mathbb{E}(\Phi(W_{\bullet,1}) \mid \mathcal{W}_{s,t})$$

Then, from Cairoli-Walsh [4], the two-parameter martingale $\Phi_{s,t}$ admits a continuous version. Now, since Φ^m is a $(\mathcal{W}_{1,t})$ -martingale, then $\Phi^m = \Phi_{1,\bullet}$ and, therefore, Φ^m also admits a continuous version. □

Remark

Here is a more direct proof of the fact that any square-integrable (\mathcal{W}_t) -martingale admits a continuous version. We now sketch this proof in three steps.

- 1- Let $(h_n)_{n \geq 0}$ be an orthonormal basis of $L^2([0, 1])$ and set, for $n \geq 0$, $t \in [0, 1]$,

$$W_t^{(n)} = \int_0^1 h_n(u) d_u W_{u,t}$$

Then $(W^{(n)}, n \geq 0)$ is a sequence of independent Brownian motions and

$$\mathcal{W}_t = \sigma\{W_s^{(n)}; n \geq 0, 0 \leq s \leq t\}$$

2- Any $X \in L^2(\mathcal{W}_1)$ admits the following representation:

$$X = c + \sum_{n=0}^{\infty} \int_0^1 H_n(s) dW_s^{(n)}$$

where (H_n) is a sequence of (\mathcal{W}_t) -predictable processes such that

$$\mathbb{E} \left[\sum_{n=0}^{\infty} \int_0^1 H_n^2(s) ds \right] < \infty$$

To prove this property, we first consider $X = \mathcal{E}_n^\varphi$ with

$$\mathcal{E}_n^\varphi = \exp \left[\sum_{k=0}^n \left(\int_0^1 \varphi_k(s) dW_s^{(k)} - \frac{1}{2} \int_0^1 \varphi_k^2(s) ds \right) \right]$$

and $\varphi = (\varphi_k)_{0 \leq k \leq n} \in (L^2([0, 1]))^{n+1}$. We then reason by density.

3- If X admits the above representation, then, for $0 \leq t \leq 1$,

$$X_t := \mathbb{E}(X | \mathcal{W}_t) = c + \sum_{n=0}^{\infty} \int_0^t H_n(s) dW_s^{(n)}$$

It is then clear, using Doob's maximal inequality, that $(X_t, t \in [0, 1])$ admits a continuous version.

3.4 Examples

In what follows, $P = (P_t)_{t \geq 0}$ denotes the Gaussian (or Heat) semigroup, and, for $0 < r \leq 1$, we denote by γ_r the normal law with variance r .

3.4.1 $\Phi(B) = f(B_r)$

Proposition 3.6 *Let $r \in (0, 1]$, $f \in L^2(\gamma_r)$ and $\Phi(B) = f(B_r)$. Then, for any $t \in [0, 1]$, we have:*

$$\Phi_t^\sharp(B) = P_{(1-t)r} f(B_{tr}) = \mathbb{E}(\Phi(B) | \mathcal{B}_{tr}) \quad (12)$$

As a consequence, $(\Phi_t^\sharp(B))$ is a (\mathcal{B}_{tr}) -martingale.

We also have:

$$\Phi_t^m(W) = P_{(1-t)r} f(W_{r,t}) \quad (13)$$

Consequently, the processes Φ^\sharp and Φ^m have the same law.

Proof

We have

$$\Phi_t^\sharp(B) = \mathbb{E}_{(\tilde{B})}[f(B_{tr} + \sqrt{1-t}\tilde{B}_r)] = \mathbb{E}[f(B_{tr} + \sqrt{(1-t)r}N)]$$

where N denotes a standard normal variable, independent of B . This yields formula (12).

Concerning

$$\Phi_t^m(W) = \mathbb{E}_{(\tilde{B})}[f(W_{r,t} + \sqrt{1-t}\tilde{B}_r)]$$

we get:

$$\Phi_t^m(W) = P_{(1-t)r}f(W_{r,t})$$

Finally, we use that $(B_{tr}, t \leq 1)$ and $(W_{r,t}, t \leq 1)$ have the same law to conclude. □

Corollary 3.6.1 *Let φ ($\varphi = \varphi(t, x)$) belong to $C^{1,2}([0, 1], \mathbb{R})$. We assume that φ is a time-space harmonic function, that is*

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} \frac{\partial^2 \varphi}{\partial x^2} = 0 \quad \text{on } [0, 1] \times \mathbb{R}$$

Let $r \in [0, 1]$. We assume that $\Phi(B) := \varphi(r, B_r) \in L^2$. Then, for any $t \in [0, 1]$,

$$\Phi_t^\sharp(B) = \varphi(tr, B_{tr})$$

We also have:

$$\Phi_t^m(W) = f(tr, W_{r,t})$$

Thus, again, the processes Φ^\sharp and Φ^m have the same law.

Proof

By Itô's formula, $(\varphi(t, B_t), t \leq 1)$ is a (\mathcal{B}_t) -martingale. So, applying the above proposition to

$$f(x) = \varphi(r, x), \quad x \in \mathbb{R}$$

we get the result. □

Particular cases

(1) $\Phi(B) = \exp(\lambda B_r - \frac{\lambda^2 r}{2})$ ($0 \leq r \leq 1$ and $\lambda \in \mathbb{R}$). Then

$$\Phi_t^\sharp(B) = \exp(\lambda B_{tr} - \frac{\lambda^2 tr}{2})$$

In this case, Φ^\sharp is thus a geometric Brownian motion.

(2) $\Phi(B) = H_n(r, B_r)$ with $r \in [0, 1]$, $n \in \mathbb{N}$ and

$$H_n(t, x) = t^{n/2} h_n\left(\frac{x}{\sqrt{t}}\right)$$

where h_n denotes the n -th Hermite polynomial. Then

$$\Phi_t^\sharp(B) = H_n(tr, B_{tr})$$

This case (2) can also be obtained from the case (1), according to the formula:

$$\exp(\lambda x - \frac{\lambda^2 t}{2}) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_n(t, x)$$

3.4.2 The set \mathcal{I}

The results in the previous subsection 3.4.1 motivated us to introduce the set

$$\mathcal{I} = \{ \Phi \in L^2 ; (\Phi_t^\sharp(B), t \leq 1) \text{ and } (\Phi_t^m(W), t \leq 1) \text{ are identical in law} \}$$

Indeed, we have just seen in the previous subsection that $\Phi(B) = f(B_r)$, or $\Phi(B) = \varphi(r, B_r)$ for φ a time-space harmonic function, belong to \mathcal{I} . We now characterize the elements in the first Wiener chaos which belong to \mathcal{I} .

Proposition 3.7 *Let $h \in L^2([0, 1])$ and let ν_h be the norm, in $L^2([0, 1])$, of h . We assume $\nu_h \neq 0$ and we set*

$$\Phi(B) = \Phi_h(B) = \int_0^1 h(u) dB_u$$

The following properties are equivalent.

(i) $\Phi \in \mathcal{I}$

(ii) $(\Phi_t^\sharp(B), t \leq 1)$ is identical in law to $(\nu_h B_t, t \leq 1)$

(iii) For every $z \in (0, 1]$,

$$\int_0^1 h(u) h(zu) \, du = \nu_h^2$$

Condition (iii) is equivalent to the fact that the function $L : [1, \infty] \rightarrow \mathbb{R}$ defined by $L(v) = (\nu_h)^{-1} h(v^{-1})$ is a *Brownian motion preserving* function, which means that

$$B_t^L := \int_0^t L\left(\frac{t}{s}\right) \, dB_s, \quad t \leq 1$$

is still a Brownian motion.

This topic was introduced by P. Lévy and has been for example discussed in Jeulin-Yor [8]. It is a particular case of the more general family of *non-canonical Brownian motions* (see, e.g., [6, 3, 7]).

Proof

We have:

$$\Phi_t^m(W) = \int_0^1 h(u) \, d_u W_{u,t}$$

Therefore, the process $(\Phi_t^m(W), t \leq 1)$ is clearly distributed as $(\nu_h B_t, t \leq 1)$. This yields the equivalence between properties (i) and (ii).

On the other hand,

$$\Phi_t^\sharp(B) = \int_0^t h\left(\frac{u}{t}\right) \, dB_u$$

from which, the equivalence between properties (ii) and (iii) follows easily. \square

In the following, we denote by \mathcal{Y} the set of functions h satisfying the equivalent conditions of Proposition 3.7. The following corollary is an extension of Proposition 3.6, which corresponds to the case where h is the indicator function of $[0, r]$.

Corollary 3.7.1 *Let $h \in \mathcal{Y}$. If $f \in L^2(\gamma_{\nu_h^2})$, then $f(\Phi_h) \in \mathcal{I}$.*

In particular, for any $n \geq 1$, $I_n(h^{(n)}) \in \mathcal{I}$, where $h^{(n)}$ is defined on Δ_n by

$$h^{(n)}(u_1, u_2, \dots, u_n) = h(u_1) h(u_2) \cdots h(u_n)$$

Proof

Let $\Phi = f(\Phi_h)$. Then,

$$\Phi_t^\sharp(B) = \mathbb{E}_{(\tilde{B})} \left[f \left((\Phi_h)_t^\sharp(B) + \sqrt{1-t} \int_0^1 h(u) d\tilde{B}_u \right) \right] \quad \text{and}$$

$$\Phi_t^m(W) = \mathbb{E}_{(\tilde{B})} \left[f \left((\Phi_h)_t^m(W) + \sqrt{1-t} \int_0^1 h(u) d\tilde{B}_u \right) \right]$$

Then the result follows from Proposition 3.7.

In particular, we can take for f the n -th Hermite polynomial, which yields the second part of the corollary. □

3.4.3 $\Phi = \int_0^1 F_u^\sharp h(u) du$

We consider a Borel function h on $[0, 1]$ and $F \in L^2$. We assume

$$\int_0^1 \|F_u^\sharp\|_{L^2} |h(u)| du < \infty \tag{14}$$

and we set

$$\Phi = \int_0^1 F_u^\sharp h(u) du$$

Proposition 3.8 For $t \in (0, 1]$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^t F_u^\sharp h\left(\frac{u}{t}\right) du$$

Proof

By Proposition 2.1, ii),

$$(F_u^\sharp)_t^\sharp = F_{ut}^\sharp$$

Therefore

$$\Phi_t^\sharp = \int_0^1 F_{ut}^\sharp h(u) du = \frac{1}{t} \int_0^t F_u^\sharp h\left(\frac{u}{t}\right) du$$

□

Proposition 3.9 *Suppose (without loss of generality) that $\mathbb{E}(F) = 0$. We assume that h is an absolutely continuous function on $(0, 1]$ satisfying*

$$\int_0^1 u^{3/2} |h'(u)| \, du < \infty$$

Then the condition (14) is satisfied and the process Φ^\sharp is continuous and of finite variation on $[0, 1]$. Moreover, the variation on $[0, 1]$ is square-integrable.

Proof

We have

$$\int_0^1 \sqrt{u} |h(u)| \, du \leq |h(1)| + \int_0^1 s^{3/2} |h'(s)| \, ds < \infty$$

On the other hand, as $\mathbb{E}(F) = 0$, we have by Proposition 2.1, i),

$$\|F_u^\sharp\|_{L^2} \leq \sqrt{u} \|F\|_{L^2}$$

Consequently, the condition (14) is satisfied.

We have, for $0 < u \leq t \leq 1$,

$$h\left(\frac{u}{t}\right) = h(1) - \int_u^t \frac{u}{s^2} h'\left(\frac{u}{s}\right) \, ds$$

Then, by the previous proposition,

$$\begin{aligned} \Phi_t^\sharp &= \frac{1}{t} \left(h(1) \int_0^t F_u^\sharp \, du - \int_0^t \frac{1}{s^2} \left(\int_0^s F_u^\sharp u h'\left(\frac{u}{s}\right) \, du \right) \, ds \right) \\ &= \frac{1}{t} \left(h(1) \int_0^t F_u^\sharp \, du - \int_0^t \left(\int_0^1 F_{u_s}^\sharp u h'(u) \, du \right) \, ds \right) \end{aligned}$$

Therefore, Φ_t^\sharp is absolutely continuous on $(0, 1]$ and

$$\frac{d}{dt} \Phi_t^\sharp = \frac{1}{t} \left(h(1) F_t^\sharp - \int_0^1 F_{u_t}^\sharp u h'(u) \, du - \Phi_t^\sharp \right)$$

Hence,

$$\left\| \frac{d}{dt} \Phi_t^\sharp \right\|_{L^2} \leq \frac{2}{\sqrt{t}} \|F\|_{L^2} \left(|h(1)| + \int_0^1 u^{3/2} |h'(u)| \, du \right)$$

which yields the result. □

As a consequence of the previous proposition and of Lemma 3.2, under the above hypotheses, Φ^\sharp is a 1-martingale, but it is not a 2-martingale (unless $\Phi = 0$).

Particular case

A particular case is the case $h = 1$. Let $F \in L^2$. We set $\Phi = \int_0^1 F_u^\sharp du$. Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^t F_u^\sharp du$$

and $\Phi_0^\sharp = \mathbb{E}(F)$. By Proposition 3.9, Φ^\sharp is a continuous process with finite variation on $[0, 1]$.

If $F(B) = \exp(\lambda B_1 - \frac{\lambda^2}{2})$, then by what we saw in 3.4.1, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^t \exp(\lambda B_u - \frac{\lambda^2 u}{2}) du$$

and $\Phi_0^\sharp = 1$. Proposition 3.3 for this particular case was shown by P. Carr et al [5] by a completely different method. The proof given later in Baker-Yor [1] is at the origin of our present generalization.

We now give a density result.

Proposition 3.10 *We set*

$$\mathcal{U} = \left\{ \Phi = \int_0^1 F_u^\sharp h(u) du ; F \in L^2, h \in C^1([0, 1]) \right\}$$

Then, for any $\Phi \in \mathcal{U}$, Φ^\sharp is continuous and of finite variation on $[0, 1]$, and \mathcal{U} is dense in L^2 .

Proof

By Proposition 3.9, we only need to prove the density. For this purpose, we consider:

$$\Phi_n = \int_0^1 F_u^\sharp h_n(u) du$$

with $h_n \in C^1([0, 1])$, $h_n = 0$ on $[0, 1 - 1/n]$, $h_n \geq 0$ and $\int_0^1 h_n(x) dx = 1$. We then have by Proposition 2.1, iii),

$$\lim_{n \rightarrow \infty} \Phi_n = F$$

in L^2 .

□

$$3.4.4 \quad \Phi = \int_0^1 F_u^\# h(u) dB_u$$

We begin with a general result.

Proposition 3.11 *Let $\Phi \in L^2$ given by its predictable representation*

$$\Phi = a + \int_0^1 H_u dB_u$$

Then, for any $t \in (0, 1]$,

$$\Phi_t^\# = a + \int_0^t (H_{u/t})_t^\# dB_u$$

Proof

We have easily, according to formula (10),

$$\begin{aligned} \Phi_t^\#(B) &= a + \int_0^1 \mathbb{E}_{(\tilde{B})}[H_u(B_{t\bullet} + \sqrt{1-t}\tilde{B})] dB_{tu} \\ &= a + \int_0^1 (H_u)_t^\# dB_{tu} = a + \int_0^t (H_{u/t})_t^\# dB_u \end{aligned}$$

□

Corollary 3.11.1 *Let $F \in L^2$ and let h be a Borel function on $[0, 1]$ such that*

$$\int_0^1 \|F_u^\#\|_{L^2}^2 |h(u)|^2 du < \infty$$

We consider

$$\Phi = \int_0^1 F_u^\# h(u) dB_u$$

Then, for $0 < t \leq 1$,

$$\Phi_t^\# = \int_0^t F_u^\# h\left(\frac{u}{t}\right) dB_u$$

Proof

This is a direct consequence of the previous proposition and of Proposition 2.1, ii).

□

Corollary 3.11.2 *Let $F \in L^2$, $r \in [0, 1]$ and $\Phi = \int_0^r F_u^\sharp dB_u$. Then*

$$\Phi_t^\sharp = \int_0^{tr} F_u^\sharp dB_u$$

As a consequence, Φ^\sharp is a (\mathcal{B}_{tr}) -martingale.

We now show that the functions Φ of the previous form generate the whole space L^2 .

Proposition 3.12 *Let*

$$\mathcal{H} = \left\{ a + \int_0^r F_u^\sharp dB_u ; a \in \mathbb{R}, r \in [0, 1], F \in L^2 \right\}$$

Then the vector space spanned by \mathcal{H} is dense in L^2 .

Proof

Let $G = b + \int_0^1 H_u dB_u$ be orthogonal to \mathcal{H} in L^2 . Then

$$\forall a \in \mathbb{R}, \forall r \in [0, 1], \forall F \in L^2, \quad ab + \int_0^r \mathbb{E}(F_u^\sharp H_u) du = 0$$

Therefore, $b = 0$ and, for every $F \in L^2$, $\mathbb{E}(F_u^\sharp H_u) = 0$ for almost every $u \in [0, 1]$. As L^2 is separable, for almost every $u \in [0, 1]$, $\mathbb{E}(F_u^\sharp H_u) = 0$ for every $F \in L^2$.

Now, by the expression of F_u^\sharp on the Brownian chaoses, we have, for u fixed in $(0, 1]$,

$$\{F_u^\sharp ; F \in L^2\} = \left\{ \sum_{n \geq 0} I_n^u(\varphi_n) ; \sum_{n \geq 0} u^{-n} \|\varphi_n\|_{\Delta_n(u)}^2 < \infty \right\}$$

In particular, $\{F_u^\sharp ; F \in L^2\}$ is dense in $L^2(\mathcal{B}_u)$. Hence, for almost every $u \in [0, 1]$, $H_u = 0$ and, finally, $G = 0$. □

3.4.5 $\Phi = L_r^a$

In this last example, we take as Φ the local time of B at $a \in \mathbb{R}$ and at time $r \in (0, 1]$: $\Phi = L_r^a$.

Proposition 3.13 For $0 \leq t < 1$,

$$\Phi_t^\sharp = \frac{1}{\sqrt{2\pi(1-t)}} \int_0^r \exp\left(-\frac{(B_{ts}-a)^2}{2s(1-t)}\right) \frac{1}{\sqrt{s}} ds \quad (15)$$

The process Φ^\sharp is continuous and of finite variation on $[0, 1)$.

Proof

By the occupation times formula, for any $f \in C_c(\mathbb{R})$ ¹

$$\int_0^r f(B_s) ds = \int_{-\infty}^{+\infty} f(a) L_r^a da$$

By Proposition 3.6, we get then, for $t \in [0, 1]$,

$$\int_0^r P_{(1-t)s} f(B_{ts}) ds = \int_{-\infty}^{+\infty} f(a) (L_r^a)_t^\sharp da$$

and the formula (15) follows by identification.

The continuity of Φ^\sharp on $[0, 1)$ is clear on the formula (15).

By change of variable, for $t \in (0, 1)$,

$$\Phi_t^\sharp = \frac{1}{\sqrt{2\pi t(1-t)}} \int_0^{tr} \exp\left(-\frac{(B_u-a)^2 t}{2u(1-t)}\right) \frac{1}{\sqrt{u}} du$$

Therefore, Φ^\sharp is of class C^1 on $(0, 1)$ and we can write its derivative as a sum of four terms ℓ_t^i , $1 \leq i \leq 4$ defined below:

$$\begin{aligned} \ell_t^1 &= \frac{2t-1}{2t(1-t)} (\Phi_t^\sharp - \Phi_0^\sharp) \\ \ell_t^2 &= \frac{\sqrt{r}}{t\sqrt{2\pi(1-t)}} \left[\exp\left(-\frac{(B_{tr}-a)^2}{2r(1-t)}\right) - \exp\left(-\frac{a^2}{2r(1-t)}\right) \right] \\ \ell_t^3 &= -\frac{1}{2t(1-t)^2\sqrt{2\pi(1-t)}} \times \\ &\int_0^r \left[(B_{tu}-a)^2 \exp\left(-\frac{(B_{tu}-a)^2}{2u(1-t)}\right) - a^2 \exp\left(-\frac{a^2}{2u(1-t)}\right) \right] u^{-3/2} du \\ \sqrt{2\pi} \ell_t^4 &= \frac{2t-1}{2t(1-t)} \int_0^r \exp\left(-\frac{a^2}{2u}\right) u^{-1/2} du + \frac{\sqrt{r}}{t\sqrt{(1-t)}} \exp\left(-\frac{a^2}{2r(1-t)}\right) \\ &\quad - \frac{1}{2t(1-t)^2\sqrt{(1-t)}} \int_0^r a^2 \exp\left(-\frac{a^2}{2u(1-t)}\right) u^{-3/2} du \end{aligned}$$

¹Here and in what follows, the subscript: c , means: with compact support.

By Proposition 2.1, i), $\|\ell_t^1\|_{L^2} = O(t^{-1/2})$ when t tends to 0. On the other hand, it is not difficult to see that $\|\ell_t^2\|_{L^2}$ and $\|\ell_t^3\|_{L^2}$ are $O(t^{-1/2})$ if $a \neq 0$ and $O(1)$ if $a = 0$, when t tends to 0. By integration by parts in the last integral, we can write $\sqrt{2\pi} \ell_t^4$ as the difference of the two following terms:

$$\frac{1}{2t(1-t)} \int_0^r \left[(1-t)^{-1/2} \exp\left(-\frac{a^2}{2u(1-t)}\right) - (1-2t) \exp\left(-\frac{a^2}{2u}\right) \right] u^{-1/2} du$$

and

$$\frac{\sqrt{r}}{(1-t)\sqrt{(1-t)}} \exp\left(-\frac{a^2}{2r(1-t)}\right)$$

Hence,

$$\sqrt{2\pi} \lim_{t \rightarrow 0} \ell_t^4 = \frac{1}{4} \int_0^r u^{-3/2} (5u - a^2) \exp\left(-\frac{a^2}{2u}\right) du - \sqrt{r} \exp\left(-\frac{a^2}{2r}\right)$$

Finally, $\|\frac{d}{dt} \Phi_t^\sharp\|_{L^2}$ is $O(t^{-1/2})$ when t tends to 0, which entails that Φ^\sharp is of finite variation on any interval $[0, s]$ with $0 < s < 1$, the variation being square-integrable on any such interval. □

4 Semigroup Q

4.1 Definition of Q

In this section, we are interested in the family $Q = (Q_h)_{h \geq 0}$ of operators in L^2 , defined by

$$\forall h \geq 0 \quad Q_h = R_{e^{-h}}$$

where the operators R_t were defined in Section 2. As a direct consequence of Proposition 2.1 and Corollary 2.3.1, we have:

Proposition 4.1 $Q = (Q_h)_{h \geq 0}$ is a strongly continuous semigroup of Markovian operators in L^2 .

4.2 Infinitesimal generator of Q

In this subsection, we look for a description of the infinitesimal generator \mathcal{A} , of the semigroup Q . The domain of \mathcal{A} will be denoted by $\text{dom}(\mathcal{A})$.

We first introduce another notation: If φ_n is a C^1 function on the interior of Δ_n (denoted by $\text{int}(\Delta_n)$), we denote by $\widehat{\varphi}_n$ the function defined by

$$\forall u \in \text{int}(\Delta_n) \quad \widehat{\varphi}_n(u) = \sum_{j=1}^n \frac{\partial \varphi_n}{\partial u_j}(u) u_j = \varphi'_n(u) \cdot u$$

We also denote by $\Delta'_n = \{u \in \Delta_n ; u_1 \neq 0\}$

Theorem 4.2 *We denote by $D_{\mathcal{A}}$ the space of functions $\Phi = \sum_{n \geq 0} I_n(\varphi_n) \in L^2$ such that*

- 1- for any $n \geq 1$, φ_n is continuous on Δ'_n and of class C^1 on $\text{int}(\Delta_n)$,
- 2- $\forall n \geq 1, \forall v \in \Delta_{n-1}, \varphi_n(1, v) = 0$,
- 3- $\sum_{n \geq 1} \|\widehat{\varphi}_n\|_{\Delta_n}^2 < \infty$.

Then $D_{\mathcal{A}} \subset \text{dom}(\mathcal{A})$, and,

$$\forall \Phi \in D_{\mathcal{A}} \quad \mathcal{A}\Phi = \sum_{n \geq 1} I_n(\widehat{\varphi}_n)$$

Proof

Let $\Phi \in D_{\mathcal{A}}$. We have, by formula (4) in Section 2,

$$Q_h \Phi(B) = \mathbb{E}(\Phi) + \sum_{n \geq 1} \int_{\Delta_n(e^{-h})} \varphi_n(e^h u) d^{(n)}B_u$$

Now, for $n \geq 1$ and $u \in \Delta_n(e^{-h})$ with $u_1 \neq 0$, we have by hypotheses 1 and 2:

$$\varphi_n(e^h u) = - \int_h^{-\log u_1} \widehat{\varphi}_n(e^k u) dk = - \int_{u_1}^{e^{-h}} \frac{1}{s} \widehat{\varphi}_n\left(\frac{1}{s} u\right) ds$$

Therefore,

$$\int_{\Delta_n(e^{-h})} \varphi_n(e^h u) d^{(n)}B_u = - \int_0^{e^{-h}} \frac{1}{s} \left[\int_{\Delta_n(s)} \widehat{\varphi}_n\left(\frac{1}{s} u\right) d^{(n)}B_u \right] ds$$

The result then follows by taking the derivative with respect to h at $h = 0$. □

We now can complete the description of the infinitesimal generator \mathcal{A} .

Theorem 4.3 *The infinitesimal generator \mathcal{A} is the closure of its restriction to $D_{\mathcal{A}}$.*

Proof

We proceed in three steps.

1) Let

$$\Psi = \sum_{n \geq 0} I_n(\psi_n) \in L^2$$

such that, for every $n \geq 1$, $\psi_n \in C_c^1(\text{int}(\Delta_n))$ and let $\ell \in C_c^1((0, \infty))$. We set

$$\Phi = \int_0^\infty Q_h \Psi \ell(h) dh$$

We have

$$\begin{aligned} \Phi(B) &= \sum_{n \geq 0} \int \ell(h) \left(\int_{\Delta_n(e^{-h})} \psi_n(e^h u) d^{(n)} B_u \right) dh \\ &= \sum_{n \geq 0} \int_{\Delta_n} \left(\int_0^{-\log u_1} \ell(h) \psi_n(e^h u) dh \right) d^{(n)} B_u \end{aligned}$$

Therefore, for $n \geq 1$,

$$\varphi_n(u) = \int_0^{-\log u_1} \ell(h) \psi_n(e^h u) dh$$

Consequently, Φ satisfies the conditions 1 and 2 of the statement of Theorem 4.2 and, for $n \geq 1$,

$$\begin{aligned} \widehat{\varphi}_n(u) &= \int_0^{-\log u_1} \ell(h) \widehat{\psi}_n(e^h u) dh = \int_0^{-\log u_1} \ell(h) \frac{\partial}{\partial h} [\psi_n(e^h u)] dh \\ &= - \int_0^{-\log u_1} \ell'(h) \psi_n(e^h u) dh \end{aligned}$$

Then, setting $C = \int_0^\infty (\ell'(h))^2 dh$, we have:

$$\begin{aligned} \|\widehat{\varphi}_n\|_{\Delta_n}^2 &\leq C \int_{\Delta_n} \left[\int_0^{-\log u_1} \psi_n^2(e^h u) dh \right] du \\ &= C \int_0^\infty \left[\int_{\Delta_n(e^{-h})} \psi_n^2(e^h u) du \right] dh = \frac{C}{n} \|\psi_n\|_{\Delta_n}^2 \end{aligned}$$

Therefore, Φ also satisfies the condition 3 of Theorem 4.2. Thus, $\Phi \in D_{\mathcal{A}}$. Moreover,

$$I_n(\widehat{\varphi}_n) = - \int_0^\infty \ell'(h) \left(\int_{\Delta_n(e^{-h})} \psi_n(e^h u) d^{(n)} B_u \right) dh$$

and consequently

$$\mathcal{A}\Phi = - \int_0^\infty Q_h \Psi \ell'(h) \, dh$$

2) Assume now that $\Psi \in L^2$ and $\ell \in C_c^1((0, \infty))$, and set as before

$$\Phi = \int_0^\infty Q_h \Psi \ell(h) \, dh$$

It is easy to see directly that

$$\Phi \in \text{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\Phi = - \int_0^\infty Q_h \Psi \ell'(h) \, dh$$

We can approximate Ψ in L^2 by a sequence $\Psi_p = \sum_{n \geq 0} I_n(\psi_{n,p})$ such that $\psi_{n,p} \in C_c^1(\text{int}(\Delta_n))$. We set

$$\Phi_p = \int_0^\infty Q_h \Psi_p \ell(h) \, dh$$

Then, by the first step, $\Phi_p \in D_{\mathcal{A}}$, $\mathcal{A}\Phi_p = - \int_0^\infty Q_h \Psi_p \ell'(h) \, dh$, and therefore

$$\lim_{p \rightarrow \infty} (\Phi_p, \mathcal{A}\Phi_p) = (\Phi, \mathcal{A}\Phi)$$

in $L^2 \times L^2$.

3) Finally, let $\Phi \in \text{dom}(\mathcal{A})$. We consider a sequence (ℓ_p) in $C_c^1((0, \infty))$ such that, for every p ,

$$\ell_p \geq 0, \quad \ell_p(h) = 0 \text{ on } [1/p, \infty), \quad \int_0^\infty \ell_p(h) \, dh = 1$$

We set

$$\Phi_p = \int_0^\infty Q_h \Phi \ell_p(h) \, dh$$

We have clearly

$$\Phi_p \in \text{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\Phi_p = \int_0^\infty Q_h \mathcal{A}\Phi \ell_p(h) \, dh$$

and therefore

$$\lim_{p \rightarrow \infty} (\Phi_p, \mathcal{A}\Phi_p) = (\Phi, \mathcal{A}\Phi)$$

in $L^2 \times L^2$.

The result follows from both previous approximations. □

We now give a large subset of $\text{dom}(\mathcal{A})$; it consists in the functions appearing in Proposition 3.9.

Proposition 4.4 *We assume that ℓ is an absolutely continuous function on $(0, 1]$ satisfying*

$$\int_0^1 u^{3/2} |\ell'(u)| \, du < \infty$$

Let $F \in L^2$ such that $\mathbb{E}(F) = 0$ and

$$\Phi = \int_0^1 F_u^\# \ell(u) \, du$$

Then

$$\Phi \in \text{dom}(\mathcal{A}) \quad \text{and} \quad \mathcal{A}\Phi = \int_0^1 F_u^\# [\ell(u) + u \ell'(u)] \, du - \ell(1)F$$

Proof

We saw, in the proof of Proposition 3.9, that

$$\|F_u^\#\|_{L^2} \leq \sqrt{u} \|F\|_{L^2} \quad \text{and} \quad \int_0^1 u^{1/2} |\ell(u)| \, du < \infty$$

In particular, $\Phi \in L^2$. We have, for $h > 0$,

$$\frac{1}{h} (Q_h \Phi - \Phi) = \frac{1}{h} \int_0^1 (F_{e^{-h}u}^\# - F_u^\#) \ell(u) \, du$$

Therefore,

$$\frac{1}{h} (Q_h \Phi - \Phi) = \frac{1}{h} \left(\int_0^{e^{-h}} F_u^\# [e^h \ell(e^h u) - \ell(u)] \, du - \int_{e^{-h}}^1 F_u^\# \ell(u) \, du \right)$$

Therefore, letting h tend to 0, we get the result. □

Remark

The previous result still holds if $\mathbb{E}(F) \neq 0$, provided we assume that ℓ satisfies the additional condition:

$$\int_0^1 |\ell(u)| \, du < \infty$$

4.3 A Markovian process with semigroup (Q_h)

In this subsection, we shall associate a Markov process with the semigroup Q . We adopt, for this subsection, the following notation.

We denote by $(W(s, t), s \geq 0, t \geq 0)$ a standard Brownian sheet, and we set, for $h \geq 0$,

$$\widehat{\mathcal{W}}_h = \sigma\{W(u, v); 0 \leq u \leq 1, 1 \leq v \leq e^h\}$$

We define a process $(Y^h, h \geq 0)$ taking values in the Wiener space E by:

$$\forall u \in [0, 1] \quad Y_u^h = W(e^{-h}u, e^h)$$

Proposition 4.5 *For $h \geq 0, k \geq 0$, and for any $\Phi \in L^2$, one has:*

$$\mathbb{E}[\Phi(Y^{h+k}) | \widehat{\mathcal{W}}_k] = Q_h \Phi(Y^k)$$

Proof

We have

$$\Phi(Y^{h+k}) = \Phi(W(e^{-(h+k)}\bullet, e^k) + (W(e^{-(h+k)}\bullet, e^{h+k}) - W(e^{-(h+k)}\bullet, e^k)))$$

Therefore,

$$\mathbb{E}[\Phi(Y^{h+k}) | \widehat{\mathcal{W}}_k] = \mathbb{E}_{(\tilde{B})}[\Phi(Y_{e^{-h}\bullet}^k + \sqrt{1 - e^{-h}} \tilde{B})]$$

where \tilde{B} denotes a Brownian motion independent of W . Hence the result follows by the definition of Q_h and formula (9). □

We remark that the process Y^h is nothing else but $\sigma_{e^{-h}}(X^h)$, where (X^h) denotes the classical Ornstein-Uhlenbeck process in the Wiener space, and σ_t denotes as previously the scaling operator with parameter t .

5 Spaces \mathcal{M} , \mathcal{V} and \mathcal{S}

In this section, we are interested in description and properties of the following spaces:

$$\begin{aligned} \mathcal{M} &= \{\Phi \in L^2; \Phi^\sharp \text{ is a } (\mathcal{B}_t)\text{-martingale}\} \\ \mathcal{V} &= \{\Phi \in L^2; \Phi^\sharp \text{ is a continuous process with finite variation on } [0, 1]\} \\ \mathcal{S} &= \{\Phi \in L^2; \Phi^\sharp \text{ is a } (\mathcal{B}_t)\text{-semi-martingale}\} \end{aligned}$$

5.1 Space \mathcal{M}

5.1.1 On Itô's integrand for $\Phi \in \mathcal{M}$

Theorem 5.1 *Let $\Phi \in L^2$ given by its predictable representation*

$$\Phi = a + \int_0^1 H_u dB_u$$

Then $\Phi \in \mathcal{M}$ if and only if the following condition is fulfilled:

There exists a version of H which is L^2 -continuous on $[0, 1)$ and satisfies

$$\forall u \in [0, 1), \forall t \in [0, 1], \quad (H_u)_t^\# = H_{ut}$$

Proof

By Proposition 3.11, $\Phi \in \mathcal{M}$ if and only if

$$\forall t \in (0, 1] \quad (H_{u/t})_t^\# = H_u \text{ for almost every } u \in [0, t]$$

or

$$\forall t \in (0, 1] \quad (H_u)_t^\# = H_{ut} \text{ for almost every } u \in [0, 1]$$

The condition of the theorem is therefore sufficient.

Conversely, suppose that $\Phi \in \mathcal{M}$. Then, by Fubini's theorem, for almost every $u \in (0, 1]$,

$$(H_u)_t^\# = H_{ut} \text{ for almost every } t \in [0, 1]$$

or

$$(H_u)_{t/u}^\# = H_t \text{ for almost every } t \in [0, u]$$

Then, considering a sequence (u_n) tending to 1 and such that the above property holds for each u_n , we see, according to Theorem 3.1, i), that there exists a version of H which is L^2 -continuous on each interval $[0, u_n]$ and hence on $[0, 1)$. For such a version,

$$\forall t \in (0, 1] \quad (H_u)_t^\# = H_{ut} \text{ for every } u \in [0, 1)$$

Letting t tend to 0, we also have

$$(H_u)_0^\# = H_0 \text{ for every } u \in [0, 1)$$

Thus, the condition of the theorem is necessary. □

5.1.2 Definition of the space \mathcal{N}

We now introduce

$$\mathcal{N} = \left\{ \Phi = a + \int_0^1 F_u^\# dB_u ; F \in L^2 \text{ and } a \in \mathbb{R} \right\}$$

As a consequence of Theorem 5.1 or of Corollary 3.11.2, we have: $\mathcal{N} \subset \mathcal{M}$.

The following proposition clarifies the situation in the framework of the example 3.4.1.

Proposition 5.2 *Let $f \in L^2(\gamma_1)$ and $\Phi(B) = f(B_1)$. Then $\Phi \in \mathcal{N}$ if and only if the function f is absolutely continuous on \mathbb{R} and its derivative f' belongs to $L^2(\gamma_1)$. In this case,*

$$\Phi(B) = \mathbb{E}_{\gamma_1}(f) + \int_0^1 F_u^\# dB_u$$

with $F(B) = f'(B_1)$.

Proof

Let $(h_n)_{n \geq 0}$ be the sequence of Hermite polynomials. As $f \in L^2(\gamma_1)$, f admits the following expansion in $L^2(\gamma_1)$:

$$f = \sum_{n \geq 0} a_n h_n \quad \text{with} \quad \sum_{n \geq 0} \frac{1}{n!} a_n^2 < \infty$$

Let $\Phi(B) = f(B_1)$. Then

$$\Phi = a_0 + \int_0^1 H_s dB_s$$

with

$$H_s = \sum_{n \geq 1} a_n \int_{\Delta_{n-1}(s)} d^{(n-1)} B_u$$

Therefore

$$\|H_s\|_{L^2}^2 = \sum_{n \geq 1} \frac{1}{(n-1)!} a_n^2 s^{n-1}$$

Consequently, $\Phi \in \mathcal{N}$ if and only if $\sum_{n \geq 1} \frac{1}{(n-1)!} a_n^2 < \infty$.

Equivalently, $\Phi \in \mathcal{N}$ if and only if f belongs to the domain of the canonical Dirichlet form on $L^2(\gamma_1)$, that is if and only if f is an absolutely continuous function on \mathbb{R} such that f and f' belong to $L^2(\gamma_1)$ (see, e.g., Bouleau-Hirsch

[2]).

In this case, we have $H_s = F_s^\sharp$ with

$$\begin{aligned} F(B) &= \sum_{n \geq 1} a_n \int_{\Delta_{n-1}} d^{(n-1)} B_u \\ &= \sum_{n \geq 1} a_n h_{n-1}(B_1) = f'(B_1) \end{aligned}$$

□

5.1.3 Comparison of \mathcal{M} and \mathcal{N}

Theorem 5.3 *The following properties hold:*

i) $\mathcal{N} \subset \mathcal{M}$

ii) $\mathcal{N} \neq \mathcal{M}$

iii) \mathcal{M} is the closure of \mathcal{N} in L^2

Proof

As it was already mentioned, the property i) is contained in Corollary 3.11.2. It also is a consequence of Theorem 5.1.

The property ii) is a direct consequence of the previous proposition 5.2 (consider for example $\Phi(B) = f(B_1)$ with f the indicator function of \mathbb{R}_+).

As \mathcal{M} is closed, the property i) entails that \mathcal{M} contains the closure of \mathcal{N} . Suppose then that $\Phi \in \mathcal{M}$ and consider its predictable representation fulfilling the condition of Theorem 5.1. We set, for $0 < v < 1$,

$$\Phi^{(v)} = a + \int_0^1 H_{uv} dB_u$$

By definition, $\Phi^{(v)} \in \mathcal{N}$ and

$$\|\Phi^{(v)} - \Phi\|_{L^2}^2 = \int_0^1 \|H_{uv} - H_u\|_{L^2}^2 du$$

Now, for $u \in [0, 1)$,

$$\lim_{v \rightarrow 1} \|H_{uv} - H_u\|_{L^2}^2 = 0 \quad \text{and}$$

$$\forall 0 < v < 1 \quad \|H_{uv}\|_{L^2}^2 = \|(H_u)_v^\sharp\|_{L^2}^2 \leq \|H_u\|_{L^2}^2$$

Therefore

$$\lim_{v \rightarrow 1} \|\Phi^{(v)} - \Phi\|_{L^2}^2 = 0$$

□

5.1.4 Comparison of \mathcal{M} and \mathcal{I}

In Subsection 3.4.2, we introduced the set

$$\mathcal{I} = \{ \Phi \in L^2 ; (\Phi_t^\sharp(B), t \leq 1) \text{ and } (\Phi_t^m(W), t \leq 1) \text{ are identical in law} \}$$

and we gave examples of functions in \mathcal{I} . It is immediate, from the discussion following Proposition 3.7, that if h is assumed regular, then Φ_h in that proposition belongs to \mathcal{M} if and only if it is constant. Thus, \mathcal{I} is not contained in \mathcal{M} . Conversely, a natural question is to decide whether the set \mathcal{M} is contained in \mathcal{I} . First, we note the following simple identity.

Lemma 5.4 *For any bounded Borel function $h : [0, 1] \rightarrow \mathbb{R}$ and $\Phi \in \mathcal{M}$, there is the identity*

$$\mathbb{E} \left[\left(\int_0^1 h(t) d\Phi_t^m \right)^2 \right] = \mathbb{E} \left[\left(\int_0^1 h(t) d\Phi_t^\sharp \right)^2 \right] \quad (16)$$

Proof

As (Φ_t^m) and (Φ_t^\sharp) are square-integrable continuous martingales, we have:

$$\mathbb{E} \left[\left(\int_0^1 h(t) d\Phi_t^m \right)^2 \right] = \mathbb{E} \left[\int_0^1 h^2(t) d\langle \Phi^m \rangle_t \right] = \int_0^1 h^2(t) d_t \mathbb{E}[(\Phi_t^m)^2]$$

and similarly

$$\mathbb{E} \left[\left(\int_0^1 h(t) d\Phi_t^\sharp \right)^2 \right] = \mathbb{E} \left[\int_0^1 h^2(t) d\langle \Phi^\sharp \rangle_t \right] = \int_0^1 h^2(t) d_t \mathbb{E}[(\Phi_t^\sharp)^2]$$

Consequently the identity (16) holds, since Φ^m and Φ^\sharp have the same one dimensional marginals.

□

Despite Lemma 5.4, the following proposition shows that, in general, the elements of \mathcal{N} do not belong to \mathcal{I} .

Proposition 5.5 *Let $0 < a < 1$. We set*

$$\Phi(B) = \int_0^1 B_{au} dB_u$$

Then, $\Phi \in \mathcal{N}$ and $\Phi \notin \mathcal{I}$. More precisely, if $0 < t < 1$,

$$\mathbb{E}[(\Phi)^2 \Phi_t^\sharp] \neq \mathbb{E}[(\Phi_1^m)^2 \Phi_t^m]$$

Proof

Obviously, since $B_{au} = F_u^\sharp(B)$ with $F(B) = B_a$, $\Phi \in \mathcal{N}$.

We have:

$$\Phi_t^\sharp(B) = \int_0^t B_{au} dB_u \quad \text{and} \quad \Phi_t^m(W) = \int_0^1 W_{au,t} d_u W_{u,t}$$

Now, a tedious computation yields:

$$\mathbb{E}[(\Phi)^2 \Phi_t^\sharp] = \frac{1}{3} [2a^3 t^3 + (t \wedge a)^3 + 3t^2(a - a \wedge t)]$$

whereas:

$$\mathbb{E}[(\Phi_1^m)^2 \Phi_t^m] = a^3 t^2$$

which entails

$$\mathbb{E}[(\Phi)^2 \Phi_t^\sharp] > \mathbb{E}[(\Phi_1^m)^2 \Phi_t^m]$$

□

5.2 Space \mathcal{V}

Proposition 5.6 *We have*

$$\text{dom}(\mathcal{A}) \subset \mathcal{V}$$

Moreover, if $\Phi \in \text{dom}(\mathcal{A})$, the variation of Φ^\sharp on $[0, 1]$ is square-integrable.

Proof

Let $\Phi \in \text{dom}(\mathcal{A})$. Then, for $h \geq 0$,

$$Q_h \Phi = \Phi + \int_0^h Q_k \mathcal{A} \Phi dk$$

So, after change of variable, we get for $t > 0$

$$\Phi_t^\# = \Phi + \int_t^1 (\mathcal{A}\Phi)_s^\# \frac{1}{s} ds$$

As $\mathbb{E}(\mathcal{A}\Phi) = 0$, then $\|(\mathcal{A}\Phi)_s^\#\|_{L^2} \leq \sqrt{s} \|\mathcal{A}\Phi\|_{L^2}$ and the result follows. More precisely, we obtain that the L^2 -norm of the variation of $\Phi^\#$ on $[0, 1]$ is smaller than $(2 \|\mathcal{A}\Phi\|_{L^2})$.

□

As a consequence of the previous proposition and of Proposition 4.4, we obtain again Proposition 3.9.

5.3 Space \mathcal{S}

In this subsection, we give a sufficient condition, based on the chaos expansion, entailing that a function $\Phi \in L^2$ belongs to \mathcal{S} .

We keep the notation of Section 4 and, for $n \geq 1$ and for φ_n a function on Δ_n , we denote by $\tilde{\varphi}_n$ the function defined on Δ'_n by

$$\tilde{\varphi}_n(u) = \varphi_n\left(\frac{1}{u_1} u\right)$$

We remark that

$$\|\tilde{\varphi}_n\|_{\Delta_n}^2 = \frac{1}{n} \|\varphi_n(1, \bullet)\|_{\Delta_{n-1}}^2$$

Theorem 5.7 *Let $\Phi = \sum_{n \geq 0} I_n(\varphi_n) \in L^2$ such that*

- 1- *for any $n \geq 1$, φ_n is continuous on Δ'_n , φ_n is of class C^1 on $\text{int}(\Delta_n)$ and $\varphi_n(1, \bullet)$ is of class C^1 on $\text{int}(\Delta_{n-1})$,*
- 2- $\sum_{n \geq 1} \|\tilde{\varphi}_n\|_{\Delta_n}^2 < \infty$,
- 3- $\sum_{n \geq 1} \|\hat{\varphi}_n\|_{\Delta_n}^2 < \infty$.

Then $\Phi \in \mathcal{S}$. More precisely, $\Phi = \Phi_1 + \Phi_2$ with

$$\Phi_1 = \mathbb{E}(\Phi) + \sum_{n \geq 1} I_n(\tilde{\varphi}_n) \in \mathcal{M} \quad \text{and}$$

$$\Phi_2 = \sum_{n \geq 1} I_n(\varphi_n - \tilde{\varphi}_n) \in D_{\mathcal{A}} \quad (\subset \text{dom}(\mathcal{A}) \subset \mathcal{V})$$

Proof

We can write

$$\Phi_1 = \mathbb{E}(\Phi) + \int_0^1 H_s dB_s$$

with, for $s \in (0, 1)$,

$$H_s = \sum_{n \geq 1} \int_{\Delta_{n-1}(s)} \varphi_n \left(1, \frac{1}{s} v \right) d^{(n)}B_v$$

Hence, we see that Φ_1 satisfies the condition of Theorem 5.1.

On the other hand, it is clear, by the definition of $D_{\mathcal{A}}$ given in Theorem 4.2, that $\Phi_2 \in D_{\mathcal{A}}$. We can then apply the results of this section to conclude that $\Phi \in \mathcal{S}$. □

Remark

If we replace, in the statement of the previous theorem, the condition 2 by the stronger condition:

$$\sum_{n \geq 1} n \|\tilde{\varphi}_n\|_{\Delta_n}^2 < \infty$$

then $\Phi_1 \in \mathcal{N}$.

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