A construction of processes with one-dimensional martingale marginals, associated with a Lévy process, via its Lévy sheet

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Abstract We give some adequate extension, in the framework of a general Lévy process, of our previous construction of processes with one-dimensional martingale marginals, done originally in the set-up of Brownian motion. The Lévy process framework allows us to streamline our previous arguments, as well as to reach a larger class of such processes, even in the Brownian case. We give some illustrations of our construction when the Lévy process is either a Gamma process, or a Poisson process. We also work in the fractional Brownian and stable frameworks.

Key words 1-martingale; convex order; Lévy process; Lévy sheet; space-time harmonic function; fractional Brownian motion; fractional stable process

1 Introduction

1.1 Convex order increase and 1-martingales

This paper is devoted to investigations about two apparently different classes of processes, which are:

(C_1) the class of processes $(U_t\ ,\ t \geq 0)$ which are *increasing in the convex order*, that is: for any $g : \mathbb{R} \longrightarrow \mathbb{R}$ convex, the function: $t \longrightarrow \mathbb{E}[g(U_t)]$ is increasing;

(C_2) the class of processes $(V_t\ ,\ t \geq 0)$ which are *1-martingales*, that is: there exists, on possibly another probability space, a martingale $(M_t\ ,\ t \geq 0)$ such that, for any given $t \geq 0$, $V_t \overset{\text{(law)}}{=} M_t$.

In fact, these two classes coïncide, as it gradually emerged from the papers by Strassen (1965), Doob (1968), and Kellerer (1972). See, especially, Kellerer ([7, p. 120]).
However, the proofs offered in these three papers of the identity between the two classes (C_1) and (C_2) are not constructive, and it is an interesting question, given a process $(U_t\ ,\ t \geq 0)$, or the family of its marginals $(\mu_t\ ,\ t \geq 0)$, which are increasing in the convex order, to find, as explicitly and concretely as possible, a martingale $(M_t\ ,\ t \geq 0)$ which admits the same one-dimensional marginals $(\mu_t\ ,\ t \geq 0)$ as $(U_t\ ,\ t \geq 0)$. A connected question is to exhibit large classes of 1-martingales, and so to obtain large classes of processes which are increasing for the convex order.
These aims have already been the topic of the papers by Madan-Yor [8], Baker-Yor [2], Hirsch-Yor [5].
In the present paper we work towards these goals in a general Lévy process framework.

1.2 The Brownian "guiding example"

In our previous paper [5], we worked in a Brownian motion framework, and used in an essential manner the Wiener (or Brownian) sheet $(W_{u,t}\ ;\ u \geq 0, t \geq 0)$ in order to construct martingales with respect to

$$\mathcal{W}_t = \sigma\{W_{u,s}\ ;\ u \geq 0, s \leq t\}\ , \quad t \geq 0$$

The two key properties we used are:

a) $(W_{\bullet,t}\ ,\ t \geq 0)$ is a Lévy process, taking values in $C([0,\infty);\mathbb{R})$ (in fact, it may be called a $C([0,\infty);\mathbb{R})$-Brownian motion);

b) for any fixed $t \geq 0$, $\quad B_{t\bullet} \overset{\text{(law)}}{=} W_{\bullet,t}$.

The "guiding example" in Baker-Yor [2] and Hirsch-Yor [5] has been the identity in law, which follows from b): for fixed $t > 0$,

$$\text{(G)} \quad \frac{1}{t}\int_0^t \exp\left(\lambda\, B_s - \frac{\lambda^2 s}{2}\right)\,\mathrm{d}s \overset{\text{(law)}}{=} \int_0^1 \exp\left(\lambda\, W_{u,t} - \frac{\lambda^2 ut}{2}\right)\,\mathrm{d}u$$

and the fact that the RHS in (G) is a $(\mathcal{W}_t)$-martingale.

In this paper, we develop some adequate extensions of (G), with Brownian motion being replaced by a general Lévy process (then, λ often needs to be assumed purely imaginary).

Thus, for this purpose, to a Lévy process $(L_t\ ,\ t \geq 0)$, we associate a Lévy sheet $(X_{u,t}\ ;\ u \geq 0, t \geq 0)$ and the exact analogues of a) and b) are satisfied. This is the content of Section 2.

1.3 Extending (G) in the Lévy framework

We now explain how (G) above may be developed in the Lévy framework; we do so in the hope that it will facilitate the reader's understanding of our construction of martingale processes $(\Phi_t^m(X))$ and 1-martingale processes $(\Phi_t^\sharp(L))$, as indicated briefly in Subsection 1.4 below, and developed thoroughly in Sections 3 and 4 of the paper.

Let $\mathbb{D}_0$ be the Skorokhod space consisting of all càdlàg functions ε from $\mathbb{R}_+$ into $\mathbb{R}$ such that $\varepsilon(0) = 0$. Searching for some adequate extension of (G), we would like to find a reasonable class of functionals $U(\varepsilon, s)$ $(\varepsilon \in \mathbb{D}_0\ ,\ s \geq 0)$ such that the process:

$$V_t = \frac{1}{t}\ \int_0^t U(L_{s\bullet}, s)\,\mathrm{d}s\ , \quad t \geq 0$$

is a 1-martingale. We show in particular (see Proposition 4.7) that this is the case for $U(\varepsilon, s) = f(\varepsilon(1), s)$, where $f(x, s)$ is a *space-time harmonic function for L*. In general, concerning V_t, we note that, from b) written for the pair (L, X) instead of (B, W), we get, for fixed t,

$$V_t = \int_0^1 U(L_{u\,t\bullet}, u\,t)\,\mathrm{d}u \overset{\text{(law)}}{=} \int_0^1 U(X_{u\,\bullet,t}, u\,t)\,\mathrm{d}u$$

Thus, in order to show that $(V_t\ ,\ t \geq 0)$ is a 1-martingale, it suffices to find U such that, for any given $u \in (0, 1)$, the process

$$(U(X_{u\,\bullet,t}, u\,t)\ ,\ t \geq 0)$$

is a $(\mathcal{X}_t)$-martingale, where:

$$\mathcal{X}_t = \sigma\{X_{v,s}\ ;\ v \geq 0, s \leq t\}$$

Already, a large class of such functionals U may be obtained by taking (ψ denoting the characteristic exponent of the Lévy process L):

$$U(\varepsilon, s) = \exp\left(\mathrm{i} \int_0^1 h(v)\, \mathrm{d}\varepsilon(v) + s \int_0^1 \psi(h(v))\, \mathrm{d}v\right) \tag{1}$$

for, say, bounded Borel h's. Consequently, the process:

$$\begin{aligned} H_t &= \frac{1}{t} \int_0^t \exp\left(\mathrm{i} \int_0^1 h(v)\, \mathrm{d}_v L_{s\,v} + s \int_0^1 \psi(h(v))\, \mathrm{d}v\right) \mathrm{d}s \\ &= \frac{1}{t} \int_0^t \exp\left(\mathrm{i} \int_0^s h\left(\frac{v}{s}\right)\, \mathrm{d}L_v + \int_0^s \psi\left(h\left(\frac{v}{s}\right)\right)\, \mathrm{d}v\right) \mathrm{d}s \end{aligned} \tag{2}$$

is a 1-martingale.

To present real-valued variants of this construction, we assume that L is a subordinator τ, so that its Lévy-Khintchine representation is:

$$\mathbb{E}[\exp(-\lambda\, \tau_s)] = \exp(-s\, \phi(\lambda))\ , \quad s \geq 0,\ \lambda \geq 0$$

Then, we may modify the previous formula (2) as:

$$K_t = \frac{1}{t} \int_0^t \exp\left(-\int_0^s k\left(\frac{v}{s}\right)\, \mathrm{d}\tau_v + \int_0^s \phi\left(k\left(\frac{v}{s}\right)\right)\, \mathrm{d}v\right) \mathrm{d}s \tag{3}$$

for k a nonnegative bounded Borel function.

We believe that the reader who kept with us throughout this construction should not find the more general set-up for constructing 1-martingales, as it is developed below, either too difficult or too abstract.

1.4 A systematic construction of 1-martingales

We equip the Skorokhod space $\mathbb{D}_0$ with the law $\mathbb{P}$ of L.
In agreement with the preceding discussion in 1.3, we associate, in Sections 3 and 4, to a general functional $\Phi \in L^1(\mathbb{P})$ two processes, the first one being defined in terms of X, and the second one in terms of L: for $0 \leq t \leq 1$,

$$\Phi_t^m(X) = \Pi_{1-t}\Phi(X_{\bullet,t}) \quad \text{and} \quad \Phi_t^\sharp(L) = \Pi_{1-t}\Phi(L_{t\,\bullet})$$

where $(\Pi_s\ ,\ s \geq 0)$ is the semigroup of the $\mathbb{D}_0$-valued Lévy process $(X_{\bullet,t}\ ,\ t \geq 0)$. Two key properties of Φ^m and $\Phi^\sharp$ are:

i) $(\Phi_t^m \ , \ t \leq 1)$ is a $(\mathcal{X}_t)$-martingale;

ii) for any fixed $t \leq 1$, $\quad \Phi_t^m \overset{\text{(law)}}{=} \Phi_t^\sharp$.

Consequently, $(\Phi_t^\sharp \ , \ t \leq 1)$ is a 1-martingale; hence, it is also increasing in the convex order.

1.5 Considering space-time harmonic functions for $(X_{\bullet,t})_{t\geq 0}$

Another manner to express the above property i) is to say that $\Pi_{1-t}\Phi(\varepsilon)$ is a *space-time harmonic function* of $(\varepsilon, t) \in \mathbb{D}_0 \times [0,1]$, for $(X_{\bullet,t})$.
It is then natural to look for some suitable extension of the discussion made in the previous subsection 1.4. This is easy indeed: start with a generic space-time harmonic function $F(\varepsilon, t)$ on $\mathbb{D}_0 \times \mathbb{R}_+$ and consider both processes:

$$F_t^m = F(X_{\bullet,t}, t) \quad \text{and} \quad F_t^\sharp = F(L_{t\bullet}, t)$$

They satisfy:

i') $(F_t^m \ , \ t \geq 0)$ is a $(\mathcal{X}_t)$-martingale;

ii') for any fixed $t \geq 0$, $\quad F_t^m \overset{\text{(law)}}{=} F_t^\sharp$.

1.6 Further examples of 1-martingales

In Section 5, we exhibit examples of 1-martingales defined from stochastic integrals with respect to L. They are closely related to the extension of (G) we discussed in Subsection 1.3.

1.7 Extension to fractional Brownian and α-stable processes

Finally, in Section 6, we show that a slight variation of our method allows to prove that, if $(B_s^H \ , \ s \geq 0)$ denotes the fractional Brownian motion with Hurst index H, then:

$$\frac{1}{t} \int_0^t \exp\left(\lambda\, B_s^H - \frac{\lambda^2}{2}\, s^{2H}\right) \mathrm{d}s \ , \quad t \geq 0$$

is a 1-martingale.
We also present extensions when B^H is replaced by any fractional α-stable process.

2 From a Lévy process L to its Lévy sheet X

In this section, we shall precise our framework and the notation.

2.1 The Lévy-Khintchine representation of L

We start with a real-valued Lévy process $(L_t\ ,\ t \geq 0)$ starting from 0. We denote by ψ its characteristic exponent:

$$\forall \lambda \in \mathbb{R},\ \forall t \geq 0, \quad \mathbb{E}\left[\exp(\mathrm{i}\,\lambda\, L_t)\right] = \exp(-t\,\psi(\lambda))$$

One has (*Lévy-Khintchine formula*):

$$\psi(\lambda) = \sigma^2\,\frac{\lambda^2}{2} + \mathrm{i}\,\gamma\,\lambda + \int \left(1 - \mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\,\lambda\,x\,1_{|x|\leq 1}\right)\ \nu(\mathrm{d}x)$$

with $\sigma, \gamma \in \mathbb{R}$ and ν a positive measure on $\mathbb{R} \setminus \{0\}$ such that

$$\int \frac{x^2}{1+x^2}\ \nu(\mathrm{d}x) < \infty$$

We refer e.g. to Bertoin [1] for a deep study of Lévy processes.

2.2 The Skorokhod space

We denote by $\mathbb{D}_0$ the Skorokhod space consisting of all càdlàg functions ε from $\mathbb{R}_+$ into $\mathbb{R}$ such that $\varepsilon(0) = 0$ (we refer, for example, to Jacod-Shiryaev [6, VI-1]). The space $\mathbb{D}_0$ is equipped with the probability $\mathbb{P}$ which is the law of L. We often identify L with the coordinate process on $\mathbb{D}_0$.
We denote by $(\mathcal{F}_t)$ the natural filtration of L on $(\mathbb{D}_0, \mathbb{P})$ and we set $\mathcal{F} = \mathcal{F}_\infty$. Thus, $\mathcal{F}$ is the Borel σ-field of $\mathbb{D}_0$ completed with respect to $\mathbb{P}$.

2.3 The X-integral of a rectangle R

If $(X_{s,t}\ ;\ s \geq 0, t \geq 0)$ is a real-valued two-parameter process and if

$$R = (s_1, s_2] \times (t_1, t_2]\ , \quad s_1 < s_2,\ t_1 < t_2$$

is a rectangle, we set

$$\Delta_R X = X_{s_2,t_2} - X_{s_1,t_2} - X_{s_2,t_1} + X_{s_1,t_1}$$

and we denote by $|R|$ the area of R:

$$|R| = (s_2 - s_1)(t_2 - t_1)$$

2.4 Defining the Lévy sheet X

The following results, for which we refer for example to Dalang-Walsh [3, Section 2], are essential for our purpose.

Theorem 2.1 *There exists a real-valued two-parameter process $X = (X_{s,t}\ ;\ s \geq 0, t \geq 0)$ satisfying the following properties:*

1)

$$\forall s, t \geq 0, \quad X_{s,0} = X_{0,t} = 0$$

2) Almost surely, for any $s, t \geq 0$, $X_{s,\bullet}$ and $X_{\bullet,t}$ are càdlàg functions on $\mathbb{R}_+$.

3) For all finite sets of disjoint rectangles $R^1, \cdots, R^n$, the random variables $\Delta_{R^1} X, \cdots, \Delta_{R^n} X$ are independent.

4) For any rectangle R,

$$\Delta_R X \overset{\text{(law)}}{=} L_{|R|}$$

The process X will be called *the Lévy sheet associated with L.*
Let, for $t \geq 0$,

$$\mathcal{X}_t = \sigma\{X_{u,v}\ ;\ u \geq 0\ ,\ 0 \leq v \leq t)$$

We summarize, in the following theorem, some straightforward consequences of Theorem 2.1, which we will need in the sequel.

Theorem 2.2 *a) Let $0 \leq t_1 \leq t_2$. Then the process $(X_{s,t_2} - X_{s,t_1}\ ,\ s \geq 0)$ is a Lévy process starting from 0, independent of $\mathcal{X}_{t_1}$, and having the same law as $(L_{(t_2-t_1)\,s}\ ,\ s \geq 0)$.*
In particular, for any fixed $t \geq 0$,

$$X_{\bullet,t} \overset{\text{(law)}}{=} L_{t\,\bullet}$$

b) There is the equality in law:

$$(X_{s,t}\ ;\ s,t \geq 0) \overset{\text{(law)}}{=} (X_{t,s}\ ;\ s,t \geq 0)$$

Thus, a) may be stated with the roles of s and t exchanged.

2.5 Remark

In the sequel, the following elementary fact shall play some important role: Let $\widetilde{L}$ be an independent copy of L. Then, for any $A, B \geq 0$,

$$L_{A\bullet} + \widetilde{L}_{B\bullet} \overset{\text{(law)}}{=} L_{(A+B)\bullet}$$

In particular, this shows that the $\mathbb{D}_0$-valued random variable $L_\bullet$ is infinitely divisible. Theorem 2.2, a), then states that the Lévy sheet X may be understood as the $\mathbb{D}_0$-valued Lévy process $(X_{\bullet,t}\ ,\ t \geq 0)$ such that:

$$X_{\bullet,1} \overset{\text{(law)}}{=} L_\bullet$$

3 The processes $(\Phi_t^m(X)\ ,\ 0 \leq t \leq 1)$

3.1 Some equivalent formulae

To any $\Phi \in L^1(\mathbb{P})$, we associate a process $\Phi^m(X) = (\Phi_t^m(X)\ ,\ 0 \leq t \leq 1)$ by

$$\Phi_t^m(X) = \mathbb{E}\left[\Phi(X_{\bullet,1}) \mid \mathcal{X}_t\right]\ , \quad 0 \leq t \leq 1 \tag{4}$$

By definition, $\Phi^m(X)$ is thus a $(\mathcal{X}_t)$-martingale.
In what follows, we often will denote $\Phi_t^m(X)$ (resp. $\Phi^m(X)$) simply by Φ_t^m (resp. Φ^m).

Theorem 3.1 *For $0 \leq t \leq 1$, the following alternative formulae hold:*

$$\Phi_t^m = \mathbb{E}_{\widetilde{X}}\left[\Phi(X_{\bullet,t} + \widetilde{X}_{\bullet,1-t})\right] \tag{5}$$

$$\Phi_t^m = \mathbb{E}_{\widetilde{L}}\left[\Phi(X_{\bullet,t} + \widetilde{L}_{(1-t)\bullet})\right] \tag{6}$$

$$\Phi_t^m = \Pi_{1-t}\Phi(X_{\bullet,t}) \tag{7}$$

where

- *in (5), $\widetilde{X}$ is an independent copy of X, and $\mathbb{E}_{\widetilde{X}}$ means integrating with respect to $\widetilde{X}$;*
- *in (6), $\widetilde{L}$ is a copy of L, independent of X, and $\mathbb{E}_{\widetilde{L}}$ means integrating with respect to $\widetilde{L}$;*
- *in (7), $(\Pi_t(\varepsilon, \mathrm{d}\eta)\ ,\ t \geq 0)$ denotes the semigroup of $(X_{\bullet,t}\ ,\ t \geq 0)$ viewed as a $\mathbb{D}_0$-valued Lévy process.*

Proof

We have:

$$\Phi(X_{\bullet,1}) = \Phi(X_{\bullet,t} + (X_{\bullet,1} - X_{\bullet,t}))$$

Then, formulae (5) and (6) follow directly from Theorem 2.2.
We obtain formula (7) from (4) simply by the definition of the semigroup (Π_t).

□

3.2 Interpretation in terms of space-time harmonic functions

There is another way to understand the previous definition.
Let $r > 0$. A function F on $\mathbb{D}_0 \times [0, r]$ is called a *space-time harmonic function for* $(X_{\bullet,t}\ ,\ 0 \leq t \leq r)$, if the process

$$(F(X_{\bullet,t}, t)\ ,\ 0 \leq t \leq r)$$

is a martingale. Likewise, a function F on $\mathbb{D}_0 \times [0, +\infty)$ is called a *space-time harmonic function for* $(X_{\bullet,t}\ ,\ t \geq 0)$ if the process

$$(F(X_{\bullet,t}, t)\ ,\ t \geq 0)$$

is a martingale.
Let us mention that H. Föllmer [4] determined the nonnegative space-time harmonic functions for $(W_{\bullet,t}\ ,\ t \geq 0)$ where W is the Brownian sheet.
The definition (4) may be written as:

$$\Phi_t^m = F^{\Phi}(X_{\bullet,t}, t) \tag{8}$$

where F^{Φ}, defined on $\mathbb{D}_0 \times [0, 1]$, is the space-time harmonic function for $(X_{\bullet,t}\ ,\ 0 \leq t \leq 1)$ such that $F^{\Phi}(\varepsilon, 1) = \Phi(\varepsilon)$.
We note that, from formulae (6), (7) and (8), one obtains:

$$F^{\Phi}(\varepsilon, t) = \mathbb{E}_{\widetilde{L}}[\Phi(\varepsilon + \widetilde{L}_{(1-t)\,\bullet})] = \Pi_{1-t}\Phi(\varepsilon) \tag{9}$$

where $\widetilde{L}$ is an independent copy of L.
We are then led to exhibit such space-time harmonic functions.

3.3 Some examples

In the sequel, μ_t denotes the law of L_t.

Proposition 3.2 *Let $u_0 = 0 < u_1 < \cdots < u_n$ and $u = (u_1, \cdots, u_n)$. We set:*

$$\nu_t^u = \bigotimes_{j=1}^{n} \mu_{t(u_j - u_{j-1})}$$

We consider $f \in L^1(\nu_1^u)$ and

$$\Phi(\varepsilon) = f(\varepsilon(u_1), \varepsilon(u_2) - \varepsilon(u_1), \cdots, \varepsilon(u_n) - \varepsilon(u_{n-1}))$$

Then

$$F^\Phi(\varepsilon, t) = \int_{\mathbb{R}^n} f(\varepsilon(u_1)+y_1, \varepsilon(u_2)-\varepsilon(u_1)+y_2, \cdots, \varepsilon(u_n)-\varepsilon(u_{n-1})+y_n)\ \nu_{1-t}^u(\mathrm{d}y)$$

Proof

This is a straightforward consequence of formula (9), since ν_t^u is the law of

$$(L_{tu_1}, L_{tu_2} - L_{tu_1}, \cdots, L_{tu_n} - L_{tu_{n-1}})$$

□

Corollary 3.2.1 *Let $r > 0$ and $f \in L^1(\mu_r)$. We set, for $(x,t) \in \mathbb{R} \times [0,1]$,*

$$\widetilde{f}(x,t) = \int_{\mathbb{R}} f(x+y)\ \mu_{r(1-t)}(\mathrm{d}y) \tag{10}$$

If $\Phi(\varepsilon) = f(\varepsilon(r))$, then

$$F^\Phi(\varepsilon, t) = \widetilde{f}(\varepsilon(r), t)$$

Moreover, $(\widetilde{f}(L_{r\,t}, t)\ ,\ 0 \le t \le 1)$ is an $(\mathcal{F}_{r\,t})$-martingale.

Proof

We apply Proposition 3.2 to: $n = 1$, $u_1 = r$. Moreover, we have clearly by (10),

$$\widetilde{f}(L_{r\,t}, t) = \mathbb{E}[f(L_r) \mid \mathcal{F}_{r\,t}]$$

□

Since $(\widetilde{f}(L_{r\,t}, t)\ ,\ 0 \le t \le 1)$ is an $(\mathcal{F}_{r\,t})$-martingale, the functions $\widetilde{f}$ defined by formula (10) may be called *space-time harmonic function for* $(L_{r\,t}\ ,\ 0 \leq t \leq 1)$. We give hereafter an "infinitesimal characterisation" of some of such space-time harmonic functions. We first introduce the space $C_0(\mathbb{R})$ of continuous real functions tending to 0 at infinity. We denote by (P_t) the semigroup on $C_0(\mathbb{R})$ defined by:

$$\forall f \in C_0(\mathbb{R}),\ \forall t \geq 0,\ \forall x \in \mathbb{R}, \quad P_t f(x) = \int f(x+y)\ \mu_t(\mathrm{d}y)$$

Then, $\widetilde{f}(x,t)$ in (10) is $P_{r(1-t)}f(x)$, and (P_t) is a strongly continuous semigroup with infinitesimal generator denoted by A. We denote the domain of A by dom A.

Proposition 3.3 *Let* $H \in C(\mathbb{R} \times [0,1])$ *such that:*

i)

$$\lim_{x\to\infty} \sup_{t\in[0,1]} |H(x,t)| = 0$$

ii)

$$\forall t \in [0,1], \quad H(\bullet, t) \in \textit{dom } A$$

iii)

$$\frac{\partial H}{\partial t}(x,t) \quad \textit{exists on } \mathbb{R} \times [0,1]$$

iv)

$$\forall (x,t) \in \mathbb{R} \times [0,1] \quad \frac{\partial H}{\partial t}(x,t) + r\, A[H(\bullet,t)](x) = 0$$

Then, for all $(x,t) \in \mathbb{R} \times [0,1]$*,*

$$H(x,t) = P_{r(1-t)}(H(\bullet,1))(x)$$

In other words, $H = \widetilde{f}$ *with* $f = H(\bullet,1)$*.*

Proof

We set, for $a > 0$,

$$K(x,t) = P_{r(1-t)}(H(\bullet,1))(x) \quad \text{and} \quad R_a(x,t) = K(x,t) - H(x,t) + a\,t$$

Then, $R_a(x,1) = a$, $\lim_{x\to\infty} R_a(x,t) = a\,t$ uniformly with respect to $t \in [0,1]$, and

$$\forall (x,t) \in \mathbb{R} \times [0,1] \quad \frac{\partial R_a}{\partial t}(x,t) + r\,A[R_a(\bullet,t)](x) = a > 0 \tag{11}$$

Now, if the supremum of R_a on $\mathbb{R} \times [0,1]$ is $> a$, then this supremum is achieved at (x_0,t_0) with $0 \le t_0 < 1$ and $x_0 \in \mathbb{R}$. We then have, by maximum principle,

$$\frac{\partial R_a}{\partial t}(x_0,t_0) \le 0 \quad \text{and} \quad A[R_a(\bullet,t_0)](x_0) \le 0$$

This contradicts (11). Therefore, for any $(x,t) \in \mathbb{R} \times [0,1]$, $R_a(x,t) \le a$. Letting a tend to 0, we obtain $K \le H$ and, likewise, we have $H \le K$.

□

We now present another consequence of Proposition 3.2. We recall that ψ denotes the characteristic exponent of L.

Proposition 3.4 *Let $0 < u_1 < \cdots < u_n$ and $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. We set, for $1 \le j \le n-1$,*

$$\mu_j = \psi(\lambda_j + \lambda_{j+1} + \cdots + \lambda_n) - \psi(\lambda_{j+1} + \cdots + \lambda_n)$$

and $\mu_n = \psi(\lambda_n)$. Let

$$\Phi(\varepsilon) = \exp\left[\mathrm{i}\,\sum_{j=1}^{n} \lambda_j\,\varepsilon(u_j)\right]$$

Then,

$$F^\Phi(\varepsilon,t) = \exp\left[\sum_{j=1}^{n} (\mathrm{i}\,\lambda_j\,\varepsilon(u_j) - (1-t)\mu_j u_j)\right]$$

Proof

We can write

$$\Phi(\varepsilon) = \exp\left[\mathrm{i}\,\sum_{j=1}^{n} \nu_j\,(\varepsilon(u_j) - \varepsilon(u_{j-1}))\right]$$

with $\nu_j = \lambda_j + \cdots + \lambda_n$ and $u_0 = 0$. Therefore, by Proposition 3.2,

$$F^{\Phi}(\varepsilon, t) = \Phi(\varepsilon) \ \exp\left[-(1-t) \sum_{j=1}^{n} \psi(\nu_j)\,(u_j - u_{j-1})\right]$$

Moreover,

$$\sum_{j=1}^{n} \psi(\nu_j)\,(u_j - u_{j-1}) = \sum_{j=1}^{n} \mu_j\, u_j$$

□

The same kind of computation as above yields both following propositions. The notation is the same as in the previous proposition.

Proposition 3.5 *For any $t \geq 0$,*

$$\int_{\mathbb{D}_0} \exp\left[\mathrm{i} \ \sum_{j=1}^{n} \lambda_j\, \eta(u_j)\right] \ \Pi_t(\varepsilon, \mathrm{d}\eta) = \exp\left[\sum_{j=1}^{n} (\mathrm{i}\ \lambda_j\, \varepsilon(u_j) - t\, \mu_j\, u_j)\right]$$

This therefore determines $\Pi_t(\varepsilon, \mathrm{d}\eta)$ from the Fourier transforms of its finite-dimensional marginals.

Proposition 3.6 *Let, for $(\varepsilon, t) \in \mathbb{D}_0 \times \mathbb{R}_+$,*

$$F(\varepsilon, t) = \exp\left[\sum_{j=1}^{n} (\mathrm{i}\ \lambda_j\, \varepsilon(u_j) + t\, \mu_j\, u_j)\right]$$

Then F is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \geq 0)$. In other words,

$$\left(\exp\left[\sum_{j=1}^{n} (\mathrm{i}\ \lambda_j\, X_{u_j,t} + t\, \mu_j\, u_j)\right]\ ,\ t \geq 0\right)$$

is a $(\mathcal{X}_t)$-martingale.

The following proposition, which is actually a variant of Corollary 3.2.1, gives other examples of space-time harmonic functions for $(X_{\bullet,t}\ ,\ t \geq 0)$.

Proposition 3.7 *Let $f : \mathbb{R} \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ be a space-time harmonic function for $(L_t\ ,\ t \geq 0)$, i.e. :*

$$(f(L_t, t)\ ,\ t \geq 0) \ \text{is a } (\mathcal{F}_t)\text{-martingale.}$$

Then, for any $u \geq 0$,

$$F_u : (\varepsilon, t) \in \mathbb{D}_0 \times \mathbb{R}_+ \longrightarrow f(\varepsilon(u), u\, t)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \geq 0)$.

Proof

We obviously have:

$$(f(L_{u\,t}, u\,t)\ ,\ t \geq 0) \text{ is a } (\mathcal{F}_{u\,t})_{t\geq 0}\text{-martingale.}$$

Therefore, for $0 \leq s \leq t$,

$$\mathbb{E}_{\widetilde{L}}\left[f(x + \widetilde{L}_{(t-s)\,u}, u\,t)\right] = f(x, u\,s) \quad \mu_{u\,s}(\mathrm{d}x)\text{-a.s.}$$

Now, by Theorem 2.2,

$$\mathbb{E}\left[F_u(X_{\bullet,t}, t) \mid \mathcal{X}_s\right] = \mathbb{E}_{\widetilde{L}}\left[f(X_{u,s} + \widetilde{L}_{(t-s)\,u}, u\,t)\right]$$

The result then follows from the fact that the law of $X_{u,s}$ is $\mu_{u\,s}$. □

Corollary 3.7.1 *Let f be as in the previous proposition. Then the function:*

$$F : (\varepsilon, t) \in \mathbb{D}_0 \times \mathbb{R}_+ \longrightarrow \int_0^1 f(\varepsilon(u), u\,t)\,\mathrm{d}u$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ t \geq 0)$.

4 The processes $(\Phi^\sharp_t(L)\ ,\ 0 \leq t \leq 1)$

4.1 Definition and relation with $(\Phi^m_t(X)\ ,\ 0 \leq t \leq 1)$

To any $\Phi \in L^1(\mathbb{P})$, we now associate a process $\Phi^\sharp(L) = (\Phi^\sharp_t(L)\ ,\ 0 \leq t \leq 1)$ by

$$\Phi^\sharp_t(L) = \mathbb{E}_{\widetilde{L}}\left[\Phi(L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet})\right]\ , \quad 0 \leq t \leq 1 \tag{12}$$

where $\widetilde{L}$ is an independent copy of L, and $\mathbb{E}_{\widetilde{L}}$ means integrating with respect to $\widetilde{L}$.

In what follows, we will often denote $\Phi^\sharp_t(L)$ (resp. $\Phi^\sharp(L)$) simply by $\Phi^\sharp_t$ (resp. $\Phi^\sharp$). Moreover, we identify in our notation, the coordinate process on $\mathbb{D}_0$ with the process L, that is we identify $\varepsilon(\bullet)$ with $L_\bullet$. We have:

$$\Phi^\sharp_0 = \mathbb{E}(\Phi),\ \ \Phi^\sharp_1 = \Phi,\ \ \mathbb{E}(\Phi^\sharp_t) = \mathbb{E}(\Phi)$$

Theorem 4.1 *We have, with the notation of Section 3,*

$$\Phi_t^\sharp = \Pi_{1-t}\Phi(L_{t\,\bullet}) = F^\Phi(L_{t\,\bullet}, t) \tag{13}$$

In particular, for any given $t \in [0,1]$,

$$\Phi_t^\sharp \overset{\text{(law)}}{=} \Phi_t^m$$

Hence, $\Phi^\sharp$ is a 1-martingale.

Proof

Formula (13) follows directly from (6), (7), (9) and (12). Now, since $L_{t\,\bullet}$ has the same law as $X_{\bullet,t}$, we clearly have from (7) and (13): $\Phi_t^\sharp \overset{\text{(law)}}{=} \Phi_t^m$.

□

4.2 A chaos decomposition formula for $\Phi_t^\sharp$ in the Poisson case

We first recall that, among Lévy processes, only Brownian motion (with drift) and the Poisson process enjoy the chaos decomposition property. In the case $L = B$, we presented in Hirsch-Yor [5] a formula for $\Phi_t^\sharp(B)$, based on the chaos decomposition of Φ. We now derive such a formula when $L = N$ is the standard Poisson process, or rather (equivalently), when $(L_t = N_t - t\ ,\ t \geq 0)$ is the centered Poisson process. In this case, any $\Phi(L)$ which belongs to $L^2(\mathbb{P})$ may be written as:

$$\Phi(L) = \mathbb{E}(\Phi) + \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}L_{s_1} \int_0^{s_1-} \mathrm{d}L_{s_2} \cdots \int_0^{s_{n-1}-} \mathrm{d}L_{s_n}\ \varphi_n(s_1, \cdots, s_n) \tag{14}$$

with the sequence (φ_n) satisfying

$$\sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n\ \varphi_n^2(s_1, \cdots, s_n) < \infty$$

We may then deduce from formula (12) the following representation of $\Phi_t^\sharp(L)$.

Proposition 4.2 *Let $\Phi \in L^2(\mathbb{P})$ and $L = (N_t - t\ ,\ t \geq 0)$. Then, with the previous notation, the following formula holds for t fixed, $0 \leq t \leq 1$:*

$$\Phi_t^\sharp(L) = \mathbb{E}(\Phi) + \sum_{n=1}^{\infty} \int_0^\infty \mathrm{d}L_{t\,s_1} \int_0^{s_1-} \mathrm{d}L_{t\,s_2} \cdots \int_0^{s_{n-1}-} \mathrm{d}L_{t\,s_n}\ \varphi_n(s_1, \cdots, s_n)]$$

Proof

Combining formulae (12) and (14), we obtain the formula stated in the Proposition, since in formula (12), the expectation with respect to $\widetilde{L}$ of any stochastic integral involved is equal to 0, as $\widetilde{L}$ is a martingale.

Note that we might also use Proposition 5.5 below, and reason by induction on the order of the chaos.

□

4.3 Some examples

We now present examples corresponding to those given in Subsection 3.3.

Proposition 4.3 *We keep the notation and hypotheses of Corollary 3.2.1. Then, for* $0 \leq t \leq 1$,

$$\Phi_t^\sharp = \widetilde{f}(L_{r\,t}, t) \quad \textit{and} \quad \Phi_t^m = \widetilde{f}(X_{r,t}, t)$$

Consequently, in this particular case, the <u>*processes*</u> $\Phi^\sharp$ *and* Φ^m *have the same law.*

The following proposition is a direct consequence of Proposition 3.4 and of formula (13).

Proposition 4.4 *Let* $0 < u_1 < \cdots < u_n$ *and* $\lambda_1, \cdots, \lambda_n \in \mathbb{R}$. *We set as in Proposition 3.4, for* $1 \leq j \leq n-1$,

$$\mu_j = \psi(\lambda_j + \lambda_{j+1} + \cdots + \lambda_n) - \psi(\lambda_{j+1} + \cdots + \lambda_n)$$

and $\mu_n = \psi(\lambda_n)$. *Let*

$$\Phi^{\lambda,u} = \exp\left[\sum_{j=1}^{n} \big(\mathrm{i}\,\lambda_j\,L_{u_j} + \mu_j\,u_j\big)\right]$$

Then, for $0 \leq t \leq 1$,

$$(\Phi^{\lambda,u})_t^\sharp = \Phi^{\lambda,t\,u} = \exp\left[\sum_{j=1}^{n} \big(\mathrm{i}\,\lambda_j\,L_{t\,u_j} + t\,\mu_j\,u_j\big)\right]$$

Corollary 4.4.1 *Let, for $a > 0$,*

$$\Delta_n^a = \{u = (u_1, \cdots, u_n)\ ;\ 0 < u_1 < \cdots < u_n \leq a\}$$

We set

$$\Phi = \int_{\Delta_n^a} \Phi^{\lambda,u}\, \mathrm{d}u = \int_{\Delta_n^a} \exp\left[\sum_{j=1}^{n} \big(\mathrm{i}\,\lambda_j\, L_{u_j} + \mu_j\, u_j\big)\right]\, \mathrm{d}u$$

Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = t^{-n} \int_{\Delta_n^{ta}} \Phi^{\lambda,u}\, \mathrm{d}u = t^{-n} \int_{\Delta_n^{ta}} \exp\left[\sum_{j=1}^{n} \big(\mathrm{i}\,\lambda_j\, L_{u_j} + \mu_j\, u_j\big)\right]\, \mathrm{d}u$$

In the particular case $n = 1$, we have the following extension of our guiding example in Subsection 1.2.

Corollary 4.4.2 *Let $\lambda \in \mathbb{R}$ and $a > 0$. If*

$$\Phi = \int_0^a \exp\left(\mathrm{i}\,\lambda\, L_u + \psi(\lambda)\, u\right)\, \mathrm{d}u$$

then, for any $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^{at} \exp\left(\mathrm{i}\,\lambda\, L_u + \psi(\lambda)\, u\right)\, \mathrm{d}u$$

and $\Phi_0^\sharp = a$.

In some cases, it is possible to replace the above exponentials with purely imaginary arguments by real-valued exponentials.

Lemma 4.5 *Suppose that for some $r > 0$ and $\lambda \in \mathbb{R}$,*

$$\mathbb{E}\left[\exp(\lambda\, L_r)\right] < \infty$$

Then, there exists $\phi(\lambda) \in \mathbb{R}$ such that

$$\forall t \geq 0 \quad \mathbb{E}\left[\exp(\lambda\, L_t)\right] = \exp\left(t\, \phi(\lambda)\right)$$

Proof

Properties of Lévy processes imply that, for any $q \in \mathbb{Q}_+$,

$$\mathbb{E}\left[\exp(\lambda\, L_{q\,r})\right] = \exp\left(q\, r\, \phi(\lambda)\right)$$

with $\phi(\lambda) = r^{-1}\ \log\left\{\mathbb{E}\left[\exp(\lambda\, L_r)\right]\right\}$.
By the right continuity of L and Fatou's Lemma,

$$\forall t \geq 0 \quad \mathbb{E}\left[\exp(\lambda\, L_t)\right] \leq \exp\left(t\, \phi(\lambda)\right) < \infty$$

Therefore the function

$$\ell : t \geq 0 \longrightarrow \mathbb{E}\left[\exp(\lambda\, L_t)\right] \in \mathbb{R}$$

is a l.s.c function satisfying

$$\forall t, s \geq 0 \quad \ell(t+s) = \ell(t)\, \ell(s)$$

Therefore ℓ is an exponential and

$$\forall t \geq 0 \quad \mathbb{E}\left[\exp(\lambda\, L_t)\right] = \exp\left(t\, \phi(\lambda)\right)$$

□

Proposition 4.6 *Suppose the condition in Lemma 4.5 is satisfied and let, for $u > 0$,*

$$\Phi^u = \exp(\lambda\, L_u - u\, \phi(\lambda))$$

Then, for $0 \leq t \leq 1$,

$$(\Phi^u)^\sharp_t = \exp(\lambda\, L_{t\,u} - t\, u\, \phi(\lambda))$$

As a consequence, if

$$\Phi = \int_0^a \exp(\lambda\, L_u - u\, \phi(\lambda))\ \mathrm{d}u$$

then, for $0 < t \leq 1$,

$$\Phi^\sharp_t = \frac{1}{t}\ \int_0^{at} \exp(\lambda\, L_u - u\, \phi(\lambda))\ \mathrm{d}u$$

According to Subsection 1.5, we may, by Proposition 3.6, extend the definition of the processes $(\Phi_t^\sharp\ ,\ 0 \leq t \leq 1)$ in Proposition 4.4, Corollary 4.4.1, Corollary 4.4.2 and Proposition 4.6, to any $t \geq 0$, by the same formulae, and the processes thus defined are 1-martingales on $\mathbb{R}_+$. Other examples of 1-martingales on $\mathbb{R}_+$ are obtained as a direct consequence of Corollary 3.7.1:

Proposition 4.7 *Let f be, as in Proposition 3.7, a space-time harmonic function for $(L_t\ ,\ t \geq 0)$. Then*

$$\left(\frac{1}{t}\int_0^t f(L_s, s)\,\mathrm{d}s\ ,\ t \geq 0\right)$$

is a 1-martingale.

Proof

We write this process as:

$$\left(\int_0^1 f(L_{t\,u}, u\,t)\,\mathrm{d}u\ ,\ t \geq 0\right)$$

Now, for fixed $t \geq 0$,

$$\int_0^1 f(L_{t\,u}, u\,t)\,\mathrm{d}u \overset{\text{(law)}}{=} \int_0^1 f(X_{u,t}, u\,t)\,\mathrm{d}u$$

and we may apply Corollary 3.7.1.

□

Here are two examples of application of the above proposition, for two particular cases of Lévy processes, namely $(\gamma_t\ ,\ t \geq 0)$ a standard Gamma process, and $(N_t\ ,\ t \geq 0)$ a standard Poisson process.
We recall (see, e.g., Schoutens [9]) the following generating function:

$$(1+\lambda)^u \exp(-\lambda\, g) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\,\widetilde{C}_n(u,g)\ ,\quad u, g, \lambda \geq 0 \tag{15}$$

where $\{\widetilde{C}_n(u,g)\ ,\ n \in \mathbb{N}\}$ denotes the sequence of Charlier polynomials. Now, we note that, with an obvious terminology: for every $\lambda \geq 0$,

$$\{(1+\lambda)^t \exp(-\lambda\,\gamma_t)\ ,\ t \geq 0\} \text{ is a Gamma martingale,}$$

$$\text{and } \{(1+\lambda)^{N_t} \exp(-\lambda\, t)\ ,\ t \geq 0\} \text{ is a Poisson martingale.}$$

Consequently, from (15), for any $n \in \mathbb{N}$, $\{\widetilde{C}_n(t, \gamma_t)\ ,\ t \geq 0\}$ and $\{\widetilde{C}_n(N_t, t)\ ,\ t \geq 0\}$ are, respectively, a Gamma martingale, and a Poisson martingale. Finally, from Proposition 4.7, we obtain that, for any $n \in \mathbb{N}$,

$$\left(\frac{1}{t}\int_0^t \widetilde{C}_n(s, \gamma_s)\, \mathrm{d}s\ ,\ t \geq 0\right) \quad \text{and} \quad \left(\frac{1}{t}\int_0^t \widetilde{C}_n(N_s, s)\, \mathrm{d}s\ ,\ t \geq 0\right)$$

are 1-martingales.

4.4 Some properties of the map: $\Phi \longrightarrow \Phi^\sharp$

In this subsection, we present some general results concerning the map: $\Phi \longrightarrow \Phi^\sharp$. They extend those obtained by Hirsch-Yor [5] in the Brownian setting, that is when $L = B$ is a Brownian motion.

In what follows, we denote, for $1 \leq p < \infty$, simply by L^p the L^p-space with respect to $\mathbb{P}$. For $0 \leq r \leq \infty$, $L^p(\mathcal{F}_r)$ denotes the subspace of L^p consisting of those functions which are $\mathcal{F}_r$-measurable. $\|\ \|_p$ denotes the L^p-norm.

Theorem 4.8 *Let $1 \leq p < \infty$, $r \in [0, \infty]$ and $\Phi \in L^p(\mathcal{F}_r)$. Then,*

1) for any $t \in [0,1]$, $\Phi^\sharp_t \in L^p(\mathcal{F}_{t\,r})$ (with the usual convention: $t\,\infty = \infty$ if $t \neq 0$ and $0\,\infty = 0$);

2) for any $t \in [0,1]$, $\|\,\Phi^\sharp_t\,\|_p \leq \|\,\Phi\,\|_p$;

3) the map:

$$t \in [0,1] \longrightarrow \Phi^\sharp_t \in L^p$$

is continuous;

4) for any $t, s \in [0,1]$,

$$(\Phi^\sharp_t)^\sharp_s = \Phi^\sharp_{t\,s}$$

Proof

Property 1) is clear by definition of $\Phi^\sharp$ (formula (12)).

We also have by (12):

$$|\Phi^\sharp_t|^p \leq \mathbb{E}_{\widetilde{L}}\left[|\Phi|^p(L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet})\right]$$

Therefore,

$$\|\,\Phi^\sharp_t\,\|_p^p \leq \mathbb{E}_{(L,\widetilde{L})}\left[|\Phi|^p(L_{t\,\bullet} + \widetilde{L}_{(1-t)\,\bullet})\right]$$

By the remark in Subsection 2.5, the right hand side is equal to $\|\,\Phi\,\|_p^p$.

Let now $h_1, \cdots, h_n$ be functions with compact support in $[0, \infty)$, integrable with respect to the Lebesgue measure, and let φ be a bounded continuous function on $\mathbb{R}^n$. We consider

$$\Phi = \varphi\left(\int_0^\infty L_u\, h_1(u)\, \mathrm{d}u, \cdots, \int_0^\infty L_u\, h_n(u)\, \mathrm{d}u\right)$$

Then

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}}\left[\varphi\left(\int_0^\infty L_{tu}\, h_1(u)\, \mathrm{d}u + \int_0^\infty \widetilde{L}_{(1-t)u}\, h_1(u)\, \mathrm{d}u, \cdots\right)\right]$$

Since the paths of L and $\widetilde{L}$ have a countable set of discontinuities, a dominated convergence argument yields:

$$\forall t_0 \geq 0 \quad \lim_{t \to t_0} \Phi_t^\sharp = \Phi_{t_0}^\sharp \quad \text{a.s.}$$

and, therefore, L^p-continuity also holds. Now, the functions Φ of the above kind are dense in L^p, which, thanks to Property 2), implies Property 3) .

By formula (12) again,

$$(\Phi_t^\sharp)_s^\sharp = \mathbb{E}_{(\widehat{L}, \widetilde{L})}\left[\Phi(L_{ts\bullet} + \widehat{L}_{t(1-s)\bullet} + \widetilde{L}_{(1-t)\bullet})\right]$$

where $(L, \widehat{L}, \widetilde{L})$ are three independent copies. According to Subsection 2.5, $\widehat{L}_{t(1-s)\bullet} + \widetilde{L}_{(1-t)\bullet}$ has the same law as $L_{(1-ts)\bullet}$ and is independent of L. Property 4) therefore follows from (12).

□

4.5 Some further examples

4.5.1

The following example is the analogue of that in Hirsch-Yor [5, § 3.4.3].

Proposition 4.9 *Let* $F \in L^1(\mathbb{P})$ *and* ℓ *be a Borel function on* $[0, 1]$.

1) Assume

$$\int_0^1 |\ell(u)|\, \mathrm{d}u < \infty \tag{16}$$

We consider

$$\Phi = \int_0^1 F_u^\sharp\, \ell(u)\, \mathrm{d}u$$

Then, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \frac{1}{t} \int_0^t F_u^\sharp \, \ell\left(\frac{u}{t}\right) \, \mathrm{d}u \tag{17}$$

2) Assume that ℓ *is absolutely continuous on* $]0,1]$ *and*

$$\int_0^1 |\ell'(u)| \, u \, \mathrm{d}u < \infty \tag{18}$$

Then (16) holds and $\Phi^\sharp$ *defined by (17) is a process with finite variation on* $(0,1]$.

Proof We have, by Property 4) in Theorem 4.8:

$$\Phi_t^\sharp = \int_0^1 F_{u\,t}^\sharp \, \ell(u) \, \mathrm{d}u$$

and therefore, formula (17) holds.

The proof of Property 2) is similar to the proof of Proposition 3.9 in Hirsch-Yor [5]. We actually have, on $(0,1]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\sharp = \frac{1}{t} \left[F_t^\sharp \, \ell(1) - \int_0^1 F_{ut}^\sharp \, [\ell(u) + u \, \ell'(u)] \, \mathrm{d}u \right]$$

In particular, for $0 < a < 1$,

$$\int_a^1 \| \frac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\sharp \|_1 \, \mathrm{d}t < \infty$$

□

4.5.2

We extend, in this paragraph, the first part of Proposition 3.13 in Hirsch-Yor [5].

We assume:

$$\int_{\mathbb{R}} \frac{1}{1 + \Re\psi(z)} \, \mathrm{d}z < \infty \tag{19}$$

Let $a \in \mathbb{R}$ and $r > 0$. We take as functional Φ, the local time of L at level a and time r (see Bertoin [1, Chapter V], where condition (19) is shown to ensure the existence of these local times).

Proposition 4.10 *We have, for $t \in [0,1)$,*

$$\Phi_t^\sharp = \frac{1}{2\pi} \int_0^r \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}z \ \exp\left[\mathrm{i}\,(L_{st} - a)z - s(1-t)\psi(z)\right]$$

Proof

Denote, for $\epsilon > 0$, by φ_ϵ the indicator function of the interval $[-\epsilon, +\epsilon]$. We set

$$\Phi_\epsilon = \frac{1}{2\epsilon} \int_0^r \varphi_\epsilon(L_s - a)\ \mathrm{d}s$$

By Corollary 3.2.1 and Proposition 4.3, we have for $0 \le t < 1$,

$$(\Phi_\epsilon)_t^\sharp = \frac{1}{2\epsilon} \int_0^r \mathrm{d}s \int \varphi_\epsilon(L_{st} - a + y)\ \mu_{s(1-t)}(\mathrm{d}y)$$

A computation by Fourier transform, taking into account the assumption (19), yields:

$$(\Phi_\epsilon)_t^\sharp = \frac{1}{2\pi} \int_0^r \mathrm{d}s \int_{-\infty}^{+\infty} \mathrm{d}z \ \exp\left[\mathrm{i}\,(L_{st} - a)z - s(1-t)\psi(z)\right] \frac{\sin(\epsilon z)}{\epsilon z} \qquad (20)$$

We know (Bertoin [1, Chapter V]) that

$$\lim_{\epsilon \to 0} \Phi_\epsilon = \Phi \quad \text{in } L^2$$

Therefore,

$$\Phi_t^\sharp = \lim_{\epsilon \to 0} (\Phi_\epsilon)_t^\sharp \quad \text{in } L^2$$

Consequently, taking the limit as ϵ tends to 0 in (20), we obtain by dominated convergence, thanks to (19), the announced result.

□

4.6 A Markov semigroup related to the processes $\Phi^\sharp$

We can interpret the family of maps: $\Phi \longrightarrow \Phi_t^\sharp$, indexed by $t \le 1$, as defined in Theorem 4.8, in terms of a Markovian semigroup (Q_h), where t and h are related by: $t = \mathrm{e}^{-h}$.

Let $1 \le p < \infty$. We set, for $\Phi \in L^p(\mathbb{P})$ and $h \ge 0$,

$$Q_h \Phi = \Phi^\sharp_{\mathrm{e}^{-h}}$$

In other words, we have with the notation of Subsection 4.1,

$$Q_h\Phi = \mathbb{E}_{\widetilde{L}}\left[\Phi(L_{\mathrm{e}^{-h}\bullet} + \widetilde{L}_{(1-\mathrm{e}^{-h})\bullet})\right] \;, \quad h \geq 0$$

We now may state Theorem 4.8 in the following way.

Proposition 4.11 *$Q = (Q_h \;,\; h \geq 0)$ is a Markovian strongly continuous semigroup on L^p. Moreover,*

$$\forall h \geq 0,\; \forall \Phi \in L^p, \quad \| Q_h\Phi \|_p \leq \| \Phi \|_p$$

and

$$\lim_{h\to\infty} Q_h\Phi = \mathbb{E}(\Phi) \quad \text{in } L^p$$

A Markov process can now be associated with the semigroup Q.

Theorem 4.12 *We define a $\mathbb{D}_0$-valued process: $(Y^h \;,\; h \geq 0)$, by*

$$Y^h_\bullet = X_{\mathrm{e}^{-h}\bullet, \mathrm{e}^h}$$

Then, if $\Phi \in L^1(\mathbb{P})$, for all $h, k \geq 0$,

$$\mathbb{E}\left[\Phi(Y^{h+k}) \mid \mathcal{X}_{\mathrm{e}^h}\right] = Q_k\Phi(Y^h)$$

Proof

We write $\Phi(Y^{h+k})$ as:

$$\Phi\left(X_{\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^h} + X_{\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^{h+k}} - X_{\mathrm{e}^{-(h+k)}\bullet, \mathrm{e}^h}\right)$$

Therefore, by Theorem 2.2,

$$\begin{aligned}\mathbb{E}\left[\Phi(Y^{h+k}) \mid \mathcal{X}_{\mathrm{e}^h}\right] &= \mathbb{E}_{\widetilde{L}}\left[\Phi(Y^h_{\mathrm{e}^{-k}\bullet} + \widetilde{L}_{(1-\mathrm{e}^{-k})\bullet})\right] \\ &= Q_k\Phi(Y^h)\end{aligned}$$

□

5 Examples of processes Φ^m and $\Phi^\sharp$ obtained from stochastic integrals with respect to L

5.1 Definition of the stochastic integrals with respect to L

It is well-known that any Lévy process L can be written as the sum of three independent Lévy processes $L^{(1)}$, $L^{(2)}$, $L^{(3)}$ where:

- $L^{(1)}$ is a Brownian motion with drift,
- $L^{(2)}$ is a compound Poisson process with jumps of size at least 1,
- $L^{(3)}$ is a pure-jump martingale having only jumps of size less than 1.

We are interested in this section in functionals Φ defined from stochastic integrals $\int_0^\infty H_s \, \mathrm{d}L_s$. (We identify, as previously, L with the coordinate process on $\mathbb{D}_0$.)

If L is Brownian, these functionals where studied in Hirsch-Yor [5].

If L is a compound Poisson process, stochastic integrals are ordinary Stieltjes integrals and the study is rather simple.

Consequently, we concentrate our attention, in this section, on the third part of L. A little more generally, we assume in the rest of this section, that the characteristic exponent ψ of L is

$$\psi(\lambda) = \int \left(1 - \mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\,\lambda\, x\right) \, \nu(\mathrm{d}x)$$

with ν a positive measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$0 < a := \int x^2 \, \nu(\mathrm{d}x) < \infty$$

As a consequence,

$$\mathbb{E}(L_t) = 0 \quad \text{and} \quad \mathbb{E}(L_t^2) = a\, t$$

We denote in what follows, by $\mathcal{H}$ the space of $(\mathcal{F}_s)$-predictable processes $H = (H_s \, , \, s \geq 0)$ such that

$$\int_0^\infty \mathbb{E}(H_s^2) \, \mathrm{d}s < \infty$$

We then obtain easily:

Proposition 5.1 *Let $H \in \mathcal{H}$. Then, the stochastic integral*

$$\int_0^\infty H_s \, \mathrm{d}L_s$$

is well defined as an element of L^2 and

$$\mathbb{E}\left[\left(\int_0^\infty H_s \, \mathrm{d}L_s\right)^2\right] = a \int_0^\infty \mathbb{E}(H_s^2) \, \mathrm{d}s \tag{21}$$

Moreover,

$$\left(\int_0^t H_s \, \mathrm{d}L_s \, , \; t \geq 0\right)$$

is an $(\mathcal{F}_t)$-martingale.

5.2 Examples of processes Φ^m and $\Phi^\sharp$ defined from Wiener type integrals with respect to L

We study in this subsection, functionals of Wiener type integrals $\int_0^\infty h(s) \, \mathrm{d}L_s$ with $h \in L^2(\mathbb{R}_+)$.

Proposition 5.2 *Let $h \in L^2(\mathbb{R}_+)$. Then,*

$$\left(\int_0^\infty h(s) \, \mathrm{d}_s X_{s,t} \, , \; t \geq 0\right)$$

is a Lévy process whose characteristic exponent is

$$\psi^h(\lambda) = \int_0^\infty \psi(\lambda \, h(s)) \, \mathrm{d}s$$

Proof

We first remark:

$$|\psi(\lambda)| \leq \frac{a}{2} \lambda^2$$

The integral $\int_0^\infty \psi(\lambda \, h(s)) \, \mathrm{d}s$ is therefore convergent.

By approximation of h by simple functions, we get:

$$\begin{aligned} \mathbb{E}\left[\exp\left(\mathrm{i} \, \lambda \int_0^\infty h(s) \, \mathrm{d}_s X_{s,t}\right)\right] &= \mathbb{E}\left[\exp\left(\mathrm{i} \, \lambda \int_0^\infty h(s) \, \mathrm{d}_s L_{t\,s}\right)\right] \\ &= \exp\left(-t \int_0^\infty \psi(\lambda \, h(s)) \, \mathrm{d}s\right) \end{aligned}$$

The rest of the proposition follows from the properties of the Lévy sheet X (Theorem 2.2).

□

We denote in the sequel, by μ_t^h the law of $\int_0^\infty h(s)\ \mathrm{d}_s X_{s,t}$. The following extension of Proposition 4.3 holds.

Proposition 5.3 *Let $h \in L^2(\mathbb{R}_+)$ and $f \in L^1(\mu_1^h)$. We set, for $(x,t) \in \mathbb{R} \times [0,1]$,*

$$\widetilde{f}(x,t) = \int_{\mathbb{R}} f(x+y)\ \mu_{1-t}^h(\mathrm{d}y)$$

Let

$$\Phi = f\left(\int_0^\infty h(s)\ \mathrm{d}L_s\right)$$

Then, for $0 \le t \le 1$,

$$\Phi_t^\sharp = \widetilde{f}\left(\int_0^\infty h(s)\ \mathrm{d}_s L_{t\,s}\,, t\right) \quad \text{and} \quad \Phi_t^m = \widetilde{f}\left(\int_0^\infty h(s)\ \mathrm{d}_s X_{s,t}\,, t\right)$$

In particular, the function:

$$(L,t) \in \mathbb{D}_0 \times [0,1] \longrightarrow \widetilde{f}\left(\int_0^\infty h(s)\ \mathrm{d}L_s\,, t\right)$$

is a space-time harmonic function for $(X_{\bullet,t}\ ,\ 0 \le t \le 1)$.

Proof

By formulae (6) and (12),

$$\Phi_t^m = \mathbb{E}_{\widetilde{L}}\left[f\left(\int_0^\infty h(s)\ \mathrm{d}_s X_{s,t} + \int_0^\infty h(s)\ \mathrm{d}_s \widetilde{L}_{(1-t)\,s}\right)\right]$$

and

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}}\left[f\left(\int_0^\infty h(s)\ \mathrm{d}_s L_{t\,s} + \int_0^\infty h(s)\ \mathrm{d}_s \widetilde{L}_{(1-t)\,s}\right)\right]$$

Now, as $\widetilde{L}_{(1-t)\bullet} \overset{\text{(law)}}{=} X_{\bullet,1-t}$, we have the announced result.

□

Corollary 5.3.1 *Let $h \in L^2(\mathbb{R}_+)$ and*

$$\Phi = \exp\left(\mathrm{i}\int_0^\infty h(s)\ \mathrm{d}L_s + \int_0^\infty \psi(h(s))\ \mathrm{d}s\right)$$

Then, for $0 \le t \le 1$,

$$\Phi_t^m = \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s X_{s,t} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)$$

and

$$\Phi_t^\sharp = \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)$$

The above corollary leads to the example presented in Subsection 1.3. Likewise, Propositions 3.4 and 4.4 appear as a particular case of Corollary 5.3.1, taking

$$h(s) = \sum_{j=1}^n \lambda_j\, 1_{[0,u_j]}(s)$$

Finally, by the same kind of arguments as previously, we obtain the following extension of Proposition 3.6.

Proposition 5.4 *Let* ρ *be a bounded signed measure on* $L^2(\mathbb{R}_+)$ *such that*

$$\int_{L^2(\mathbb{R}_+)} \exp\left(\lambda\int_0^\infty h^2(s)\,\mathrm{d}s\right)\,|\rho(\mathrm{d}h)| < \infty$$

for every $\lambda > 0$. *Let* F *be defined (almost surely) on* $\mathbb{D}_0 \times \mathbb{R}_+$ *by*

$$F(L,t) = \int_{L^2(\mathbb{R}_+)} \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}L_s + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)\,\rho(\mathrm{d}h)$$

Then F *is a space-time harmonic function for* $(X_{\bullet,t}\ ,\ t \ge 0)$. *In particular, setting for* $t \ge 0$:

$$F_t^\sharp = \int_{L^2(\mathbb{R}_+)} \exp\left(\mathrm{i}\int_0^\infty h(s)\,\mathrm{d}_s L_{t\,s} + t\int_0^\infty \psi(h(s))\,\mathrm{d}s\right)\,\rho(\mathrm{d}h)$$

the process $(F_t^\sharp\ ,\ t \ge 0)$ *is a 1-martingale.*

5.3 Processes $\Phi^\sharp$ for Φ a stochastic integral with respect to L

Proposition 5.5 *Let* $H \in \mathcal{H}$ *and*

$$\Phi = \int_0^\infty H_s\,\mathrm{d}L_s$$

Then, for $0 \le t \le 1$,

$$\Phi_t^\sharp = \int_0^\infty (H_s)_t^\sharp \, \mathrm{d}_s L_{t\,s} \tag{22}$$

As a consequence,

$$\| \Phi_t^\sharp \|_2 \le \sqrt{t} \, \| \Phi \|_2 \tag{23}$$

Proof

We first remark that $((H_s)_t^\sharp \ , \ s \ge 0)$ is $(\mathcal{F}_{ts})_{s\ge 0}$-predictable: it is indeed enough to consider left-continuous, bounded, adapted processes H, for which the property is clear by the definition (formula (12)). Then, (22) also follows easily from the definition. Now, by formula (21) and Property 2) in Theorem 4.8,

$$\| \Phi_t^\sharp \|_2^2 = t\, a \int_0^\infty \mathbb{E}\left[((H_s)_t^\sharp)^2 \right] \, \mathrm{d}s \le t\, a \int_0^\infty \mathbb{E}(H_s^2) \, \mathrm{d}s = t \, \| \Phi \|_2^2$$

which yields (23).

□

The following corollary improves, in some cases, upon Proposition 4.9.

Corollary 5.5.1 *Let* $H \in \mathcal{H}$ *and* $F = \int_0^\infty H_s \, \mathrm{d}L_s$. *Let* ℓ *be an absolutely continuous function on* $(0,1]$ *such that*

$$\int_0^1 |\ell'(u)| \, u^{3/2} \, \mathrm{d}u < \infty \tag{24}$$

and let

$$\Phi = \int_0^1 F_u^\sharp \, \ell(u) \, \mathrm{d}u$$

Then $\Phi^\sharp$ *is an absolutely continuous process on* $[0,1]$ *and its variation belongs to* L^2.

Proof The proof is similar to that of Proposition 3.9 in [5]. By Proposition 5.5, formula (23),

$$\| F_t^\sharp \|_2 \le \sqrt{t} \, \| F \|_2 \tag{25}$$

Therefore, hypothesis (24) is sufficient to entail, as in Proposition 4.9, that $\Phi^\sharp$ is absolutely continuous on $(0,1]$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi_t^\sharp = \frac{1}{t} \left[F_t^\sharp \, \ell(1) - \int_0^1 F_{ut}^\sharp \, [\ell(u) + u \, \ell'(u)] \, \mathrm{d}u \right]$$

Consequently, by (25),

$$\| \frac{\mathrm{d}}{\mathrm{d}t} \Phi_t^\sharp \|_2 \leq \frac{\| F \|_2}{\sqrt{t}} \left[|\ell(1)| + \int_0^1 |u^{1/2}\ell(u) + u^{3/2}\ell'(u)| \,\mathrm{d}u \right]$$

and hence the announced result follows.

□

The following result is an extension of Hirsch-Yor [5, Theorem 5.1]. The proof is quite similar and will be omitted.

Theorem 5.6 *Let $H \in \mathcal{H}$ and $\Phi = \int_0^1 H_s \;\mathrm{d}L_s$. Then $\Phi^\sharp$ is an $(\mathcal{F}_t)$-martingale if and only if the following condition is fulfilled:*

There exists a version of H which is L^2-continuous on $[0,1)$ and satisfies:

$$\forall u \in [0,1),\ \forall t \in [0,1], \quad (H_u)_t^\sharp = H_{u\,t}$$

In particular, if $F \in L^2(\mathcal{F}_1)$ and $\Phi = \int_0^1 F_s^\sharp \;\mathrm{d}L_s$, then $\Phi^\sharp$ is an $(\mathcal{F}_t)$-martingale.

6 Examples related to fractional α-stable processes

6.1 Fractional stable processes

In this section, we fix $\alpha \in (0,2]$, and we consider a $\mathbb{R}^2$-valued Lévy process: $L = (L^1, L^2)$, where L^1 and L^2 are two independent copies of a strictly α-stable $\mathbb{R}$-valued Lévy process with characteristic exponent:

$$\psi(\lambda) = c\,|\lambda|^\alpha (1 + \mathrm{i}\, b\,\mathrm{sign}(\lambda)), \quad c > 0 \text{ and } b \in \mathbb{R}$$

Let $H \in (0,1)$ and $\gamma = H - \frac{1}{\alpha}$. Following for example Samorodnitsky-Taqqu [10, Definition 7.4.1], we define a fractional stable motion $L^{\alpha,H}$, setting, for $t \geq 0$,

$$L_t^{\alpha,H} = \int_0^t (t-s)^\gamma \;\mathrm{d}L_s^1 + \int_0^\infty [(t+s)^\gamma - s^\gamma] \;\mathrm{d}L_s^2$$

(Here, for simplicity, we consider the process as defined only on $\mathbb{R}_+$, and not on $\mathbb{R}$ as usual.)

It is easy to see that this process has the scaling property of index H^{-1}, which means that, for any $k > 0$,

$$L_{k\bullet}^{\alpha,H} \overset{\text{(law)}}{=} k^H\, L_\bullet^{\alpha,H} \tag{26}$$

As a particular case, if $\alpha = 2$, $b = 0$ and

$$c = c_H := \left[H^{-1} + 2 \int_0^\infty [(1+s)^{H-1/2} - s^{H-1/2}]^2 \, \mathrm{d}s \right]^{-1}$$

then $L^{\alpha,H}$ is the classical fractional Brownian motion B^H with Hurst index H. In this case, L^1 and L^2 have the same law as $\sqrt{2\, c_H}\, B$, where B denotes a standard Brownian motion.
We set, for $\epsilon = \pm 1$,

$$d_\epsilon = c \left[\frac{1 + \mathrm{i}\, \epsilon\, b}{\alpha\, H} + (1 + \mathrm{i}\, \epsilon\, b \, \mathrm{sign}(\gamma)) \int_0^\infty |(1+s)^\gamma - s^\gamma|^\alpha \, \mathrm{d}s \right]$$

(for the fractional Brownian motion, $d_\epsilon = 1/2$). The following proposition, which is well-known (see [10]), can also be seen as a consequence of Proposition 5.2.

Proposition 6.1 *We have, for $\lambda \in \mathbb{R}$ and $t \geq 0$,*

$$\mathbb{E}\left[\exp(\mathrm{i}\, \lambda\, L_t^{\alpha,H}) \right] = \exp(-d_{\mathrm{sign}(\lambda)} |\lambda|^\alpha t^{\alpha\, H})$$

We denote, in the sequel, by $\mu_t^{\alpha,H}$ the law of $L_t^{\alpha,H}$.

6.2 Definition of the processes $\Phi^\sharp$ and Φ^m

We now introduce processes $\Phi^\sharp$ and Φ^m which are slightly different from those which were defined before, but better adapted to the present framework.

The Lévy process which is henceforth considered is $L = (L^1, L^2)$ presented in the previous subsection. The associated Lévy sheet is obviously $X = (X^1, X^2)$, where X^1 and X^2 are two independent Lévy sheets associated with L^1 and L^2. We consider the space $\mathbb{D}_0 \times \mathbb{D}_0$ equipped with the probability $\mathbb{P} \times \mathbb{P}$, which is the law of L. Here again, we identify the coordinate process $\varepsilon = (\varepsilon^1, \varepsilon^2)$ on $\mathbb{D}_0 \times \mathbb{D}_0$ with the process L.

As L has the scaling property of index α^{-1}, for any $t > 0$,

$$t^\gamma L_{t\,\bullet} \stackrel{\text{(law)}}{=} L_{t^{\alpha H}\,\bullet}$$

Therefore, if $\widetilde{L}$ is an independent copy of L,

$$t^\gamma L_{t\,\bullet} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet} \stackrel{\text{(law)}}{=} L_\bullet$$

Now, if $\Phi \in L^1(\mathbb{P} \times \mathbb{P})$, we set, for $0 < t \leq 1$,

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}} \left[\Phi(t^\gamma L_{t\,\bullet} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet})\right] \text{ and } \Phi_t^m = \mathbb{E}_{\widetilde{L}} \left[\Phi(X_{\bullet, t^{\alpha H}} + \widetilde{L}_{(1-t^{\alpha H})\,\bullet})\right]$$

where, in the last equality, $\widetilde{L}$ is assumed independent of X.
We also set $\Phi_0^\sharp = \Phi_0^m = \mathbb{E}(\Phi)$.

The processes $(\Phi_t^\sharp\,,\ 0 \leq t \leq 1)$ and $(\Phi_t^m\,,\ 0 \leq t \leq 1)$ have quite similar properties as before (Sections 3, 4, 5). In particular:

Theorem 6.2 *The process $(\Phi_t^m\,,\ 0 \leq t \leq 1)$ is a $(\mathcal{X}_{t^{\alpha H}})$-martingale and, for $0 \leq t \leq 1$,*

$$\Phi_t^\sharp \overset{\text{(law)}}{=} \Phi_t^m$$

As a consequence, $\Phi^\sharp$ is a 1-martingale.

6.3 Some examples

In the sequel, we consider some simple functionals of $L^{\alpha,H}$.

Proposition 6.3 *Let $r > 0$ and $f \in L^1(\mu_r^{\alpha,H})$. We consider $\Phi = f(L_r^{\alpha,H})$. Then, for $0 \leq t \leq 1$,*

$$\Phi_t^\sharp = \int f(L_{t\,r}^{\alpha,H} + (1 - t^{\alpha H})^{1/\alpha}\, y)\ \mu_r^{\alpha,H}(\mathrm{d}y)$$

Proof

Let $a > 0$. Then we obtain by change of variable:

$$\forall t \geq 0, \quad \int_0^t (t-s)^\gamma\ \mathrm{d}L^1_{a\,s} + \int_0^\infty [(t+s)^\gamma - s^\gamma]\ \mathrm{d}L^2_{a\,s} = a^{-\gamma}\, L^{\alpha,H}_{a\,t}$$

Therefore, by the definition of $\Phi_t^\sharp$:

$$\Phi_t^\sharp = \mathbb{E}_{\widetilde{L}^{\alpha,H}} \left[f\left(L_{t\,r}^{\alpha,H} + (1 - t^{\alpha H})^{-\gamma}\, \widetilde{L}^{\alpha,H}_{(1-t^{\alpha H})\,r}\right)\right]$$

where $\widetilde{L}^{\alpha,H}$ denotes an independent copy of $L^{\alpha,H}$. Now, by the scaling property (26),

$$(1 - t^{\alpha H})^{-\gamma}\, \widetilde{L}^{\alpha,H}_{(1-t^{\alpha H})\,r} \overset{\text{(law)}}{=} (1 - t^{\alpha H})^{1/\alpha}\, \widetilde{L}^{\alpha,H}_r$$

which gives the announced result.

□

As a straightforward consequence of Propositions 6.1 and 6.3, we obtain the following important example.

Proposition 6.4 *We set, for $\lambda \in \mathbb{R}$ and $u \geq 0$,*

$$\Phi^{\lambda,u} = \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\,|\lambda|^\alpha\, u^{\alpha H}\right)$$

Then, for $0 \leq t \leq 1$,

$$(\Phi^{\lambda,u})_t^\sharp = \Phi^{\lambda,t\,u}$$

Corollary 6.4.1 *For every $\lambda \in \mathbb{R}$ and $r > 0$, the process*

$$\left(\frac{1}{t}\int_0^t \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\,|\lambda|^\alpha\, u^{\alpha H}\right)\,\mathrm{d}u\;,\; 0 \leq t \leq r\right)$$

is a 1-martingale.

Proof

Let

$$\Phi = \int_0^r \Phi^{\lambda,u}\,\mathrm{d}u$$

Then, by Proposition 6.4, we have for $0 \leq t \leq 1$,

$$\Phi_t^\sharp = \int_0^r \Phi^{\lambda,t\,u}\,\mathrm{d}u = \frac{1}{t}\int_0^{rt} \Phi^{\lambda,u}\,\mathrm{d}u$$

Therefore, the process

$$\left(\frac{1}{rt}\int_0^{rt} \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\,|\lambda|^\alpha\, u^{\alpha H}\right)\,\mathrm{d}u\;,\; 0 \leq t \leq 1\right)$$

is a 1-martingale, which, after replacing rt by t, is the announced result.

□

In the case of the fractional Brownian motion with Hurst index H ($\alpha = 2$, $d_\epsilon = 1/2$), the previous Proposition 6.4 and Corollary 6.4.1 can be extended to any $\lambda \in \mathbb{C}$. In particular, we have the following extension of our guiding example of Subsection 1.2:

Proposition 6.5 *For every $\lambda \in \mathbb{R}$ and $r > 0$, the process*

$$\left(\frac{1}{t}\int_0^t \exp\left(\lambda\, B_u^H - \frac{\lambda^2}{2}\,u^{2H}\right)\,\mathrm{d}u\;,\; 0 \leq t \leq r\right)$$

is a 1-martingale.

Finally, we remark that Corollary 6.4.1 and Proposition 6.5 extend to $t \geq 0$ instead of $0 \leq t \leq r$:

Proposition 6.6 *1) For every $\lambda \in \mathbb{R}$, the process*

$$\left(\frac{1}{t}\int_0^t \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H} + d_{\mathrm{sign}(\lambda)}\,|\lambda|^\alpha\, u^{\alpha H}\right)\,\mathrm{d}u\ ,\ t \geq 0\right)$$

is a 1-martingale.

2) For every $\lambda \in \mathbb{R}$, the process

$$\left(\frac{1}{t}\int_0^t \exp\left(\lambda\, B_u^H - \frac{\lambda^2}{2}\, u^{2H}\right)\,\mathrm{d}u\ ,\ t \geq 0\right)$$

is a 1-martingale.

Proof

We prove, for example, property 1).
Let, for $\lambda \in \mathbb{R}$ and $t \geq 0$,

$$M_t = \int_0^1 \exp\left(\mathrm{i}\,\lambda\, L_u^{\alpha,H}(X_{\bullet,t^{\alpha H}}) + d_{\mathrm{sign}(\lambda)}\,|\lambda|^\alpha\,(t\,u)^{\alpha H}\right)\,\mathrm{d}u$$

where $L_u^{\alpha,H}(X_{\bullet,t^{\alpha H}})$ means that, in the expression of $L_u^{\alpha,H}$, one has replaced the process $L_\bullet$ by the process $X_{\bullet,t^{\alpha H}}$. Then $(M_t\ ,\ t \geq 0)$ is a $(\mathcal{X}_{t^{\alpha H}})$-martingale: this comes, by Proposition 6.1, from the fact that, if $0 \leq s \leq t$,

$$X_{\bullet,t^{\alpha H}} - X_{\bullet,s^{\alpha H}} \overset{\text{(law)}}{=} (t^{\alpha H} - s^{\alpha H})^{1/\alpha}\, L_\bullet$$

Now, by the scaling properties, for $t \geq 0$ fixed,

$$\left(L_{t\,u}^{\alpha,H}\ ,\ 0 \leq u \leq 1\right) \overset{\text{(law)}}{=} \left(L_u^{\alpha,H}(X_{\bullet,t^{\alpha H}})\ ,\ 0 \leq u \leq 1\right)$$

and the result follows easily.

□

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