

Moderate deviations for centered additive functionals of recurrent Harris processes having general state space

Eva LÖCHERBACH*, Dasha LOUKIANOVA†

February 4, 2009

Abstract

Let X be a Harris recurrent strong Markov process with general Polish state space E , having invariant measure μ . In this paper we derive non asymptotic deviation bounds for

$$P_x \left(\left| \int_0^t f(X_s) ds \right| \geq t^{\frac{1}{2} + \eta} \varepsilon \right)$$

in the positive recurrent case, for nice functions f with $\mu(f) = 0$. We generalize these bounds to the fully null-recurrent case where we obtain an exponential rate of convergence which is expressed in terms of the deterministic equivalent of the process. The main ingredient of the proof is Nummelin splitting in continuous time which allows to introduce regeneration times for the process.

Key words : Harris recurrence, Nummelin splitting, continuous time Markov processes, special functions, additive functionals, large deviations, deviation inequalities, deterministic equivalent for additive functionals.

MSC 2000 : 60 J 55, 60 J 35, 60 F 10, 62 M 05

1 Introduction

Consider a Harris recurrent strong Markov process $X = (X_t)_{t \geq 0}$ with invariant measure μ , taking values in a Polish space E .

*LAMA UMR-CNRS 8050, Faculté de Sciences et Technologie, Université Paris-Est Val de Marne, 61 avenue du Général de Gaulle, 94010 Créteil Cedex, France. E-mail: locherbach@univ-paris12.fr

†Département de Mathématiques, Université d'Evry-Val d'Essonne, Bd François Mitterrand, 91025 Evry Cedex, France. E-mail: dasha.loukianova@univ-evry.fr

If the total mass of μ is finite, X is called positive recurrent, null-recurrent otherwise. In the case of positive recurrence it is well known that for all f such that $\mu(f) = 0$ we have a central limit theorem for

$$\left(\frac{1}{\sqrt{n}} \int_0^{nt} f(X_s) ds \right)_{t \geq 0} \quad (1.1)$$

as n goes to ∞ , see for instance Touati ([27]).

In the null-recurrent case, the re-normalization \sqrt{n} of (1.1) has to be changed. For that sake we have to consider what is called *deterministic equivalent* of additive functionals. The deterministic equivalent has been introduced for Markov chains by Chen [5] and then been generalized to the context of continuous time diffusion models by Loukianova and Loukianov [17] and to any continuous time recurrent Markov process by Löcherbach and Loukianova in [15]. It is a deterministic function $t \mapsto v(t)$ such that $v(t) \rightarrow \infty$ as $t \rightarrow \infty$ and such that for any integrable additive functional A_t ,

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi(1/M \leq A_t/v(t) \leq M) = 1$$

for any initial measure π . $v(t)$ can be defined as follows. Take any fixed positive special function g of the process having $\mu(g) > 0$ (see definition 2.4 below for the exact definition of special functions, for strong Feller processes, any bounded function having compact support is special) and define

$$v(t) := E_\eta \left(\int_0^t g(X_s) ds \right),$$

where η is an arbitrary initial measure. Then the strong Chacon-Ornstein theorem implies that for any other special function g' and any other initial measure η' ,

$$\lim_{t \rightarrow \infty} \frac{E_\eta \left(\int_0^t g(X_s) ds \right)}{E_{\eta'} \left(\int_0^t g'(X_s) ds \right)} = \frac{\mu(g)}{\mu(g')}.$$

Hence the deterministic equivalent is unique up to a constant in the sense that for two choices of the deterministic equivalent, v and v' , we have that $\lim_{t \rightarrow \infty} v(t)/v'(t) = c$, where c is a positive constant. In *regular* models, $v(t) \sim t^\alpha l(t)$, where l is a function that varies slowly at infinity. For example, for Brownian motion in dimension one, we have $\alpha = 1/2$. The explosion rate v_t is in general slower than the ergodic (positive recurrent) rate t .

In the null-recurrent regular case, we have convergence in law of

$$\left(\frac{1}{\sqrt{v(n)}} \int_0^{nt} f(X_s) ds \right)_{t \geq 0} \quad (1.2)$$

to $B \circ W^\alpha$, where B is a one-dimensional Brownian motion and W^α the Mittag-Leffler process of index α , $0 < \alpha \leq 1$, see Touati ([27]) and Höpfner and Löcherbach ([13]) for similar results for martingale additive functionals. In the fully null-recurrent case (i.e. null-recurrent, but not regular) no convergence in law holds true, which is a consequence of the famous theorem of Darling-Kac.

Continuing in the spirit of the limit theorems of (1.1) and (1.2), the aim of this paper is to study moderate deviations of additive functionals $\int_0^t f(X_s) ds$ for a certain class of centered functions

f (the class of bounded special functions that will be introduced below). More precisely we study the deviation bounds for

$$P_x \left(\left| \int_0^t f(X_s) ds \right| \geq t^{\frac{1}{2} + \eta} x \right)$$

for $\eta > 0$ in the positive recurrent case, and in the general null-recurrent case, including the non-regular case, of

$$P_x \left(\left| \int_0^t f(X_s) ds \right| \geq v_t^{\frac{1}{2} + \eta} x \right).$$

Moderate deviations for additive functionals of Markov chains, i.e. in the discrete time case, have been extensively studied over the last decade, and we refer the reader to the work of Guillin ([11]), Djellout and Guillin ([8]), Chen and Guillin ([6]) for a survey on the subject. In the case of time continuous observations, less results are known. In the ergodic situation, Guillin and Liptser ([12]) have studied the moderate deviations of

$$\int_0^t f(X_s) ds$$

where X is a multidimensional ergodic diffusion and f any centered function belonging to $L^1(\mu)$. They use techniques of the stochastic calculus which are well-adapted to the particular diffusion case. However, the general case of recurrent processes in continuous time that are not necessarily ergodic has not yet been treated.

In the present paper we propose to study the question of moderate deviations in the most general situation, i.e. for any recurrent Markov process taking values in any Polish state space, without restrictions on the quality of recurrence. As a counterpart of this general approach, we have to restrict attention to the class of test functions which are the bounded centered special functions. In the case of strong Feller processes, that means that we consider centered functions being of compact support.

Our approach is based on the regeneration method. Regeneration times allow to split the trajectory of the process into i.i.d. excursions. In the one-dimensional case, such regeneration times are usually introduced as successive visits to recurrent points. Our aim is, however, to work in the frame of a general state space. In this general frame, recurrent points usually do not exist, and we use Nummelin splitting in continuous time, as developed in Löcherbach and Loukianova [15], in order to overcome this difficulty. Even if the technical details concerning Nummelin splitting are somehow cumbersome, the evident advantage of this method is that it is conceptually very easy (regeneration means that we are able to work with i.i.d. variables) and that it works in any dimension and for any state space.

For positive recurrent processes X , we obtain the following deviation upper bound. For any bounded special function f with $\mu(f) = 0$, we have

$$P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq t^{\frac{1}{2} + \eta} x \right) \leq C_1 \exp \left(-C_2 t^{2\eta} (x^2 \wedge x) \right) + R_t, \quad (1.3)$$

where π is any initial measure, for any $0 < \eta \leq \frac{1}{2}$. Here, C_1 and C_2 are explicit positive constants. R_t is a remainder term which is of order $O(\exp(-\sqrt{t}))$. In the general null-recurrent case, including the non-regular case, we have

$$P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq v(t)^{\frac{1}{2} + \eta} x \right) \leq C_1 \exp \left(-C_2 v(t)^\eta (x^2 \wedge x) \right) + R_t, \quad (1.4)$$

R_t being of order $O(\exp(-\sqrt{v(t)}))$. Although, (1.3) and (1.4) are not stated as moderate deviations results in the most general form, they represent a first and important step towards the study of the problem in the fully general case of any recurrent process.

The problem of obtaining such non-asymptotic bounds is of major importance for many applications. Let us cite just some of them : model selection or other non asymptotic problems for statistics of Harris processes, particle approximations of Gibbs measures, Our work is strongly motivated by applications to statistics (see for example [19]), in particular model selection, and it is important for such applications to obtain bounds that are valid for fixed t such that the constants involved are as explicit as possible. This is what we try to achieve in this paper.

Finally, let us cite in this context a recent work by Loukianova et al. [18] dealing with the same type of questions in the more restricted, but more concrete framework of positive recurrent diffusions in dimension one. In this case regeneration times are hitting times, and a precise control of these hitting times is provided. These kinds of results are of course not available within a general frame such as in the present article.

The paper is organized as follows. Since we are using heavily the method of Nummelin splitting and the concept of deterministic equivalent, section 2 gives a review of all known results concerning this technique that will be needed in the sequel. Section 3 gives the deviation inequality in its most general form, for both positive and null-recurrent cases. We also state a technical lemma which is a sort of generalization of Kac's formula, but more cumbersome in the context of Nummelin splitting. In section 4, finally, we interpret our general result in the positive and in the general null-recurrent case. In the case of null-recurrent but regular models, i.e. in the case when $v_t \sim t^\alpha$ for some $0 < \alpha < 1$, some finer control of the Laplace transform of the length of the regeneration time yields better results than in the general null-recurrent case. Finally, section 5 gives an idea of possible applications of our results in the framework of some interacting particle systems.

2 Notation

Consider a probability space $(\Omega, \mathcal{A}, (P_x)_x)$. Let $X = (X_t)_{t \geq 0}$ be a processes defined on $(\Omega, \mathcal{A}, (P_x)_x)$ which is strong Markov, taking values in a locally compact Polish space (E, \mathcal{E}) , with càdlàg paths. $X_0 = x$ P_x -almost surely. We write L for the generator and $(P_t)_t$ for the semi group of X and we suppose that X is recurrent in the sense of Harris, with invariant measure μ , unique up to multiplication with a constant. This means that for any set $A \in \mathcal{E}$ such that $\mu(A) > 0$, $\limsup_{t \rightarrow \infty} 1_A(X_t) = 1$ almost surely. Moreover, we shall write $(\mathcal{F}_t)_t$ for the filtration generated by the process.

We impose the following condition on the transition semi-group $(P_t)_t$ of X :

Assumption 2.1 *There exists a sigma-finite positive measure Λ on (E, \mathcal{E}) such that for every $t > 0$, $P_t(x, dy) = p_t(x, y)\Lambda(dy)$, where $(t, x, y) \mapsto p_t(x, y)$ is jointly measurable.*

Example 2.2 1. In general, it is difficult to check whether a given Markov process is recurrent or not, and the most used criterion for recurrence is the existence of a so-called Lyapunov-function for the generator of the process, see for example Meyn and Tweedie ([20], [21], [22]).

We say that $V \in \text{dom}(L)$ is a Lyapunov-function, if $V \geq 1$ and if there exists a constant $a > 0$, a constant b and a closed petite set C such that for all x ,

$$LV(x) \leq -aV(x) + b1_C(x).$$

The existence of Lyapunov-functions implies exponential ergodicity. This concept can be extended to obtain slower rates of convergence, still in the positive recurrent case.

2. In the context of interacting particle systems, (see Galves et al. ([10] and section 5 below), recurrence can be shown via arguments using the dual process of the system.

2.1 On the deterministic equivalent of additive functionals

In the general recurrent not necessarily positive recurrent case, rates of convergence of additive functionals are given by what is called *deterministic equivalent* of additive functionals. This object has been introduced for Markov chains by Chen [5] and then been generalized to the context of continuous time diffusion models by Loukianova and Loukianov [17] and to any continuous time recurrent Markov process by Löcherbach and Loukianova in [15]. In the context of one dimensional diffusion models, similar ideas to the notion of *deterministic equivalent* have also been developed in Delattre et al. ([7]). We start by resuming the most relevant results of ([15]) on the deterministic equivalent that will be needed in the sequel. We first recall the notion of additive functionals.

Definition 2.3 An additive functional of the process X is a $\bar{\mathbb{R}}_+$ -valued, adapted process $A = (A_t)_{t \geq 0}$ such that

1. Almost surely, the process is non-decreasing, right-continuous, having $A_0 = 0$.
2. For any $s, t \geq 0$, $A_{s+t} = A_t + A_s \circ \theta_t$ almost surely. Here, θ denotes the shift operator.

Examples for additive functionals are $A_t = \int_0^t f(X_s)ds$ where f is a positive measurable function. Such an additive functional is said to be *integrable*, if $\mu(f) < \infty$. The deterministic equivalent of any integrable additive functional is a deterministic function $v \mapsto v(t)$ such that $v(0) = 0$, $v(\cdot)$ is non-decreasing and $v(t) \rightarrow \infty$ as $t \rightarrow \infty$. It satisfies that for any integrable additive functional A_t , $A_t/v(t)$ is bounded and bounded away from zero in probability. In order to define the deterministic equivalent, we have to recall the notion of a special function (see also [25], [4]):

Definition 2.4 A measurable function $f : E \rightarrow \mathbb{R}_+$ is called *special* if for all bounded and positive measurable functions h such that $\mu(h) > 0$, the function

$$x \mapsto E_x \int_0^\infty \exp \left[- \int_0^t h(X_s)ds \right] f(X_t)dt$$

is bounded.

Note that in the case of strong Feller processes having locally compact Polish state space, any bounded function having compact support is special.

By [15], any special function g of X with $\mu(g) > 0$ defines a version of the deterministic equivalent via

$$v(t) = E_\pi \int_0^t g(X_s) ds, \quad (2.5)$$

for any arbitrary initial measure π . $v(t)$ is called deterministic equivalent due to the following result (corollary 2.19 of [15]).

Theorem 2.5 *For any additive functional A of the process having $E_\mu(A_1) \in]0, \infty[$, for any initial measure π , we have*

$$\lim_{M \rightarrow \infty} \liminf_{t \rightarrow \infty} P_\pi \left(\frac{1}{M} \leq \frac{1}{v(t)} A_t \leq M \right) = 1.$$

Remark 2.6 *The deterministic equivalent is unique up to a constant : for two choices of the deterministic equivalent, v and v' , we have that $\lim_{t \rightarrow \infty} v(t)/v'(t) = c$, where c is a positive constant.*

In the sequel, depending on the situation, we shall fix a suitable choice of $v(t)$. In the positive recurrent case, evidently $v(t) = t$, up to multiplication with a constant. In order to avoid too cumbersome notation, we sometimes also write $v_t = v(t)$.

2.2 On Nummelin splitting in continuous time

The proof of the deviation inequality is based on a very simple idea : the use of regeneration times that allow to divide the trajectory of the process into (almost) i.i.d. excursions. In the one-dimensional case, regeneration times are introduced as successive visits to recurrent points. For Harris recurrent Markov processes with general state space, points are in general not recurrent. That is why we have to use the Nummelin splitting in continuous time, as developed in [15], in order to introduce a *recurrent atom* for the process. An *atom* is a set that, roughly speaking, behaves as a point for the process. Once a recurrent atom exists, we can introduce regeneration times that split the trajectory of the process into (almost) i.i.d. excursions.

We recall briefly the construction of Nummelin splitting in continuous time.

Introduce a sequence $(\sigma_n)_{n \geq 1}$ of i.i.d. $exp(1)$ -waiting times, independent of the process X itself. Let $T_0 := 0$, $T_n := \sigma_1 + \dots + \sigma_n$ and $\bar{X}_n := X_{T_n}$. Then the chain $\bar{X} = (\bar{X}_n)_n$ is recurrent in the sense of Harris and its one-step transition kernel $U^1(x, dy) := \int_0^\infty e^{-t} P_t(x, dy) dt$ satisfies the minorization condition

$$U^1(x, dy) \geq \alpha 1_C(x) \nu(dy), \quad (2.6)$$

where $0 < \alpha < 1$, $\mu(C) > 0$ and ν a probability measure equivalent to $\mu(\cdot \cap C)$ (cf [25], [13], proposition 6.7). The set C can be chosen to be compact.

Remark 2.7 *In some cases, the measure of assumption 2.1 Λ satisfies $\Lambda \sim \mu$, and the densities $p_t(x, y)$ are explicitly known, for example in the case of k -dimensional diffusions, under suitable regularity assumptions. If one can specify some set C and some time interval $[s, t]$ such that*

$$\inf_{(x,y) \in C \times C, u \in [s,t]} p_u(x, y) > 0, \quad \Lambda(C) > 0,$$

(w.l.o.g. also $\Lambda(C) \leq 1$) then (2.6) holds true with

$$\alpha = [e^{-s} - e^{-t}] \Lambda(C) \left(\inf_{(x,y) \in C \times C, u \in [s,t]} p_u(x, y) \wedge 1 \right) \quad \text{and } \nu = \Lambda(\cdot \cap C) / \Lambda(C).$$

In particular, for multi-dimensional diffusions satisfying Hörmander's condition on some set Γ , the classical results of Kusuoka and Stroock [14] allow us to conclude that any choice of a compact set $C \subset \Gamma$ will be possible.

Then it is possible to define on an extension of the original space $(\Omega, \mathcal{A}, (P_x))$ a Markov process $Z = (Z_t)_{t \geq 0}$, taking values in $E \times [0, 1] \times E$ such that the T_n are jump times of the process and such that under P_x , $((Z_t^1)_t, (T_n)_n)$ has the same distribution as $((X_t)_t, (T_n)_n)$. Here are the details of this construction.

First of all, define the following transition kernel $Q((x, u), dy)$ from $E \times [0, 1]$ to E :

$$Q((x, u), dy) = \begin{cases} \nu(dy) & \text{if } (x, u) \in C \times [0, \alpha] \\ \frac{1}{1-\alpha} (U^1(x, dy) - \alpha \nu(dy)) & \text{if } (x, u) \in C \times]\alpha, 1] \\ U^1(x, dy) & \text{if } x \notin C \end{cases} \quad (2.7)$$

We now recall the construction of $Z_t = (Z_t^1, Z_t^2, Z_t^3)$ taking values in $E \times [0, 1] \times E$ as given in [15]. Write $u^1(x, x') := \int_0^\infty e^{-t} p_t(x, x') dt$. Let $Z_0^1 = X_0 = x$. Choose Z_0^2 according to the uniform distribution U on $[0, 1]$. On $\{Z_0^2 = u\}$, choose $Z_0^3 \sim Q((x, u), dx')$. Then inductively in $n \geq 0$, on $Z_{T_n} = (x, u, x')$:

1. Choose a new jump time σ_{n+1} according to

$$e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt \quad \text{on } \mathbb{R}_+,$$

where we define $0/0 := a/\infty := 1$, for any $a \geq 0$, and put $T_{n+1} := T_n + \sigma_{n+1}$.

2. On $\{\sigma_{n+1} = t\}$, put $Z_{T_n+s}^2 := u$, $Z_{T_n+s}^3 := x'$ for all $0 \leq s < t$.
3. For every $s < t$, choose

$$Z_{T_n+s}^1 \sim \frac{p_s(x, y) p_{t-s}(y, x')}{p_t(x, x')} \Lambda(dy).$$

Choose $Z_{T_n+s}^1 := x_0$ for some fixed point $x_0 \in E$ on $\{p_t(x, x') = 0\}$. Moreover, given $Z_{T_n+s}^1 = y$, on $s + u < t$, choose

$$Z_{T_n+s+u}^1 \sim \frac{p_u(y, y') p_{t-s-u}(y', x')}{p_{t-s}(y, x')} \Lambda(dy').$$

Again, on $\{p_{t-s}(y, x') = 0\}$, choose $Z_{T_n+s+u}^1 = x_0$.

4. At the jump time T_{n+1} , choose $Z_{T_{n+1}}^1 := Z_{T_n}^3 = x'$. Choose $Z_{T_{n+1}}^2$ independently of $Z_s, s < T_{n+1}$, according to the uniform law U . Finally, on $\{Z_{T_{n+1}}^2 = u'\}$, choose $Z_{T_{n+1}}^3 \sim Q((x', u'), dx'')$.

Note that by construction, given the initial value of Z at time T_n , the evolution of the process Z^1 during $[T_n, T_{n+1}[$ does not depend on the chosen value of $Z_{T_n}^2$.

We will write P_π for the measure related to X , under which X starts from the initial measure $\pi(dx)$, and \mathbb{P}_π for the measure related to Z , under which Z starts from the initial measure $\pi(dx) \otimes U(du) \otimes Q((x, u), dy)$. In the same spirit we denote E_π the expectation with respect to P_π and \mathbb{E}_π the expectation with respect to \mathbb{P}_π . Moreover, we shall write \mathbb{F} for the filtration generated by Z , \mathbb{G} for the filtration generated by the first two coordinates Z^1 and Z^2 of the process, and \mathbb{F}^X for the sub-filtration generated by X interpreted as first coordinate of Z .

Write

$$A := C \times [0, \alpha] \times E.$$

A is the recurrent atom of the process. Now we put

$$S_0 := 0, R_0 := 0, S_{n+1} := \inf\{T_m > R_n : Z_{T_m} \in A\}, R_{n+1} := \inf\{T_m : T_m > S_{n+1}\}.$$

Then the sequence of \mathbb{F} -stopping times R_n generalizes the notion of life-cycle decomposition in the following sense.

Proposition 2.8 [Proposition 2.13 of [15]]

- a) Z_{R_n+} is independent of \mathcal{F}_{S_n-} for all $n \geq 1$.
- b) $Z_{R_n} \sim \nu(dx)U(du)Q((x, u), dx')$ for all $n \geq 1$.
- c) The sequence of $(Z_{R_n})_{n \geq 1}$ is i.i.d.

Proposition 2.9 [Proposition 2.20 of [15]]

Let A_t be any integrable additive functional of X . Then, up to multiplication by a constant, for any initial measure π and any $n \geq 1$,

$$\mathbb{E}_\pi(A_{R_{n+1}} - A_{R_n}) = \mathbb{E}_\nu(A_{R_1}) = E_\mu(A_1).$$

From now on, we shall fix a version μ of the invariant measure such that always

$$\mu(f) = \mathbb{E}_\pi \int_{R_1}^{R_2} f(X_s) ds. \quad (2.8)$$

Moreover, we have the following:

Proposition 2.10 [Proposition 4.4 of [16]]

Let f be a measurable μ -integrable function. Put

$$\xi_n := \int_{R_{n-1}}^{R_n} f(X_s) ds, n \geq 1.$$

Then the sequence $(\xi_n)_n$ is a stationary ergodic sequence under \mathbb{P}_ν . Moreover, for $n \geq 2$, ξ_n is independent of $\mathcal{F}_{R_{n-2}}$.

Remark 2.11 *In the usual one-dimensional case, regeneration times R_n allow to split in such a way that ξ_n is independent of ξ_{n-1} . Our situation, however, is more complicated, since by construction, at a regeneration time, Z_{R_n} depends on the state of the process one jump time before, i.e. on Z_{S_n} . This is due to the structure of continuous time and due to the use of Markov bridges (step 3. of the construction).*

Finally, let us recall the following useful result.

Proposition 2.12 *[Proposition 2.16 of [15]]*

Let f be a special function of the process X . Then

$$C(f) := \sup_{x \in E} \mathbb{E}_x \int_0^{R_1} |f|(X_s) ds < +\infty.$$

3 The deviation inequality

In this section, we state our main result which is a deviation inequality for

$$\int_0^t f(X_s) ds,$$

where f is a special function of the process having $\mu(f) = 0$. Write

$$N_t := \sup\{n : R_n \leq t\}.$$

We first state the deviation inequality in the most general version, independently of the quality of recurrence (positive or null).

Theorem 3.1 *Let f be a bounded special function such that $\mu(f) = 0$. Recall that*

$$C(f) = \sup_x \mathbb{E}_x \int_0^{R_1} |f|(X_s) ds$$

and put $K(f) := \|f\|_\infty + C(f)$. Then there exists t_0 , such that for all $x > 0$, for any initial measure π , for any $0 \leq \eta \leq \frac{1}{2}$, $0 \leq \delta \leq 2\eta$, for any fixed choice of a deterministic equivalent v_t and for all $t \geq t_0$,

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq v_t^{\frac{1}{2} + \eta} x \right) &\leq 4 \exp \left(-\frac{1}{42} \frac{1}{\max(K^2(f), K(f))} v_t^{2\eta - \delta} (x^2 \wedge x) \right) \\ &\quad + 4e \exp \left(-\frac{1}{6K(f)} v_t^{\frac{1}{2} + \eta} x \right) + 4\mathbb{P}_\pi(N_t > v_t^{1+\delta}). \end{aligned}$$

Here, t_0 is given by the equation $(v_{t_0})^{\frac{1}{2} + \delta - \eta} = 1$.

Remark 3.2 δ should be chosen equal to 0 in the positive recurrent case, and in this case we can even choose η equal to zero. However, we then have to choose $v_t = \frac{2}{m}t$ where $m = \mathbb{E}(R_2 - R_1)$, see section 4.1 below. In the null-recurrent case, both η and δ have to be chosen strictly positive. The exact control of $\mathbb{P}_\pi(N_t > v_t^{1+\delta})$ depends on the quality of recurrence and is different in positive recurrent and null-recurrent situations. We refer the reader to section 4.

Proof Write $\xi_n := \int_{R_{n-1}}^{R_n} f(X_s) ds$, then the $\xi_n, n \geq 2$, are identically distributed random variables having mean zero, such that ξ_n and ξ_{n+2} are independent. It can be proven that for any $p \geq 1$, $\mathbb{E}(|\xi_n|^p) < \infty$, by the properties of a special function. More precisely, we have that for any $n \geq 1$,

$$\mathbb{E}(|\xi_n|^p) \leq p!K(f)^p, \quad (3.9)$$

which will be shown in proposition 3.4 below. Thus for $\lambda > 0$, sufficiently small ($\lambda < K(f)^{-1}$ suffices),

$$Z(\lambda) := \mathbb{E}_\pi(e^{\lambda\xi_n}), \quad n \geq 2, \quad (3.10)$$

exists and is finite (and does not depend on n).

Since the ξ_n are not independent, but only 2-independent, we have to proceed in the following way. Firstly, define a sequence $\xi_n^{(1)}$ by

$$\xi_n^{(1)} = \begin{cases} \xi_n & \text{if } n \text{ odd} \\ 0 & \text{elseif} \end{cases}. \quad (3.11)$$

Then define a second sequence $\xi_n^{(2)}$ by

$$\xi_n^{(2)} = \begin{cases} \xi_n & \text{if } n \text{ even} \\ 0 & \text{elseif} \end{cases}.$$

Note that ξ_n is not independent of $\mathcal{F}_{R_{n-1}}$, but it is independent of $\mathcal{F}_{R_{n-2}}$. That is why we introduce the following two sub-filtrations, associated to the sum of odd and the sum of even terms. Let

$$\mathcal{G}_n^{(1)} := \sigma\{R_k, \xi_k^{(1)} : k \leq n, k \text{ odd}\},$$

and

$$\mathcal{G}_n^{(2)} := \sigma\{R_k, \xi_k^{(2)} : k \leq n, k \text{ even}\}.$$

Moreover, let

$$N_t^{(1)} := \sup\{n : n \text{ odd}, R_n \leq t\}, \quad N_t^{(2)} := \sup\{n : n \text{ even}, R_n \leq t\}.$$

Then it is immediate that $N_t^{(1)} + 2$ is a $(\mathcal{G}_n^{(1)})_n$ -stopping time and $N_t^{(2)} + 2$ a $(\mathcal{G}_n^{(2)})_n$ -stopping time. Moreover, we evidently have that $N_t^{(1)} \leq N_t, N_t^{(2)} \leq N_t$.

Now, for λ sufficiently small, let

$$M_n^1 := \exp\left(\lambda \sum_{k=2}^n \xi_k^{(1)}\right) \cdot Z(\lambda)^{-[n-1]/2}, \quad M_n^2 := \exp\left(\lambda \sum_{k=2}^n \xi_k^{(2)}\right) \cdot Z(\lambda)^{-[n/2]},$$

where $[\cdot]$ denotes the integer part of a real number. Then $(M_n^1)_n$ and $(M_n^2)_n$ are discrete $\mathcal{G}_n^{(1)}$ –martingales ($\mathcal{G}_n^{(2)}$ –martingales, respectively). Hence using Doob’s stopping rule for positive super-martingales we get

$$\mathbb{E}_\pi \left[\exp\left(\lambda \sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)}\right) \exp\left(-[(N_t^{(1)} + 1)/2] \log Z(\lambda)\right) \right] \leq 1 \quad (3.12)$$

and also

$$\mathbb{E}_\pi \left[\exp\left(\lambda \sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)}\right) \exp\left(-[(N_t^{(2)} + 2)/2] \log Z(\lambda)\right) \right] \leq 1. \quad (3.13)$$

Now, we proceed as follows. Evidently,

$$\mathbb{P}_\pi \left(\int_0^{R_{N_t+2}} f(X_s) ds \geq v_t^{\frac{1}{2}+\eta} x \right) \leq I_0 + I_1 + I_2,$$

where $I_0 = \mathbb{P}_\pi(\xi_1 \geq v_t^{\frac{1}{2}+\eta} x/3)$,

$$I_1 = \mathbb{P}_\pi \left(\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2}+\eta} x/3 \right), \quad I_2 = \mathbb{P}_\pi \left(\sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)} \geq v_t^{\frac{1}{2}+\eta} x/3 \right).$$

We start with a study of the first term I_0 . Let

$$\tilde{Z}(\lambda) = \mathbb{E}_\pi e^{\lambda \xi_1}.$$

Then, for $0 < \lambda < K(f)^{-1}$,

$$\mathbb{P}_\pi[\xi_1 \geq v_t^{\frac{1}{2}+\eta} x/3] \leq \exp\left(-\left(v_t^{\frac{1}{2}+\eta} \lambda x/3 - \log \tilde{Z}(\lambda)\right)\right).$$

But due to (3.9), we have that $\log \tilde{Z}(\lambda) \leq \frac{\lambda K(f)}{1-\lambda K(f)}$, and thus, taking $\lambda = \frac{1}{2} K(f)^{-1}$,

$$I_0 = \mathbb{P}_\pi[\xi_1 \geq v_t^{\frac{1}{2}+\eta} x K(f)/3] \leq e \exp\left(-\frac{1}{6K(f)} v_t^{\frac{1}{2}+\eta} x\right). \quad (3.14)$$

Let us now turn to the study of I_1 and I_2 .

Note that for any $k > 0$,

$$I_1 \leq \mathbb{P}_\pi \left(\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2}+\eta} x/3; N_t \leq k \right) + \mathbb{P}_\pi(N_t > k).$$

By (3.9), we have that for any $0 < \lambda < K(f)^{-1}$,

$$Z(\lambda) \leq 1 + \sum_{n \geq 2} [\lambda K(f)]^n.$$

Hence

$$Z(\lambda) \leq 1 + \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}, \quad \text{thus } \log Z(\lambda) \leq \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}.$$

Hence for any $0 < \lambda < K(f)^{-1}$, using (3.12) and recalling that $N_t^{(1)} \leq N_t$,

$$\begin{aligned}
\mathbb{P}_\pi\left[\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2}+\eta} x/3 ; N_t \leq k\right] &\leq \mathbb{P}_\pi\left[\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2}+\eta} x/3 ; N_t^{(1)} \leq k\right] \\
&\leq \mathbb{P}_\pi\left[M_{N_t^{(1)}+2}^1 \geq \exp(\lambda v_t^{\frac{1}{2}+\eta} x/3 - \log Z(\lambda)[(N_t^{(1)} + 1)/2]) ; N_t^{(1)} \leq k\right] \\
&\leq \mathbb{P}_\pi\left[M_{N_t^{(1)}+2}^1 \geq \exp(\lambda v_t^{\frac{1}{2}+\eta} x/3 - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} [(N_t^{(1)} + 1)/2]) ; N_t^{(1)} \leq k\right] \\
&\leq \mathbb{P}_\pi\left[M_{N_t^{(1)}+2}^1 \geq \exp(\lambda v_t^{\frac{1}{2}+\eta} x/3 - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} [(k + 1)/2])\right] \\
&\leq \exp\left(-\left(v_t^{\frac{1}{2}+\eta} \lambda x \frac{1}{3} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} k\right)\right).
\end{aligned}$$

In the same way we get that

$$\mathbb{P}_\pi\left[\sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)} \geq v_t^{\frac{1}{2}+\eta} x/3 ; N_t \leq k\right] \leq \exp\left(-\left(v_t^{\frac{1}{2}+\eta} \lambda x \frac{1}{3} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} k\right)\right).$$

Now, take

$$k = v_t^{1+\delta}. \quad (3.15)$$

Then we have that

$$\mathbb{P}_\pi\left[\sum_{k=2}^{N_t^{(1)}+2} \xi_k^{(1)} \geq v_t^{\frac{1}{2}+\eta} x/3 ; N_t \leq v_t^{1+\delta}\right] \leq \exp\left(-v_t^{1+\delta} h(x)\right), \quad (3.16)$$

$$\mathbb{P}_\pi\left[\sum_{k=2}^{N_t^{(2)}+2} \xi_k^{(2)} \geq v_t^{\frac{1}{2}+\eta} x/3 ; N_t \leq v_t^{1+\delta}\right] \leq \exp\left(-v_t^{1+\delta} h(x)\right),$$

where

$$h(x) := \sup_{0 < \lambda < K(f)^{-1}} \left(\lambda \frac{x}{3v_t^{\frac{1}{2}+\delta-\eta}} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} \right). \quad (3.17)$$

It can be shown, see for example Birgé and Massart ([3]), lemma 8, pages 366 and 367, that

$$\sup_{0 < \lambda < 1/v} \left(\lambda x - \frac{\lambda^2 v^2}{1 - \lambda v} \right) \geq \frac{x^2}{2vx + 4v^2}. \quad (3.18)$$

This is seen as follows. A simple calculus shows that

$$\sup_{0 < \lambda < 1/v} \left(\lambda x - \frac{\lambda^2 v^2}{1 - \lambda v} \right) = \lambda^* x - \frac{(\lambda^*)^2 v^2}{1 - \lambda^* v},$$

where

$$\lambda^* = \frac{1}{v} \left(1 - \sqrt{\frac{v}{x+v}} \right) < \frac{1}{v}.$$

It follows that

$$\lambda^* x - \frac{(\lambda^*)^2 v^2}{1 - \lambda^* v} = \left(\sqrt{\frac{1}{x+v}} - 1 \right)^2 = \frac{x^2}{xv + 2v^2 + 2v^2(1 + \frac{xy}{v^2})^{1/2}},$$

and using $(1+x)^{1/2} \leq 1+x/2$, one gets the desired inequality.

Using (3.17) and (3.18), we get for any $t \geq t_0$ such that $(v_{t_0})^{\frac{1}{2}+\delta-\eta} \geq 1$,

$$h(x) \geq \frac{1}{3} \frac{x^2}{2xK(f) + 12K^2(f)} v_t^{-1+2\eta-2\delta} \geq \frac{1}{42} \frac{x^2 \wedge x}{\max(K^2(f), K(f))} v_t^{-1+2\eta-2\delta}. \quad (3.19)$$

Note that the right hand side of (3.19) tends to zero at speed $v_t^{-1+2\eta-2\delta}$. However, this right hand side has to be multiplied with $v_t^{1+\delta}$, compare to (3.16), which yields the term $v_t^{2\eta-\delta}$ which does not tend to zero since by assumption, $\delta \leq 2\eta$. Thus, together with (3.14),

$$\begin{aligned} \mathbb{P}_\pi \left[\int_0^{R_{N_t+2}} f(X_s) ds \geq v_t x \right] &\leq 2 \exp \left(-\frac{1}{42} \frac{x^2 \wedge x}{\max(K^2(f), K(f))} v_t^{2\eta-\delta} \right) \\ &+ e \exp \left(-\frac{1}{6K(f)} v_t^{\frac{1}{2}+\eta} x \right) + 2\mathbb{P}_\pi(N_t > v_t^{1+\delta}). \end{aligned}$$

Applying the same argument to $-f$ instead of f yields

$$\begin{aligned} \mathbb{P}_\pi \left(\left| \int_0^{R_{N_t+2}} f(X_s) ds \right| \geq v_t x \right) &\leq 4 \exp \left(-\frac{1}{42} \frac{x^2 \wedge x}{\max(K^2(f), K(f))} v_t^{2\eta-\delta} \right) \\ &+ 2e \exp \left(-\frac{1}{6K(f)} v_t^{\frac{1}{2}+\eta} x \right) + 4\mathbb{P}_\pi(N_t > v_t^{1+\delta}). \end{aligned} \quad (3.20)$$

Moreover, note that

$$\begin{aligned} \mathbb{P}_\pi \left(\left| \int_t^{R_{N_t+2}} f(X_s) ds \right| \geq v_t^{\frac{1}{2}+\eta} x \right) &\leq \mathbb{P}_\pi \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \\ &+ \mathbb{P}_\pi \left(\left| \int_{R_{N_t+1}}^{R_{N_t+2}} f(X_s) ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \end{aligned}$$

and, using Markov's property with respect to \mathcal{F}_t and to $\mathcal{F}_{R_{N_t+1}}$,

$$\mathbb{P}_\pi \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \leq \mathbb{E}_\pi \mathbb{E}_{X_t} e^{\lambda \int_0^{R_1} |f|(X_s) ds} e^{-v_t^{\frac{1}{2}+\eta} \lambda x / 2}. \quad (3.21)$$

Write

$$\tilde{Z}(\lambda) = \mathbb{E}_\pi \mathbb{E}_{X_t} \left(\exp \lambda \int_0^{R_1} |f|(X_s) ds \right).$$

As before we have that

$$\log \tilde{Z}(\lambda) \leq \frac{\lambda K(f)}{1 - \lambda K(f)},$$

and thus, taking $\lambda = \frac{1}{2}K(f)^{-1}$,

$$\mathbb{P}_\pi \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \leq e \exp \left(-\frac{1}{4K(f)} v_t^{\frac{1}{2}+\eta} x \right).$$

In the same way we get that

$$\mathbb{P}_\pi \left(\left| \int_{R_{N_t+1}}^{R_{N_t+2}} f(X_s) ds \right| \geq \frac{1}{2} v_t^{\frac{1}{2}+\eta} x \right) \leq e \exp \left(-\frac{1}{4K(f)} v_t^{\frac{1}{2}+\eta} x \right).$$

•

Remark 3.3 1. *The most delicate point in the above proof is (3.19) which gives a lower bound converging to zero. It is important to be able control the speed of convergence of this expression to zero. If we decided to work with the Legendre transform, then the above proof remains true by replacing (3.17) by*

$$h(x) = \sup_{\lambda > 0} \left(\lambda \frac{x}{v_t^{\frac{1}{2}+\delta-\eta}} - \log Z(\lambda) \right).$$

In this form it is evident that $h(x) = h_t(x) \rightarrow 0$ as $t \rightarrow \infty$, and developing $Z(\lambda) = 1 + \frac{\sigma^2}{2}\lambda^2 + o(\lambda^2)$, where

$$\sigma^2 := \mathbb{E} \left(\int_{R_1}^{R_2} f(X_s) ds \right)^2,$$

would yield the same speed of convergence. However, using this approach we would not be able to put hands on the exact form of the constants appearing in the remainder term $o(\lambda^2)$, and that is why we decided to use the upper bound of $\log Z(\lambda)$ as proposed in the proof.

2. *The bounds obtained in the above theorem are in general not optimal. But they are valid in a non-asymptotic framework and they have the advantage that they involve only known quantities. The above theorem does not use the Legendre transform of $\xi_n = \int_{R_{n-1}}^{R_n} f(X_s) ds$ since in general $Z(\lambda)$ defined in (3.10) and hence the associated Legendre transform are not known : they depend on the abstract regeneration times R_n . That is why the above deviation inequality involves the constant $K(f)$. In the following, we give some examples on how to put hands on this constant $K(f)$.*

- *By proposition 2.16 of [15] we know the following. Recall the definition of the set C of (2.6), see also remark 2.7. Let S be the first jump time of a Poisson process having rate $1_C(X_s)$, and let K be a constant such that for all x ,*

$$E_x \int_0^S |f|(X_s) ds \leq K, \quad |f(x)| \leq K.$$

Then we have that

$$C(f) \leq K + \frac{3K}{\alpha} \text{ and hence } K(f) \leq 2K + \frac{3K}{\alpha}.$$

- We are now going to explore the relationship between $C(f)$ and the invariant measure $\mu(|f|)$ in some special cases.

Suppose that the process X is strong Feller and positive recurrent. Let F be a compact set such that the support of f is contained in F . Let

$$T_C := \inf\{t \geq 0 : X_t \in C\}$$

be the entrance time in the set C . Recall that the measure ν is concentrated on C . Then

$$C(f) = \sup_{x \in F} E_x \int_0^{R_1} |f|(X_s) ds \leq \left[\|f\|_\infty \sup_{x \in F} E_x T_C \right] + \sup_{x \in F \cap C} E_x \int_0^{R_1} |f|(X_s) ds. \quad (3.22)$$

By continuity of the map $x \mapsto E_x \int_0^{R_1} |f|(X_s) ds$, there exists $x_0 \in C \cap F$ such that

$$\sup_{x \in F \cap C} E_x \int_0^{R_1} |f|(X_s) ds = E_{x_0} \int_0^{R_1} |f|(X_s) ds.$$

Moreover, there exists $\varepsilon > 0$, such that for all $x \in B_\varepsilon(x_0)$,

$$E_x \int_0^{R_1} |f|(X_s) ds \geq \frac{1}{2} E_{x_0} \int_0^{R_1} |f|(X_s) ds.$$

Then we have that

$$\begin{aligned} \mu(|f|) &= E_\nu \int_0^{R_1} |f|(X_s) ds \geq \int_{B_\varepsilon(x_0)} \nu(dx) E_x \int_0^{R_1} |f|(X_s) ds \\ &\geq \frac{1}{2} \cdot \nu(B_\varepsilon(x_0)) \cdot E_{x_0} \int_0^{R_1} |f|(X_s) ds. \end{aligned} \quad (3.23)$$

Putting together (3.22) and (3.23), we conclude that

$$C(f) \leq \left[\|f\|_\infty \sup_{x \in F} E_x T_C \right] + \frac{2}{\nu(B_\varepsilon(x_0))} \mu(|f|). \quad (3.24)$$

- Suppose that X is a positive recurrent one-dimensional diffusion process

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t.$$

In the case of dimension one, we can avoid Nummelin splitting since successive visits of recurrent points allow to split the trajectory into i.i.d. excursions. More precisely, let $a < b$ be two recurrent points of X and define a sequence of stopping times $(S_n)_n, (R_n)_n$ as follows. $S_0 = R_0 = 0$,

$$S_1 = \inf\{t \geq 0 : X_t = b\}, \quad R_1 = \inf\{t \geq S_1 : X_t = a\}, \quad (3.25)$$

and for any $n \geq 1$, $S_{n+1} = R_n + S_1 \circ \theta_{R_n}$, $R_{n+1} = R_n + R_1 \circ \theta_{R_n}$. Using this sequence $(R_n)_n$, under additional regularity assumptions (see [19] for the details), we have that

$$C(f) \leq \kappa \mu(|f|),$$

for any function f having compact support. Here, the constant κ is explicitly known.

In the above proof of Theorem 3.1, we made use of the following proposition.

Proposition 3.4 *Let f be a bounded special function. Put $K(f) := \|f\|_\infty + C(f)$. Then we have for any initial measure π and any $n \geq 1$, that*

$$\mathbb{E}_\pi(|\xi_n|^p) \leq p!K(f)^p. \quad (3.26)$$

In particular, we obtain for any $0 < \lambda < K(f)^{-1}$,

$$\mathbb{E}_\pi(e^{\lambda\xi_n}) \leq 1 + \sum_{p \geq 1} \lambda^p K(f)^p = \frac{1}{1 - \lambda K(f)},$$

and if $\mu(f) = 0$,

$$\mathbb{E}_\pi(e^{\lambda\xi_n}) \leq 1 + \sum_{p \geq 2} \lambda^p K(f)^p = 1 + \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)}.$$

Proof Evidently, we have that

$$\begin{aligned} \xi_n^p &= \int_{R_{n-1}}^{R_n} \dots \int_{R_{n-1}}^{R_n} f(X_{t_1}) \dots f(X_{t_p}) dt_1 \dots dt_p \\ &\leq p! \int_{R_{n-1}}^{R_n} \dots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_p\}} |f|(X_{t_1}) \dots |f|(X_{t_p}) dt_1 \dots dt_p. \end{aligned} \quad (3.27)$$

Taking expectation and conditional expectation with respect to $\mathcal{F}_{t_{p-1}}$, we get

$$\begin{aligned} &\mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \dots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_p\}} |f|(X_{t_1}) \dots |f|(X_{t_p}) dt_1 \dots dt_p \\ &= \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \dots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_{p-1}\}} |f|(X_{t_1}) \dots |f|(X_{t_{p-1}}) \\ &\quad \mathbb{E}_{Z_{t_{p-1}}} \left[\int_0^{R_1} |f|(X_s) ds \right] dt_1 \dots dt_{p-1}. \end{aligned} \quad (3.28)$$

But note that for any fixed z , for $M = \|f\|_\infty$,

$$\begin{aligned} \mathbb{E}_z \left[\int_0^{R_1} |f|(X_s) ds \right] &\leq M \mathbb{E}_z(T_1) + \mathbb{E}_z \int_{T_1}^{R_1} |f|(X_s) ds \\ &\leq M \mathbb{E}_z(T_1) + \mathbb{E}_z(\mathbb{E}_{Z_{T_1}^1} \int_0^{R_1} |f|(X_s) ds) \\ &\leq M \mathbb{E}_z(T_1) + C(f). \end{aligned}$$

(Compare to proposition 2.16 of [15].) But by construction, for $z = (x, u, x')$,

$$\mathbb{E}_z(T_1) = \int_0^\infty t e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt,$$

and this expression does only depend on x and x' . According to proposition 4.1 of [16], we have that

$$\mathcal{L}(Z_t^3 | Z_t^1 = x)(dx') = u^1(x, x') \Lambda(dx').$$

Taking now conditional expectation in (3.28) with respect to $\mathcal{F}_{t_{p-2}}$ and putting all these results together, we obtain

$$\begin{aligned}
& \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_p\}} |f|(X_{t_1}) \cdots |f|(X_{t_p}) dt_1 \cdots dt_p \\
&= \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_{p-1}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-1}}) \mathbb{E}_{Z_{t_{p-1}}} \left[\int_0^{R_1} |f|(X_s) ds \right] dt_1 \cdots dt_{p-1} \\
&\leq \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_{p-1}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-1}}) \left[M \mathbb{E}_{Z_{t_{p-1}}}(T_1) + C(f) \right] dt_1 \cdots dt_{p-1} \\
&= \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_{p-2}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-2}}) \\
&\quad \mathbb{E}_{Z_{t_{p-2}}} \left[\int_0^{R_1} |f|(X_s) (M \mathbb{E}_{Z_s}(T_1) + C(f)) ds \right] dt_1 \cdots dt_{p-2} \\
&= \mathbb{E}_\pi \int_{R_{n-1}}^{R_n} \cdots \int_{R_{n-1}}^{R_n} 1_{\{t_1 \leq \dots \leq t_{p-2}\}} |f|(X_{t_1}) \cdots |f|(X_{t_{p-2}}) \\
&\quad \mathbb{E}_{Z_{t_{p-2}}} \left[\int_0^{R_1} |f|(X_s) (M + C(f)) ds \right] dt_1 \cdots dt_{p-2},
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}
\mathbb{E}_z \int_0^{R_1} |f|(X_s) \mathbb{E}_{Z_s}(T_1) ds &= \mathbb{E}_z \int_0^\infty 1_{\{s < R_1\}} |f|(X_s) \left(\int u^1(X_s, x') \Lambda(dx') \mathbb{E}_{(X_s, x')}(T_1) \right) ds \\
&= \mathbb{E}_z \int_0^\infty 1_{\{s < R_1\}} |f|(X_s) ds,
\end{aligned}$$

since

$$\int u^1(x, x') \Lambda(dx') \int_0^\infty t e^{-t} \frac{p_t(x, x')}{u^1(x, x')} dt = 1.$$

Taking successively conditional expectations with respect to $\mathcal{F}_{t_{p-3}}, \dots, \mathcal{F}_{t_1}$ yields the result. •

4 Control of the number of life-cycles before time t

Theorem 3.1 is stated in the most general situation. That is why the number of life cycles before time t , N_t , appears. This control depends on the quality of recurrence.

4.1 The positive recurrent case

In the case that X is positive recurrent, we have that

$$m := \mathbb{E}(R_2 - R_1) < +\infty.$$

Then it is possible to take $\eta = \delta = 0$ and

$$v_t = \frac{2}{m} \cdot t.$$

In this case, concerning the last term $\mathbb{P}_\pi(N_t > v_t)$ in Theorem 3.1, we have :

Proposition 4.1 *Suppose that X is positive recurrent. Then*

$$\mathbb{P}_\pi(N_t > v_t) \leq 2 \exp\left(-\frac{1}{2}v_t \sup_\lambda h_t(\lambda)\right),$$

where

$$h_t(\lambda) := -\left(1 - \frac{m}{t}\right) \log \hat{F}(\lambda) - \lambda m/2$$

and where

$$\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2 - R_1)})$$

is the Laplace transform of the length of a life cycle. Moreover we have for any $u < m$ that

$$\Lambda^*(u) = \sup_{\lambda > 0} [-\lambda u - \log \hat{F}(\lambda)] > 0. \quad (4.29)$$

Proof of Proposition 4.1 Writing $v_t = \frac{2}{m}t$ and $k_t := [v_t]$, for any $\lambda > 0$,

$$\mathbb{P}_\pi(N_t > v_t) = \mathbb{P}_\pi(R_{k_t} \leq t) \leq \mathbb{P}_\pi(R_{k_t} - R_1 \leq t) \leq \mathbb{P}_\pi(e^{-\lambda(R_{k_t} - R_1)} \geq e^{-\lambda t}).$$

But, using the definition of $\xi_n^{(1)}$ and $\xi_n^{(2)}$ as in (3.11), with $f = 1$, and the same technique as above,

$$\begin{aligned} \mathbb{P}_\pi(e^{-\lambda(R_{k_t} - R_1)} \geq e^{-\lambda t}) &\leq \mathbb{P}_\pi(e^{-\lambda \sum_{n=2}^{k_t} \xi_n^{(1)}} \geq e^{-\lambda t/2}) + \mathbb{P}_\pi(e^{-\lambda \sum_{n=2}^{k_t} \xi_n^{(2)}} \geq e^{-\lambda t/2}) \\ &\leq 2\hat{F}(\lambda)^{(k_t-1)/2} e^{\lambda t/2} \leq 2\hat{F}(\lambda)^{(v_t-2)/2} e^{\lambda t/2}, \end{aligned} \quad (4.30)$$

where

$$\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2 - R_1)})$$

is the Laplace transform of the length of a life cycle. So write

$$h_t(\lambda) := -\left(1 - \frac{2}{v_t}\right) \log \hat{F}(\lambda) - \lambda \frac{t}{v_t}.$$

Then

$$\mathbb{P}_\pi(N_t > v_t) \leq 2e^{-\frac{1}{2}v_t \sup_\lambda h_t(\lambda)}.$$

Let us finally show that

$$\Lambda^*(u) = \sup_{\lambda > 0} [-\lambda u - \log \hat{F}(\lambda)] > 0$$

for any $u < m$. This is evident using the fact that for $\lambda \rightarrow 0$,

$$-\log \hat{F}(\lambda) \sim 1 - \hat{F}(\lambda) \sim m\lambda.$$

This concludes the proof. •

A straightforward calculus shows that, for all $t \geq 4m$ (which implies that $1 - \frac{2}{v_t} \geq \frac{3}{4}$),

$$\sup_\lambda h_t(\lambda) \geq \frac{3}{4} \sup_\lambda \left(-\frac{2}{3}m\lambda - \log \hat{F}(\lambda)\right) = \frac{3}{4} \Lambda^*\left(\frac{2}{3}m\right).$$

Hence we get the following corollary of Theorem 3.1.

Theorem 4.2 *Let X be positive recurrent. Let f be a bounded special function such that $\mu(f) = 0$. Let*

$$\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda(R_2 - R_1)})$$

be the Laplace transform of the length of a life cycle and put

$$m = \mathbb{E}(R_2 - R_1).$$

Write

$$B(f) := \max(K^2(f), K(f)).$$

Then we have the following results.

1. *For any $0 < \eta \leq \frac{1}{2}$, for all x , for any initial measure π and for all $t > 4m$,*

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq t^{\frac{1}{2} + \eta} \left(\frac{2}{m} \right)^{\frac{1}{2} + \eta} x \right) \\ \leq 4 \exp \left(-t^{2\eta} \frac{1}{42m B(f)} (x^2 \wedge x) \right) + 4e \exp \left(-\frac{2^{\frac{1}{2} + \eta}}{6 K(f) m^{\frac{1}{2} + \eta}} t^{\frac{1}{2} + \eta} x \right) \\ + 8 \exp \left(-t \frac{3}{4m} \Lambda^* \left(\frac{2}{3} m \right) \right), \end{aligned}$$

where $\Lambda^ \left(\frac{2}{3} m \right) > 0$ as in (4.29).*

2. *For all x and for any initial measure π , for all $t > 4m$,*

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq \sqrt{t} x \frac{\sqrt{2}}{\sqrt{m}} \right) \leq 4 \exp \left(-\frac{1}{42 B(f)} (x^2 \wedge x) \right) \\ + 4e \exp \left(-\frac{\sqrt{2}}{6 K(f) \sqrt{m}} \sqrt{t} x \right) + 8 \exp \left(-t \frac{3}{4m} \Lambda^* \left(\frac{2}{3} m \right) \right). \end{aligned}$$

Remark 4.3 *1. The above theorem has to be compared with the results obtained by Loukianova et al. ([18]). Note that we are in a different situation since we consider centered functions that are supposed to be special, i.e. satisfying $\sup_x \mathbb{E}_x \int_0^{R_1} |f|(X_s) ds < \infty$. If f is any positive bounded function having compact support, then f is certainly special, but $\bar{f} := f - \mu(f)$ in general won't (constants are special only if the underlying process is uniformly ergodic). However, our method works for any function f such that exponential moments (3.10) are finite. Thus, if, for λ sufficiently small,*

$$\mathbb{E}_\pi(e^{\lambda(R_2 - R_1)}) + \mathbb{E}_\pi(e^{\lambda R_1}) < \infty,$$

then our result holds also for any function $\bar{f} = f - \mu(f)$, where f is positive, bounded and of compact support. In particular, in the uniform ergodic case, our theorem applies to any function of the type $f - \mu(f)$, with f positive and bounded.

2. Note that even in the case when considering not exponentially ergodic processes, i.e. the situation where $R_2 - R_1$ does not possess moments of any order, we get exponential convergence rates. This is due to the fact that we consider special functions only.

3. Our result holds in any dimension and under any starting measure π .

4.2 The null-recurrent case

In the null-recurrent case, we cannot expect to obtain as good convergence rates as in the positive recurrent case and we have to take $\delta > 0$ and $\eta > 0$ strictly positive. We fix – for this section – the following choice of a deterministic equivalent.

$$v_t^* = \mathbb{E}_\nu(N_t) + 1, \quad (4.31)$$

where $\nu = \mathcal{L}(Z_{R_n}^1)$ is given in (2.6).

Remark 4.4 *Note that in (4.31) we do not use the usual form of the deterministic equivalent as chosen in (2.5). But we have the following comparison result :*

1. For the choice of $v_t = E_\nu \int_0^t g(X_s) ds$ as in (2.5), we have

$$v_t \leq C(g) + \mu(g) \cdot v_t^*. \quad (4.32)$$

This can be seen as follows. Since $g > 0$, we have clearly that

$$\begin{aligned} v_t &\leq \sum_{n \geq 0} E_\nu \left(\mathbf{1}_{\{R_n \leq t\}} \int_{R_n}^{R_{n+1}} g(X_s) ds \right) \\ &\leq C(g) + \sum_{n \geq 1} E_\nu \left(\mathbf{1}_{\{R_{n-1} \leq t\}} \int_{R_n}^{R_{n+1}} g(X_s) ds \right). \end{aligned}$$

Using Markov's property with respect to $\mathcal{F}_{R_{n-1}}$ in the last expression and (2.8), we obtain that

$$v_t \leq C(g) + \mu(g)[E_\nu(N_t) + 1],$$

and thus,

$$v_t \leq C(g) + \mu(g) \cdot v_t^*.$$

2. Moreover, we have the following lower bound.

$$v_t \geq \mu(g) \cdot v_t^* - 2C(g). \quad (4.33)$$

Indeed,

$$\begin{aligned} v_t &\geq \left(\sum_{n \geq 0} E_\nu \left(\mathbf{1}_{\{R_n \leq t\}} \int_{R_{n+1}}^{R_{n+2}} g(X_s) \right) \right) - E_\nu \left(\int_{R_{N_t}}^{R_{N_t+2}} g(X_s) \right) \\ &\geq [E_\nu(N_t) + 1] \mu(g) - 2C(g) = v_t^* \cdot \mu(g) - 2C(g), \end{aligned}$$

since $E_\nu \left(\int_{R_{N_t}}^{R_{N_t+2}} g(X_s) \right) \leq 2C(g)$.

The following proposition is important in order to control the deviations for N_t .

Proposition 4.5

$$\mathbb{E}_\pi(N_t^p) \leq p!(v_t^*)^p.$$

Proof We have that

$$N_t = \sum_{n \geq 1} 1_{\{R_n \leq t\}},$$

and thus

$$N_t^p = p! \sum_{1 \leq n_1 \leq n_2 \leq \dots \leq n_p} 1_{\{R_{n_1} \leq t\}} \cdot \dots \cdot 1_{\{R_{n_p} \leq t\}} = p! \sum_{1 \leq n_1 \leq n_2 \leq \dots \leq n_p} 1_{\{R_{n_p} \leq t\}}.$$

Hence, using successively Markov's property with respect to $\mathcal{F}_{R_{n_p-1}}$ etc, we get

$$\mathbb{E}_\pi(N_t^p) \leq p! \sum_{1 \leq n_1 \leq n_2 \leq \dots \leq n_{p-1}} \mathbb{E}_\pi(1_{\{R_{n_{p-1}} \leq t\}} \mathbb{E}_{Z_{R_{n_{p-1}}}}(\sum_{n_p \geq 1} 1_{\{R_{n_p} \leq t\}})).$$

But note that for any n ,

$$\begin{aligned} \mathbb{E}_{Z_{R_n}}(\sum_{k \geq 1} 1_{\{R_k \leq t\}}) &\leq 1 + \mathbb{E}_{Z_{R_n}}(\sum_{k \geq 2} 1_{\{R_k \leq t\}}) \leq 1 + E_{Z_{R_n}}(E_{Z_{R_1}} \sum_{k \geq 1} 1_{\{R_k \leq t\}}) \\ &= 1 + \mathbb{E}_\nu \sum_{k \geq 1} 1_{\{R_k \leq t\}} = v^*(t), \end{aligned}$$

since $\mathcal{L}(Z_{R_1}) = \nu$. Induction on n then yields the assertion. •

We get the following corollary.

Corollary 4.6 *We have that*

$$\mathbb{P}_\pi(N_t > (v_t^*)^{1+\delta}) \leq 2 \exp\left(-\frac{1}{2}(v_t^*)^\delta\right). \quad (4.34)$$

Proof By proposition 4.5, we have that

$$\mathbb{E}_\pi(\exp(\lambda N_t)) \leq \frac{1}{1 - \lambda v_t^*},$$

for λ sufficiently small. Choosing $\lambda = (v_t^*)^{-1}/2$, we get that

$$\mathbb{P}_\pi(N_t > (v_t^*)^{1+\delta}) \leq e^{-\lambda(v_t^*)^{1+\delta}} \mathbb{E}_\pi(\exp(\lambda N_t)) \leq e^{-(v_t^*)^\delta/2} \frac{1}{1 - \lambda v_t^*} \leq 2 e^{-(v_t^*)^\delta/2}. \quad \bullet$$

Choosing $\delta = \eta$, we can thus rewrite the result of theorem 3.1 in the following way.

Theorem 4.7 *Suppose that X is null-recurrent. Let f be a bounded special function such that $\mu(f) = 0$. Recall that*

$$K(f) = \|f\|_\infty + \sup_x \mathbb{E}_x \int_0^{R_1} |f|(X_s) ds, \quad B(f) = \max(K^2(f), K(f)).$$

Then there exists t_0 , such that for all $t \geq t_0$, for all x , for any $0 < \eta \leq \frac{1}{2}$, for any initial measure π and for v_t^ as in (4.31),*

$$\begin{aligned} \mathbb{P}_\pi\left(\left|\int_0^t f(X_s) ds\right| \geq (v_t^*)^{\frac{1}{2}+\eta} x\right) &\leq 4 \exp\left(-\frac{1}{42 B(f)} (v_t^*)^\eta (x^2 \wedge x)\right) \\ &\quad + 4e \exp\left(-\frac{1}{6K(f)} (v_t^*)^{\frac{1}{2}+\eta} x\right) + 8 \exp\left(-\frac{1}{2}(v_t^*)^\eta\right). \end{aligned}$$

t_0 is given by the solution of $(v_{t_0}^*)^{1/2} = 1$.

4.3 Some discussions concerning the regular case

The above theorem 4.7 is far from being optimal in the null-recurrent but regular case. Here, in accordance with Theorem 3.15 of Höpfner and Löcherbach, [13], we call a process *regular*, if for $0 < \alpha < 1$ and a function l varying slowly at ∞ , the following is true. For any measurable and positive function g with $0 < \mu(g) < \infty$, we have regular variation of resolvents

$$R_{1/t}g(x) = E_x \left(\int_0^\infty e^{-\frac{1}{t}s} g(X_s) ds \right) \sim t^\alpha \frac{1}{l(t)} \mu(g) \text{ as } t \rightarrow \infty, \quad (4.35)$$

for μ -almost all x . Here we do not consider the case $\alpha = 1$. Then (4.35) is equivalent to the following :

$$\mathbb{P}(R_2 - R_1 > x) \sim \frac{x^{-\alpha} l(x)}{\Gamma(1 - \alpha)} \text{ as } x \rightarrow \infty. \quad (4.36)$$

This will be shown in proposition 4.12 below.

Thus we are in the situation where $R_2 - R_1$ belongs to the domain of attraction of a stable law. In this case a finer control of N_t is possible and it is well-known that

$$v(t) \sim t^\alpha \frac{1}{l(t)} \text{ as } t \rightarrow \infty,$$

see for instance Höpfner and Löcherbach [13], theorem 5.6.A. Moreover,

$$1 - \hat{F}(\lambda) \sim \lambda^\alpha l(1/\lambda) \text{ as } \lambda \rightarrow 0, \quad (4.37)$$

see for instance Bingham et al. ([2], corollary 8.1.7).

Then we get the following version of the deviation theorem.

Theorem 4.8 *Suppose that (4.36) holds. Let f be a bounded special function such that $\mu(f) = 0$. Then there exists $t_0 \geq 0$, see (4.41) below, a function L varying slowly at infinity, such that for all x and for any initial measure π , for all $t \geq t_0$, for all $0 < \eta \leq \frac{\alpha}{2}$,*

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq t^{\frac{\alpha}{2} + \eta} x \right) &\leq 4 \exp \left(-\frac{1}{42 B(f)} t^{2\eta/(2-\alpha)} (x^2 \wedge x) L(t) \right) \\ &+ 4e \exp \left(-\frac{1}{6K(f)} t^{\frac{\alpha}{2} + \eta} x \right) + 8 \exp \left(-\frac{1}{2} t^{2\eta/(2-\alpha)} \Lambda_t^* \right), \end{aligned}$$

where

$$\Lambda_t^* = \sup_{\lambda > 0} \left[-\log \hat{F}(\lambda) t^{\alpha - \alpha \frac{2\eta}{2-\alpha}} \frac{1}{L(t)} - \lambda t^{1 - \frac{2\eta}{2-\alpha}} \right] > 0$$

is positive and does not depend on t asymptotically :

$$\liminf_{t \rightarrow \infty} \Lambda_t^* \geq (1 - \alpha) \alpha^{\alpha/(1-\alpha)} > 0.$$

Remark 4.9 Note that in the above theorem, for $\eta = \alpha/2$, we obtain a rate of decay of the order of $\exp(-t^{\alpha/(2-\alpha)})$. For $\alpha = 2$, this gives the rate of convergence $\exp(-t^{1/3})$ which is better than the rate $\exp(-t^{1/4})$ obtained in theorem 4.7 (for $\eta = \frac{1}{2}$).

Proof The proof is a slight modification of the proof of theorem 3.1.

We go back to the proof of theorem 3.1. We fix the following choice of k (compare to (3.15)) :

$$k = k_t = 2 + t^\gamma \frac{1}{l(1/\lambda_t)}, \text{ where } \gamma = \alpha + 2\eta \frac{1-\alpha}{2-\alpha}. \quad (4.38)$$

Here, $\lambda_t \rightarrow 0$ will be defined in (4.40) below.

As in (4.30), we have for any $\lambda > 0$,

$$\mathbb{P}_\pi(N_t > k) \leq 2e^{-\frac{1}{2}[-\log \hat{F}(\lambda)(k-2) - \lambda t]},$$

and therefore, due to the choice (4.38), we have to find

$$\sup_\lambda \left[-\log \hat{F}(\lambda) t^\gamma \frac{1}{l(1/\lambda_t)} - \lambda t \right].$$

Writing $L(t) = l(1/\lambda_t)$, this last expression can be rewritten as

$$t^{\frac{\gamma-\alpha}{1-\alpha}} \sup_\lambda \left[-\log \hat{F}(\lambda) t^{\alpha \frac{1-\gamma}{1-\alpha}} \frac{1}{L(t)} - \lambda t^{\frac{1-\gamma}{1-\alpha}} \right] = t^{\frac{2\eta}{2-\alpha}} \Lambda_t^*.$$

(Note that $\frac{1-\gamma}{1-\alpha} = 1 - \frac{2\eta}{2-\alpha}$.) Write for simplicity

$$s_t := t^{\frac{1-\gamma}{1-\alpha}} = t^{1 - \frac{2\eta}{2-\alpha}}.$$

Then,

$$\Lambda_t^* = \sup_{\lambda > 0} \left[-\log \hat{F}(\lambda) \frac{1}{l(1/\lambda_t)} s_t^\alpha - \lambda s_t \right],$$

and we have to show that Λ_t^* is positive. Note that since $R_2 - R_1$ does not possess any moments, we are not able to develop $\hat{F}(\lambda)$ near 0 in the usual way. But we can use (4.37). That's why we take λ of the form $\lambda_t \rightarrow 0$ at a speed that will be precised in (4.40) below. Note that $\log \hat{F}(\lambda_t) = \log [1 - (1 - \hat{F}(\lambda_t))]$. Due to (4.37) we have that

$$-\log \hat{F}(\lambda_t) \frac{1}{l(1/\lambda_t)} s_t^\alpha - \lambda_t s_t \sim \lambda_t^\alpha s_t^\alpha - \lambda_t s_t,$$

and hence

$$\Lambda_t^* \geq -\log \hat{F}(\lambda_t) \frac{1}{l(1/\lambda_t)} s_t^\alpha - \lambda_t s_t \sim \lambda_t^\alpha s_t^\alpha - \lambda_t s_t. \quad (4.39)$$

Maximizing $\lambda^\alpha s_t^\alpha - \lambda s_t$ with respect to λ suggests the choice

$$\lambda_t = \alpha^{1/(1-\alpha)} s_t^{-1}. \quad (4.40)$$

For this choice we get that

$$\lambda_t^\alpha s_t^\alpha - \lambda_t s_t = (1 - \alpha)\alpha^{\alpha/(1-\alpha)} > 0.$$

This implies that Λ_t^* is strictly positive and that

$$\liminf_{t \rightarrow \infty} \Lambda_t^* \geq (1 - \alpha)\alpha^{\alpha/(1-\alpha)} > 0.$$

We continue the proof following the lines of the proof of theorem 3.1. (3.16) is true with $v_t^{\frac{1}{2}+\eta}$ replaced by $t^{\frac{\alpha}{2}+\eta}$ and $v_t^{1+\delta}$ by k_t . Here,

$$h(x) = \sup_{\lambda > 0} \left(\lambda \frac{x}{3k_t t^{-\frac{\alpha}{2}-\eta}} - \frac{\lambda^2 K(f)^2}{1 - \lambda K(f)} \right).$$

Write for short $n_t := k_t t^{-\alpha/2-\eta}$, $n_t \rightarrow \infty$ as $t \rightarrow \infty$. (Recall that $k_t \sim t^\gamma$ for $\gamma > \alpha$, and that $\alpha/2 + \eta \leq \alpha$.) Let

$$t_0 \text{ such that for all } t \geq t_0, n_t \geq 1. \quad (4.41)$$

Then for $t \geq t_0$, with $B(f) = \max(K^2(f), K(f))$,

$$h(x) \geq \frac{1}{3} x^2 L(t)^2 \left[\frac{1}{2xK(f) + 12K^2(f)} \right] t^{-2\gamma+\alpha+2\eta} \geq \frac{x^2 \wedge x}{42B(f)} L(t)^2 t^{-2\gamma+\alpha+2\eta}.$$

•

Remark 4.10 *Note that in the regular case, we even have convergence in law of $(\frac{1}{\sqrt{v_n}} \int_0^{nt} f(X_s) ds)_t$ to $\sigma B \circ W^\alpha$, where B is a one-dimensional Brownian motion and W^α the Mittag-Leffler process of index α , i.e. the process inverse of the stable subordinator of index α , see for instance Touati [27].*

There are various examples where the exact form of the Laplace transform is known. Brownian motion in dimension one is the most famous example.

Example 4.11 *Let X be the one-dimensional standard Brownian motion. In this case, we can define the regeneration times without Nummelin splitting in the following very simple way. Let*

$$R_n = \inf\{t > S_n : X_t = 0\}, S_n = \inf\{t > R_{n-1} : X_t = 1\}, R_0 = 0.$$

Then

$$\hat{F}(\lambda) = e^{-2\sqrt{2\lambda}},$$

see Revuz-Yor ([26]). In this case it is possible to take $v_t = \sqrt{t}$ (see for example Höpfner and Löcherbach ([13]), theorem 5.6.A.). Note that here, $(R_n - R_{n-1})_{n \geq 1}$ are independant. Thus we get for any $k \in \mathbb{N}$,

$$P_\pi(N_t \geq k) \leq \exp\left(-\left[-(k-1) \log \hat{F}(\lambda) - \lambda t\right]\right),$$

and thus

$$P_\pi \left(N_t \geq v_t^{1+\delta} \right) = P_\pi \left(N_t \geq t^{(1+\delta)/2} \right) \leq \exp \left(- \left[- (t^{(1+\delta)/2} - 2) \log \hat{F}(\lambda) - \lambda t \right] \right).$$

Choose t_0 such that $t_0^{(1+\delta)/2} - 2 \geq \frac{1}{2} t_0^{(1+\delta)/2}$. Hence, for all $t \geq t_0$,

$$P_\pi \left(N_t \geq t^{(1+\delta)/2} \right) \leq \exp \left(-h_t(\lambda) \right),$$

where

$$h_t(\lambda) = -\frac{1}{2} t^{(1+\delta)/2} \log \hat{F}(\lambda) - \lambda t = t^{(1+\delta)/2} \sqrt{2\lambda} - \lambda t.$$

Then a simple calculus shows that

$$\sup_{\lambda > 0} h_t(\lambda) = \frac{1}{2} t^\delta.$$

We conclude that

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq t^{1/2} x \right) &\leq 4 \exp \left(-\frac{1}{42B(f)} t^{(1-\delta)/2} (x^2 \wedge x) \right) \\ &\quad + 4e \exp \left(-\frac{1}{6K(f)} \sqrt{tx} \right) + 4 \exp \left(-\frac{1}{2} t^\delta \right). \end{aligned}$$

Choosing finally $\delta = 1/3$, which equilibrates the terms $(1-\delta)/2$ and δ , we conclude that

$$\begin{aligned} P_\pi \left(\left| \int_0^t f(X_s) ds \right| \geq t^{1/2} x \right) &\leq C \left[\exp \left(-t^{1/3} \frac{x^2 \wedge x}{42B(f)} \right) \right. \\ &\quad \left. + \exp \left(-\frac{1}{6K(f)} t^{1/2} x \right) + \exp \left(-\frac{1}{2} t^{1/3} \right) \right], \end{aligned}$$

where $C = 4e$.

We conclude this section with the following proposition showing that the notion of regularity (4.36) is intrinsic of the process and does not depend on the concrete splitting we are using.

Proposition 4.12 *Suppose that for $0 < \alpha < 1$ and a function l varying slowly at ∞ , the following is true. For any measurable and positive function g with $0 < \mu(g) < \infty$, we have regular variation of resolvents*

$$R_{1/t} g(x) = E_x \left(\int_0^\infty e^{-\frac{1}{t}s} g(X_s) ds \right) \sim t^\alpha \frac{1}{l(t)} \mu(g) \text{ as } t \rightarrow \infty, \quad (4.42)$$

for μ -almost all x . Then we have

$$\mathbb{P}(R_2 - R_1 > x) \sim \frac{x^{-\alpha} l(x)}{\Gamma(1-\alpha)} \text{ as } x \rightarrow \infty. \quad (4.43)$$

Proof By theorem 3.15 of [13], we know that (4.42) implies weak convergence

$$\frac{(A_{tn})_{t \geq 0}}{n^\alpha/l(n)} \rightarrow E_\mu(A_1)W^\alpha,$$

for any additive functional A_t of the process having $0 < E_\mu(A_1) < \infty$. This convergence holds true in $D(\mathbb{R}_+, \mathbb{R})$, under P_π for any initial measure π . W^α is the Mittag-Leffler process of index α .

From now on we fix ν as initial measure. Hence we have in particular weak convergence of $(N_{tn})/(n^\alpha/l(n))$, under \mathbb{P}_ν . Now, write $\mathbf{v}(t) = t^\alpha/l(t)$ and let $\mathbf{a}(n)$ be its asymptotic inverse, i.e. $\mathbf{a}(n) \sim n^{1/\alpha}\tilde{l}(n)$. Note that $\mathbf{a}(n)/n \rightarrow \infty$ since $\alpha < 1$. N_t is the inverse process of the sequence of regeneration times R_n , which implies that

$$\frac{R_n}{\mathbf{a}(n)} \text{ converges weakly as } n \rightarrow \infty, \text{ under } \mathbb{P}_\nu. \quad (4.44)$$

But $R_n = R_1 + (R_2 - R_1) + \dots + (R_n - R_{n-1})$. We would like to deduce from this that necessarily (4.43) is true. Unfortunately, due to the complex definition of the Nummelin splitting, the $(R_k - R_{k-1})_k$ are not independent, but they have all the same law. Independence holds true only after ‘‘nearly’’ exponential times, the reason for this being the choice of the new jump times in step 1. of the splitting algorithm.

That is why we proceed as follows. Define for any $k \geq 0$, $\tilde{T}_k = \inf\{T_n : T_n > R_k\}$ the first jump of the process after the k -th regeneration time. Then it is straightforward to show that under \mathbb{P}_ν , $\tilde{T}_k - R_k$ is exponentially distributed with parameter 1, and that the random times $\tilde{T}_k - R_k, k \geq 0$, are independent. Hence,

$$\frac{\sum_{k=0}^{n-1}(\tilde{T}_k - R_k)}{n} \rightarrow 1, \text{ which implies that } \frac{\sum_{k=0}^{n-1}(\tilde{T}_k - R_k)}{\mathbf{a}(n)} \rightarrow 0$$

almost surely. This together with (4.44) yields weak convergence of

$$\frac{\sum_{k=0}^{n-1}(R_{k+1} - \tilde{T}_k)}{\mathbf{a}(n)}. \quad (4.45)$$

But $R_{k+1} - \tilde{T}_k, k \geq 0$, is an i.i.d. sequence of random variables under \mathbb{P}_ν , thus (4.45) implies that necessarily the law of $R_{k+1} - \tilde{T}_k$ belongs to the domain of attraction of a stable law, see for instance Feller, [9], XIII.6. This implies that

$$\mathbb{P}_\nu(R_2 - \tilde{T}_1 > x) \sim x^{-\alpha}L(x)$$

for some function L varying slowly at infinity. Since

$$\begin{aligned} \mathbb{P}_\nu(R_2 - \tilde{T}_1 > x) &\leq \mathbb{P}_\nu(R_2 - R_1 > x) \leq \mathbb{P}_\nu(R_2 - \tilde{T}_1 > x/2) + \mathbb{P}_\nu(\tilde{T}_1 - R_1 > x/2) \\ &= \mathbb{P}_\nu(R_2 - \tilde{T}_1 > x/2) + e^{-x/2}, \end{aligned}$$

we deduce that

$$x \mapsto \mathbb{P}_\nu(R_2 - R_1 > x) \text{ varies regularly at infinity with index } \alpha.$$

This implies (4.43). •

5 Application to some interacting particle systems

In some applications it is interesting to know at which speed empirical means $\frac{1}{t} \int_0^t f(X_s) ds$ converge to the – in general unknown – invariant measure $\mu(f)$. This is most often the case in statistical applications.

Consider for instance the following interacting particle system which has been studied in Galves et al. [10]. Particles are on positions (sites) of \mathbb{Z}^d , and any particle has either spin $+1$ or -1 . So let $A = \{-1, +1\}$ and let $E = A^{\mathbb{Z}^d}$. Any element of E shall be called configuration of the system. Configurations will be denoted by letters η, ξ, ζ . Any element $i \in \mathbb{Z}^d$ is called a site. For any site i let η^i be the modified configuration $\eta^i(i) = -\eta(i), \eta^i(j) = \eta(j)$ for all $j \neq i$. We associate to any site i and any configuration η a spin-flip-rate $c_i(\eta) \geq 0$. Roughly speaking, site i will change its spin at rate $c_i(\eta)$ whenever the overall configuration of particles is η . We suppose that

$$\sup_{\eta} c_i(\eta) \leq M_i$$

for some constant M_i . Then the associated interacting particle system is a Markov process X on E having generator

$$Lf(\eta) = \sum_{i \in \mathbb{Z}^d} c_i(\eta)[f(\eta^i) - f(\eta)].$$

Note that X is a process having interactions of infinite range. In order to show that such a process exists, one has to impose a continuity condition on the spin-flip rates : Let $V_i(k) = \{j \in \mathbb{Z}^d; 0 \leq \|j - i\| \leq k\}$, where $\|j\| = \sum_{u=1}^d |j_u|$ is the usual L_1 -norm of \mathbb{Z}^d . Then we suppose that

$$\sup_{i \in \mathbb{Z}^d} \sup_{\eta(V_i(k)) = \zeta(V_i(k))} |c_i(\eta) - c_i(\zeta)| \rightarrow 0,$$

as $k \rightarrow \infty$.

In Galves et al. [10] the following criterion for recurrence of the system has been given :

Proposition 5.1 (Theorem 3 of Galves et al.) *There exists a sequence $\lambda_i(k), k \geq -1$, associated to the spin-flip rates c_i such that : If*

$$\sup_{i \in \mathbb{Z}^d} \sum_{k \geq 0} |V_i(k)| \lambda_i(k) \leq 1, \tag{5.46}$$

then the process is recurrent. If the above sum is strictly less than 1, then the process is uniformly exponentially ergodic. Here, $|V_i(k)|$ is the number of sites belonging to $V_i(k)$.

Remark 5.2 *The above theorem relies on the construction of a backward dual process $C_s^{(i)}, s \geq 0$, such that for any site $i \in \mathbb{Z}^d$, $C_s^{(i)}$ denotes the set of sites at time $-s$ that have to be known in order to determine the spin of site i at time 0. The cardinal of $C_s^{(i)}$ can be compared to a classical branching process in continuous time, having reproduction mean $\sum_{k \geq 0} |V_i(k)| \lambda_i(k) < 1$, and thus being subcritical. We refer the reader to Galves et al. ([10]) for the details.*

The sequence $\lambda_i(k)$ can be constructed explicitly, we refer the reader to Galves et al. (2008) for the details. In [10], we were mainly concerned with the issue of perfect simulation of the

invariant measure μ of the process, under condition (5.46), based on the precise knowledge of the spin-flip rates c_i . Suppose now, that we are in the converse situation, observing the process over some time interval $[0, t]$, without knowledge of the spin-flip rates, and that we want to deduce information about the associated invariant measure (and for example on the associated spin-flip rates, a posteriori). This is the classical situation of statistical inference.

Note that assumption 2.1 is satisfied and that the process is strongly Feller, due to the continuity assumption.

We apply Theorem 3.1 with $\eta = \frac{1}{2}$. Since the process is uniformly exponentially ergodic, any constant function is special. In particular, we have for any bounded cylindrical function f , i.e. depending only on a finite number of sites of \mathbb{Z}^d , since $f - \mu(f)$ is special, that

$$P\left(\left|\frac{1}{t}\int_0^t f(X_s)ds - \mu(f)\right| > x\right) \leq ce^{-Ct}, \quad (5.47)$$

for some constants c, C . This is a first step on the way towards estimating spin-flip rates and associated interaction schemes for interacting particles.

References

- [1] Athreya, K.B., Ney, P. A new approach to the limit theory of recurrent Markov chains. Trans. Am. Math. Soc. 245 (1978), 493-501.
- [2] Bingham, N.H.; Goldie, C.M.; Teugels, Jozef L. *Regular variation*. Encyclopedia of Mathematics and its applications, Vol. 27. Cambridge etc.: Cambridge University Press. XIX (1987).
- [3] Birgé, Lucien; Massart, Pascal. Minimum contrast estimators on sieves: Exponential bounds and rates of convergence. Bernoulli 4, No.3, 329-375 (1998).
- [4] Brancovan, M. Fonctionnelles additives spéciales des processus récurrents au sens de Harris. Z. Wahrscheinlichkeitstheor. Verw. Geb. 47 (1979), 163–194.
- [5] Chen, X. How often does a Harris recurrent Markov chain recur? Ann. Probab. 27 (1999), 1324–1346.
- [6] Chen, X., Guillin, A.. The functional moderate deviations for Harris recurrent Markov chains and applications. Annales de l’Institut Henri Poincaré (Prob. Stat.) 40 (2004), 89–124.
- [7] Delattre, S., Hoffmann, M., Kessler, M.. Dynamics adaptive estimation of a scalar diffusion. Preprint 762 of Laboratoire de Probabilités et modèles aléatoires, october 2002.
- [8] Djellout, H., Guillin, A.. Moderate deviations of Markov Chains with atom. Stoch. Proc. Appl. 95 (2001), 203-217.
- [9] Feller, W.: *An introduction to probability theory and its applications*. Vol. 2 Wiley: New York 1971.

- [10] Galves, A., Garcia, N.L., Löcherbach, E.(2008): Perfect simulation and finitary coding for multicolor systems with interactions of infinite range. Submitted to SPA, arxiv.org/abs/0809.3494.
- [11] Guillin, A. Uniform moderate deviations of functional empirical processes of Markov Chains. *Probability and Mathematical Statistics* 20 (2000), 237-260.
- [12] Guillin, A., Liptser, R. Examples of moderate deviation principle for diffusion processes. *Discrete and Continuous Dynamical Systems (B)* 6 (2006), 803-828.
- [13] Höpfner, R., Löcherbach, E.: *Limit theorems for null recurrent Markov processes*. *Memoirs AMS* **161**, Number 768, 2003.
- [14] Kusuoka, S.; Stroock, D. Applications of the Malliavin calculus. III. *J. Fac. Sci., Univ. Tokyo, Sect. I A* 34, 391-442 (1987).
- [15] Löcherbach, E., Loukianova, D.. On Nummelin splitting for continuous time Harris recurrent Markov processes and application to kernel estimation for multi-dimensional diffusions. *Stoch. Proc. Appl.* 118 (2008), No. 8, 1301-1321.
- [16] Löcherbach, E., Loukianova, D.. The law of iterated logarithm for additive functionals and martingale additive functionals of Harris recurrent Markov processes. To appear in *Stoch. Proc. Appl.* 2009.
- [17] Loukianova, D. Loukianov, O.. Deterministic equivalent of additive functionals of recurrent diffusions and drift estimation. *Statistical Inference for Stochastic processes* 111 (2008), No 2, 107–121.
- [18] Löcherbach, E., Loukianova, D., Loukianov, O.. Deviation bounds in ergodic theorem for positively recurrent one-dimensional diffusions and integrability of hitting times. *Manuscript*, 2009.
- [19] Löcherbach, E., Loukianova, D., Loukianov, O.. Penalized nonparametric drift estimation in a continuous time one-dimensional diffusion process. *Manuscript*, 2009.
- [20] Meyn, S.P., Tweedie, R.L. Generalized resolvents and Harris recurrence of Markov processes. Cohn, Harry (ed.), *Doebelin and modern probability. Proceedings of the Doebelin conference '50 years after Doebelin: development in the theory of Markov chains, Markov processes, and sums of random variables'*. Providence, RI: American Mathematical Society. *Contemp. Math.* 149 (1993), 227-250.
- [21] Meyn, S.P., Tweedie, R.L.. A survey of Foster-Lyapunov techniques for general state space Markov processes. In *Proceedings of the Workshop on Stochastic Stability and Stochastic Stabilization*, Metz, France, June 1993.
- [22] Meyn, S.P., Tweedie, R.L.. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. Appl. Probab.* 25 (1993), 487–548.
- [23] Nummelin, E. A splitting technique for Harris recurrent Markov chains. *Z. Wahrscheinlichkeitstheorie Verw. Geb.* 43 (1978), 309–318.

- [24] Nummelin, E.: *General irreducible Markov chains and non-negative operators*. Cambridge University Press, Cambridge, England, 1984.
- [25] Revuz, D.: *Markov chains*. Revised edition. Amsterdam: North Holland 1984.
- [26] Revuz, D., Yor, M.: *Continuous martingales and Brownian motion*. 3rd edition, Grundlehren der Mathematischen Wissenschaften 293. Berlin: Springer 2005.
- [27] Touati, A.. Loi fonctionnelle du logarithme itéré pour les processus de Markov récurrents. Ann. Probab. 18, No.1 (1990), 140-159.