

Polynomial bounds in the Ergodic Theorem for positive recurrent one-dimensional diffusions and integrability of hitting times.

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Abstract

Let X be a one dimensional positive recurrent diffusion with invariant measure μ . We say that the degree of recurrence of X is polynomial of order $p \geq 1$, if for all x, a we have $\mathbf{E}_x T_a^p < \infty$ and $\mathbf{E}_x T_a^{p+1} = \infty$, where T_a is the hitting time of a . We give sufficient conditions on the coefficients of X in order to have a degree of recurrence at least equal to p . For such a diffusion, we derive non asymptotic deviation bounds

$$\mathbf{P}_\nu \left(\left| \frac{1}{t} \int_0^t f(X_s) ds - \mu(f) \right| \geq \varepsilon \right) \leq K(p) \frac{1}{t^{p/2}} \frac{1}{\varepsilon^p} A(f)^p$$

where ν is an initial distribution, f bounded or bounded and compactly supported and $A(f) = \|f\|_\infty$ when f is bounded and $A(f) = \mu(|f|)$ when f is bounded and compactly supported. We also give a polynomial control of $\mathbf{E}_x T_a^p$ from above and below under assumptions similar to those used in [15] and [1].

Key words : Recurrence, Additive functionals, Chacon-Ornstein theorem, Diffusion process, Polynomial convergence, Hitting times, Kac formula, Deviations inequalities.

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1 Introduction

We consider the solution of the one-dimensional stochastic differential equation

$$dX_t = \beta(X_t) dt + \sigma(X_t) dW_t,$$

with arbitrary initial data.

We suppose that X is recurrent, in the sense that hitting times satisfy $P_x(T_y < \infty) = 1$ for all x, y , where $T_y = \inf\{t \geq 0, X_t = y\}$. Such a diffusion admits a unique (up to multiplication with constants) invariant measure μ . If the total mass of this measure is finite, X is called positive recurrent, and null recurrent otherwise. The Ergodic Theorem tells us in the case of positive recurrence that for all x and for all f such that $\mu(f) < \infty$, and for all $\varepsilon > 0$

$$\mathbf{P}_x \left(\left| \frac{1}{t} \int_0^t f(X_s) ds - \mu(f) \right| \geq \varepsilon \right) \rightarrow 0$$

as t goes to $+\infty$. The aim of this paper is to study the rate of convergence in this Ergodic Theorem. More precisely, we are aiming at non-asymptotic bounds. This is of major importance for many applications, see for example model selection or other non asymptotic problems for statistics of diffusions, a priori bounds for large or moderate deviations in averaging principle, concentration for particular approximations of granular media equations and many other examples.

For Markov chains such kind of deviation inequalities were obtained by Bertail and Cléménçon [2] using the regeneration method.

For continuous time Markov processes and in the case when the speed of convergence is exponential, the asymptotic bound in the Ergodic theorem was firstly obtained by Wu [16]. This work has been continued by Cattiaux and Guillin [4]. The approach of these authors relies on the use of functional inequalities for μ like the Poincaré- or the log-Sobolev-inequality. In this way they obtain an asymptotically sharp exponential bound, in the spirit of the large deviation principle, for a process starting from the invariant probability μ .

For one dimensional ergodic diffusion processes, using stochastic calculus and control of the moments of X , the exponential bound in the Ergodic Theorem was obtained by Galtchouk and Pergamenshchikov, see [10], under the condition of constant diffusion coefficient and drift bounded from above and below by linear functions. Their bound is uniform in time, initial distribution and drift, where these parameters are all supposed to belong to some restricted class.

However, in concrete situations, conditions assuming exponential speed of convergence can be rather restrictive. In this paper we consider the case when only polynomial speed of convergence is guaranteed, and give the most weak conditions on the coefficients for this case which seem to be available.

Our approach is based on the regeneration method. The conditions we impose are formulated in terms of integrability of the regeneration time, i.e. in terms of integrability of hitting times T_y . To formulate these conditions we start by introducing the notion of polynomial degree

of recurrence. It is well known that X is positive recurrent if and only if $E_x T_y < \infty$ for all x, y . It is natural to extend this definition and to relate the integrability of T_y to the degree of recurrence : We say that the degree of recurrence of X is polynomial of order $p \geq 1$, if for all x, y we have $\mathbf{E}_x T_y^p < \infty$ and $\mathbf{E}_x T_y^{p+1} = \infty$. In section 2 we explain that $\mathbf{E}_x T_y^p < \infty$ or $= \infty$ simultaneously for all couples x, y .

In this spirit, for bounded or bounded and compactly supported functions f , we show in Theorem 4.3 the following deviation inequality : If for some $p > 1$ the moment $E_x T_y^p < \infty$ for some (and hence for all) x, y , we prove that for all $\varepsilon > 0$,

$$\mathbf{P}_x(|\frac{1}{t} \int_0^t f(X_s) ds - \mu(f)| > \varepsilon) \leq K(p, x) \varepsilon^{-p} t^{-p/2} A(f)^p.$$

Here $A(f) = \|f\|_\infty$ when f is bounded and $A(f) = \mu(|f|)$ when f is bounded and of compact support. The constant K depends on x only through the corresponding moments of the hitting time. If in addition ν is such that $E_\nu T_y^p = \int E_x T_y^p d\nu(x) < \infty$, we have a polynomial rate of convergence in the Ergodic Theorem under \mathbf{P}_ν . To be able to check this condition for given ν , we show in section 5 that $E_x T_y^p$ is controlled from above and below by two polynomes, under assumptions close to those given by Veretennikov, [15], and by Balaji and Ramasubramanian, see [1]. Finally, to show the deviation inequality for $\frac{1}{t} \int_0^t f(X_s) ds$ we need an auxiliary polynomial deviation inequality for the number of regeneration times before time t , denoted by N_t . This is the content of section 4.1.

Remark that the idea to relate the integrability of regeneration times to the rate of convergence is not new and appears naturally whenever the regeneration method is applied, see for example Roberts and Tweedie, [14], and Douc, Guillin and Moulines [6] and the references therein.

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2 Degree of recurrence in terms of integrability of hitting times.

Let X_t be a one-dimensional diffusion process given by

$$dX_t = \beta(X_t) dt + \sigma(X_t) dW_t.$$

We need the continuity of β, σ and conditions to guarantee the existence of a non-exploding solution. For this sake we assume:

Assumption 2.1 1. For all x , $\sigma^2(x) > 0$.

2. β and σ are locally Lipschitz, and $|\sigma(x)| + |\beta(x)| \leq C(1 + |x|)$, for some $C > 0$.

In this section we show how the notion of degree of recurrence can be explained in terms of existence of moments of hitting times (see definition 2.5). To do this we need to formulate a particular version of Kac's moment formula (compare to [9]). We refer also to [11] for a simpler version. We have to introduce the following notation before stating Kac's formula. Let

$$s(x) = \exp\left(-2 \int_0^x \frac{\beta(u)}{\sigma^2(u)} du\right), \quad m(x) = \frac{1}{\sigma^2(x)s(x)},$$

and $S(x) = \int_0^x s(t)dt$. We need the following assumption which is in fact an assumption of recurrence for the diffusion.

Assumption 2.2 *S is a space-transform, i.e. $\lim_{x \rightarrow +\infty} S(x) = +\infty$ and $\lim_{x \rightarrow -\infty} S(x) = -\infty$.*

Moreover, we define

$$G(-\infty, b, x, \xi) = 2 \begin{cases} (S(b) - S(\xi)), & x \leq \xi \leq b \\ (S(b) - S(x)), & -\infty < \xi \leq x \end{cases} \quad (2.1)$$

and

$$G(a, +\infty, x, \xi) = 2 \begin{cases} (S(\xi) - S(a)), & a \leq \xi \leq x \\ (S(x) - S(a)), & x \leq \xi < \infty. \end{cases} \quad (2.2)$$

Proposition 2.3 *Under assumptions 2.1 and 2.2, we have for all $a < x < b$,*

$$\mathbf{E}_x T_b^n = n \int_{-\infty}^{+\infty} G(-\infty, b, x, \xi) \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi \quad (2.3)$$

and

$$\mathbf{E}_x T_a^n = n \int_{-\infty}^{+\infty} G(a, +\infty, x, \xi) \mathbf{E}_\xi T_a^{n-1} m(\xi) d\xi \quad (2.4)$$

Remark 2.4 *The expressions (2.3) and (2.4) are always defined, because all functions we integrate are positive. In the next theorem we discuss the issue of finiteness of these terms. Remark also that s, m , and S are continuous. We even have $S \in C^2(\mathbb{R})$ and $LS = 0$, where $L = \sigma^2 * \Delta/2 - \nabla * \beta$ is the generator of the semigroup of X .*

Proof For $a < x < b$ let us consider

$$T_{a,b} = \{t \geq 0; X_t \notin [a, b]\}. \quad (2.5)$$

We start by showing the theorem for $n = 1$. Consider the following differential equation with boundary conditions:

$$\begin{cases} Lu_1(x) = -1, & a \leq x \leq b \\ u_1(a) = u_1(b) = 0, \end{cases} \quad (2.6)$$

and more general the equation

$$\begin{cases} Lu(x) = -f, & a \leq x \leq b \\ u_1(a) = u_1(b) = 0, \end{cases} \quad (2.7)$$

where f is a continuous function on $[a, b]$. Let G be the Green potential kernel associated to the stopping time $T_{a,b}$, defined by

$$G(a, b, x, \xi) = 2 \begin{cases} \frac{(S(x)-S(a))(S(b)-S(\xi))}{S(b)-S(a)} & a \leq x \leq \xi \leq b \\ \frac{(S(b)-S(x))(S(\xi)-S(a))}{S(b)-S(a)} & a \leq \xi \leq x \leq b \\ 0 & \text{otherwise.} \end{cases} \quad (2.8)$$

A simple calculus, using $LS = 0$, shows that if f is continuous on $[a, b]$, then

$$u(x, a, b) = \int_{-\infty}^{+\infty} G(a, b, x, \xi) f(\xi) m(\xi) d\xi \quad (2.9)$$

is a solution of equation (2.7). In particular, $u(\cdot, a, b)$ is C^2 on $[a, b]$.

Hence,

$$u_1(x, a, b) = \int_{-\infty}^{+\infty} G(a, b, x, \xi) m(\xi) d\xi \quad (2.10)$$

is a solution of (2.6). On the other hand, the Ito formula applied to $u_1 = u_1(x, a, b)$ and X_t starting from x gives

$$du_1(X_t) = -dt + dM_t; \quad u_1(X_t) = u_1(x) - t + M_t, \quad (2.11)$$

where $M_t = \int_0^t u_1(X_s) \sigma^2(X_s) ds$ is a continuous local martingale such that $M_{t \wedge T_{a,b}}$ is uniformly integrable. Doob's stopping rule gives

$$0 = u_1(X_{T_{a,b}}) = u_1(x) - T_{a,b} + M_{T_{a,b}},$$

thus

$$\mathbf{E}_x T_{a,b} = u_1(x) = u_1(x, a, b)$$

and hence

$$\mathbf{E}_x T_{a,b} = \int_{-\infty}^{+\infty} G(a, b, x, \xi) m(\xi) d\xi.$$

Using monotone convergence, we get

$$\mathbf{E}_x T_b = \lim_{a \rightarrow -\infty} \mathbf{E}_x T_{a,b}.$$

Note moreover that

$$G(-\infty, b, x, \xi) = \lim_{a \rightarrow -\infty} G(a, b, x, \xi) = 2 \begin{cases} (S(b) - S(\xi)) & x \leq \xi \leq b \\ (S(b) - S(x)) & \xi \leq x \leq b. \end{cases} \quad (2.12)$$

Moreover, for all $a < x < b$, $G(a, b, x, \xi) \leq G(-\infty, b, x, \xi)$. So, if

$$\int_{-\infty}^{+\infty} G(-\infty, b, x, \xi) m(\xi) d\xi < \infty, \quad (2.13)$$

then we can use dominated convergence in order to show

$$\begin{aligned} \mathbf{E}_x T_b &= \int_{-\infty}^{+\infty} G(-\infty, b, x, \xi) m(\xi) d\xi \\ &= 2(S(b) - S(x)) \int_{-\infty}^x m(\xi) d\xi + \int_x^b 2(S(b) - S(\xi)) m(\xi) d\xi. \end{aligned} \quad (2.14)$$

We see that the condition (2.13) is equivalent to the following

$$\int_{-\infty}^x m(\xi)d\xi < \infty. \quad (2.15)$$

If

$$\int_{-\infty}^{+\infty} G(-\infty, b, x, \xi)m(\xi)d\xi = +\infty, \quad (2.16)$$

then Fatou's lemma applies and gives

$$\mathbf{E}_x T_b = +\infty.$$

Thus the claim of the theorem is verified for $n = 1$.

The same arguments apply for arbitrary $n \geq 1$. Let $u_n(x) = u_n(x, a, b)$, for $a \leq x \leq b$, be the solution of the equation

$$\begin{cases} Lu_n = -u_{n-1} \\ u_n(a) = u_n(b) = 0. \end{cases} \quad (2.17)$$

We have

$$u_n(x, a, b) = \int_{-\infty}^{+\infty} G(a, b, x, \xi)u_{n-1}(\xi, a, b)m(\xi)d\xi. \quad (2.18)$$

On the other hand, using Ito's formula we get

$$\begin{aligned} d(t^{n-1}u_1(X_t) + (n-1)t^{n-2}u_2(X_t) + (n-1)(n-2)t^{n-3}u_3(X_t) + \dots + (n-1)!tu_n(X_t)) \\ = -t^{n-1}dt + (n-1)!u_{n-1}(X_t)dt + M_t, \end{aligned} \quad (2.19)$$

where M is a continuous local martingale such that $M_{t \wedge T_{a,b}}$ is uniformly integrable. The stopping rule gives the following formula (which is known in a more general case as Kac's formula, see [9])

$$\mathbf{E}_x T_{a,b}^n = n!u_n(x, a, b), \quad (2.20)$$

and hence

$$\mathbf{E}_x T_{a,b}^n = n! \int_{-\infty}^{+\infty} G(a, b, x, \xi)u_{n-1}(\xi, a, b)m(\xi)d\xi = n \int_{-\infty}^{+\infty} G(a, b, x, \xi)\mathbf{E}_\xi T_{a,b}^{n-1}m(\xi)d\xi. \quad (2.21)$$

If the integral

$$\int_{-\infty}^{+\infty} G(-\infty, b, x, \xi)\mathbf{E}_\xi T_{a,b}^{n-1}m(\xi)d\xi \quad (2.22)$$

converges, using dominated convergence, we pass to the limit when $a \rightarrow \infty$, which gives

$$\mathbf{E}_x T_b^n = n \int_{-\infty}^{+\infty} G(-\infty, b, x, \xi)\mathbf{E}_\xi T_b^{n-1}m(\xi)d\xi. \quad (2.23)$$

We can rewrite this last expression as

$$\mathbf{E}_x T_b^n = n \left(2(S(b) - S(x)) \int_{-\infty}^x \mathbf{E}_\xi T_b^{n-1}m(\xi)d\xi + \int_x^b 2(S(b) - S(\xi))\mathbf{E}_\xi T_b^{n-1}m(\xi)d\xi \right). \quad (2.24)$$

If the integral in (2.22) diverges, using Fatou's lemma, we have

$$\mathbf{E}_x T_b^n = n \int_{-\infty}^{+\infty} G(-\infty, b, x, \xi) \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi = \infty.$$

The proof of (2.4) is similar to this of (2.3). •

Due to proposition 2.3, it is thus reasonable to introduce the following definition.

Definition 2.5 *We say that the diffusion X is positive recurrent of degree of recurrence n_0 , if*

$$n_0 = \sup\{n \in \mathbf{N} \quad \text{such that for all } x, y \in \mathbb{R}, \quad \mathbf{E}_x T_y^n < \infty\}.$$

This definition has to be compared with the notion of positive recurrence, where for all x, y $\mathbf{E}_x T_y < \infty$.

The following theorem is known (see [12]). We give here an independent proof based on Kac's formula.

Theorem 2.6 *Grant assumptions 2.1 and 2.2.*

1. *Let $x < b$ and $n \in \mathbf{N}^*$.*

(1i) $\mathbf{E}_x T_b^n < \infty$ *if and only if* $\int_{-\infty}^x \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi < \infty$.

(1ii) *If for one couple $x < b$, $\mathbf{E}_x T_b^n < \infty$, then for all couples $x' < b'$, $\mathbf{E}_{x'} T_{b'}^n < \infty$.
Moreover, for all b' fixed, the function $x' \rightarrow \mathbf{E}_{x'} T_{b'}^n$ is continuous in x' , for $x' < b'$.*

2. *Let $a < x$ and $n \in \mathbf{N}^*$.*

(2i) $\mathbf{E}_x T_a^n < \infty$ *if and only if* $\int_x^{+\infty} \mathbf{E}_\xi T_a^{n-1} m(\xi) d\xi < \infty$.

(2ii) *If for one couple $a < x$, $\mathbf{E}_x T_a^n < \infty$, then for all couples $a' < x'$, $\mathbf{E}_{x'} T_{a'}^n < \infty$.
Moreover, for all a' fixed, the function $x' \rightarrow \mathbf{E}_{x'} T_{a'}^n$ is continuous in x' , for $a' < x'$.*

Proof Suppose $n = 1$. Using Kac's formula,

$$\mathbf{E}_x T_b = 2(S(b) - S(x)) \int_{-\infty}^x m(\xi) d\xi + \int_x^b 2(S(b) - S(\xi)) m(\xi) d\xi. \quad (2.25)$$

The functions S and m are continuous, hence the last expression is finite if and only if $\int_{-\infty}^x m(\xi) d\xi < \infty$. The finiteness of the last integral does not depend on x nor on b . Hence, $\mathbf{E}_x T_b$ is finite or not simultaneously for all x, b such that $x < b$. If $\mathbf{E}_x T_b < \infty$, the Kac's formula (2.25) gives the continuity in $x < b$ of $\mathbf{E}_x T_b$.

Now we suppose that the claim of the theorem verified for all $k \leq n - 1$, and we show it for $k = n$.

Suppose for some fixed $x < b$, $\mathbf{E}_x T_b^n < \infty$. Then $\mathbf{E}_x T_b^{n-1} < \infty$ too. This implies by our recurrence assumption that $\mathbf{E}_{x'} T_b^{n-1}$ is finite and continuous for all $x' < b$. We use Kac's formula once more in order to get

$$\mathbf{E}_x T_b^n = n \left(2(S(b) - S(x)) \int_{-\infty}^x \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi + \int_x^b 2(S(b) - S(\xi)) \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi \right). \quad (2.26)$$

$\mathbf{E}_x T_b^n$ is finite if and only if $\int_{-\infty}^x \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi < \infty$. Using continuity of $\mathbf{E}_\xi T_b^{n-1}$, we see that for fixed b the integral $\int_{-\infty}^{x'} \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi$ converges or diverges simultaneously for all $x' < b$. Hence we obtain the following equivalence for fixed $b \in \mathbb{R}$.

$$\text{For some } x \text{ s.t. } x < b \mathbf{E}_x T_b^n < \infty \iff \text{for all } x' \text{ s.t. } x' < b \mathbf{E}_{x'} T_b^n < \infty. \quad (2.27)$$

Now let $\mathbf{E}_x T_b^n < \infty$ and fixe some b' such that $x < b < b'$. We have $\mathbf{E}_x T_{b'}^n < \infty$ if and only if $\int_{-\infty}^x \mathbf{E}_\xi T_{b'}^{n-1} m(\xi) d\xi < \infty$. Using the strong Markov property and Hölder's inequality,

$$\int_{-\infty}^x \mathbf{E}_\xi T_{b'}^{n-1} m(\xi) d\xi \leq 2^{n-2} \left(\int_{-\infty}^x \mathbf{E}_\xi T_b^{n-1} m(\xi) d\xi + \mathbf{E}_b T_{b'}^{n-1} \int_{-\infty}^x m(\xi) d\xi \right). \quad (2.28)$$

Moreover, for $x < b < b'$, $\mathbf{E}_x T_b^n \leq \mathbf{E}_x T_{b'}^n$. Therefore, the following two statements are equivalent.

$$\text{For some } x \text{ s.t. } x < b \mathbf{E}_x T_b^n < \infty \iff \text{for all } b' \text{ s.t. } x < b < b' \mathbf{E}_x T_{b'}^n < \infty. \quad (2.29)$$

The proof of point 2. of the theorem is similar. With (2.27) and (2.29), the proof is complete.

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3 Some useful results concerning life-cycle decompositions.

From now on we suppose that X is positive recurrent, i.e.

$$M := \int_{-\infty}^{\infty} m(x) dx < +\infty,$$

with $m(x) = \frac{1}{\sigma^2(x)s(x)}$. We shall denote by μ the unique invariant probability measure of the process. This probability is given by $\mu(dx) = \frac{1}{M} m(x) dx$.

In the sequel, we relate μ to regeneration times of the diffusion. For that sake, fix two points $a < b, a, b \in \mathbb{R}$. Let us define a sequence of stopping times $(S_n)_n, (R_n)_n$ as follows: $S_0 = 0, R_0 = 0$,

$$S_1 = \{t \geq 0 : X_t = b\}, \quad R_1 := \inf\{t \geq S_1 : X_t = a\},$$

and for $n \geq 1$,

$$S_{n+1} := R_n + S_1 \circ \partial_{R_n}, \quad R_{n+1} := R_n + R_1 \circ \partial_{R_n}.$$

For any measurable and bounded function $f : \mathbb{R} \rightarrow \mathbb{R}$, we put

$$\xi_n = \int_{R_n}^{R_{n+1}} f(X_s) ds, \quad n \geq 1. \quad (3.30)$$

Then we have the following proposition.

Proposition 3.1 For any initial distribution ν , the sequence $(\xi_n)_{n \geq 1}$ is an i.i.d. sequence under \mathbf{P}_ν . For all $n \geq 1$, the law of ξ_n under \mathbf{P}_ν is equal to the law of $\int_0^{R_1} f(X_s) ds$ under P_a .

This last proposition is well known and easy to show using the strong Markov property. Remark that in particular the sequence $(R_{k+1} - R_k); k = 1, 2, \dots$ is an i.i.d. sequence with common distribution equal to the law of R_1 under P_a . Now suppose that f has compact support. Denote

$$C(f) := \sup_x E_x \int_0^{R_1} |f|(X_s) ds.$$

Proposition 3.2 Grant assumptions 2.1 and 2.2. If f is measurable bounded with compact support, then $C(f) < \infty$.

Remark 3.3 In an abstract setting of Markov processes, functions satisfying $C(f) < \infty$ are called special.

Proof Denote by K the support of f and let $\tau = \inf\{t \geq 0; X_t \in K\}$. Let $M > 0$ be such that $|f| \leq M$. Then,

$$\begin{aligned} C(f) &= E_x \left(E_x \left(\int_{\tau \wedge R_1}^{R_1} |f(X_s)| ds \middle| \mathcal{F}_\tau \right) \right) \leq E_x \left(E_{X_\tau} \int_0^{R_1} |f(X_s)| ds \right) \\ &\leq \sup_{x \in K} E_x \int_0^{R_1} |f(X_s)| ds \leq M \sup_{x \in K} E_x R_1. \end{aligned} \quad (3.31)$$

Note that X is positive recurrent, $\int_{-\infty}^{+\infty} m(\xi) d\xi < \infty$, and we can use the theorem (2.6) for $n = 1$. Hence $x \rightarrow E_x R_1$ is continuous, and thus $\sup_{x \in K} E_x R_1 < \infty$. •

The following proposition extends the uniform in x integrability property of the first life cycle for special functions and will play an important role in the sequel.

Proposition 3.4 Let f be a bounded measurable function with compact support. Then for any $p \geq 1$, $\sup_x E_x (\int_0^{R_1} |f(X_s)| ds)^p \leq p! C(f)^p$. In particular, $\mathbf{E}_\nu \xi^p \leq p! C(f)^p$ for any initial distribution ν .

Proof We will first consider the case $p = 2$, the general case can be obtained in the same way.

$$\begin{aligned} \left(\int_0^{R_1} |f(X_s)| ds \right)^2 &= \int_0^{R_1} \int_0^{R_1} |f(X_s)| |f(X_u)| ds du \\ &= 2! \int_0^{R_1} ds |f(X_s)| \int_s^{R_1} |f(X_u)| du \\ &= 2! \int_0^\infty ds \left(|f(X_s)| \mathbf{1}_{\{0 < s < R_1\}} \int_s^{R_1} |f(X_u)| du \right) \\ &\leq 2! \int_0^\infty ds \left(|f(X_s)| \mathbf{1}_{\{0 < s < R_1\}} \int_s^{R_1 \circ \theta_s} |f(X_u)| du \right). \end{aligned} \quad (3.32)$$

Taking expectation and using Markov's property in the last integral gives an upper bound

$$\begin{aligned} \mathbf{E}_x \left(\int_0^{R_1} |f(X_s)| ds \right)^2 &\leq 2! \int_0^\infty ds \mathbf{E}_x \left[|f(X_s)| \mathbf{1}_{\{0 < s < R_1\}} \mathbf{E}_x \left(\int_s^{R_1 \circ \theta_s} |f(X_u)| du \middle| \mathcal{F}_s \right) \right] \\ &= 2! \int_0^\infty ds \mathbf{E}_x \left[|f(X_s)| \mathbf{1}_{\{0 < s < R_1\}} \mathbf{E}_{X_s} \left(\int_s^{R_1} |f(X_u)| du \right) \right] \leq 2! C(f)^2. \end{aligned} \quad (3.33)$$

Applying this argument p times successively yields the result for arbitrary $p \geq 2$. •

Sometimes, the following estimates can be useful. They are obtained using local time, hence the result is typically one-dimensional in spirit.

Lemma 3.5 *For any function f having compact support K ,*

$$C(f) \leq k \mu(|f|),$$

where k is a constant that is given by

$$k := M \sup_{y \in K} s(y) \sup_{y \in K} \sup_x E_x L_y^{R_1}. \quad (3.34)$$

Proof Using assumption 2.1, σ^2 , s , m are continuous and strictly positive. Using the occupation time formula,

$$\sup_x \mathbf{E}_x \int_0^{R_1} |f(X_s)| ds \leq \int_{-\infty}^{+\infty} |f(y)| \frac{1}{\sigma^2(y)} \sup_x \mathbf{E}_x L_{R_1}^y dy \leq M \sup_{y \in K} s(y) \sup_{y \in K} \sup_x E_x L_y^{R_1} \mu(|f|). \quad (3.35)$$

It suffices to show that $\sup_{y \in K} \sup_x E_x L_{R_1}^y$ is finite. We start by showing that for all $y \in \mathbb{R}$ $\sup_x E_x L_{R_1}^y = E_y L_{R_1}^y$, which can be seen as follows:

$$\begin{aligned} E_x L_{R_1}^y &= E_x L_{R_1}^y \mathbf{1}_{(R_1 > T_y)} \leq E_x [\mathbf{1}_{(R_1 > T_y)} E_x (L_{T_y + R_1 \circ \theta_{T_y}}^y | F_{T_y})] \\ &\leq P(R_1 > T_y) E_y L_{R_1}^y \leq E_y L_{R_1}^y. \end{aligned} \quad (3.36)$$

Hence $\sup_x E_x L_{R_1}^y = E_y L_{R_1}^y$. Let $c = \inf K$, $d = \sup K$. Now for $y \in K$ we write

$$E_y L_{R_1}^y \leq E_y L_{R_1}^c + E_y |L_{R_1}^y - L_{R_1}^c| \leq E_c L_{R_1}^c + E_y |L_{R_1}^y - L_{R_1}^c|.$$

But

$$|L_{R_1}^y - L_{R_1}^c| \leq |y| + \left| \int_0^{R_1} \mathbf{1}_{(c < X_s < y)} \sigma(X_s) dW_s \right| + \int_0^{R_1} \mathbf{1}_{(c < X_s < d)} |\beta(X_s)| ds. \quad (3.37)$$

Taking expectation with respect to E_y and taking $\sup_{y \in K}$, using continuity of β and $y \rightarrow \mathbf{E}_y R_1$ (theorem (2.6)), we only need to show that

$$\sup_{y \in K} E_y \left| \int_0^{R_1} \mathbf{1}_{c < X_s < y} \sigma(X_s) dW_s \right| < \infty. \quad (3.38)$$

By norm inclusion and isometry,

$$\begin{aligned} E_y |\int_0^{R_1} \mathbf{1}_{c < X_s < y} \sigma(X_s) dW_s| &< (E_y (\int_0^{R_1} \mathbf{1}_{(c < X_s < y)} \sigma(X_s) dW_s)^2)^{1/2} \\ &\leq (E_y (\int_0^{R_1} \mathbf{1}_{(c < X_s < d)} \sigma^2(X_s) ds))^{1/2}, \end{aligned} \quad (3.39)$$

and using continuity of σ^2 and of $y \rightarrow \mathbf{E}_y R_1$ we see that we have $\sup_{y \in K} E_y (\int_0^{R_1} \mathbf{1}_{(c < X_s < d)} \sigma^2(X_s) ds) < \infty$. •

We now define the point process associated to the life cycle decomposition R_n . Let $N_0 = 0$ and put for $t > 0$,

$$N_t = \sup\{n : R_n \leq t\} = \sum_{n=1}^{\infty} \mathbf{1}_{\{R_n \leq t\}}.$$

Then the processes $(N_t)_{t \geq 0}$ and $(R_n)_{n \in \mathbb{N}}$ are mutually inverse in the following sense.

$$\{N_t \geq n\} = \{R_n \leq t\} \quad \text{and} \quad \{N_t \leq n\} = \{R_n \geq t\}. \quad (3.40)$$

Lemma 3.6 *The quantities $E_a R_1$ and $E_\mu N_1$ are positive and finite, and for any initial distribution ν the followings statements hold.*

1. $\lim_{n \rightarrow \infty} R_n/n = E_a R_1 \quad \mathbf{P}_\nu - a.s.$
2. $\lim_{t \rightarrow \infty} N_t/t = E_\mu N_1 \quad \mathbf{P}_\nu - a.s.$
3. $E_a R_1 = 1/E_\mu N_1$.

Proof The finiteness of $E_a R_1$ follows from positive recurrence. Statement 1. is the strong law of large numbers since we can write

$$R_n/n = R_1/n + 1/n \sum_{k=1}^{n-1} (R_{k+1} - R_k).$$

Using the recurrence property, $R_1 < \infty$ *a.s.* and hence $R_1/n \rightarrow 0$ *a.s.* Using proposition 3.1 the variables $R_{k+1} - R_k, k \geq 1$, are i.i.d. and equal in law to R_1 under P_a . To prove the third statement we write:

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \lim_{n \rightarrow \infty} \frac{N_{R_n}}{R_n} = \lim_{n \rightarrow \infty} \frac{n}{R_n}.$$

Statement 2. follows from the Ergodic Theorem : $(N_t)_t$ is an integrable additive functional of X , hence $\lim_{t \rightarrow \infty} N_t/t = E_\mu N_1/E_\mu 1 = E_\mu N_1$ *a.s.* •

The following proposition will be useful in the sequel:

Proposition 3.7 *Denote $l := E_\mu(N_1)$. Then for any initial measure ν ,*

$$E_\nu \int_{R_1}^{R_2} f(X_s) ds = \frac{\mu(f)}{l} = \mu(f) E_a R_1.$$

Proof Using the Ergodic Theorem, almost surely,

$$\mu(f) = \lim_{t \rightarrow \infty} \frac{\int_0^t f(X_s) ds}{t} = \lim_{n \rightarrow \infty} \frac{\int_0^{R_n} f(X_s) ds}{R_n}.$$

On the other hand, using the strong law of large numbers,

$$\lim_{n \rightarrow \infty} \frac{\int_0^{R_n} f(X_s) ds}{R_n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n} \int_0^{R_n} f(X_s) ds}{R_n/n} = \frac{E_\nu \int_{R_1}^{R_2} f(X_s) ds}{E_a R_1}.$$

•

4 The deviation inequalities

4.1 Deviations for $(N_t/t)_{t \geq 0}$

This section is devoted to the study of deviations of $(N_t/t)_{t \geq 0}$ around its limit value $E_\mu(N_1)$. The control of deviations of $(N_t/t)_{t \geq 0}$ will permit us to control the deviations of other additive functionals. We recall that $l = E_\mu(N_1)$.

Theorem 4.1 *Grant assumptions 2.1 and 2.2. Let ν be any initial distribution, $0 < \varepsilon < 1$ and $p \geq 1$. Suppose that $E_\nu(R_1)^{p/2} < \infty$ and $E_\nu(R_2 - R_1)^p = E_a R_1^p < \infty$. Then there exists a positive constant $C(l, p, \varepsilon, \nu)$ such that the following inequality holds.*

$$\mathbf{P}_\nu(|N_t/t - l| > l\varepsilon) \leq C(l, p, \varepsilon, \nu) \frac{1}{\varepsilon^p} \frac{1}{t^{p/2}}, \quad (4.41)$$

where $C(l, p, \varepsilon, \nu)$ is given by

$$C(l, p, \varepsilon, \nu) = 2^{p+2} (\mathbf{E}_\nu |R_1 - 1/l|^{p/2} + C_p^p \mathbf{E}_\nu |\bar{\eta}_2|^{p/2})$$

and where $\bar{\eta}_2 = R_2 - R_1 - \frac{1}{l}$.

Remark 4.2 *Using $E_\nu(R_1)^p \leq 2^{p-1} (E_\nu T_b^p + E_b T_a^p)$, we can see that the hypothesis of the theorem 4.1 are satisfied if $E_b T_a^p < \infty$, $E_a T_b^p < \infty$ and $E_\nu T_b^{p/2} < \infty$. In particular, if for all $x, y \in \mathbb{R}$, $\mathbf{E}_x(T_y)^p < \infty$, then the hypotheses of the theorem are satisfied for any $x \in \mathbb{R}$ and initial measure of the form $\nu = \delta_{\{x\}}$.*

Proof We write

$$\mathbf{P}_\nu(|N_t/t - l| > l\varepsilon) \leq \mathbf{P}_\nu(N_t/t > l(1 + \varepsilon)) + \mathbf{P}_\nu(N_t/t < l(1 - \varepsilon)). \quad (4.42)$$

Put

$$\bar{\eta}_k = -1(R_k - R_{k-1} - 1/l.)$$

For the first term of (4.42), we have

$$\begin{aligned}
\mathbf{P}_\nu(N_t/t > l(1 + \varepsilon)) &= \mathbf{P}_\nu(N_t \geq [tl(1 + \varepsilon)] + 1) = \mathbf{P}_\nu(R_{[tl(1+\varepsilon)]+1} \leq t) = \\
&= \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1+\varepsilon)]+1} (R_k - R_{k-1}) \leq t\right) \\
&= \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1+\varepsilon)]+1} (R_k - R_{k-1} - 1/l) \leq t\left(1 - \frac{[tl(1 + \varepsilon)] + 1}{tl}\right)\right) \\
&\leq \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1+\varepsilon)]+1} (R_k - R_{k-1} - 1/l) \leq t\left(1 - \frac{tl(1 + \varepsilon)}{tl}\right)\right) \\
&\leq \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1+\varepsilon)]+1} (R_k - R_{k-1} - 1/l) \leq -t\varepsilon\right) \\
&\leq \mathbf{P}_\nu(R_1 - 1/l \leq -t\varepsilon/2) + \mathbf{P}_\nu\left(\sum_{k=2}^{[tl(1+\varepsilon)]+1} \bar{\eta}_k \geq t\varepsilon/2\right). \tag{4.43}
\end{aligned}$$

We use BGD inequalities to bound the second term of the previous inequality. For $k \geq 2$ ($\bar{\eta}_k$) are i.i.d. centered random variables, and by hypothesis $\mathbf{E}_\nu|\bar{\eta}_k|^p < \infty$. Let $M_0 = 0$ and $M_n = \sum_{k=2}^{n+1} \bar{\eta}_k$. This is a martingale such that $[M]_n = \sum_{k=2}^{n+1} \bar{\eta}_k^2$. Denote $M_n^* = \sup_{k \leq n+1} M_k$. By BGD inequality, there exists an absolute constant C_p such that

$$\|M_n^*\|_p \leq C_p \| [M]_n^{1/2} \|_p \quad .$$

It follows, using Hölder's inequality,

$$\mathbf{E}_\nu(M_n^*)^p \leq C_p^p \mathbf{E}_\nu\left(\sum_{k=2}^{n+1} \bar{\eta}_k^2\right)^{p/2} \leq C_p^p n^{p/2} \mathbf{E}_\nu \frac{1}{n} \sum_{k=2}^{n+1} |\bar{\eta}_k|^p = C_p^p n^{p/2} \mathbf{E}_\nu |\bar{\eta}_2|^p. \tag{4.44}$$

Using (4.44) for the last term of (4.43), we can write

$$\begin{aligned}
\mathbf{P}_\nu\left(\sum_{k=2}^{[tl(1+\varepsilon)]+1} \bar{\eta}_k \geq t\varepsilon/2\right) &\leq \mathbf{P}_\nu\left(\sup_{k \leq [tl(1+\varepsilon)]} |M_k| \geq t\varepsilon/2\right) \leq \\
&\leq \frac{2^p \mathbf{E}_\nu(M_{[tl(1+\varepsilon)]}^*)^p}{t^p \varepsilon^p} \leq \frac{2^p C_p^p \{ [tl(1 + \varepsilon)] \}^{p/2} \mathbf{E}_\nu |\bar{\eta}_2|^p}{t^p \varepsilon^p} \leq 2^p C_p^p \mathbf{E}_\nu |\bar{\eta}_2|^p \frac{\{l(1 + \varepsilon)\}^{p/2}}{\varepsilon^p} \frac{1}{t^{p/2}}. \tag{4.45}
\end{aligned}$$

In an analogous way, using (4.44) we treat the second term in (4.42):

$$\begin{aligned}
\mathbf{P}_\nu(N_t/t < l(1 - \varepsilon)) &= \mathbf{P}_\nu(N_t \leq [tl(1 - \varepsilon)]) = \mathbf{P}_\nu(R_{[tl(1-\varepsilon)]} \geq t) = \\
&= \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1-\varepsilon)]} (R_k - R_{k-1} - 1/l) \geq t - \frac{[tl(1 - \varepsilon)]}{l}\right) \\
&\leq \mathbf{P}_\nu\left(\sum_{k=1}^{[tl(1-\varepsilon)]} (R_k - R_{k-1} - 1/l) \geq t\varepsilon\right) \\
&\leq \mathbf{P}_\nu\left(R_1 - \frac{1}{l} \geq t\varepsilon/2\right) + \mathbf{P}_\nu\left(\sum_{k=2}^{[tl(1-\varepsilon)]} (R_k - R_{k-1} - \frac{1}{l}) \geq t\varepsilon/2\right) \\
&\leq \frac{2^{p/2}\mathbf{E}_\nu|R_1 - 1/l|^{p/2}}{(t\varepsilon)^{p/2}} + \frac{2^p C_p^p ([tl(1 - \varepsilon)] - 1)^{p/2} \mathbf{E}_\nu|\bar{\eta}_2|^p}{(t\varepsilon)^p} \\
&\leq 2^p C_p^p \mathbf{E}_\nu|\bar{\eta}_2|^p \frac{(l(1 - \varepsilon))^{p/2}}{\varepsilon^p} \frac{1}{t^{p/2}} + \frac{2^{p/2}\mathbf{E}_\nu|R_1 - 1/l|^{p/2}}{(t\varepsilon)^{p/2}}. \tag{4.46}
\end{aligned}$$

Plugging in (4.42) the upper bounds (4.45) together with (4.46), we obtain:

$$\begin{aligned}
\mathbf{P}_\nu(|N_t/t - l| > l\varepsilon) &\leq 2 \frac{2^{p/2}\mathbf{E}_\nu|R_1 - 1/l|^{p/2}}{(t\varepsilon)^{p/2}} + 2^p C_p^p \mathbf{E}_\nu|\bar{\eta}_2|^p l^{p/2} \left[\frac{(1 + \varepsilon)^{p/2} + (1 - \varepsilon)^{p/2}}{\varepsilon^p} \right] \frac{1}{t^{p/2}} \\
&\leq 2^{p+2}(\mathbf{E}_\nu|R_1 - 1/l|^{p/2} + C_p^p \mathbf{E}_\nu|\bar{\eta}_2|^p l^{p/2}) \frac{1}{\varepsilon^p} \frac{1}{t^{p/2}} = C(p) \frac{1}{\varepsilon^p} \frac{1}{t^{p/2}}. \tag{4.47}
\end{aligned}$$

•

4.2 Rate of convergence in the Ergodic Theorem.

We apply the results of the previous section to get a bound on the rate of convergence in the Ergodic Theorem for additive functionals $\int_0^t f(X_s)ds$, where f is such that $\mu(f) \neq 0$. We consider two situations. First the case where f is bounded, second the case when f is bounded and of compact support. Recall that $C(f) = \sup_x E_x \int_0^{R_1} |f|(X_s)ds$ in the last case. In the first case, our bound depends on f through M , where $M = \|f\|_\infty$. In the second case, our bound depends on f through $\mu(f)$ and $C(f)$. Actually, in the second case we only need the finiteness of $\mu(f)$ and $C(f)$. Such conditions are often used in the study of recurrent diffusions. Functions f with $C(f)$ finite are called special (with respect to the process (X) and to the decomposition in the life cycles (R_n)). It is known that for strong Feller diffusions all bounded functions with compact support are special with respect to all decompositions. That's why we work in the second case with this class of functions. Finally recall that $C(f) \leq Cste \cdot \mu(f)$ (recall lemma 3.5).

To summarize, the dependence on f can be explained in terms of its sup-norm or its $L^1(\mu)$ -norm. We give both possibilities, because in concrete applications – even for bounded f having compact support – it is important to dispose of both upper bounds.

Theorem 4.3 *Grant assumptions 2.1 and 2.2. Let f be a measurable bounded function, or measurable bounded with compact support. Suppose that $\mu(f) \neq 0$. Let ν be some initial distribution. Suppose that for some $p \geq 1$, $\mathbf{E}_\nu R_1^p < \infty$ and $\mathbf{E}_\nu (R_2 - R_1)^p = \mathbf{E}_a (R_1)^p < \infty$.*

(a) *For a bounded function f such that $\|f\|_\infty \leq M$, $0 < M < \infty$, we have the following polynomial bound:*

$$\mathbf{P}_\nu \left(\left| \frac{1}{t} \int_0^t f(X_s) ds - \mu(f) \right| > \varepsilon \right) \leq K(p) \frac{1}{t^{p/2}} \frac{1}{\varepsilon^p} M^p, \quad (4.48)$$

where $K = K(p)$ does not depend on f , t , ε .

(b) *For a bounded function f with compact support we have:*

$$\mathbf{P}_\nu \left(\left| \frac{1}{t} \int_0^t f(X_s) ds - \mu(f) \right| > \varepsilon \right) \leq K(p) \frac{1}{t^{p/2}} \frac{1}{\varepsilon^p} (C(f) \vee |\mu(f)|)^p, \quad (4.49)$$

where $K = K(p)$ does not depend on f , t , ε .

Proof For $n \geq 1$, $\xi_n = \int_{R_n}^{R_{n+1}} f(X_s) ds$ are i.i.d. Using proposition 3.1, the law of ξ_n , $n \geq 1$, does not depend on the initial distribution and is equal to the law of ξ_0 under \mathbf{P}_a . Recall in virtue of proposition 3.7 that $\mathbf{E}_a \xi_0 = \mu(f)/l$. Denote

$$\Omega_t = \left\{ \left| \frac{N_t}{t} - l \right| \leq \frac{l\varepsilon}{4} \right\}. \quad (4.50)$$

Suppose without loss of generality that $\mu(f) > 0$. We start by deriving deviation bound for

$$\mathbf{P}_\nu \left(\left| \int_0^t f(X_s) ds - t\mu(f) \right| > t\mu(f)\varepsilon \right). \quad (4.51)$$

We shall use the following decomposition.

$$\begin{aligned} & \mathbf{P}_\nu \left(\left| \int_0^t f(X_s) ds - t\mu(f) \right| > t\mu(f)\varepsilon \right) \\ & \leq \mathbf{P}_\nu \left(\left| \int_0^t f(X_s) ds - t\mu(f) \right| > t\mu(f)\varepsilon ; \Omega_t \right) + \mathbf{P}_\nu (\Omega_t^c) \\ & \leq \mathbf{P}_\nu \left(\left| \int_0^{R_1} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4} \right) + \mathbf{P}_\nu \left(\left| \int_{R_1}^{R_{N_t+1}} f(X_s) ds - N_t \frac{\mu(f)}{l} \right| > \frac{t\varepsilon\mu(f)}{4} ; \Omega_t \right) \\ & + \mathbf{P}_\nu \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4} ; \Omega_t \right) + \mathbf{P}_\nu (\Omega_t^c) \\ & = A + B + C + D. \end{aligned} \quad (4.52)$$

In the sequel, $K(p)$ means a constant that may change from line to line but that does not depend on t, ε, f . This constant depends only on the process, the choice of life cycles and p . For the term A , we have

(A1) in the case when f is bounded,

$$\mathbf{P}_\nu \left(\left| \int_0^{R_1} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4} \right) \leq \mathbf{P}_\nu \left(R_1 > \frac{t\varepsilon\mu(f)}{4M} \right) \leq \frac{E_\nu R_1^p}{t^p} \left(\frac{4M}{\varepsilon\mu(f)} \right)^p, \quad (4.53)$$

(A2) and in the case when f is bounded and compactly supported

$$\begin{aligned} \mathbf{P}_\nu \left(\left| \int_0^{R_1} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4} \right) &\leq \frac{E_\nu (\int_0^{R_1} |f(X_s)| ds)^p}{t^p} \left(\frac{4}{\varepsilon\mu(f)} \right)^p \\ &\leq \frac{(p)! C(f)^p}{t^p} \left(\frac{4}{\varepsilon\mu(f)} \right)^p. \end{aligned} \quad (4.54)$$

In the sequel we need $\mathbf{E}_\nu |\xi_k - \mu(f)/l|^p < \infty$, which can be seen as follows: if f is bounded,

$$\mathbf{E}_\nu |\xi_k - \mu(f)/l|^p < 2^p (\mathbf{E}_\nu |\xi_k|^p + (\mu(f)/l)^p) < 2^p M^p \mathbf{E}_\nu (R_2 - R_1)^p + 2^p (\mu(f)/l)^p < \infty,$$

if f is bounded and compactly supported,

$$\mathbf{E}_\nu |\xi_k - \mu(f)/l|^p < 2^p (\mathbf{E}_\nu |\xi_k|^p + (\mu(f)/l)^p) < 2^p p! C(f)^p + 2^p (\mu(f)/l)^p < \infty.$$

Now we treat the term B , which is the main term of the decomposition.

$$\begin{aligned} B &= \mathbf{P}_\nu \left(\left| \int_{R_1}^{R_{N_t+1}} f(X_s) ds - N_t \mu(f)/l \right| > \frac{t\varepsilon\mu(f)}{4}; \Omega_t \right) \\ &= \mathbf{P}_\nu \left(\left| \sum_{k=1}^{N_t} \xi_k - N_t \mu(f)/l \right| > \frac{t\varepsilon\mu(f)}{4}; \Omega_t \right) \\ &\leq \mathbf{P}_\nu \left(\left| \sum_{k=1}^{N_t} (\xi_k - \mathbf{E}_x \xi_k) \right| > \frac{t\varepsilon\mu(f)}{4}; \quad |N_t/t - l| \leq l\varepsilon/4 \right) \\ &\leq \mathbf{P}_\nu \left(\sup_{n \leq [tl(1+\varepsilon/4)]} \left| \sum_{k=1}^n (\xi_k - \mathbf{E}_x \xi_k) \right| > \frac{t\varepsilon\mu(f)}{4} \right) \\ &\leq \frac{C_p^p 4^p [tl(1+\varepsilon/4)]^{p/2} \mathbf{E}_\nu |\xi_1 - \mathbf{E}_x \xi_1|^p}{\mu(f)^p \varepsilon^{pt}}, \end{aligned} \quad (4.55)$$

where we have used BDG inequality for the martingale $\sum_{k=1}^n (\xi_k - \mathbf{E}_\nu \xi_k)$.

(B1) In the case where f is bounded we have:

$$B \leq \frac{C_p^p 4^p [tl(1+\varepsilon/4)]^{p/2}}{\mu(f)^p \varepsilon^{pt}} 2^p M^p (\mathbf{E}_\nu |R_2 - R_1|^p + 1/l^p) \leq K(p) M^p \frac{1}{t^{p/2}} \frac{1}{(\varepsilon\mu(f))^p}, \quad (4.56)$$

where $K(p) = C_p^p 16^p l^{p/2} (\mathbf{E}_\nu |R_2 - R_1|^p + 1/l^p)$.

(B2) In the case where f is bounded with compact support we have:

$$B \leq \frac{C_p^p 4^p [tl(1 + \varepsilon/4)]^{p/2}}{\mu(f)^p \varepsilon^p t^p} 2^p (p! C(f)^p + (\frac{\mu(f)}{l})^p) \leq K(p) (C(f) \vee |\mu(f)|)^p \frac{1}{t^{p/2}} \frac{1}{(\varepsilon \mu(f))^p}, \quad (4.57)$$

where $K(p) = C_p^p 16^p l^{p/2} (p! + 1/l^p)$.

For the term C we can write:

$$\begin{aligned} C &= \mathbf{P}_\nu \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4}; \Omega_t \right) \\ &\leq \sum_{k=1}^{[tl(1+\varepsilon/4)]} \mathbf{P}_\nu \left(\left| \int_t^{R_{N_t+1}} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4}; N_t = k \right) \\ &\leq \sum_{k=1}^{[tl(1+\varepsilon/4)]} \mathbf{P}_\nu \left(\left| \int_{R_k}^{R_{k+1}} f(X_s) ds \right| > \frac{t\varepsilon\mu(f)}{4} \right) \\ &\leq tl(1 + \varepsilon/4) \frac{\mathbf{E}_\nu(\int_{R_k}^{R_{k+1}} f(X_s) ds)^p}{t^p} \left(\frac{4}{\varepsilon\mu(f)} \right)^p. \end{aligned} \quad (4.58)$$

(C1) In the case of bounded f we get

$$C \leq K(p) \frac{1}{t^{p/2}} \frac{M^p}{(\varepsilon\mu(f))^p}, \quad (4.59)$$

where $K(p) = 2l4^p \mathbf{E}_a R_1^p$.

(C2) In the case of bounded f with compact support we get

$$C \leq K(p) \frac{1}{t^{p/2}} \frac{C(f)^p}{(\varepsilon\mu(f))^p}, \quad (4.60)$$

where $K(p) = 2l4^p p!$

For the term D , we use the theorem 4.1 :

$$D \leq K(p) (\mu(f))^p \frac{1}{t^{p/2}} \frac{1}{(\varepsilon\mu(f))^p}.$$

Finally, putting together (4.53), (4.56) and (4.59), we obtain for f bounded

$$\mathbf{P}_\nu \left(\left| \int_0^t f(X_s) ds - t\mu(f) \right| > t\mu(f)\varepsilon \right) \leq K(p) M^p \frac{1}{t^{p/2}} \frac{1}{(\varepsilon\mu(f))^p}. \quad (4.61)$$

In the same way, putting together (4.54), (4.57) and (4.60), we obtain for f bounded with compact support

$$\mathbf{P}_\nu \left(\left| \int_0^t f(X_s) ds - t\mu(f) \right| > t\mu(f)\varepsilon \right) \leq K(p) (C(f) \vee |\mu(f)|)^p \frac{1}{t^{p/2}} \frac{1}{(\varepsilon\mu(f))^p}. \quad (4.62)$$

We now replace $\varepsilon\mu(f)$ by ε , and f by $-f$ if $\mu(f) < 0$ and the theorem follows. •

5 Existence of hitting times moments

The question of existence of moments of hitting times arises in various problems and is widely studied in the literature, see Fitzsimmons and Pitman [9], Carmona and Klein [3], Darling and Siegert [5], Veretennikov[15], Balaji and Ramasubramanian [1], Ditlevsen[8], Deaconu and Wantz[7] and the references therein. In this subsection we explore some sufficient and necessary conditions for existence of polynomial moments of hitting times and give lower and upper bounds on these moments. Recall that we consider a diffusion given by

$$dX_t = \beta(X_t) dt + \sigma(X_t) dW_t$$

under assumptions guaranteeing the continuity of coefficients, the non-degeneracy of σ and the existence of non-exploding solution, see assumption 2.1.

In addition to this standard assumption we suppose throughout this section the following:

Assumption 5.1 *There exist positive constants σ_0 and σ_1 such that*

$$\sigma_0 \leq |\sigma(x)| \leq \sigma_1, \text{ for all } x \in \mathbb{R}. \quad (5.63)$$

We also impose some conditions on $\beta(x)$ for $|x| > M_0$, where M_0 is large enough. The two complementary cases of interest will be the following.

Assumption 5.2 *There exists a constant $r > 0$ such that*

$$x\beta(x) < -r \quad \text{for } |x| > M_0. \quad (5.64)$$

Assumption 5.3 *There exists a constant $R > 0$ such that*

$$0 > x\beta(x) > -R \quad \text{for } |x| > M_0. \quad (5.65)$$

It is known (see e.g. [1]) that under these assumptions for $|x| > |a|$ $\mathbf{E}_x T_a^p$ is finite for $p < r/\sigma_1^2 + 1/2$ and infinite for $p > R/\sigma_0^2 + 1/2$, (see also [15]) but we need a finer control on $\mathbf{E}_x T_a^p$ to be able to estimate $\mathbf{E}_\nu T_a^p$.

Recall that the scale function of X_t is given by

$$S(x) = \int_0^x s(t) dt, \quad \text{where } s(t) = \exp\left(-2 \int_0^t \frac{\beta(u)}{\sigma^2(u)} du\right).$$

It is easily seen that $\int_{-\infty}^{\infty} s(x) dx = +\infty = \int_{-\infty}^{\infty} s(x) dx$, which implies that X_t is recurrent, with speed density $m(\xi) = \frac{1}{s(\xi)\sigma^2(\xi)}$.

In the following lemma we estimate the moments of the speed measure of X_t . Put

$$I_p(x, a) = \int_x^\infty \frac{(\xi - a)^p}{\sigma^2(\xi)s(\xi)} d\xi,$$

and let $p^* = \sup\{p > 0 : I_p(x, 0) < \infty\}$.

Lemma 5.4 *Suppose that $M_0 < a < x$, then*

- *under the assumption 5.2, $p^* \geq 2r/\sigma_1^2 - 1$, and $s(x)I_p(x, 0) \leq x^{p+1}/(2r - (p+1)\sigma_1^2)$ for any $p < 2r/\sigma_1^2 - 1$.*
- *under the assumption 5.3, $p^* \leq 2R/\sigma_0^2 - 1$, and $s(x)I_p(x, a) \geq (x-a)^{p+1}/(2R - (p+1)\sigma_0^2)$ for any $0 < p < p^*$.*

Proof Under the assumption 5.2, $1/s(x) < C|x|^{-2r/\sigma_1^2}$, so the speed measure of X_t admits moments of any order $p < 2r/\sigma_1^2 - 1$. Moreover, $\lim_{\xi \rightarrow \infty} \xi^{p+1}/s(\xi) = 0$.

Note that $1/s(\xi)$ is decreasing on $[M_0, +\infty[$, and that

$$d\left(\frac{1}{s(x)}\right) = 2\frac{\beta(x)}{\sigma^2(x)s(x)} dx.$$

Integration by parts gives

$$\begin{aligned} I_p(x, 0) &= \int_x^\infty \frac{\xi^p d\xi}{\sigma^2(\xi)s(\xi)} = \frac{1}{2} \int_x^\infty \frac{\xi^p}{\beta(\xi)} d\left(\frac{1}{s(\xi)}\right) \leq -\frac{1}{2r} \int_x^\infty \xi^{p+1} d\left(\frac{1}{s(\xi)}\right) = \\ &= -\frac{1}{2r} \left[\frac{\xi^{p+1}}{s(\xi)} \Big|_x^\infty - (p+1) \int_x^\infty \frac{\xi^p}{s(\xi)} d\xi \right] \leq \frac{1}{2r} \left[\frac{x^{p+1}}{s(x)} + (p+1)\sigma_1^2 I_p(x, 0) \right] \end{aligned}$$

and thus the first assertion follows.

On the other hand, under the assumption 5.3, $1/s(x) > C|x|^{-2R/\sigma_0^2}$ for large $|x|$, hence $p^* \leq 2R/\sigma_0^2 - 1$. Suppose that $p < p^*$, then, for any $\varepsilon > 0$ there exists a sequence $y_k \rightarrow \infty$ such that $(y_k - a)^{p+1}/s(y_k) < \varepsilon$. Choose $y > x$ such that

$$(y - a)^{p+1}/s(y) < \varepsilon \quad \text{and} \quad \int_y^\infty \frac{(\xi - a)^p d\xi}{\sigma^2(\xi)s(\xi)} < \varepsilon.$$

Then

$$\begin{aligned} I_p(x, a) &= \int_x^\infty \frac{(\xi - a)^p d\xi}{\sigma^2(\xi)s(\xi)} \geq \int_x^y \frac{(\xi - a)^p d\xi}{\sigma^2(\xi)s(\xi)} = \frac{1}{2} \int_x^y \frac{(\xi - a)^p}{\beta(\xi)} d\left(\frac{1}{s(\xi)}\right) \geq \\ &= -\frac{1}{2R} \int_x^y \xi(\xi - a)^p d\left(\frac{1}{s(\xi)}\right) \geq -\frac{1}{2R} \int_x^y (\xi - a)^{p+1} d\left(\frac{1}{s(\xi)}\right) = \\ &= -\frac{1}{2R} \left[\frac{(\xi - a)^{p+1}}{s(\xi)} \Big|_x^y - (p+1) \int_x^y \frac{(\xi - a)^p}{s(\xi)} d\xi \right] \geq \\ &= \frac{1}{2R} \left[\frac{(x - a)^{p+1}}{s(x)} - \varepsilon + (p+1)\sigma_0^2(I_p(x, a) - \varepsilon) \right], \end{aligned}$$

whence

$$s(x)I_p(x, a) \geq \frac{1}{2R - (p+1)\sigma_0^2} \left[(x - a)^{p+1} - \varepsilon(1 + (p+1)\sigma_0^2)s(x) \right].$$

Since ε can be chosen arbitrary small, we get the second point of the lemma.

Remark 5.5 Under the assumption 5.3, for $n \geq 0$ we have

$$\begin{aligned}
I_n(x) &= \int_x^\infty \frac{(\xi - a)^n}{\sigma^2(\xi)s(\xi)} d\xi = \frac{1}{2} \int_x^\infty \frac{(\xi - a)^n}{\beta(\xi)} d\left(\frac{1}{s(\xi)}\right) \geq \\
&\quad - \frac{1}{2R} \int_x^\infty \xi(\xi - a)^n d\left(\frac{1}{s(\xi)}\right) = -\frac{1}{2R} \int_x^\infty (\xi - a + a)(\xi - a)^n d\left(\frac{1}{s(\xi)}\right) = \\
&\quad - \frac{1}{2R} \left[\frac{(\xi - a)^{n+1} + a(\xi - a)^n}{s(\xi)} \Big|_x^\infty - (n+1) \int_x^\infty \frac{(\xi - a)^n}{s(\xi)} d\xi - na \int_x^\infty \frac{(\xi - a)^{n-1}}{s(\xi)} d\xi \right] \geq \\
&\quad \frac{1}{2R} \frac{(x - a)^{n+1} + a(x - a)^n}{s(x)} + \frac{(n+1)\sigma_0^2}{2R} I_n(x) + \frac{na\sigma_0^2}{2R} I_{n-1}(x),
\end{aligned}$$

where $I_{-1}(x) = 0$. Hence

$$s(x)I_n(x) \geq \frac{1}{2R - (n+1)\sigma_0^2} \left((x - a)^{n+1} + a(x - a)^n + na\sigma_0^2 s(x)I_{n-1}(x) \right),$$

and for $n = 0$ we get

$$s(x)I_0(x) \geq \frac{1}{2R - \sigma_0^2} ((x - a) + a) = \tilde{P}_1(x - a).$$

Now, for $n > 0$ we obtain by induction

$$s(x)I_n(x) \geq \frac{1}{2R - (n+1)\sigma_0^2} \left((x - a)^{n+1} + a(x - a)^n + na\sigma_0^2 \tilde{P}_n(x - a) \right) = \tilde{P}_{n+1}(x - a).$$

•

Theorem 5.6 Under the assumption 5.2, for $M_0 < a < x$, or $x < a < -M_0$, the fact that $n < r/\sigma_1^2 + 1/2$ implies that

$$\mathbf{E}_x T_a^n \leq P_{2n}(x) = \frac{x^{2n}}{r_n} + x^{2n-1}O(1),$$

where P_{2n} is a polynomial of degree $2n$ and $r_n = \prod_{k=1}^n (2r - (2k - 1)\sigma_1^2)$. Hence $\mathbf{E}_x T_a^n < \infty$ for all x, a .

Under the assumption 5.3, for $M_0 < a < x$, or $x < a < -M_0$,

- if $n < p^*/2 + 1$ then

$$\mathbf{E}_x T_a^n \geq \frac{(x - a)^{2n}}{R_n},$$

where $R_n = \prod_{k=1}^n (2R - (2k - 1)\sigma_0^2)$.

- if $n > R/\sigma_0^2 + 1/2$, then $\mathbf{E}_x T_a^n = \infty$, and hence $\mathbf{E}_x T_a^n = \infty$ for all x, a .

Proof We start with the lower bound. Recall that $p^* \leq 2R/\sigma_0^2 - 1$. By the Kac formula, for $n \geq 1$,

$$(\mathbf{E}_x T_a^n)' = 2ns(x) \int_x^\infty \frac{\mathbf{E}_\xi T_a^{n-1}}{\sigma^2(\xi)s(\xi)} d\xi,$$

where the derivative is taken with respect to x . So for $n = 1$ we get

$$(\mathbf{E}_x T_a)' = 2s(x) \int_x^\infty \frac{d\xi}{\sigma^2(\xi)s(\xi)} \geq \frac{2(x-a)}{2R - \sigma_0^2},$$

whence

$$\mathbf{E}_x T_a \geq \int_a^x \frac{2(\xi-a)}{2R - \sigma_0^2} d\xi = \frac{(x-a)^2}{R_1}.$$

By induction, for $n > 1$,

$$(\mathbf{E}_x T_a^n)' \geq 2ns(x) \int_a^x \frac{(\xi-a)^{2n-2}}{R_{n-1}\sigma^2(\xi)s(\xi)} d\xi \geq \frac{2n(x-a)^{2n-1}}{R_{n-1}(2R - (2n-1)\sigma_0^2)},$$

whence

$$\mathbf{E}_x T_a^n \geq \frac{2n}{R_n} \int_a^x (\xi-a)^{2n-1} d\xi = \frac{(x-a)^{2n}}{R_n}.$$

The upper bound is proved in a similar way. The case $x < a < -M_0$ follows by symmetry. The fact that, under condition 5.3, if $n > R/\sigma_0^2 + 1/2$, then $\mathbf{E}_x T_a^n = \infty$ follows from [1], Theorem 3. Using theorem 2.6, the moment $\mathbf{E}_x T_a^n$ is finite or infinite simultaneously for all couples x, a . •

Remark 5.7 1. Under assumption 5.2, in order to check the condition $\mathbf{E}_\nu T_a^n < \infty$ in the case when $\mathbf{E}_x T_a^n < \infty$ for all x , we can use the continuity in x of $\mathbf{E}_x T_a^n < \infty$ and the polynomial bound of the previous theorem.

2. The constants r and R of the assumptions 5.2, 5.3 can be replaced respectively by $-\liminf_{x \rightarrow \infty} x\beta(x)$ and $-\limsup_{x \rightarrow \infty} x\beta(x)$.

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