Looking for martingales associated to a self-decomposable law

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Abstract We construct martingales whose 1-dimensional marginals are those of a centered self-decomposable variable multiplied by some power of time t. Many examples involving quadratic functionals of Bessel processes are discussed.

Key words Convex order, Self-decomposable law, Sato process, Karhunen-Loeve representation, Perturbed Bessel process, Ray-Knight theorem.

1 Introduction, Motivation

1.1

We first introduce some notation which will be used throughout our paper.

If A and B are two random variables, $A \overset{\text{d}}{=} B$ means that these variables have the same law.

If $(X_t\ ,\ t \geq 0)$ and $(Y_t\ ,\ t \geq 0)$ are two processes, $(X_t) \overset{\text{(1.d)}}{=} (Y_t)$ means that the processes $(X_t\ ,\ t \geq 0)$ and $(Y_t\ ,\ t \geq 0)$ have the same one-dimensional marginals, that is, for any fixed t, $X_t \stackrel{\mathrm{d}}{=} Y_t$.

If $(X_t\ ,\ t \geq 0)$ and $(Y_t\ ,\ t \geq 0)$ are two processes, $(X_t) \stackrel{\mathrm{(d)}}{=} (Y_t)$ means that the two processes are identical in law.

All random variables and processes which will be considered are assumed to be real valued.

1.2

In a number of applied situations involving randomness, it is a quite difficult problem to single out a certain stochastic process $(Y_t\ ,\ t \geq 0)$, or rather its law, which is coherent with the real-world data.

In some cases, it is already nice to be able to consider that the one-dimensional marginals of (Y_t) are accessible. The random situation being studied may suggest, for instance, that:

(i) there exists a martingale (M_t) such that

$$(Y_t) \stackrel{\mathrm{(1.d)}}{=} (M_t)$$

(this hypothesis may indicate some kind of "equilibrium" with respect to time),

(ii) there exists $H > 0$ such that

$$(Y_t) \stackrel{\mathrm{(1.d)}}{=} (t^H\, Y_1)$$

(there is a "scaling" property involved in the randomness).

It is a result due to Kellerer [12] that (i) is satisfied for a given process (Y_t) if and only if this process is increasing in the convex order, that is: it is integrable ($\forall t \geq 0$, $\mathbb{E}[|Y_t|] < \infty$), and for every convex function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$,

$$t \geq 0 \longrightarrow \mathbb{E}[\varphi(Y_t)] \in (-\infty, +\infty]$$

is increasing.

In the sequel, we shall use the acronym PCOC for such processes, since, in French, the name of such processes becomes: <u>P</u>rocessus <u>C</u>roissant pour l'<u>O</u>rdre <u>C</u>onvexe.

A martingale (M_t) which has the same one-dimensional marginals as a PCOC is said to be *associated* to this PCOC. Note that several different martingales may be associated to a given PCOC. We shall see several striking occurrences of this in our examples.

Roynette [19] has exhibited two large families, $(F1)$ and $(F2)$, of PCOC's: If (N_s) is a martingale satisfying some integrability condition, then

$$\left(\frac{1}{t}\int_0^t N_s\,\mathrm{d}s\ ,\ t\geq 0\right)\ \text{is a PCOC in } (F1);$$

$$\left(\int_0^t (N_s - N_0)\,\mathrm{d}s\ ,\ t\geq 0\right)\ \text{is a PCOC in } (F2).$$

1.3

It is a non-trivial problem to exhibit, for either of these PCOC's, an associated martingale. We have been able to do so concerning some examples in $(F1)$, in the Brownian context, with the help of the Brownian sheet, in our paper [8]. Concerning the class $(F2)$, note that, considering a trivial filtration, it follows that $(t\,X)$, where X is a centered random variable, is a PCOC. Even with this reduction, it is not obvious to find a martingale which is associated to $(t\,X)$. In order to exhibit examples, we were led to introduce the class (S) of processes (Y_t) satisfying the above condition (ii) and such that Y_1 is a self-decomposable integrable random variable. It is a result due to Sato (see Sato [20, Chapter 3, Sections 15-17]) that, if $(Y_t)\in(S)$, then there exists a process (U_t) which has independent increments, is *H-self-similar* ($\forall c>0$, $(U_{ct})\overset{(\mathrm{d})}{=}(c^H\,U_t)$) and satisfies $Y_1\overset{\mathrm{d}}{=}U_1$. This process (U_t), which is unique in law, will be called the *H-Sato process associated* to Y_1. Clearly, then $(U_t-\mathbb{E}[U_t])$ is a H-self-similar martingale which is associated to the PCOC (V_t) defined by: $V_t=Y_t-\mathbb{E}[Y_t]$. Moreover, $(V_t)\overset{(1.\mathrm{d})}{=}(t^H\,(Y_1-\mathbb{E}[Y_1]))$.

We note that the self-decomposability property has also been used in Madan-Yor [15, Theorem 4,Theorem 5] in a very different manner than in this paper, to construct martingales with one-dimensional marginals those of $(t\,X)$.

1.4

We look for some interesting processes in the class (S), in a Brownian framework.

Example 1 A most simple example is the process:

$$Y_t := \int_0^t B_s\,\mathrm{d}s \quad , \quad t\geq 0$$

Then,

$$\left(\int_0^t B_s \, \mathrm{d}s\right) \overset{(1.\mathrm{d})}{=} \left(\int_0^t s \, \mathrm{d}B_s\right)$$

and the RHS is a centered (3/2)-Sato process. Moreover the process (Y_t) obviously belongs to the class $(F2)$.

Example 2 The process

$$V_1(t) := \int_0^t (B_s^2 - s) \, \mathrm{d}s \quad , \quad t \geq 0$$

and more generally the process

$$V_N(t) := \int_0^t (R_N^2(s) - Ns) \, \mathrm{d}s \quad , \quad t \geq 0$$

where $(R_N(s))$ is a Bessel process of dimension $N > 0$ starting from 0, belongs to the family $(F2)$ and is 2-self-similar. We show in Section 4 that the centered 2-Sato process:

$$\frac{N^2}{4} \int_0^{\tau_t} 1_{\left(|B_s| \leq \frac{2}{N} \ell_s\right)} \, \mathrm{d}s - \frac{Nt^2}{2} \quad , \quad t \geq 0$$

where (ℓ_s) is the local time in 0 of the Brownian motion B, and

$$\tau_t = \inf\{s \; ; \; \ell_s > t\}$$

is a martingale associated to the PCOC V_N.

Example 3 We extend our discussion of Example 2 by considering, for $N > 0$ and $K > 0$, the process:

$$V_{N,K}(t) := \frac{1}{K^2} \int_0^t s^{2\left(\frac{1}{K} - 1\right)} (R_N^2(s) - Ns) \, \mathrm{d}s \quad , \quad t \geq 0$$

Then, in Section 5, a centered $(2/K)$-Sato process (and hence a martingale) associated to the PCOC $V_{N,K}$ may be constructed from the process of first hitting times of a perturbed Bessel process $R_{K,1-\frac{N}{2}}$ as defined and studied first in Le Gall-Yor [13, 14] and then in Doney-Warren-Yor [5]. We remark that, if $0 < K < 2$, then the process

$$V_{N,K}(t^{\frac{K}{2-K}}) \quad , \quad t \geq 0$$

belongs to $(F2)$.

Example 4 In Section 6, we generalize again our discussion by considering the process

$$V_N^{(\mu)}(t) := \int_{(0,\infty)} (R_N^2(ts) - Nts)\, \mathrm{d}\mu(s) \quad , \quad t \geq 0$$

for μ a nonnegative measure on $(0, \infty)$ such that $\int_{(0,\infty)} s\ \mathrm{d}\mu(s) < \infty$. We show that $V_N^{(\mu)}$ is a PCOC to which we are able to associate two very different martingales. The first one is purely discontinuous and is a centered 1-Sato process, the second one is continuous. The method of proof is based on a Karhunen-Loeve type decomposition (see, for instance, [4] and the references therein, notably Kac-Siegert [11]). For this, we need to develop a precise spectral study of the operator $K^{(\mu)}$ defined on $L^2(\mu)$ by :

$$K^{(\mu)} f(t) = \int_{(0,\infty)} f(s)\, t \wedge s\ \mathrm{d}\mu(s)$$

This spectral study is certainly classical, but we present it for the convenience of the reader.

1.5

It is still an open problem (for the authors) whether the PCOC's (in $(F2)$):

$$\int_0^t H_n(B_s, s)\ \mathrm{d}s \quad , \quad t \geq 0$$

defined from the 2 variables Hermite polynomials

$$H_n(x, s) = s^{n/2}\, h_n(x/\sqrt{s})$$

lend themselves to our method, namely: is it true that the random variable

$$\int_0^1 H_n(B_s, s)\ \mathrm{d}s$$

is self decomposable for $n = 3, 4, \cdots$.

1.6

We now present more precisely the organisation of our paper:

- in Section 2, we recall some basic results about various representations of self-decomposable variables, and we complete the discussion of Subsection 1.3 above;

- in Section 3, we consider the simple situation, as in Subsection 1.3, where $Y_t = R_N^2(t)$, for R_N a Bessel process of dimension N starting from 0;

- the contents of Sections 4, 5, 6 have already been discussed in the above Subsection 1.4;

- in a short final Section 7, we prove some negative results concerning further self-decomposability properties for squared Bessel processes: indeed, it is well-known, and goes back to Shiga-Watanabe [21], that $R_N^2(\bullet)$, considered as a random variable taking values in $C(\mathbb{R}_+, \mathbb{R}_+)$ is infinitely divisible. Furthermore, in the present paper, we exploit the self-decomposability of $\int_{(0,\infty)} R_N^2(s)\, \mathrm{d}\mu(s)$ for any positive measure μ. It then seemed natural to wonder about the self-decomposability of $R_N^2(\bullet)$, but this property is ruled out, as the 2-dimensional vectors: $(R_N^2(t_1), R_N^2(t_1 + t_2))$ are not self-decomposable.

2 Sato processes and PCOC's

2.1 Self-decomposability and Sato processes

We recall, in this subsection, some general facts concerning the notion of self-decomposability. We refer the reader, for background, complements and references, to Sato [20, Chapter 3].

A random variable X is said to be *self-decomposable* if, for each u with $0 < u < 1$, there is the equality in law:

$$X \overset{\mathrm{d}}{=} u\, X + \widehat{X}_u$$

for some variable $\widehat{X}_u$ independent of X.
On the other hand, an *additive process* $(U_t\ ,\ t \geq 0)$ is a stochastically continuous process with càdlàg paths, independent increments, and satisfying $U_0 = 0$.
An additive process (U_t) which is H*-self-similar* for some $H > 0$, meaning that, for each $c > 0$, $(U_{ct}) \overset{(\mathrm{d})}{=} (c^H\, U_t)$, will be called a *Sato process* or, more precisely, a H*-Sato process*.

The following theorem, for which we refer to Sato's book [20, Chapter 3, Sections 16-17], gives characterizations of the self-decomposability property that we state in the following theorem:

Theorem 2.1 *Let X be a real valued random variable. Then, X is self-decomposable if and only if one of the following equivalent properties is satisfied:*

1) X is infinitely divisible and its Lévy measure is $\dfrac{h(x)}{|x|}\,\mathrm{d}x$ with h increasing on $(-\infty, 0)$ and decreasing on $(0, +\infty)$.

2) There exists a Lévy process $(C_s\ ,\ s \geq 0)$ such that

$$X \stackrel{\mathrm{d}}{=} \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s\ .$$

3) For any (or some) $H > 0$, there exists a H-Sato process $(U_t\ ,\ t \geq 0)$ such that $X \stackrel{\mathrm{d}}{=} U_1$.

In 2) (resp. 3)) the Lévy process (C_s) (resp. the H-Sato process (U_t)) is uniquely determined in law by X, and will be said to be *associated* with X. We note that, if $X \geq 0$, then the function h vanishes on $(-\infty, 0)$, (C_s) is a subordinator and (U_t) is an increasing process.

The relation between (C_s) and (U_t) was precised by Jeanblanc-Pitman-Yor [9, Theorem 1]:

Theorem 2.2 *If (U_t) is a H-Sato process, then the formulae:*

$$C_s^{(-)} = \int_{\mathrm{e}^{-s}}^1 r^{-H}\,\mathrm{d}U_r \quad \textit{and} \quad C_s^{(+)} = \int_1^{\mathrm{e}^s} r^{-H}\,\mathrm{d}U_r \quad , \quad s \geq 0$$

define two independent and identically distributed Lévy processes from which $(U_t\ ,\ t \geq 0)$ can be recovered by:

$$U_t = \int_{-\log t}^\infty \mathrm{e}^{-sH}\,\mathrm{d}C_s^{(-)} \quad \textit{if} \quad 0 \leq t \leq 1$$

and

$$U_t = U_1 + \int_0^{\log t} \mathrm{e}^{sH}\,\mathrm{d}C_s^{(+)} \quad \textit{if} \quad t \geq 1\ .$$

In particular, the Lévy process associated with the self-decomposable random variable U_1 is

$$C_s = C_{s/H}^{(-)} \quad , \quad s \geq 0\ .$$

2.2 Sato processes and PCOC's

We recall (see Subsection 1.2) that a PCOC is an integrable process which is increasing in the convex order. On the other hand, a process $(V_t\ ,\ t \geq 0)$ is said to be a *1-martingale* if there exists, on some filtered probability space, a martingale $(M_t\ ,\ t \geq 0)$ such that $(V_t) \stackrel{(1.d)}{=} (M_t)$. Such a martingale M is said to be *associated* with V. It is a direct consequence of Jensen's inequality that, if V is a 1-martingale, then V is a PCOC. As indicated in Subsection 1.2, the converse holds true (Kellerer [12]).

The following proposition, which is central in the following, summarizes the method sketched in Subsection 1.3.

Proposition 2.3 *Let $H > 0$. Suppose that $Y = (Y_t\ ,\ t \geq 0)$ satisfies:*

(a) Y_1 is an integrable self-decomposable random variable;

(b) $(Y_t) \stackrel{(1.d)}{=} (t^H\, Y_1)$.

Then the process

$$V_t := Y_t - t^H\, \mathbb{E}[Y_1] \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_t := U_t - t^H\, \mathbb{E}[Y_1] \quad , \quad t \geq 0$$

where (U_t) denotes the H-Sato process associated with Y_1 according to Theorem 2.1.

3 About the process $(R_N^2(t)\ ,\ t \geq 0)$

In the sequel, we denote by $(R_N(t)\ ,\ t \geq 0)$ the Bessel process of dimension $N > 0$, starting from 0.

3.1 Self-decomposability of $R_N^2(1)$

As is well-known (see, for instance, Revuz-Yor [18, Chapter XI]) one has

$$\mathbb{E}[\exp(-\lambda\, R_N^2(1))] = (1 + 2\,\lambda)^{-N/2}\,.$$

In other words,

$$R_N^2(1) \stackrel{\mathrm{d}}{=} 2\,\gamma_{N/2}$$

where, for $a > 0$, γ_a denotes a gamma random variable of index a. Now,

$$\frac{N}{2} \log(1 + 2\lambda) = \frac{N}{2} \int_0^\infty (1 - \mathrm{e}^{-\lambda t}) \frac{\mathrm{e}^{-t/2}}{t} \, \mathrm{d}t \, .$$

Then, $R_N^2(1)$ satisfies the property 1) in Theorem 2.1 with

$$h(x) = \frac{N}{2} 1_{(0,\infty)}(x) \, \mathrm{e}^{-x/2}$$

and it is therefore self-decomposable.

The process R_N^2 is 1-self-similar and $\mathbb{E}[R_N^2(1)] = N$. By Proposition 2.3, the process

$$V_t^N := R_N^2(t) - t\,N \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_t^N := U_t^N - t\,N \quad , \quad t \geq 0$$

where (U_t^N) denotes the 1-Sato process associated with $R_N^2(1)$ by Theorem 2.1.

We remark that, in this case, the process (V_t^N) itself is a *continuous martingale* and therefore obviously a PCOC. In the following subsections, we give two expressions for the process (U_t^N). As we will see, this process is purely discontinuous with finite variation; consequently, the martingales (V_t^N) and (M_t^N), which have the same one-dimensional marginals, do not have the same law.

3.2 Expression of (U_t^N) from a compound Poisson process

We denote by $(\Pi_s \, , \; s \geq 0)$ the compound Poisson process with Lévy measure:

$$1_{(0,\infty)}(t) \, \mathrm{e}^{-t} \, \mathrm{d}t \, .$$

This process allows to compute the distributions of a number of perpetuities

$$\int_0^\infty \mathrm{e}^{-\Lambda_s} \, \mathrm{d}\Pi_s$$

where (Λ_s) is a particular Lévy process, independent of Π; see, e.g., Nilsen-Paulsen [17]. In the case $\Lambda_s = r\,s$, the following result seems to go back at least to Harrison [7].

Proposition 3.1 *The Lévy process* (C_s^N) *associated with the self-decomposable random variable* $R_N^2(1)$ *in the sense of Theorem 2.1 is*

$$C_s^N = 2\,\Pi_{Ns/2} \quad , \quad s \geq 0\,.$$

Proof

We set $C_s^N = 2\,\Pi_{Ns/2}$. Then,

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s^N\right)\right] = \exp\left(-\frac{N}{2}\int_0^\infty F(2\,\lambda\,\mathrm{e}^{-s})\,\mathrm{d}s\right)$$

with, for $x > 0$,

$$F(x) = \int_0^\infty (1 - \mathrm{e}^{-tx})\,\mathrm{e}^{-t}\,\mathrm{d}t = \frac{x}{1+x}\,.$$

Consequently,

$$\mathbb{E}\left[\exp\left(-\lambda \int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}C_s^N\right)\right] = (1 + 2\,\lambda)^{-N/2}\,,$$

which proves the result.

□

By application of Theorem 2.2 we get:

Corollary 3.1.1 *Let* $\Pi^{(+)}$ *and* $\Pi^{(-)}$ *two independent copies of the Lévy process* Π. *Then*

$$U_t^N = 2\int_{-\frac{N}{2}\log t}^\infty \mathrm{e}^{-2s/N}\,\mathrm{d}\Pi_s^{(-)} \quad \textit{if} \quad 0 \leq t \leq 1$$

and

$$U_t^N = U_1^N + 2\int_0^{\frac{N}{2}\log t} \mathrm{e}^{2s/N}\,\mathrm{d}\Pi_s^{(+)} \quad \textit{if} \quad t \geq 1\,.$$

3.3 Expression of (U_t^N) from the local time of a perturbed Bessel process

There is by now a wide literature on perturbed Bessel processes, a notion originally introduced by Le Gall-Yor [13, 14], and then studied by Chaumont-Doney [3], Doney-Warren-Yor [5]. We also refer the interested reader to Doney-Zhang [6].

We first introduce the perturbed Bessel process $(R_{1,\alpha}(t)\ ,\ t \geq 0)$ starting from 0, for $\alpha < 1$, as the nonnegative continuous strong solution $(R_t\ ,\ t \geq 0)$ of the equation

$$R_t = B_t + \frac{1}{2}\, L_t(R) + \alpha\, M_t(R) \tag{1}$$

where $L_t(R)$ is the semi-martingale local time of R in 0 at time t, and

$$M_t(R) = \sup_{0 \leq s \leq t} R_s\,,$$

(B_t) denoting a standard linear Brownian motion starting from 0. (The strong solution property has been established in Chaumont-Doney [3].)
It is clear that the process $R_{1,0}$ is nothing else but the Bessel process R_1 (reflected Brownian motion).
We also denote by $T_t(R)$ the hitting time:

$$T_t(R) = \inf\{s\ ;\ R_s > t\}\,.$$

We set $L_{T_t}(R)$ for $L_{T_t(R)}(R)$.
Finally, in the sequel, we set

$$\alpha_N = 1 - \frac{N}{2}\,.$$

Proposition 3.2 *For any $\alpha < 1$, the process $(L_{T_t}(R_{1,\alpha})\ ,\ t \geq 0)$ is a 1-Sato process, and we have*

$$(U_t^N) \overset{\text{(d)}}{=} (L_{T_t}(R_{1,\alpha_N}))\,.$$

Proof

By the uniqueness in law of the solution to the equation (1), the process $R_{1,\alpha}$ is (1/2)-self-similar. As a consequence, the process $(L_{T_t}(R_{1,\alpha})\ ,\ t \geq 0)$ is 1-self-similar.

On the other hand, the pair $(R_{1,\alpha}, M(R_{1,\alpha}))$ is strong Markov (see Doney-Warren-Yor [5, p. 239]). As

$$R_{1,\alpha}(u) = M_u(R_{1,\alpha}) = t \quad \text{if} \quad u = T_t(R_{1,\alpha})\,,$$

the fact that $(L_{T_t}(R_{1,\alpha})\ ,\ t \geq 0)$ is an additive process follows from standard arguments.

Finally, we need to prove:

$$R_N^2(1) \overset{\text{d}}{=} L_{T_1}(R_{1,\alpha_N})\,.$$

We denote below R_{1,α_N} by R, and $L_t(R)$, $T_t(R)$, $M_t(R) \cdots$ are simply denoted respectively by L_t, T_t, $M_t \cdots$ As a particular case of the "balayage formula" (Yor [22]) we deduce from equation (1), that:

$$\exp(-\lambda\, L_t)\, R_t \;=\; \int_0^t \exp(-\lambda\, L_s)\, \mathrm{d}R_s$$
$$= \int_0^t \exp(-\lambda\, L_s)\, \mathrm{d}B_s \;+\; \frac{1-\exp(-\lambda\, L_t)}{2\,\lambda} + \alpha_N \int_0^t \exp(-\lambda\, L_s)\, \mathrm{d}M_s\,.$$

Hence,

$$\begin{aligned}\exp(-\lambda\, L_t)\,(1+2\,\lambda\, R_t) \;&=\; 1+2\,\lambda \int_0^t \exp(-\lambda\, L_s)\, \mathrm{d}B_s \\ &\quad +2\,\lambda\,\alpha_N \int_0^t \exp(-\lambda\, L_s)\, \mathrm{d}M_s\,.\end{aligned}$$

By time changing, we get:

$$\int_0^{T_t} \exp(-\lambda\, L_s)\, \mathrm{d}M_s = \int_0^t \exp(-\lambda\, L_{T_u})\, \mathrm{d}u\,.$$

Therefore, the optional stopping theorem yields:

$$\mathbb{E}[\exp(-\lambda\, L_{T_t})]\,(1+2\,\lambda\, t) = 1+2\,\lambda\,\alpha_N \int_0^t \mathbb{E}[\exp(-\lambda\, L_{T_u})]\, \mathrm{d}u\,.$$

Setting

$$\varphi_\lambda(t) = \mathbb{E}[\exp(-\lambda\, L_{T_t})]\,,$$

we obtain:

$$\varphi_\lambda(t) = \frac{1}{1+2\,\lambda\, t} + \frac{2\,\lambda\,\alpha_N}{1+2\,\lambda\, t} \int_0^t \varphi_\lambda(u)\, \mathrm{d}u\,.$$

Consequently

$$\varphi_\lambda(t) = (1+2\,\lambda\, t)^{-N/2}\,.$$

Therefore,

$$\mathbb{E}[\exp(-\lambda\, L_{T_1})] = (1+2\,\lambda)^{-N/2} = \mathbb{E}[\exp(-\lambda\, R_N^2(1)]\,,$$

which proves the desired result.

□

4 About the process $\left(\int_0^t R_N^2(s)\,\mathrm{d}s\ ,\ t \geq 0\right)$

4.1 A class of Sato processes

Let $(\ell_t\ ,\ t \geq 0)$ be the local time in 0 of a linear Brownian motion $(B_t\ ,\ t \geq 0)$ starting from 0. We denote, as usual, by $(\tau_t\ ,\ t \geq 0)$ the inverse of this local time:

$$\tau_t = \inf\{s \geq 0\ ;\ \ell_s > t\}\,.$$

Proposition 4.1 *Let $f(x,u)$ be a Borel function on $\mathbb{R}_+ \times \mathbb{R}_+$ such that*

$$\forall t > 0 \qquad \int\int_{\mathbb{R}_+ \times [0,t]} |f(x,u)|\,\mathrm{d}x\,\mathrm{d}u < \infty\,. \tag{2}$$

Then the process $A^{(f)}$ defined by:

$$A_t^{(f)} = \int_0^{\tau_t} f(|B_s|, \ell_s)\,\mathrm{d}s \quad , \quad t \geq 0$$

is an integrable additive process. Furthermore,

$$\mathbb{E}[A_t^{(f)}] = 2\int\int_{\mathbb{R}_+ \times [0,t]} f(x,u)\,\mathrm{d}x\,\mathrm{d}u\,.$$

Proof

Assume first that f is nonnegative. Then,

$$A_t^{(f)} = \sum_{0 \leq u \leq t} \int_{\tau_{u-}}^{\tau_u} f(|B_s|, u)\,\mathrm{d}s\,.$$

By the theory of excursions (Revuz-Yor [18, Chapter XII, Proposition 1.10]) we have

$$\mathbb{E}[A_t^{(f)}] = \int_0^t \mathrm{d}u \int n(\mathrm{d}\varepsilon) \int_0^{V(\varepsilon)} \mathrm{d}s\ f(|\varepsilon_s|, u)$$

where n denotes the Itô measure of Brownian excursions and $V(\varepsilon)$ denotes the life time of the excursion ε. The entrance law under n is given by:

$$n(\varepsilon_s \in \mathrm{d}x\ ;\ s < V(\varepsilon)) = (2\pi s^3)^{-1/2}\,|x|\,\exp(-x^2/(2s))\,\mathrm{d}x\,.$$

Therefore

$$\mathbb{E}[A_t^{(f)}] = 2\int_0^t \mathrm{d}u \int_0^\infty \mathrm{d}x\ f(x,u)\,.$$

The additivity of the process $A^{(f)}$ follows easily from the fact that, for any $t \geq 0$, $(B_{\tau_t+s}\ ,\ s \geq 0)$ is a Brownian motion starting from 0, which is independent of $\mathcal{B}_{\tau_t}$ (where $(\mathcal{B}_u)$ is the natural filtration of B).

□

Corollary 4.1.1 *We assume that f is a Borel function on $\mathbb{R}_+ \times \mathbb{R}_+$ satisfying (2) and which is m-homogeneous for $m > -2$, meaning that*

$$\forall a > 0,\ \forall (x,u) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad f(ax, au) = a^m f(x,u)\,.$$

Then the process $A^{(f)}$ is a $(m+2)$-Sato process.

Proof

This is a direct consequence of the scaling property of the Brownian motion.

□

4.2 A particular case

Let $N > 0$. We denote by $A^{(N)}$ the process $A^{(f)}$ with

$$f(x,u) = \frac{N^2}{4}\, 1_{(x \leq \frac{2}{N}\, u)}\,.$$

By Proposition 4.1, $(A_t^{(N)})$ is an integrable process and

$$\mathbb{E}[A_t^{(N)}] = \frac{N\, t^2}{2}\,.$$

We now consider the process Y_N defined by

$$Y_N(t) = \int_0^t R_N^2(s)\, \mathrm{d}s \quad , \quad t \geq 0\,.$$

Theorem 4.2 *The process $A^{(N)}$ is a 2-Sato process and*

$$(Y_N(t)) \overset{(1.\mathrm{d})}{=} (A_t^{(N)})\,.$$

Proof

It is a direct consequence of Corollary 4.2.1 that $A^{(N)}$ is a 2-Sato process.

By Mansuy-Yor [16, Theorem 3.4, p.38], the following extension of the Ray-Knight theorem holds:
For any $u > 0$,

$$(L_{\tau_u}^{a-(2u/N)}\ ,\ 0 \leq a \leq (2u/N)) \overset{(\mathrm{d})}{=} (R_N^2(a)\ ,\ 0 \leq a \leq (2u/N))$$

where L_t^x denotes the local time of the semi-martingale $(|B_s| - \frac{2}{N}\,\ell_s\ ,\ s \geq 0)$ in x at time t.
We remark that

$$s \in [0, \tau_t] \Longrightarrow |B_s| - \frac{2}{N}\,\ell_s \geq -\frac{2t}{N}\,.$$

Therefore, the occupation times formula entails:

$$A_t^{(N)} = \frac{N^2}{4} \int_{-2t/N}^{0} L_{\tau_t}^x\ \mathrm{d}x = \frac{N^2}{4} \int_0^{2t/N} L_{\tau_t}^{x-(2t/N)}\ \mathrm{d}x\,.$$

Thus, by the above mentioned extension of the Ray-Knight theorem,

$$(A_t^{(N)}) \overset{(1.\mathrm{d})}{=} \left(\frac{N^2}{4} \int_0^{2t/N} R_N^2(s)\ \mathrm{d}s \right)\,.$$

The scaling property of R_N also yields the identity in law:

$$(A_t^{(N)}) \overset{(1.\mathrm{d})}{=} \left(\int_0^t R_N^2(s)\ \mathrm{d}s \right)\,,$$

and the result follows from the definition of Y_N.

□

We may now apply Proposition 2.3 to get:

Corollary 4.2.1 *The process V_N defined by:*

$$V_N(t) = Y_N(t) - \frac{Nt^2}{2} \quad , \quad t \geq 0$$

is a PCOC and an associated martingale is M_N defined by:

$$M_N(t) = A_t^{(N)} - \frac{Nt^2}{2} \quad , \quad t \geq 0\,.$$

Moreover, M_N is a centered 2-Sato process.

4.3 Representation of $A^{(N)}$ as a process of hitting times

Theorem 4.3 *The process $A^{(N)}$ is identical in law to the process*

$$T_t(R_{1,\alpha_N}) \quad , \quad t \geq 0$$

where R_{1,α_N} denotes the perturbed Bessel process defined in Subsection 3.3 and

$$T_t(R_{1,\alpha_N}) = \inf\{s\ ;\ R_{1,\alpha_N}(s) > t\}\,.$$

The proof can be found in Le Gall-Yor [14]. Nevertheless, for the convenience of the reader, we give again the proof below. A more general result, based on Doney-Warren-Yor [5], shall also be stated in the next section.

Proof

In this proof, we adopt the following notation: (B_t) still denotes a standard linear Brownian motion starting from 0, $S_t = \sup_{0\leq s\leq t} B_s$ and $\sigma_t = \inf\{s\ ;\ B_s > t\}$. Moreover, for $a < 1$ and $t \geq 0$, we set

$$X_t^a = \int_0^t 1_{(B_s > a\,S_s)}\,\mathrm{d}s \quad \text{and} \quad Z_t^a = \inf\{s\ ;\ X_s^a > t\}\,.$$

Lemma 4.3.1 *Let $a < 1$. Then*

$$\sup_{0\leq s\leq t}(B_s - a\,S_s)^+ = (1-a)\,S_t\,.$$

Proof

Since $a < 1$, we have, for $0 \leq s \leq t$,

$$(B_s - a\,S_s)^+ \leq (1-a)\,S_s \leq (1-a)\,S_t\,.$$

Moreover, there exists $s_t \in [0,t]$ such that $B_{s_t} = S_t$ and therefore $S_{s_t} = S_t$. Hence, $B_{s_t} - a\,S_{s_t} = (1-a)\,S_t$.

□

Lemma 4.3.2 *Let $a < 1$ and $\alpha = -a/(1-a)$. We set*

$$R_t^a = (B_t - a\,S_t)^+ \quad \text{and} \quad U_t^a = R_{Z_t^a}^a\,.$$

Then the processes U^a and $R_{1,\alpha}$ are identical in law.

Proof

By Tanaka's formula,

$$R_t^a = \int_0^t 1_{(R_s^a>0)}\, \mathrm{d}R_s^a + \frac{1}{2}\, L_t(R^a)$$

where $L_t(R^a)$ denotes the local time of the semi-martingale R^a in 0 at time t. Now,

$$\int_0^t 1_{(R_s^a>0)}\, \mathrm{d}R_s^a = \int_0^t 1_{(B_s - a\, S_s>0)}\, \mathrm{d}(B_s - a\, S_s)\,.$$

If $s > 0$ belongs to the support of $\mathrm{d}S_s$, then $B_s = S_s$ and, since $a < 1$, $B_s - a\, S_s > 0$. Therefore,

$$R_t^a = \int_0^t 1_{(B_s - a\, S_s>0)}\, \mathrm{d}B_s - a\, S_t + \frac{1}{2}\, L_t(R^a)\,.$$

By Lemma 4.3.1, $-a\, S_t = \alpha\, M_t(R^a)$ where

$$M_t(R^a) = \sup_{0\le s\le t} R_s^a\,.$$

Consequently,

$$U_t^a = \int_0^{Z_t^a} 1_{(B_s - a\, S_s>0)}\, \mathrm{d}B_s + \frac{1}{2}\, L_{Z_t^a}(R^a) + \alpha\, M_{Z_t^a}(R^a)\,.$$

The process

$$\int_0^{Z_t^a} 1_{(B_s - a\, S_s>0)}\, \mathrm{d}B_s \quad , \quad t \ge 0$$

is a continuous martingale whose bracket is t, therefore it is a Brownian motion.

On the other hand, it is easy to see that

$$L_{Z_t^a}(R^a) = L_t(U^a) \quad \text{and} \quad M_{Z_t^a}(R^a) = M_t(U^a)\,.$$

Therefore, the process U^a is a solution to equation (1), which obviously is continuous and nonnegative.

□

By Lévy's theorem, the process $A^{(N)}$ is identical in law to the process

$$\frac{N^2}{4} \int_0^{\sigma_t} 1_{(B_s>(1-\frac{2}{N})\, S_s)}\, \mathrm{d}s \quad , \quad t \ge 0\,.$$

By the scaling property of B, the above process has the same law as

$$\int_0^{\sigma_{Nt/2}} 1_{(B_s > (1-\frac{2}{N})\, S_s)}\, \mathrm{d}s = X^{1-\frac{2}{N}}_{\sigma_{Nt/2}} \quad , \quad t \geq 0\,.$$

Now,

$$X^{1-\frac{2}{N}}_{\sigma_{Nt/2}} = \inf\{X^{1-\frac{2}{N}}_u \ ; \ S_u > \frac{Nt}{2}\}$$

and, by Lemma 4.3.1,

$$X^{1-\frac{2}{N}}_{\sigma_{Nt/2}} = \inf\{X^{1-\frac{2}{N}}_u \ ; \ R^{1-\frac{2}{N}}_u > t\} = \inf\{v \ ; \ U^{1-\frac{2}{N}}_v > t\}\,.$$

The result then follows from Lemma 4.3.2.

□

Corollary 4.3.1 *The process*

$$T_t(R_{1,\alpha_N}) \quad , \quad t \geq 0$$

is a 2-Sato process and

$$\left(\int_0^t R_N^2(s)\, \mathrm{d}s\right) \overset{(1.\mathrm{d})}{=} \left(T_t(R_{1,\alpha_N})\right)\,.$$

5 About the process $\left(\frac{1}{K^2}\int_0^t s^{\frac{2(1-K)}{K}}\, R_N^2(s)\, \mathrm{d}s\ ,\ t \geq 0\right)$

In this section we extend Corollary 4.3.1. We fix two positive real numbers N and K. We first recall some important results on general perturbed Bessel processes $R_{K,\alpha}$ with $\alpha < 1$.

5.1 Perturbed Bessel processes

We follow, in this subsection, Doney-Warren-Yor [5]. We first recall the definition of the process $R_{K,\alpha}$ with $K > 0$ and $\alpha < 1$.
The case $K = 1$ was already introduced in Subsection 3.3. For $K > 1$, $R_{K,\alpha}$ is defined as a continuous nonnegative solution to

$$R_t = B_t + \frac{K-1}{2}\int_0^t \frac{1}{R_s}\, \mathrm{d}s + \alpha\, M_t(R)\,, \qquad (3)$$

and, for $0 < K < 1$, $R_{K,\alpha}$ is defined as the square root of a continuous nonnegative solution to

$$X_t = 2 \int_0^t \sqrt{X_s}\, \mathrm{d}B_s + K\,t + \alpha\, M_t(X)\,. \tag{4}$$

We note that, for any $K > 0$, $(R_{K,0}(t)) \overset{(\mathrm{d})}{=} (R_K(t))$. As in the case $K = 1$, for any $K > 0$, the pair $(R_{K,\alpha}, M(R_{K,\alpha}))$ is strong Markov.
We denote, as before,

$$T_t(R_{K,\alpha}) = \inf\{s\ ;\ R_{K,\alpha}(s) > t\}\,.$$

The following theorem, due to Doney-Warren-Yor ([5, Theorem 5.2, p. 246]) is an extension of the Ciesielski-Taylor theorem and of the Ray-Knight theorem.
Theorem 5.1 *1)*

$$\int_0^\infty 1_{(R_{K+2,\alpha}(s) \leq 1)}\, \mathrm{d}s \overset{\mathrm{d}}{=} T_1(R_{K,\alpha})$$

2)

$$(L_\infty^a(R_{K+2,\alpha})\ ,\ a \geq 0) \overset{(\mathrm{d})}{=} (\frac{a^{1-K}}{K}\, R_{2(1-\alpha)}(a^K)\ ,\ a \geq 0)$$

5.2 Identification of the Sato process associated to $Y_{N,K}$

We denote, for $N > 0$ and $K > 0$, by $Y_{N,K}$ the process:

$$Y_{N,K}(t) = \frac{1}{K^2} \int_0^t s^{\frac{2(1-K)}{K}}\, R_N^2(s)\, \mathrm{d}s \quad , \quad t \geq 0\,.$$

We also recall the notation:

$$\alpha_N = 1 - \frac{N}{2}\,.$$

Theorem 5.2 *The process*

$$T_{t^{1/K}}(R_{K,\alpha_N}) \quad , \quad t \geq 0$$

is a $(2/K)$-Sato process and

$$(Y_{N,K}(t)) \overset{(1.\mathrm{d})}{=} (T_{t^{1/K}}(R_{K,\alpha_N}))\,.$$

Proof

In the following proof, we denote R_{K,α_N} simply by R, and we set T_t and M_t for, respectively, $T_t(R)$ and $M_t(R)$.

The first part of the statement follows from the (1/2)-self-similarity of R and from the strong Markovianity of (R, M), taking into account that, for any $t \geq 0$,

$$R_{T_t} = M_{T_t} = t \, .$$

By occupation times formula, we deduce from 1) in Theorem 5.1,

$$\int_0^1 L_\infty^x(R_{K+2,\alpha_N}) \, \mathrm{d}x \stackrel{\mathrm{d}}{=} T_1 \, .$$

Using then 2) in Theorem 5.1, we obtain:

$$\int_0^1 L_\infty^x(R_{K+2,\alpha_N}) \, \mathrm{d}x \stackrel{\mathrm{d}}{=} \int_0^1 \frac{x^{1-K}}{K} \, R_N^2(x^K) \, \mathrm{d}x \, .$$

By change of variable, the last integral is equal to $Y_{N,K}(1)$, and hence,

$$Y_{N,K}(1) \stackrel{\mathrm{d}}{=} T_1 \, .$$

The final result now follows by self-similarity.

□

Corollary 5.2.1 *The process*

$$V_{N,K}(t) := Y_{N,K}(t) - \frac{N}{2K} \, t^{2/K} \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$M_{N,K}(t) := T_{t^{1/K}}(R_{K,\alpha_N}) - \frac{N}{2K} \, t^{2/K} \quad , \quad t \geq 0 \, ,$$

which is a centered $(2/K)$-Sato process.

Finally, we have proven, in particular, that for any $\rho > -2$ and any $N > 0$, the random variable

$$\int_0^1 s^\rho \, R_N^2(s) \, \mathrm{d}s$$

is self-decomposable. This result will be generalized and made precise in the next section, using completely different arguments.

6 About the random variables $\int R_N^2(s)\,\mathrm{d}\mu(s)$

In this section, we fix a measure μ on $\mathbb{R}_+^* = (0,\infty)$ such that

$$\int_{\mathbb{R}_+^*} s\,\mathrm{d}\mu(s) < \infty\,.$$

6.1 Spectral study of an operator

We associate with μ an operator $K^{(\mu)}$ on $E = L^2(\mu)$ defined by

$$\forall f \in E \quad K^{(\mu)} f(t) = \int_{\mathbb{R}_+^*} f(s)\, t \wedge s\,\mathrm{d}\mu(s)$$

where $\wedge$ denotes the infimum. Though the spectral study of this operator is certainly classical, we give the details for the convenience of the reader.

Lemma 6.1 *The operator $K^{(\mu)}$ is a nonnegative symmetric Hilbert-Schmidt operator.*

Proof

As a consequence of the obvious inequality:

$$(t\wedge s)^2 \leq t\,s\,,$$

we get

$$\int\int_{(\mathbb{R}_+^*)^2} (t\wedge s)^2\,\mathrm{d}\mu(t)\,\mathrm{d}\mu(s) \leq \left(\int_{\mathbb{R}_+^*} s\,\mathrm{d}\mu(s)\right)^2\,,$$

and therefore $K^{(\mu)}$ is a Hilbert-Schmidt operator.
On the other hand, denoting by $(\bullet,\bullet)_E$ the scalar product in E, we have:

$$(K^{(\mu)} f, g)_E = \mathbb{E}\left[\int f(t)\,B_t\,\mathrm{d}\mu(t)\,\int g(t)\,B_t\,\mathrm{d}\mu(t)\right]$$

where B is a standard Brownian motion starting from 0. This entails that $K^{(\mu)}$ is nonnegative symmetric.

□

Lemma 6.2 *Let $\lambda \in \mathbb{R}$. Then λ is an eigenvalue of $K^{(\mu)}$ if and only if $\lambda > 0$ and there exists $f \in L^2(\mu)$, $f \neq 0$, such that:*

i)

$$\lambda f'' + f \cdot \mu = 0 \quad \textit{in the distribution sense on } \mathbb{R}_+^* \tag{5}$$

ii) f *admits a representative which is absolutely continuous on* $\mathbb{R}_+$*,* f' *admits a representative which is right-continuous on* $\mathbb{R}_+^*$*;*
(In the sequel, f and f' respectively always denote such representatives.)

iii)

$$f(0) = 0 \quad \textit{and} \quad \lim_{t\to\infty} f'(t) = 0 \,.$$

Proof
Let $f \in L^2(\mu)$ and $g = K^{(\mu)} f$. We have, for μ-a.e. $t > 0$,

$$g(t) = \int_0^t \mathrm{d}u \int_{(u,\infty)} f(s)\, \mathrm{d}\mu(s) \,. \tag{6}$$

Thus g admits a representative (still denoted by g) which is absolutely continuous on $\mathbb{R}_+$ and $g(0) = 0$. Moreover, g' admits a representative which is right-continuous on $\mathbb{R}_+^*$ and is given by:

$$g'(t) = \int_{(t,\infty)} f(s)\, \mathrm{d}\mu(s) \,. \tag{7}$$

In particular

$$|g'(t)| \leq t^{-1/2} \left[\int_{(t,\infty)} f^2(s)\, \mathrm{d}\mu(s) \int_{(t,\infty)} s\, \mathrm{d}\mu(s) \right]^{1/2} \,. \tag{8}$$

Hence:

$$\lim_{t\to\infty} g'(t) = 0 \,.$$

Besides, (7) entails:

$$g'' + f \cdot \mu = 0 \quad \text{in the distribution sense on } \mathbb{R}_+^* \,.$$

Consequently, 0 is not an eigenvalue of $K^{(\mu)}$ and the “only if” part is proven.
Conversely, let $f \in L^2(\mu)$, $f \neq 0$, and $\lambda > 0$ such that properties i),ii),iii) hold. Then

$$\lambda f'(t) = \int_{(t,\infty)} f(s)\, \mathrm{d}\mu(s) \,.$$

Hence

$$\lambda f(t) = \int_0^t \mathrm{d}u \int_{(u,\infty)} f(s)\, \mathrm{d}\mu(s) = K^{(\mu)} f(t) \,,$$

which proves the “if” part.

□

We note that, since 0 is not an eigenvalue of $K^{(\mu)}$, $K^{(\mu)}$ is actually a *positive* symmetric operator. On the other hand, by the previous proof, the functions $f \in L^2(\mu)$, $f \neq 0$, satisfying properties i),ii),iii) in the statement of Lemma 6.2, are the eigenfunctions of the operator $K^{(\mu)}$ corresponding to the eigenvalue $\lambda > 0$.

Lemma 6.3 *Let f be an eigenfunction of $K^{(\mu)}$. Then,*

$$|f(t)| = o(t^{1/2}) \quad \textit{and} \quad |f'(t)| = o(t^{-1/2})$$

when t tends to ∞.

Proof

This is a direct consequence of (8).

□

Lemma 6.4 *Let f_1 and f_2 be eigenfunctions of $K^{(\mu)}$ with respect to the same eigenvalue. Then,*

$$\forall t > 0 \quad f_1'(t)\, f_2(t) - f_1(t)\, f_2'(t) = 0\,.$$

Proof

By (5),

$$(f_1'\, f_2 - f_1\, f_2')' = 0 \quad \text{in the sense of distributions on } \mathbb{R}_+^*\,.$$

By right-continuity, there exists $C \in \mathbb{R}$ such that

$$\forall t > 0 \quad f_1'(t)\, f_2(t) - f_1(t)\, f_2'(t) = C\,.$$

Letting t tend to ∞, we deduce from Lemma 6.3 that $C = 0$.

□

Lemma 6.5 *Let f be a solution of (5) with $\lambda > 0$, and let $a > 0$. We assume as previously that f (resp. f') denotes the representative which is absolutely continuous (resp. right-continuous) on $\mathbb{R}_+^*$. If $f(a) = f'(a) = 0$, then, for any $t \geq a$, $f(t) = 0$.*

Proof

This lemma is quite classical if the measure μ admits a density with respect to the Lebesgue measure. The proof may be easily adapted to this more general case.

□

We are now able to state the main result of this section.

Theorem 6.6 *The operator $K^{(\mu)}$ is a positive symmetric compact operator whose all eigenvalues are simple (i.e. the dimension of the eigenspaces is 1).*

Proof

It only remains to prove that the eigenvalues are simple. Let then $\lambda > 0$ be an eigenvalue and let f_1 and f_2 be eigenfunctions with respect to this eigenvalue. Let $a > 0$ with $\mu(\{a\}) = 0$. By Lemma 6.4, there exist c_1 and c_2 with $c_1^2 + c_2^2 > 0$ such that, setting $f = c_1 f_1 + c_2 f_2$, we have

$$f(a) = f'(a) = 0 .$$

By Lemma 6.5, $f(t) = 0$ for any $t \geq a$. But, since $\mu(\{a\}) = 0$, f' also is left-continuous at a. Then, we may reason on $(0, a]$ as on $[a, \infty)$ and therefore we also have $f(t) = 0$ for $0 < t \leq a$. Finally,

$$c_1 f_1 + c_2 f_2 = 0 ,$$

which proves the result.

□

In the following, we denote by $\lambda_1 > \lambda_2 > \cdots$ the decreasing sequence (possibly finite) of the eigenvalues of $K^{(\mu)}$. Of course, this sequence depends on μ, which we omit in the notation. As $K^{(\mu)}$ is Hibert-Schmidt,

$$\sum_{n\geq 1} \lambda_n^2 < \infty .$$

It will be shown in Subsection 6.3 (see Theorem 6.7) that actually

$$\sum_{n\geq 1} \lambda_n < \infty ,$$

i.e. $K^{(\mu)}$ is trace-class. The following corollary plays an essential role in the sequel.

Corollary 6.6.1 *There exists a Hilbert basis $(f_n)_{n\geq 1}$ in $L^2(\mu)$ such that*

$$\forall n \geq 1 \quad K^{(\mu)} f_n = \lambda_n f_n .$$

6.2 Examples

In this subsection, we consider two particular types of measures μ.

6.2.1 $\mu = \sum_{j=1}^n a_j \, \delta_{t_j}$

Let $a_1, \cdots, a_n$ positive real numbers and $0 < t_1 < \cdots < t_n$. We denote by δ_t the Dirac measure at t and we consider, in this paragraph,

$$\mu = \sum_{j=1}^{n} a_j \, \delta_{t_j} \, .$$

By the previous study, the sequence of eigenvalues of $K^{(\mu)}$ is finite if and only if the space $L^2(\mu)$ is finite dimensional, that is if μ is of the above form. In this case, the eigenvalues of $K^{(\mu)}$ are the eigenvalues of the matrix $(m_{i,j})_{1 \le i,j \le n}$ with

$$m_{i,j} = \sqrt{a_i \, a_j} \, t_{i \wedge j} \, .$$

In particular, by the previous study, such a matrix has n distinct eigenvalues, which are > 0.

6.2.2 $\mu = C \, t^\rho \, 1_{(0,1]}(t) \, \mathrm{d}t$

In this paragraph, we consider

$$\mu = C \, t^\rho \, 1_{(0,1]}(t) \, \mathrm{d}t$$

with $C > 0$ and $\rho > -2$. By Lemma 6.2, the eigenfunctions f of $K^{(\mu)}$ associated with $\lambda > 0$ are characterized by:

$$\lambda \, f''(x) + C \, x^\rho \, f(x) = 0 \quad \text{on} \quad (0,1) \, , \tag{9}$$

$$f(0) = 0 \quad , \quad f'(1) = 0 \, .$$

We set $\sigma = (\rho + 2)^{-1}$ and $\nu = \sigma - 1$. For $a > -1$, we recall the definition of the Bessel function J_a:

$$J_a(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \, (x/2)^{a+2k}}{k! \, \Gamma(a + k + 1)} \, .$$

Then, the only function f satisfying (9) and $f(0) = 0$ is, up to a multiplicative constant,

$$f(x) = x^{1/2} \, J_\sigma \left(2\sigma \sqrt{\frac{C}{\lambda}} \, x^{1/2\sigma} \right) \, .$$

We deduce from the equality, valid for $a > 1$,

$$a\, J_a(x) + x\, J_a'(x) = x\, J_{a-1}(x)$$

that $f'(1) = 0$ if and only if

$$J_\nu\left(2\sigma\sqrt{\frac{C}{\lambda}}\right) = 0\,.$$

Denote by $(j_{\nu,k}\ ,\ k \geq 1)$ the sequence of the positive zeros of J_ν. Then the sequence $(\lambda_k\ ,\ k \geq 1)$ of eigenvalues of $K^{(\mu)}$ is given by:

$$\lambda_k = 4\, C\, (\nu + 1)^2\, j_{\nu,k}^{-2} \quad , \quad k \geq 1\,.$$

Particular case Suppose $\rho = 0$. Then $\nu = -1/2$ and

$$J_\nu(x) = J_{-1/2}(x) = \left(\frac{2}{\pi\, x}\right)^{1/2} \cos(x)\,.$$

Hence,

$$\lambda_k = 4\, C\, \pi^{-2}\, (2k - 1)^{-2} \quad , \quad k \geq 1\,.$$

6.3 Representation of $\int B_s^2\, \mathrm{d}\mu(s)$

We again consider the general setting defined in Subsection 6.1, the notation of which we keep.

In this subsection, we study the random variable

$$Y_1^{(\mu)} := \int B_s^2\, \mathrm{d}\mu(s)\,.$$

The use of the operator $K^{(\mu)}$ and of its spectral decomposition in the type of study we develop below, is called the Karhunen-Loeve decompositions method. It has a long history which goes back at least to Kac-Siegert [10, 11]. We also refer to the recent paper [4] and to the references therein.

Theorem 6.7 *The eigenvalues $(\lambda_k\ ,\ k \geq 1)$ of the operator $K^{(\mu)}$ satisfy*

$$\sum_{k \geq 1} \lambda_k = \int_{\mathbb{R}_+^*} t\, \mathrm{d}\mu(t)\ (< \infty\ ,\ \textit{by hypothesis})\,.$$

Moreover, there exists a sequence $(\Gamma_n\ ,\ n \geq 1)$ of independent normal variables such that:

$$Y_1^{(\mu)} \stackrel{\mathrm{d}}{=} \sum_{n \geq 1} \lambda_n\, \Gamma_n^2\,.$$

Proof

We deduce from Corollary 6.6.1, by the Bessel-Parseval equality,

$$Y_1^{(\mu)} = \sum_{n \geq 1} \left(\int B_s \, f_n(s) \, \mathrm{d}\mu(s) \right)^2 \quad \text{a.s.}$$

Taking the expectation, we get

$$\int_{\mathbb{R}_+^*} t \, \mathrm{d}\mu(t) = \sum_{n \geq 1} (K^{(\mu)} f_n, f_n)_E = \sum_{n \geq 1} \lambda_n \, .$$

We set, for $n \geq 1$,

$$\Gamma_n = \frac{1}{\sqrt{\lambda_n}} \int B_s \, f_n(s) \, \mathrm{d}\mu(s) \, .$$

Then $(\Gamma_n \, , \, n \geq 1)$ is a Gaussian sequence and

$$\mathbb{E}[\Gamma_n \, \Gamma_m] = \frac{1}{\sqrt{\lambda_n \, \lambda_m}} \, (K^{(\mu)} f_n, f_m)_E = \delta_{n,m}$$

where $\delta_{n,m}$ denotes Kronecker's symbol. Hence, the result follows.

□

Corollary 6.7.1 *The Laplace transform of* $Y_1^{(\mu)}$ *is*

$$F_1^{(\mu)}(t) = \prod_{n \geq 1} (1 + 2 \, t \, \lambda_n)^{-1/2} \, .$$

Proof

This is a direct consequence of the previous theorem, taking into account that, if Γ is a normal variable, then

$$\Gamma^2 \stackrel{\mathrm{d}}{=} 2 \, \gamma_{1/2} \, .$$

□

6.4 Representation of $\int R_N^2(s)\, \mathrm{d}\mu(s)$

We now consider the random variable

$$Y_N^{(\mu)} := \int R_N^2(s)\, \mathrm{d}\mu(s)\,.$$

Theorem 6.8 *There exists a sequence* $(\Theta_{N,n}\ ,\ n \geq 1)$ *of independent variables with, for any* $n \geq 1$,

$$\Theta_{N,n} \stackrel{\mathrm{d}}{=} R_N^2(1) \stackrel{\mathrm{d}}{=} 2\,\gamma_{N/2}$$

such that

$$Y_N^{(\mu)} \stackrel{\mathrm{d}}{=} \sum_{n \geq 1} \lambda_n\, \Theta_{N,n}\,. \tag{10}$$

Moreover, the Laplace transform of $Y_N^{(\mu)}$ *is*

$$F_N^{(\mu)}(t) = \prod_{n \geq 1} (1 + 2\, t\, \lambda_n)^{-N/2}\,. \tag{11}$$

Proof

It is clear, for instance from Revuz-Yor [18, Chapter XI, Theorem 1.7], that

$$F_N^{(\mu)}(t) = [F_1^{(\mu)}(t)]^N\,.$$

Therefore, (11) holds and (10) follows directly.

□

Corollary 6.8.1 *The random variable* $Y_N^{(\mu)}$ *is self-decomposable. The function* h*, which is decreasing on* $(0, \infty)$ *and associated with* $Y_N^{(\mu)}$ *in Theorem 2.1, is*

$$h(x) = \frac{N}{2} \sum_{n \geq 1} \exp\left(-\frac{1}{2\lambda_n}\, x\right)\,.$$

As a consequence, following Bondesson [1], we see that $Y_N^{(\mu)}$ is a generalized gamma convolution (GGC) whose Thorin measure is the discrete measure:

$$\frac{N}{2} \sum_{n \geq 1} \delta_{1/2\lambda_n}\,.$$

Particular case We consider here, as in Section 5, the particular case:

$$\mu = \frac{1}{K^2}\, t^{\frac{2(1-K)}{K}}\, 1_{(0,1]}(t)\, \mathrm{d}t\,.$$

Then, $Y_N^{(\mu)}$ is the random variable $Y_{N,K}(1)$ studied in Section 5. As a consequence of Paragraph 6.2.2 with

$$C = \frac{1}{K^2} \quad \text{and} \quad \rho = \frac{2}{K} - 2\,,$$

we have

$$\lambda_k = j_{\nu,k}^{-2} \quad , \quad k \geq 1$$

with $\nu = \frac{K}{2} - 1$. Moreover, by Theorem 5.2,

$$Y_2^{(\mu)} \overset{\mathrm{d}}{=} T_1(R_K)\,.$$

It is known (see for instance Borodin-Salminen [2, formula 2.0.1, p. 387]) that

$$\mathbb{E}[\exp(-t\, T_1(R_K))] = \frac{2^{-\nu}}{\Gamma(\nu+1)}\, \frac{(\sqrt{2t})^{\nu}}{I_\nu(\sqrt{2t})}$$

where I_ν denotes the modified Bessel function:

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\,\Gamma(\nu+k+1)}\,.$$

We set:

$$\widehat{I}_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\,\Gamma(\nu+k+1)}\,.$$

Therefore, by formula (11) in the case $N = 2$, we recover the following representation:

$$\widehat{I}_\nu(x) = \frac{1}{\Gamma(\nu+1)} \prod_{k\geq 1} \left(1 + \frac{x^2}{j_{\nu,k}^2}\right)\,.$$

In particular ($\nu = -1/2$),

$$\cosh(x) = \prod_{k\geq 1} \left(1 + \frac{4\,x^2}{\pi^2\,(2k-1)^2}\right)\,.$$

Likewise we obtain, for $\nu = 1/2$,

$$\frac{\sinh(x)}{x} = \prod_{k\geq 1} \left(1 + \frac{x^2}{\pi^2\,k^2}\right)\,.$$

6.5 Sato process associated to $Y_N^{(\mu)}$

Theorem 6.9 *Let (U_t^N) be the 1-Sato process associated to $R_N^2(1)$ (cf. Section 3). Then, the 1-Sato process associated to $Y_N^{(\mu)}$ is $(U_t^{(N,\mu)})$ defined by:*

$$U_t^{(N,\mu)} = \sum_{n\geq 1} \lambda_n \, U_t^{N,n} \quad , \quad t \geq 0$$

where $((U_t^{N,n})\ ,\ n \geq 1)$ denotes a sequence of independent processes such that, for $n \geq 1$,

$$(U_t^{N,n}) \overset{(\mathrm{d})}{=} (U_t^N)\,.$$

Proof

This is a direct consequence of Theorem 6.8.

□

Corollary 6.9.1 *The process*

$$V_t^{(N,\mu)} := \int_{\mathbb{R}_+^*} (R_N^2(t\,s) - N\,t\,s)\, \mathrm{d}\mu(s) \quad , \quad t \geq 0$$

is a PCOC and an associated martingale is

$$M_t^{(N,\mu)} := U_t^{(N,\mu)} - N\,t \int_{\mathbb{R}_+^*} s\, \mathrm{d}\mu(s) \quad , \quad t \geq 0\,.$$

The above martingale $(M_t^{(N,\mu)})$ is purely discontinuous. We also may associate to the PCOC $(V_t^{(N,\mu)})$ a continuous martingale, as we now state.

Theorem 6.10 *A continuous martingale associated to the PCOC $(V_t^{(N,\mu)})$ is*

$$\sum_{n\geq 1} \lambda_n \, ((R_N^{(n)})^2(t) - Nt) \quad , \quad t \geq 0$$

where $((R_N^{(n)}(t))\ ,\ n \geq 1)$ denotes a sequence of independent processes such that, for $n \geq 1$,

$$(R_N^{(n)}(t)) \overset{(\mathrm{d})}{=} (R_N(t))\,.$$

Proof

This is again a direct consequence of Theorem 6.8.

□

We can also explicit the relation between $U^{(N,\mu)}$ and $U^{(N',\mu)}$. Let $C^{(N,\mu)}$ (resp. $C^{(N',\mu)}$) be the Lévy process associated with $Y_N^{(\mu)}$ (resp. $Y_{N'}^{(\mu)}$). We see, by Laplace transform, that

$$(C_s^{(N',\mu)}) \stackrel{(\mathrm{d})}{=} (C_{N's/N}^{(N,\mu)}) \, .$$

Then, using the relations between the processes U and C given in Theorem 2.2, we obtain:

Proposition 6.11 *We have:*

$$(U_t^{(N',\mu)} \ , \ t \geq 0) \stackrel{(\mathrm{d})}{=} \left(\int_0^{t^{N'/N}} s^{\frac{N-N'}{N'}} \ \mathrm{d}U_s^{(N,\mu)} \ , \ t \geq 0 \right) \, .$$

Corollary 6.11.1 *For $N > 0$ and $K > 0$, we set, with the notation of Section 5,*

$$T_t^{N,K} = T_t(R_{K,\alpha_N}) \quad , \quad t \geq 0 \, .$$

Then, for $N > 0$, $N' > 0$ and $K > 0$, for any $t \geq 0$,

$$T_t^{N',K} = \int_0^{t^{N'/N}} s^{2\,\frac{N-N'}{N'}} \ \mathrm{d}T_s^{N,K} \, .$$

Proof

By Theorem 5.2, $(T_{t^{1/2}}^{N,K})$ is the 1-Sato process associated with $Y_N^{(\mu)}$ defined from

$$\mu = \frac{1}{K^2} \, t^{\frac{2(1-K)}{K}} \ 1_{(0,1]}(t) \ \mathrm{d}t \, .$$

□

7 Some negative results

7.1 Squared Bessel process started from $x > 0$

Let $Y_u^{N,x}$ be the value at time $u > 0$ of the squared Bessel process of dimension $N \geq 0$, starting from $x \geq 0$.

Proposition 7.1 *The random variable $Y_u^{N,x}$ is self-decomposable if and only if $x^2 \leq N\,u$.*

Proof

One has:

$$\mathbb{E}[\exp(-t\,Y_u^{N,x}) = (1+2tu)^{-N/2}\,\exp\left(-\frac{x^2\,t}{1+2tu}\right).$$

It is then easy to see that $Y_u^{N,x}$ is infinitely divisible and its Lévy measure admits on $(0,\infty)$ the density

$$\varphi(y) = \left(N + \frac{x^2}{2u^2}\,y\right)\,\frac{1}{2y}\,\mathrm{e}^{-y/2u}.$$

Hence, the result follows from the characterisation 1) in Theorem 2.1.

□

7.2 Pairs of values of a squared Bessel process

We now consider, for $N > 0$, $t_1, t_2 > 0$, the $\mathbb{R}^2$-valued random variable

$$Y := (R_N^2(t_1), R_N^2(t_1+t_2)).$$

(Recall that $R_N(0) = 0$.)
For such an $\mathbb{R}^2$-valued random variable, we can also define the notion of self-decomposability as in Section 2. Theorem 2.1, suitably modified, is still valid.

Proposition 7.2 *The $\mathbb{R}^2$-valued random variable Y is not self-decomposable.*

Proof

An easy computation gives

$$\mathbb{E}[\exp(-\lambda_1\,R_N^2(t_1) - \lambda_2\,R_N^2(t_1+t_2))] = [P(\lambda)]^{-N/2}$$

with

$$P(\lambda) = 1 + 2\lambda_1 t_1 + 2\lambda_2(t_1 + t_2) + 4\lambda_1\lambda_2 t_1 t_2\,.$$

If Y were self-decomposable, we would have

$$\log(P(\lambda)) = \int\int_{(\mathbb{R}_+^*)^2} (1 - \exp(-\lambda_1 x_1 - \lambda_2 x_2))\, \frac{H(x_1, x_2)}{x_1^2 + x_2^2}\, \mathrm{d}x_1\, \mathrm{d}x_2$$

with H a decreasing function on each half line with origin $(0, 0)$. Taking the derivative with respect to λ_1, we get

$$\frac{2t_1(1 + 2\lambda_2 t_2)}{P(\lambda)} = \int\int_{(\mathbb{R}_+^*)^2} \exp(-\lambda_1 x_1 - \lambda_2 x_2)\, \frac{x_1\, H(x_1, x_2)}{x_1^2 + x_2^2}\, \mathrm{d}x_1\, \mathrm{d}x_2\,.$$

Letting λ_2 tend to ∞, we obtain

$$\frac{2t_1 t_2}{t_1 + t_2 + 2\lambda_1 t_1 t_2} = 0$$

which yields a contradiction.

□

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