

Unifying constructions of martingales associated with processes increasing in the convex order, via Lévy and Sato sheets

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Abstract: In this paper, we present a unified framework for our previous constructions of martingales with the same one-dimensional marginals as particular cases of processes increasing in the convex order. This framework encompasses our former uses of Lévy sheets, Sato sheets and self-decomposable laws. New examples of processes increasing in the convex order are also exhibited, but we do not know how to associate to them martingales with the same one-dimensional marginals.

Key words: convex order; 1-martingale; temporal and spatial PCOC's; Lévy sheet; Sato sheet; self-decomposable law; space-time harmonic function.

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1 Introduction

We first introduce some notation which will be used throughout our paper.

If A and B are two random variables,

$$A \stackrel{d}{=} B$$

means that these variables have the same law.

If $(X_\alpha, \alpha \in I)$ and $(Y_\alpha, \alpha \in I)$ are two families of random variables (I being an interval of \mathbb{R}),

$$(X_\alpha, \alpha \in I) \stackrel{(1.d)}{=} (Y_\alpha, \alpha \in I)$$

means that the families $(X_\alpha, \alpha \in I)$ and $(Y_\alpha, \alpha \in I)$ have the same one-dimensional marginals, that is, for any fixed $\alpha \in I$, $X_\alpha \stackrel{d}{=} Y_\alpha$.

If $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two processes,

$$(X_t, t \geq 0) \stackrel{(d)}{=} (Y_t, t \geq 0)$$

means that the two processes are identical in law.

1.1 PCOC's and 1-martingales

Families of random variables $(Y_\alpha, \alpha \in I)$, with I an interval of \mathbb{R} , which are monotone for some order relation, play important roles in a number of applications in probability and statistics, as may be seen, for instance, from the books by Shaked and Shanthikumar [17, 18], and many research articles.

In this paper, we are concerned with the *convex order*: an \mathbb{R} -valued (resp. a \mathbb{C} -valued) family $(Y_\alpha, \alpha \in I)$ increases in the convex order if

$$\forall \alpha \in I \quad \mathbb{E}[|Y_\alpha|] < \infty,$$

and for every convex function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ (resp. $\varphi : \mathbb{C} \rightarrow \mathbb{R}$),

$$\alpha \in I \rightarrow \mathbb{E}[\varphi(Y_\alpha)] \in (-\infty, +\infty]$$

is increasing.

We call such a family $(Y_\alpha, \alpha \in I)$ a PCOC, this acronym being derived from the French name: Processus Croissant pour l'Ordre Convexe.

A family $(Y_\alpha, \alpha \in I)$ is called a *1-martingale* if there exists (on a suitable filtered probability space) a martingale $(M_\alpha, \alpha \in I)$ which has the same one-dimensional marginals as $(Y_\alpha, \alpha \in I)$, that is

$$(Y_\alpha, \alpha \in I) \stackrel{(1.d)}{=} (M_\alpha, \alpha \in I).$$

Such a martingale $(M_\alpha, \alpha \in I)$ is said to be *associated* to this family $(Y_\alpha, \alpha \in I)$. Note that several different martingales may be associated to a given family.

It is an easy consequence of Jensen's inequality that an \mathbb{R} -valued or a \mathbb{C} -valued family $(Y_\alpha, \alpha \in I)$ which is a 1-martingale, is a PCOC. A remarkable result due to Kellerer [11] states that, conversely, any \mathbb{R} -valued family $(Y_\alpha, \alpha \in I)$ which is a PCOC, is a 1-martingale. But generally, it is a difficult problem to give a concrete description of a martingale which is associated to a PCOC; this is the aim of several recent papers ([2, 7, 8, 9, 12]) and of this paper too.

1.2 The guiding example of the Asian process

Our interest in the study of PCOC's and associated martingales originated from the result by Carr, Ewald and Xiao [4] that the process:

$$(1) \quad A_t = \frac{1}{t} \int_0^t \mathcal{E}_s \, ds \quad , \quad s \geq 0 ,$$

where $\mathcal{E}_s = \exp(B_s - \frac{s}{2})$ with $(B_s, s \geq 0)$ a standard Brownian motion, is a PCOC. The proof in [4] uses the following equality for one-dimensional marginals:

$$(A_t, t \geq 0) \stackrel{(1.d)}{=} \left(\int_0^1 \exp\left(\sqrt{t} B_u - \frac{tu}{2}\right) \, du, t \geq 0 \right) ,$$

and, in fact, it is shown in [4] that the process:

$$\left(\tilde{A}_\alpha := \int_0^1 \exp\left(\alpha B_u - \frac{\alpha^2 u}{2}\right) \, du, \alpha \geq 0 \right)$$

is a PCOC. This change of variables: $t \longrightarrow \alpha = \sqrt{t}$ incited us, in the present paper, to complete our previous discussions in [7], [8] and [9], by considering "temporal" as well as "spatial" PCOC's, that is PCOC's indexed by time $t \geq 0$, versus PCOC's indexed by a "dilation" parameter $\alpha \geq 0$, in a unified framework.

1.3 Lévy sheets and Sato sheets

Baker and Yor [2] showed that $(A_t, t \geq 0)$ defined by (1) is a PCOC by introducing the Wiener sheet $(W_{u,t}; u, t \geq 0)$ and using the (elementary, but essential) fact that:

$$(2) \quad \text{for every } t \geq 0, \quad (B_{ut}, u \geq 0) \stackrel{(d)}{=} (W_{u,t}, u \geq 0) .$$

Hirsch and Yor ([7, 8]) extended the discussion in Baker and Yor [2], in a number of directions replacing, in particular, Brownian motion by a general Lévy process L , and W by the Lévy sheet (see Dalang and Walsh [5] and references therein) associated with L ; in particular, (2) extends to this set-up.

In a further paper ([9]), Hirsch and Yor used the fact that, for Y a self-decomposable random variable, then

$$(\alpha Y, \alpha \geq 0) \stackrel{(1.d)}{=} (S_\alpha, \alpha \geq 0)$$

where $(S_\alpha, \alpha \geq 0)$ is the Sato process with independent increments and scaling property of index 1, attached to Y (see [15]). If Y is centered, then $(S_\alpha, \alpha \geq 0)$ is a martingale, which is associated to the PCOC $(\alpha Y, \alpha \geq 0)$. In [9], many such processes are constructed explicitly.

In the present paper, this link between self-decomposability and Sato processes is developed at the level of a self-decomposable Lévy process $(L_t, t \geq 0)$, to which we attach a Sato sheet $(S_{\alpha,t}; \alpha, t \geq 0)$ which satisfies:

- i) for every $\alpha \geq 0$, $(\alpha L_t, t \geq 0) \stackrel{(d)}{=} (S_{\alpha,t}, t \geq 0)$;
- ii) the process $(S_{\alpha,\bullet}, \alpha \geq 0)$ has independent increments and scaling property of index 1.

1.4 A unified framework

The main purpose of the present paper is to build a unified framework involving both the Lévy sheets and Sato sheets in order to create a large class of PCOC's and their associated martingales, which encompasses all of our previous constructions.

1.5 Organization of the paper

We now detail the organization of the remainder of the paper.

Section 2: We present a unified framework in which we show how “temporal” and “spatial” PCOC's are found. Three examples, which correspond respectively to Lévy sheets, Sato sheets, and the more restricted framework of [9], are presented, and shall provide the canvas of our developments in the sequel of the paper.

Section 3: Starting from space-time harmonic functions of the Markov process $(X(\alpha), \alpha \in I)$ which underlies the unified framework of Section 2, we construct many PCOC's and associated martingales.

Section 4: We obtain a number of important properties of vector spaces of PCOC's and associated martingales constructed from the semigroup of the Markov process $(X(\alpha), \alpha \in [0, 1])$.

Sections 5 and 6: They are devoted respectively to the development of the previous results, when applied to Example 2 and Example 3 in Section 2.

Section 7: We consider again some families of processes studied in the previous sections. By a direct proof, we show that these processes are PCOC's, even without making the self-decomposability assumption. But then, we do not know how to construct associated martingales. This section may be read independently of the previous ones.

2 A unified framework

2.1 Hypotheses and notation

Let E be a Polish space equipped with its Borel σ -algebra and with a measure vector space structure.

We consider Y an E -valued random variable and we denote its law by \mathbb{P} .

We also consider a family $(\varphi_\alpha)_{\alpha \geq 0}$ of linear maps from E into E satisfying:

$$\varphi_1 = \text{id}_E \quad , \quad \varphi_0 = 0 \quad ,$$

$$\forall \alpha, \beta \geq 0 \quad \varphi_\alpha \circ \varphi_\beta = \varphi_{\alpha\beta} \quad .$$

We set $Y(\alpha) = \varphi_\alpha(Y)$ and we denote by \mathbb{P}_α the law of $Y(\alpha)$. In particular, $Y(1) = Y$ and $\mathbb{P}_1 = \mathbb{P}$.

Fundamental assumption We assume that there exists an E -valued process $(X(\alpha), \alpha \geq 0)$ such that:

H_1) the process $(X(\alpha), \alpha \geq 0)$ has independent increments,

H_2) $X(1) \stackrel{d}{=} Y$,

H_3) for every $\beta \geq 0$,

$$(\varphi_\beta(X(\alpha)), \alpha \geq 0) \stackrel{(d)}{=} (X(\alpha\beta), \alpha \geq 0).$$

We denote the filtration of the process $(X(\alpha), \alpha \geq 0)$ by $(\mathcal{X}_\alpha)_{\alpha \geq 0}$:

$$\mathcal{X}_\alpha = \sigma\{X(\beta); \beta \leq \alpha\}.$$

Remark that a consequence of H_2) and H_3) is:

$$(3) \quad \forall \alpha \geq 0 \quad X(\alpha) \stackrel{d}{=} Y(\alpha).$$

2.2 Examples

Example 1 The first example is the framework of the paper by Hirsch and Yor [8]. We take as space E the Skorohod space \mathbb{D}_0^n of càdlàg functions from \mathbb{R}_+ into \mathbb{R}^n , vanishing at 0. The generic element of E is denoted by $\varepsilon = (\varepsilon(t), t \geq 0)$. We also use the alternative notation: $\varepsilon = (\varepsilon_t, t \geq 0)$. We consider $L = (L_t, t \geq 0)$ a Lévy process in \mathbb{R}^n , starting from 0, viewed as an E -valued random variable, and we set $Y = L$.

Now, the family $(\varphi_\alpha, \alpha \geq 0)$ is defined by

$$\varphi_\alpha(\varepsilon) = \varepsilon(\alpha \bullet),$$

which means:

$$\forall t \geq 0 \quad \varphi_\alpha(\varepsilon)(t) = \varepsilon(\alpha t).$$

As a consequence,

$$Y(\alpha) = L_{\alpha \bullet} \quad (\text{i.e.: } \forall t \geq 0 \quad Y_t(\alpha) = L_{\alpha t}).$$

We now verify that the fundamental assumption in Subsection 2.1 is satisfied. Following for instance Dalang and Walsh [5, Section 2] (see also Adler, Monrad, Scissors and Wilson [1]), we introduce as in Hirsch and Yor [8, Section 2], the *Lévy sheet associated with L* , which we denote by \widehat{L} .

Theorem 2.1 *There exists an \mathbb{R}^n -valued two-parameter process*

$$\widehat{L} = (\widehat{L}_{s,t}; s \geq 0, t \geq 0)$$

satisfying the following properties:

1)

$$\forall s, t \geq 0, \quad \widehat{L}_{s,0} = \widehat{L}_{0,t} = 0 .$$

2) Almost surely, for any $s, t \geq 0$, $\widehat{L}_{s,\bullet}$ and $\widehat{L}_{\bullet,t}$ are càdlàg functions on \mathbb{R}_+ .

3) Let, for $t \geq 0$,

$$\mathcal{L}_t = \sigma\{\widehat{L}_{u,v} ; u \geq 0, 0 \leq v \leq t\} .$$

Then, for $0 \leq t_1 \leq t_2$, the process $(\widehat{L}_{s,t_2} - \widehat{L}_{s,t_1}, s \geq 0)$ is a Lévy process starting from 0, independent of \mathcal{L}_{t_1} , and which is distributed as $(L_{(t_2-t_1)s}, s \geq 0)$.

4) The two-parameter processes:

$$(\widehat{L}_{s,t} ; s, t \geq 0) \quad \text{and} \quad (\widehat{L}_{t,s} ; s, t \geq 0)$$

have the same law. Thus, 3) may be stated with the roles of s and t exchanged.

We then define the process $(X(\alpha), \alpha \geq 0)$ by:

$$\forall \alpha \geq 0 \quad X(\alpha) = \widehat{L}_{\bullet,\alpha} .$$

Properties $H_1)$ and $H_2)$ follow directly from the properties of \widehat{L} stated in the above theorem. Let $\beta \geq 0$. Then, by Property 3) in Theorem 2.1,

$$(4) \quad (\varphi_\beta(X(\alpha)) = \widehat{L}_{\beta\bullet,\alpha}, \alpha \geq 0) \quad \text{and} \quad (X(\alpha\beta) = \widehat{L}_{\bullet,\alpha\beta}, \alpha \geq 0)$$

are two E -valued processes with independent increments, and, for $0 \leq \alpha_1 \leq \alpha_2$,

$$(\widehat{L}_{\beta s, \alpha_2} - \widehat{L}_{\beta s, \alpha_1}, s \geq 0) \stackrel{(d)}{=} (L_{(\alpha_2 - \alpha_1)\beta s}, s \geq 0) \stackrel{(d)}{=} (\widehat{L}_{s, \alpha_2\beta} - \widehat{L}_{s, \alpha_1\beta}, s \geq 0),$$

which shows that the increments of both processes in (4) have the same law. Consequently, $H_3)$ is satisfied too.

The general results which will be presented in the present paper, were obtained in the particular framework of this Example 1 in the previously mentioned paper [8]. The more particular framework where L is Brownian motion was first treated in Hirsch and Yor [7].

Example 2 The space E is the same as in Example 1. We also keep the same notation.

Here again, we consider $L = (L_t, t \geq 0)$ a Lévy process in \mathbb{R}^n , starting from 0, viewed as an E -valued random variable, and we set $Y = L$.

Now, the family $(\varphi_\alpha, \alpha \geq 0)$ is defined by

$$\varphi_\alpha(\varepsilon) = \alpha \varepsilon,$$

that is:

$$\forall t \geq 0 \quad \varphi_\alpha(\varepsilon)(t) = \alpha \varepsilon(t).$$

As a consequence,

$$Y(\alpha) = \alpha L. \quad (\text{i.e: } \forall t \geq 0 \quad Y_t(\alpha) = \alpha L_t).$$

In order to obtain a process $(X(\alpha), \alpha \geq 0)$ satisfying the fundamental assumption, we furthermore assume:

(SD) L_1 is a self-decomposable \mathbb{R}^n -valued random variable.

We recall below some general facts concerning the notion of self-decomposability. We refer for background, complements and references, to Sato [15, Chapter 3].

An \mathbb{R}^n -valued random variable R is said to be *self-decomposable* if, for each c with $0 < c < 1$, there is the equality in law:

$$R \stackrel{d}{=} cR + \widehat{R}_c$$

for some variable \widehat{R}_c independent of R .

On the other hand, an *additive process* $(U_t, t \geq 0)$ is a stochastically continuous process with càdlàg paths, independent increments, which satisfies $U_0 = 0$.

An additive process (U_t) with scaling property of index 1, meaning that, for each $c > 0$, $(U_{ct}) \stackrel{(d)}{=} (cU_t)$, will be called a *Sato process*.

The following theorem, for which we refer to Sato's book [15, Chapter 3, Sections 16-17], gives two characterizations of the self-decomposability property.

Theorem 2.2 *Let R be an \mathbb{R}^n -valued random variable. Then, R is self-decomposable if and only if one of the following equivalent properties is satisfied:*

1) *There exists an \mathbb{R}^n -valued Lévy process $(C_t, t \geq 0)$ such that*

$$R \stackrel{d}{=} \int_0^\infty e^{-u} dC_u.$$

2) There exists a Sato process $(U_t, t \geq 0)$ such that $R \stackrel{d}{=} U_1$.

In 1) (resp. 2)) the Lévy process (C_t) (resp. the Sato process (U_t)) is uniquely determined in law by R , and will be said to be *associated* with R .

The construction of the process $(X(\alpha), \alpha \geq 0)$ is inspired from Jeanblanc, Pitman and Yor [10, Theorem 1]. Let $(C_t, t \geq 0)$ be the Lévy process associated with the self-decomposable random variable L_1 by Property 1) in Theorem 2.2. We denote by $(\widehat{C}_{s,t}; s \geq 0, t \geq 0)$ the Lévy sheet associated with (C_t) by Theorem 2.1. We define $(\widehat{C}_{s,t}; s \geq 0, t \leq 0)$ as an independent copy of $(-\widehat{C}_{s,-t}; s \geq 0, t \leq 0)$. Thus, we have now defined $(\widehat{C}_{s,t}; s \geq 0, t \in \mathbb{R})$. By Property 3) in Theorem 2.1, we obtain easily:

Proposition 2.1 *The E -valued process $(\widehat{C}_{\bullet,t}, t \in \mathbb{R})$ has independent increments and, for every $t_1, t_2 \in \mathbb{R}$ such that $t_1 \leq t_2$, the process*

$$(\widehat{C}_{s,t_2} - \widehat{C}_{s,t_1}, s \geq 0)$$

is a Lévy process starting from 0 and distributed as

$$(C_{(t_2-t_1)s}, s \geq 0).$$

Hence, $(\widehat{C}_{\bullet,t}, t \in \mathbb{R})$ is an E -valued Lévy process indexed by \mathbb{R} .

Corollary 2.1 *For any $a \in \mathbb{R}$, the E -valued processes*

$$(\widehat{C}_{\bullet,t+a} - \widehat{C}_{\bullet,a}, t \in \mathbb{R}) \quad \text{and} \quad (\widehat{C}_{\bullet,t}, t \in \mathbb{R})$$

have the same law.

We then define the *Sato sheet* : $(S_{\alpha,t}; \alpha, t \geq 0)$, associated to the Lévy process (L_t) satisfying (SD), by:

$$(5) \quad S_{\alpha,t} = \int_{-\log \alpha}^{+\infty} e^{-u} d_u \widehat{C}_{t,u}.$$

Theorem 2.3 *The Sato sheet : $(S_{\alpha,t}; \alpha, t \geq 0)$, associated to the Lévy process (L_t) , satisfies the following properties:*

1) For every $\alpha \geq 0$,

$$(\alpha L_t, t \geq 0) \stackrel{(d)}{=} (S_{\alpha,t}, t \geq 0).$$

2) The E -valued process $(S_{\alpha, \bullet}, \alpha \geq 0)$ has independent increments and scaling property of index 1.

Proof

By Property 4) in Theorem 2.1, the process $(S_{1,t}, t \geq 0)$ is a Lévy process and

$$S_{1,1} = \int_0^{+\infty} e^{-u} d_u \widehat{C}_{1,u} \stackrel{d}{=} \int_0^{+\infty} e^{-u} dC_u \stackrel{d}{=} L_1 .$$

Hence, Property 1) is satisfied for $\alpha = 1$.

Since, by Proposition 2.1, the E -valued process $(\widehat{C}_{\bullet,u}, u \in \mathbb{R})$ has independent increments, we deduce directly from the definition of the Sato sheet S that the E -valued process $(S_{\alpha, \bullet}, \alpha \geq 0)$ also has independent increments.

Let $c > 0$. We have by change of variables:

$$\begin{aligned} S_{c\alpha,t} &= \int_{-\log \alpha - \log c}^{+\infty} e^{-u} d_u \widehat{C}_{t,u} \\ &= c \int_{-\log \alpha}^{+\infty} e^{-v} d_v [\widehat{C}_{t,v-\log c} - \widehat{C}_{t,-\log c}] . \end{aligned}$$

The scaling property then follows from Corollary 2.1, and Property 1) is satisfied for every $\alpha \geq 0$. □

We now set

$$\forall \alpha \geq 0 \quad X(\alpha) = S_{\alpha, \bullet} .$$

Theorem 2.3 expresses exactly that the fundamental assumption is fulfilled, with $\varphi_\alpha(\varepsilon) = \alpha \varepsilon$.

In the particular case where L is the standard Brownian motion in \mathbb{R}^n , we see easily that the Sato sheet S associated to L may be defined by:

$$\forall \alpha, t \geq 0 \quad S_{\alpha,t} = W_{\alpha^2,t}$$

where $(W_{s,t}; s, t \geq 0)$ denotes the \mathbb{R}^n -valued Brownian sheet.

Example 3 We consider the following simpler framework.

E is the space \mathbb{R}^n whose generic element is denoted by $x = (x_1, \dots, x_n)$.

Y is an \mathbb{R}^n -valued self-decomposable random variable.

We set, for $\alpha \geq 0$, $\varphi_\alpha(x) = \alpha x$.

Let $(S_t, t \geq 0)$ be the Sato process associated to the self-decomposable random variable Y by Theorem 2.2. It is then obvious, by the definition of a Sato process, that setting:

$$\forall \alpha \geq 0 \quad X(\alpha) = S_\alpha ,$$

the fundamental assumption is fulfilled.

Many examples of such Sato processes have been exhibited in Hirsch and Yor [9].

In the particular case where Y is a normal \mathbb{R}^n -valued random variable, we have

$$(S_t, t \geq 0) \stackrel{(d)}{=} (B_{t^2}, t \geq 0)$$

where B denotes the standard Brownian motion in \mathbb{R}^n .

3 Space-time harmonic functions

In this section as well as in Section 4 below, we adopt the general hypotheses and notation stated in Subsection 2.1.

3.1 The semigroup associated with X

We first introduce some further notation.

As X is a process with independent increments, X is a Markov process, generally non homogeneous; we denote by

$$(Q_{\beta,\alpha}, 0 \leq \beta \leq \alpha)$$

its semigroup, defined by:

$$(6) \quad Q_{\beta,\alpha} \Phi(X(\beta)) = \mathbb{E} [\Phi(X(\alpha)) \mid \mathcal{X}_\beta] .$$

The following proposition is a direct consequence of the definition (6).

Proposition 3.1 *Suppose $0 \leq \beta \leq \alpha$ and $\Phi \in L^1(\mathbb{P}_\alpha)$. Then, $Q_{\beta,\alpha} \Phi \in L^1(\mathbb{P}_\beta)$ and*

$$\| Q_{\beta,\alpha} \Phi \|_{L^1(\mathbb{P}_\beta)} \leq \| \Phi \|_{L^1(\mathbb{P}_\alpha)} .$$

We also denote, for $0 \leq \beta \leq \alpha$, by $Z(\beta, \alpha)$ a random variable defined in law by:

$$Z(\beta, \alpha) \stackrel{d}{=} X(\alpha) - X(\beta) .$$

We recall that

$$Y(\alpha) \stackrel{d}{=} \varphi_\alpha(X(1)) \stackrel{d}{=} X(\alpha) .$$

Proposition 3.2 *The following properties hold.*

1) For $r \geq 0$ and $0 \leq \beta \leq \alpha$,

$$\varphi_r(Z(\beta, \alpha)) \stackrel{d}{=} Z(r\beta, r\alpha) .$$

2) For $0 \leq \beta \leq \alpha$,

$$Y(\alpha) \stackrel{d}{=} Y(\beta) + Z(\beta, \alpha)$$

with $Z(\beta, \alpha)$ independent of $Y(\beta)$.

3) For $0 \leq \gamma \leq \beta \leq \alpha$,

$$Z(\gamma, \beta) + Z(\beta, \alpha) \stackrel{d}{=} Z(\gamma, \alpha)$$

with $Z(\gamma, \beta)$ and $Z(\beta, \alpha)$ assumed to be independent.

Proof

Property 1) follows from the linearity of φ_r and from H_3 .

Taking (3) into account, we have by H_1):

$$Y(\beta) + Z(\beta, \alpha) \stackrel{d}{=} X(\beta) + (X(\alpha) - X(\beta)) = X(\alpha) \stackrel{d}{=} Y(\alpha) .$$

Property 3) also follows directly from H_1 . □

We may now give a useful expression for the semigroup $(Q_{\beta, \alpha})$.

Proposition 3.3 *If $0 \leq \beta \leq \alpha$ and $\Phi \in L^1(\mathbb{P}_\alpha)$, then for \mathbb{P}_β -almost every ε ,*

$$Q_{\beta, \alpha}\Phi(\varepsilon) = \mathbb{E}[\Phi(\varepsilon + Z(\beta, \alpha))] .$$

Proof

By H_1) we have:

$$\Phi(X(\alpha)) = \Phi(X(\beta) + Z(\beta, \alpha))$$

with $Z(\beta, \alpha)$ independent of \mathcal{X}_β . Therefore,

$$\mathbb{E}[\Phi(X(\alpha)) \mid \mathcal{X}_\beta] = \mathbb{E}_{Z(\beta, \alpha)}[\Phi(X(\beta) + Z(\beta, \alpha))]$$

where $\mathbb{E}_{Z(\beta, \alpha)}$ means integrating with respect to $Z(\beta, \alpha)$, $X(\beta)$ being frozen. □

3.2 Definition and general properties of space-time harmonic functions

Definition 3.1 Let I be an interval of \mathbb{R}_+ . A function

$$F : E \times I \longrightarrow \mathbb{C}$$

is called a *space-time harmonic function with respect to* $(X(\alpha), \alpha \in I)$ if the process $(F(X(\alpha), \alpha), \alpha \in I)$ is a (\mathcal{X}_α) -martingale.

We introduce the following notation. If I is an interval of \mathbb{R}_+ and F is a function on $E \times I$, we denote, for $\alpha \in I$, by $F^{(\alpha)}$ the function defined on E by

$$F^{(\alpha)}(\varepsilon) = F(\varepsilon, \alpha).$$

We also use this notation for classes of functions with respect to a.s. equality (under some probability).

As a direct consequence of the definitions, we obtain:

Proposition 3.4 *A function*

$$F : E \times I \longrightarrow \mathbb{C}$$

is a space-time harmonic function with respect to $(X(\alpha), \alpha \in I)$ *if and only if:*

i) for every $\alpha \in I$

$$F^{(\alpha)} \in L^1(\mathbb{P}_\alpha),$$

ii) for every $\alpha, \beta \in I$ *with* $\beta \leq \alpha$,

$$Q_{\beta, \alpha} F^{(\alpha)} = F^{(\beta)} \quad \mathbb{P}_\beta\text{-a.s.}$$

Theorem 3.1 *Let*

$$F : E \times I \longrightarrow \mathbb{C}$$

be a space-time harmonic function with respect to $(X(\alpha), \alpha \in I)$. *Then, for every* $r > 0$, *the function*

$$F_r : (\varepsilon, \alpha) \longrightarrow F(\varphi_r(\varepsilon), r\alpha)$$

is a space-time harmonic function with respect to $(X(\alpha), \alpha \in r^{-1}I)$.

Proof

We first remark that, by definition,

$$F_r(\varepsilon, \alpha) = F^{(r\alpha)} \circ \varphi_r(\varepsilon) .$$

We shall use the characterization given in Proposition 3.4.
Let $r\alpha \in I$. We have:

$$F_r(Y(\alpha), \alpha) = F(Y(r\alpha), r\alpha) .$$

Therefore

$$\| F_r^{(\alpha)} \|_{L^1(\mathbb{P}_\alpha)} = \| F^{(r\alpha)} \|_{L^1(\mathbb{P}_{r\alpha})} < \infty .$$

Suppose $r\alpha, r\beta \in I$ with $\beta \leq \alpha$. Then we have, with the notation of the proof of Proposition 3.3,

$$Q_{\beta, \alpha} F_r^{(\alpha)}(Y(\beta)) = \mathbb{E}_{Z(\beta, \alpha)} [F_r(Y(\beta) + Z(\beta, \alpha), \alpha)]$$

with $Z(\beta, \alpha)$ independent of Y . By Property 1) in Proposition 3.2,

$$F_r(Y(\beta) + Z(\beta, \alpha), \alpha) = F(Y(r\beta) + Z(r\beta, r\alpha), r\alpha)$$

with $Z(r\beta, r\alpha)$ independent of Y . Therefore, again by Proposition 3.3,

$$Q_{\beta, \alpha} F_r^{(\alpha)}(Y(\beta)) = Q_{r\beta, r\alpha} F^{(r\alpha)}(Y(r\beta)) .$$

Since F is a space-time harmonic function, the above RHS is equal to

$$F(Y(r\beta), r\beta) = F_r(Y(\beta), \beta) = F_r^{(\beta)}(Y(\beta)) .$$

Finally, the conditions of Proposition 3.4 are satisfied by the function F_r . □

In the sequel of this paper, the *signed* measures which may be involved are always assumed to be finite (bounded).

Corollary 3.1 *Let F be as in the previous theorem and suppose $\inf I = 0$. Let σ be a signed measure on $[0, 1]$. Then, the function:*

$$(\varepsilon, \alpha) \in E \times I \longrightarrow \int F(\varphi_r(\varepsilon), r\alpha) d\sigma(r)$$

is a space-time harmonic function with respect to $(X(\alpha), \alpha \in I)$.

3.3 A class of 1-martingales

The following proposition, which plays a central role in our constructions, actually is a straightforward consequence of Definition 3.1 and Property (3).

Proposition 3.5 *Let*

$$F : E \times I \longrightarrow \mathbb{C}$$

be a space-time harmonic function with respect to $(X(\alpha), \alpha \in I)$. Then, the process $(F(Y(\alpha), \alpha), \alpha \in I)$ is a 1-martingale (and therefore a PCOC), with associated martingale : $(F(X(\alpha), \alpha), \alpha \in I)$.

Likewise, we obtain directly from Corollary 3.1:

Proposition 3.6 *Let F be as in the previous proposition and suppose $\inf I = 0$. Let σ be a signed measure on $[0, 1]$. Then, the process:*

$$\left(\int F(Y(r, \alpha), r, \alpha) d\sigma(r), \alpha \in I \right)$$

is a 1-martingale, with associated martingale:

$$\left(\int F(\varphi_r(X(\alpha)), r, \alpha) d\sigma(r), \alpha \in I \right).$$

As a particular case corresponding to $I = (0, +\infty)$ and σ the Lebesgue measure on $[0, 1]$, we obtain the following result about “Asian type” processes. See Subsection 1.2 for the guiding example.

Corollary 3.2 *Let*

$$F : E \times (0, +\infty) \longrightarrow \mathbb{C}$$

be a space-time harmonic function with respect to $(X(\alpha), \alpha \in (0, +\infty))$. Then, the process:

$$\left(\frac{1}{\alpha} \int_0^\alpha F(Y(r), r) dr, \alpha > 0 \right)$$

is a 1-martingale (and therefore a PCOC), with associated martingale:

$$\left(\int_0^1 F(\varphi_r(X(\alpha)), r, \alpha) dr, \alpha > 0 \right).$$

4 Definition and properties of Φ^\sharp for $\Phi \in L^1(\mathbb{P})$

In this section, we restrict our attention to the case $I = [0, 1]$.

4.1 Definition of processes $\Phi^\sharp(Y)$

If $\Phi \in L^1(\mathbb{P})$, we define F^Φ on $E \times [0, 1]$ by:

$$(7) \quad \forall \alpha \in [0, 1], \quad F^\Phi(\varepsilon, \alpha) = Q_{\alpha,1}\Phi(\varepsilon) \quad \mathbb{P}_\alpha\text{-a.s.}$$

Proposition 4.1 *The function F^Φ is the only space-time harmonic function F with respect to $(X(\alpha), \alpha \in [0, 1])$, such that*

$$F(\varepsilon, 1) = \Phi(\varepsilon) \quad \mathbb{P}\text{-a.s.}$$

Proof

This is a direct consequence of Proposition 3.4 and of the semigroup property for $Q_{\alpha,\beta}$. □

If $\Phi \in L^1(\mathbb{P})$, we set, for $\alpha \in [0, 1]$,

$$(8) \quad \Phi_\alpha^\sharp = (Q_{\alpha,1}\Phi) \circ \varphi_\alpha \in L^1(\mathbb{P}) .$$

As a consequence of (7) and (8) we obtain:

$$(9) \quad \Phi_\alpha^\sharp(Y) = F^\Phi(Y(\alpha), \alpha) .$$

Proposition 4.2 *The process*

$$\Phi^\sharp(Y) = (\Phi_\alpha^\sharp(Y), \alpha \in [0, 1])$$

is a 1-martingale and therefore a PCOC. An associated martingale is:

$$(F^\Phi(X(\alpha), \alpha), \alpha \in [0, 1]) .$$

Proof

Using (9), the statement is a direct consequence of Proposition 4.1 and Proposition 3.5. □

4.2 Properties of the map $\Phi \longrightarrow \Phi^\sharp$

In the following theorem, we gather the main features of the map $\Phi \longrightarrow \Phi^\sharp$.

Theorem 4.1 *The following properties hold.*

1) For every $\Phi \in L^1(\mathbb{P})$,

$$\Phi_0^\sharp = \mathbb{E}(\Phi) \quad \text{and} \quad \Phi_1^\sharp = \Phi .$$

2) If $p \in [1, +\infty]$ and $\Phi \in L^p(\mathbb{P})$, then, for every $\alpha \in [0, 1]$,

$$\Phi_\alpha^\sharp \in L^p(\mathbb{P}) \quad \text{and} \quad \|\Phi_\alpha^\sharp\|_{L^p(\mathbb{P})} \leq \|\Phi\|_{L^p(\mathbb{P})} .$$

3) For every $\alpha, \beta \in [0, 1]$ and $\Phi \in L^1(\mathbb{P})$,

$$(\Phi_\alpha^\sharp)_\beta^\sharp = \Phi_{\alpha\beta}^\sharp .$$

Proof

We have, by the definition (8) and Proposition 3.3,

$$\Phi_0^\sharp = Q_{0,1}\Phi(0) = \mathbb{E}[\Phi(X(1))] = \mathbb{E}(\Phi)$$

and

$$\Phi_1^\sharp = Q_{1,1}\Phi = \Phi .$$

Using (9), we obtain that if $p \in [1, +\infty)$ and $\Phi \in L^p(\mathbb{P})$, then, for every $\alpha \in [0, 1]$,

$$\mathbb{E} [|\Phi_\alpha^\sharp(Y)|^p] = \mathbb{E} [|F^\Phi(X(\alpha), \alpha)|^p] .$$

Since F^Φ is a space-time harmonic function with respect to $(X(\alpha), \alpha \in [0, 1])$, the map:

$$\alpha \in [0, 1] \longrightarrow \|\Phi_\alpha^\sharp\|_{L^p(\mathbb{P})}$$

is increasing, which entails Property 2) for $p < \infty$, since $\Phi_1^\sharp = \Phi$.

The case $p = +\infty$ is obvious.

We have, by the definition (8) and Proposition 3.3,

$$\begin{aligned} (\Phi_\alpha^\sharp)_\beta^\sharp(Y) &= \mathbb{E}_{Z(\beta,1)} [\Phi_\alpha^\sharp(Y(\beta) + Z(\beta,1))] \\ &= \mathbb{E}_{Z(\beta,1), \tilde{Z}(\alpha,1)} \left[\Phi(Y(\alpha\beta) + \varphi_\alpha(Z(\beta,1)) + \tilde{Z}(\alpha,1)) \right] \end{aligned}$$

with $(Y, Z(\beta,1), \tilde{Z}(\alpha,1))$ independent. Now, by Proposition 3.2,

$$\varphi_\alpha(Z(\beta,1)) + \tilde{Z}(\alpha,1) \stackrel{d}{=} Z(\alpha\beta, \alpha) + \tilde{Z}(\alpha,1) \stackrel{d}{=} Z(\alpha\beta, 1) .$$

Thus,

$$(\Phi_\alpha^\#)_\beta^\#(Y) = \Phi_{\alpha\beta}^\#(Y) .$$

□

As a direct consequence of Property 3) in Theorem 4.1, there is the following Corollary which is nothing else but Proposition 3.6 for the space-time harmonic function F^Φ .

Corollary 4.1 *Let $\Phi \in L^1(\mathbb{P})$ and let σ be a signed measure on $[0, 1]$. We set*

$$\Psi = \int \Phi_r^\# d\sigma(r) .$$

Then

$$\Psi_\alpha^\# = \int \Phi_{r\alpha}^\# d\sigma(r) .$$

Consequently,

$$\left(\int \Phi_{r\alpha}^\#(Y) d\sigma(r) , \alpha \in [0, 1] \right)$$

is a 1-martingale.

4.3 The semigroup $(\Pi_u , u \geq 0)$

For $u \geq 0$ and $\Phi \in L^1(\mathbb{P})$, we set

$$\Pi_u \Phi = \Phi_{e^{-u}}^\# .$$

By Theorem 4.1, $(\Pi_u , u \geq 0)$ is a Markovian semigroup on $L^1(\mathbb{P})$. Theorem 4.2 below describes the associated Markov process.

We set, for $u \in \mathbb{R}$:

$$\mathcal{Y}_u = \varphi_{e^{-u}}(X(e^u)) .$$

Proposition 4.3 *The process*

$$(\mathcal{Y}_u , u \in \mathbb{R})$$

is an E -valued stationary process such that

$$\forall u \in \mathbb{R} \quad \mathcal{Y}_u \stackrel{d}{=} Y .$$

Proof

Let $u \in \mathbb{R}$. Then

$$\begin{aligned} (\mathcal{Y}_{u+v}, v \in \mathbb{R}) &= (\varphi_{e^{-(u+v)}}(X(e^{u+v})), v \in \mathbb{R}) \\ &\stackrel{(d)}{=} (\varphi_{e^{-(u+v)}}(\varphi_{e^u}(X(e^v))), v \in \mathbb{R}) \\ &= (\mathcal{Y}_v, v \in \mathbb{R}). \end{aligned}$$

Since $\mathcal{Y}_0 = X(1) \stackrel{d}{=} Y$, the proof is complete. □

Theorem 4.2 *The process*

$$\{(\mathcal{Y}_u)_{u \geq 0}, (\mathcal{X}_{e^u})_{u \geq 0}\}$$

is a Markov process with state space E and semigroup (Π_u) , that is:

$$\forall u, v \geq 0, \forall \Phi \in L^1(\mathbb{P}), \quad \mathbb{E}[\Phi(\mathcal{Y}_{u+v}) \mid \mathcal{X}_{e^u}] = \Pi_v \Phi(\mathcal{Y}_u).$$

Proof

We have

$$\begin{aligned} \mathcal{Y}_{u+v} &= \varphi_{e^{-u-v}}(X(e^u) + X(e^{u+v}) - X(e^u)) \\ &= \varphi_{e^{-v}}(\mathcal{Y}_u) + Z(e^{-v}, 1) \end{aligned}$$

with $Z(e^{-v}, 1)$ independent of \mathcal{X}_{e^u} . Therefore

$$\begin{aligned} \mathbb{E}[\Phi(\mathcal{Y}_{u+v}) \mid \mathcal{X}_{e^u}] &= Q_{e^{-v}, 1} \Phi(\varphi_{e^{-v}}(\mathcal{Y}_u)) \\ &= \Phi_{e^{-v}}^\#(\mathcal{Y}_u) = \Pi_v \Phi(\mathcal{Y}_u). \end{aligned}$$

□

5 Study of Example 2

This section is devoted to a study of consequences of the previous general results, in the particular framework of Example 2 presented in Subsection 2.2. The general hypotheses and notation of the previous sections, as well as the particular ones introduced for Example 2 of Subsection 2.2, are in force throughout this section.

5.1 The process $Z(\beta, \alpha)$

We denote by \cdot the scalar product on \mathbb{R}^n . We denote by $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$, the characteristic exponent of the Lévy process L , defined by:

$$\mathbb{E}[\exp(i \lambda \cdot L_t)] = \exp(-t \psi(\lambda)) .$$

We also denote by $\tilde{\psi}$ the characteristic exponent of the Lévy process C associated to the self-decomposable random variable L_1 by Property 1) in Theorem 2.2. The following relation (10) between ψ and $\tilde{\psi}$ is an easy consequence of definitions.

$$(10) \quad \psi(\lambda) = \int_0^1 \tilde{\psi}(v \lambda) v^{-1} dv .$$

Proposition 5.1 *For $0 \leq \beta \leq \alpha$, $Z(\beta, \alpha) = (Z_t(\beta, \alpha), t \geq 0)$ is a Lévy process the characteristic exponent of which is:*

$$\lambda \in \mathbb{R}^n \rightarrow \psi(\alpha \lambda) - \psi(\beta \lambda) .$$

Proof

We have by (5):

$$Z_t(\beta, \alpha) = \int_{-\log \alpha}^{-\log \beta} e^{-u} d_u \hat{C}_{t,u} .$$

Therefore, $(Z_t(\beta, \alpha), t \geq 0)$ is a Lévy process the characteristic exponent of which is:

$$\begin{aligned} \int_{-\log \alpha}^{-\log \beta} \tilde{\psi}(e^{-u} \lambda) du &= \int_{\beta}^{\alpha} \tilde{\psi}(v \lambda) v^{-1} dv \\ &= \int_0^1 [\tilde{\psi}(v \alpha \lambda) - \tilde{\psi}(v \beta \lambda)] v^{-1} dv . \end{aligned}$$

The result then follows from (10). □

5.2 Some examples of \mathbb{C} -valued 1-martingales

We now introduce an important class of space-time harmonic functions.

Proposition 5.2 *Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ be a bounded Borel function with compact support. Then the function:*

$$H : (\varepsilon, \alpha) \in E \times \mathbb{R}_+ \longrightarrow \exp \left[\mathfrak{i} \int h(s) \cdot d\varepsilon(s) + \int \psi(\alpha h(s)) ds \right]$$

(where $\int h(s) \cdot d\varepsilon(s)$ denotes a stochastic integral defined \mathbb{P} -almost surely) is a space-time harmonic function with respect to $(X(\alpha), \alpha \geq 0)$.

Proof

By definition,

$$\| H^{(\alpha)} \|_{L^1(\mathbb{P}_\alpha)} = \exp \left[\int \Re \psi(\alpha h(s)) ds \right] < \infty .$$

On the other hand, for $0 \leq \beta \leq \alpha$,

$$H^{(\alpha)}(\varepsilon + Z(\beta, \alpha)) = H^{(\alpha)}(\varepsilon) \exp \left[\mathfrak{i} \int h(s) \cdot dZ_s(\beta, \alpha) \right] .$$

Therefore, by Proposition 3.3 and Proposition 5.1,

$$Q_{\beta, \alpha} H^{(\alpha)} = H^{(\alpha)} \exp \left[- \int [\psi(\alpha h(s)) - \psi(\beta h(s))] ds \right] = H^{(\beta)} .$$

It then suffices to apply Proposition 3.4. □

Applying Proposition 3.5, we obtain:

Corollary 5.1 *Let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}^n$ be a bounded Borel function with compact support. Then,*

$$\left(\exp \left[\mathfrak{i} \alpha \int h(s) \cdot dL_s + \int \psi(\alpha h(s)) ds \right] , \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\exp \left[\mathfrak{i} \int h(s) \cdot dX_s(\alpha) + \int \psi(\alpha h(s)) ds \right] , \alpha \geq 0 \right) .$$

In the previous corollary, it is possible to take integrals with respect to functions h . For instance, the following result holds.

Corollary 5.2 *Let σ be a signed measure on \mathbb{R}_+ such that*

$$\forall r > 0 \quad \int \exp(r u) \, d|\sigma|(u) < \infty .$$

Then, for every $\lambda \in \mathbb{R}^n$,

$$\left(\int \exp [i \alpha \lambda \cdot L_u + u \psi(\alpha \lambda)] \, d\sigma(u) , \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\int \exp [i \lambda \cdot X_u(\alpha) + u \psi(\alpha \lambda)] \, d\sigma(u) , \alpha \geq 0 \right) .$$

Corollary 5.3 *For $p \in [1, \infty)$ and $\Phi \in L^p(\mathbb{P})$, the map*

$$\alpha \in [0, 1] \longrightarrow \Phi_\alpha^\sharp \in L^p(\mathbb{P})$$

is continuous. As a consequence, the semigroup (Π_u) is strongly continuous in $L^p(\mathbb{P})$.

Proof

Suppose

$$\Phi(L) = \exp \left[i \int h(s) \cdot dL_s \right]$$

with h as before. Then, by Corollary 5.1,

$$\Phi_\alpha^\sharp(L) = \exp \left[i \alpha \int h(s) \cdot dL_s + \int [\psi(\alpha h(s)) - \psi(h(s))] \, ds \right] .$$

Consequently, for this function Φ , the map

$$\alpha \in [0, 1] \longrightarrow \Phi_\alpha^\sharp \in L^p(\mathbb{P})$$

is continuous. The general result follows by density. □

We end this subsection by a general result concerning stochastic integrals.

Proposition 5.3 *Let, for $0 < a < \infty$, $(K_t(L), t \in [0, a])$ be an \mathbb{R}^n -valued bounded L -predictable process. We set*

$$\Phi(L) = \int_0^a K_t(L) \cdot dL_t .$$

Then, for $\alpha \in [0, 1]$,

$$\Phi_\alpha^\sharp(L) = \alpha \int_0^a (K_t)_\alpha^\sharp(L) \cdot dL_t .$$

Proof

By (8) and Proposition 3.3,

$$\Phi_\alpha^\sharp(L) = \mathbb{E}_{Z(\alpha, 1)} \left[\int_0^a K_t(\alpha L + Z(\alpha, 1)) \cdot (\alpha dL_t + dZ_t(\alpha, 1)) \right]$$

with $Z(\alpha, 1)$ independent of L . This yields easily the desired result. \square

5.3 The Brownian setting

An interesting particular case is the case where, as a self-decomposable Lévy process L , we take B , the \mathbb{R}^n -valued Brownian motion starting from 0.

In this case, we rather take, as space E , the Wiener space $C_0(\mathbb{R}_+, \mathbb{R}^n)$ consisting of continuous functions from \mathbb{R}_+ into \mathbb{R}^n vanishing at 0; the generic element of E is also denoted by B .

\mathbb{P} is the Wiener measure.

We saw in Subsection 2.2 that

$$X(\alpha) = W_{\alpha^2, \bullet}$$

where $(W_{s,t}; s, t \geq 0)$ denotes the \mathbb{R}^n -valued Brownian sheet. Consequently, for $0 \leq \beta \leq \alpha$,

$$Z(\beta, \alpha) \stackrel{d}{=} \sqrt{\alpha^2 - \beta^2} B$$

and, by Proposition 3.3,

$$\forall \Phi \in L^1(\mathbb{P}), \quad Q_{\beta, \alpha} \Phi(\beta B) = \mathbb{E}_{\tilde{B}} \left[\Phi(\beta B + \sqrt{\alpha^2 - \beta^2} \tilde{B}) \right] \quad \mathbb{P}\text{-a.s.}$$

where \tilde{B} denotes an independent copy of B and $\mathbb{E}_{\tilde{B}}$ means integrating with respect to \tilde{B} .

We then deduce from (8) that, for $\Phi \in L^1(\mathbb{P})$ and $\alpha \in [0, 1]$,

$$\Phi_{\alpha}^{\sharp}(B) = \mathbb{E}_{\tilde{B}} \left[\Phi(\alpha B + \sqrt{1 - \alpha^2} \tilde{B}) \right] .$$

Consequently, for $\Phi \in L^1(\mathbb{P})$ and $u \geq 0$,

$$\Pi_u \Phi(B) = \mathbb{E}_{\tilde{B}} \left[\Phi(e^{-u} B + \sqrt{1 - e^{-2u}} \tilde{B}) \right] .$$

Thus, (Π_u) is the Ornstein-Uhlenbeck semigroup on the Wiener space, given by the Mehler formula. Besides,

$$\mathcal{Y}_u = e^{-u} W_{e^{2u}, \bullet} .$$

Hence, we recover a representation of the stationary E -valued Ornstein-Uhlenbeck process (see for example Meyer [13] and Fukushima [6, Formula (1.10)]).

In this Brownian setting, Proposition 5.2, Corollary 5.1 and Corollary 5.2 admit the following real valued versions.

Proposition 5.4 *Let $h \in L^2(\mathbb{R}_+; \mathbb{R}^n)$. Then the function:*

$$H : (B, \alpha) \in E \times \mathbb{R}_+ \longrightarrow \exp \left[\int_0^{\infty} h(s) \cdot dB_s - \frac{\alpha^2}{2} \int_0^{\infty} |h|^2(s) ds \right]$$

is a space-time harmonic function with respect to $(X(\alpha), \alpha \geq 0)$.

Corollary 5.4 *Let $h \in L^2(\mathbb{R}_+; \mathbb{R}^n)$. Then,*

$$\left(\exp \left[\alpha \int_0^{\infty} h(s) \cdot dB_s - \frac{\alpha^2}{2} \int_0^{\infty} |h|^2(s) ds \right], \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\exp \left[\int_0^{\infty} h(s) \cdot d_s W_{\alpha^2, s} - \frac{\alpha^2}{2} \int_0^{\infty} |h|^2(s) ds \right], \alpha \geq 0 \right) .$$

Corollary 5.5 *Let σ be a signed measure on \mathbb{R}_+ . Then, for every $\lambda \in \mathbb{R}^n$,*

$$\left(\int \exp \left[\alpha \lambda \cdot B_u - u |\lambda|^2 \frac{\alpha^2}{2} \right] d\sigma(u), \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\int \exp \left[\lambda \cdot W_{\alpha^2, u} - u |\lambda|^2 \frac{\alpha^2}{2} \right] d\sigma(u), \alpha \geq 0 \right) .$$

In the particular case: $n = 1$, $\lambda = 1$ and σ is the Lebesgue measure on $[0, 1]$, the above corollary yields the result of Carr, Ewald and Xiao [4] as explained in Subsection 1.2.

5.4 Further examples of \mathbb{R} -valued 1-martingales

Below we give further examples for which the results of Subsection 5.2 admit real valued versions.

Proposition 5.5 *Suppose that there exist $r > 0$ and $\lambda \in \mathbb{R}^n$ such that*

$$\forall v \geq 0 \quad \mathbb{E}[\exp(v \lambda \cdot L_r)] < \infty .$$

Then, there exists a real valued function ϕ_λ on \mathbb{R}_+ such that

$$\forall v \geq 0, \forall t \geq 0, \quad \mathbb{E}[\exp(v \lambda \cdot L_t)] = \exp(t \phi_\lambda(v))$$

and the following properties hold.

1) *Let h be a nonnegative bounded Borel function on \mathbb{R}_+ with compact support. Then,*

$$\left(\exp \left[\alpha \int_0^{+\infty} h(s) \lambda \cdot dL_s - \int_0^{+\infty} \phi_\lambda(\alpha h(s)) ds \right], \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\exp \left[\int_0^{+\infty} h(s) \lambda \cdot dX_s(\alpha) - \int_0^{+\infty} \phi_\lambda(\alpha h(s)) ds \right], \alpha \geq 0 \right) .$$

2) *Let σ be a signed measure on \mathbb{R}_+ . Then,*

$$\left(\int \exp(\alpha \lambda \cdot L_u - u \phi_\lambda(\alpha)) d\sigma(u), \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\int \exp(\lambda \cdot X_u(\alpha) - u \phi_\lambda(\alpha)) d\sigma(u), \alpha \geq 0 \right) .$$

A particular case is: $n = 1$ and $L = (\tau_t, t \geq 0)$ is a subordinator satisfying (SD). We denote by ϕ the Laplace exponent of τ defined by:

$$\forall v \geq 0 \quad \mathbb{E}[\exp(-v \tau_t)] = \exp(-t \phi(v)) .$$

Then the assumption of Proposition 5.5 is fulfilled with

$$\lambda \leq 0 \quad \text{and} \quad \phi_\lambda(v) = -\phi(-\lambda v) .$$

6 Study of Example 3

This section is devoted to a study of the particular framework of Example 3 presented in Subsection 2.2. The general hypotheses and notation of the previous sections, as well as the particular ones introduced for Example 3 of Subsection 2.2, are in force throughout this section. Thus, Y denotes an \mathbb{R}^n -valued self-decomposable random variable. Since a self-decomposable random variable is infinitely divisible, there exists a Lévy process $(L_t, t \geq 0)$ such that $Y \stackrel{d}{=} L_1$. Consequently, the results stated in the following subsection may be viewed as consequences of those in the previous section. Obviously, they may also be obtained directly.

6.1 Examples of 1-martingales

As in the previous section, we denote by ψ the characteristic exponent of the Lévy process $(L_t, t \geq 0)$ such that $Y \stackrel{d}{=} L_1$. Thus,

$$\forall \lambda \in \mathbb{R}^n \quad \mathbb{E}[\exp(i \lambda \cdot Y)] = \exp(-\psi(\lambda)) .$$

We recall that, in this case, $(X(\alpha), \alpha \geq 0)$ is simply the Sato process $(S_\alpha, \alpha \geq 0)$ associated to Y by Theorem 2.2.

Proposition 6.1 *For every $\lambda \in \mathbb{R}^n$, the function*

$$H_{(\lambda)} : (x, \alpha) \in \mathbb{R}^n \times \mathbb{R}_+ \longrightarrow \exp[i \lambda \cdot x + \psi(\alpha \lambda)]$$

is a space-time harmonic function with respect to $(S_\alpha, \alpha \geq 0)$.

Corollary 6.1

$$(\exp[i \alpha \lambda \cdot Y + \psi(\alpha \lambda)], \alpha \geq 0)$$

is a 1-martingale, an associated martingale being:

$$(\exp[i \lambda \cdot S_\alpha + \psi(\alpha \lambda)], \alpha \geq 0) .$$

Integrating with respect to λ , we obtain:

Corollary 6.2 *Let σ be a signed measure on \mathbb{R}^n such that*

$$\forall \alpha \geq 0 \quad \int |\exp[\psi(\alpha \lambda)]| d|\sigma|(\lambda) < \infty .$$

Then,

$$\left(\int \exp[i \alpha \lambda \cdot Y + \psi(\alpha \lambda)] d\sigma(\lambda) , \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\int \exp[i \lambda \cdot S_\alpha + \psi(\alpha \lambda)] d\sigma(\lambda) , \alpha \geq 0 \right) .$$

The above corollary also admits the following real valued version.

Proposition 6.2 *Suppose that there exists $\lambda \in \mathbb{R}^n$ such that*

$$\forall v \geq 0 \quad \mathbb{E}[\exp(v \lambda \cdot Y)] < \infty .$$

We set:

$$\mathbb{E}[\exp(v \lambda \cdot Y)] = \exp[\phi_\lambda(v)] .$$

Let σ be a signed measure on \mathbb{R}_+ . Then,

$$\left(\int \exp[\alpha v \lambda \cdot Y - \phi_\lambda(\alpha v)] d\sigma(v) , \alpha \geq 0 \right)$$

is a 1-martingale, an associated martingale being:

$$\left(\int \exp[v \lambda \cdot S_\alpha - \phi_\lambda(\alpha v)] d\sigma(v) , \alpha \geq 0 \right) .$$

6.2 Space-time harmonic polynomials

The polynomials P_r^X We first consider any real valued random variable X such that $\mathbb{E}[|X|^p] < \infty$ for some integer $p \geq 1$. Then, the function

$$\lambda \in \mathbb{R} \longrightarrow \Psi(\lambda) := \mathbb{E}[\exp(i \lambda X)]$$

is of class C^p and does not vanish in some neighborhood of 0. We set, for $0 \leq r \leq p$,

$$(11) \quad c_r(X) = (-i)^r \frac{d^r}{d\lambda^r} \left[\frac{1}{\Psi(\lambda)} \right] (0) .$$

Lemma 6.1 *The real numbers $c_r(X)$ are defined by induction for $0 \leq r \leq p$ by:*

$$c_0(X) = 1 \quad \text{and}$$

$$\forall 1 \leq r \leq p \quad c_r(X) = - \sum_{q=0}^{r-1} \binom{r}{q} \mathbb{E}(X^{r-q}) c_q(X) .$$

Proof This is obtained by derivation of the identity:

$$\Psi(\lambda) \frac{1}{\Psi(\lambda)} = 1 ,$$

taking into account:

$$(12) \quad \forall 0 \leq q \leq p \quad \frac{d^q}{d\lambda^q} [\Psi(\lambda)](0) = \mathbb{E}[(i X)^q] .$$

□

Definition 6.1 We define, for $0 \leq r \leq p$, the polynomial P_r^X by:

$$P_r^X(x) = \sum_{q=0}^r \binom{r}{q} c_{r-q}(X) x^q .$$

A direct consequence of the above definition and of (11) is the following formula:

$$(13) \quad (-i)^r \frac{d^r}{d\lambda^r} \left[\frac{\exp(i \lambda x)}{\Psi(\lambda)} \right] (0) = P_r^X(x) .$$

An alternative definition may be obtained by induction.

Proposition 6.3 *The polynomials P_r^X are defined by induction, for $0 \leq r \leq p$, by:*

$$P_0^X = 1 \quad \text{and} \\ \forall 1 \leq r \leq p, \quad (P_r^X)' = r P_{r-1}^X \quad \text{and} \quad \mathbb{E}[P_r^X(X)] = 0 .$$

Proof

We see directly, from Definition 6.1 or from (13), that, for $1 \leq r \leq p$, $(P_r^X)' = r P_{r-1}^X$. On the other hand, it follows from (13) that:

$$\mathbb{E}[P_r^X(X)] = (-i)^r \frac{d^r}{d\lambda^r} \left[\Psi(\lambda) \frac{1}{\Psi(\lambda)} \right] (0)$$

which entails that, if $r \geq 1$, then $\mathbb{E}[P_r^X(X)] = 0$.

□

Examples Suppose that, for every $p \in \mathbb{N}$, $\mathbb{E}[|X|^p] < \infty$. Then formula (13) may be interpreted as the formal equality:

$$\frac{\exp(i \lambda x)}{\Psi(\lambda)} \sim \sum_{r=0}^{\infty} \frac{(i \lambda)^r}{r!} P_r^X(x),$$

which provides a generating series for the sequence (P_r^X) .

Suppose for instance that $X = B_t$ where B_t denotes a centered real valued Gaussian random variable with variance t . Then we obtain:

$$\forall r \geq 0 \quad P_r^{B_t}(x) = t^{r/2} h_r \left(\frac{x}{\sqrt{t}} \right)$$

with (h_r) the sequence of *Hermite polynomials* defined by:

$$h_r(x) = (-1)^r e^{x^2/2} \frac{d^r}{dx^r} \left(e^{-x^2/2} \right).$$

Suppose now that $X = \gamma_t$ where γ_t denotes a gamma variable with expectation t . Then

$$\forall r \geq 0 \quad P_r^{\gamma_t}(x) = (-1)^r \tilde{C}_r(t, x),$$

where $(\tilde{C}_r(t, x))$ denotes the sequence of *Charlier polynomials* (see for instance Schoutens [16]).

Space-time harmonic polynomials We now consider the setting stated at the beginning of Subsection 6.1. We assume furthermore $n = 1$ and $\mathbb{E}[|Y|^p] < \infty$ for some integer $p \geq 1$. Our interest in the polynomials P_r^Y stems from the following result.

Proposition 6.4 *For $0 \leq r \leq p$, the function*

$$(x, \alpha) \in \mathbb{R} \times \mathbb{R}_+ \longrightarrow \alpha^r P_r^Y(\alpha^{-1} x) \in \mathbb{R}$$

(which, by convention, is equal to x^r if $x \in \mathbb{R}$ and $\alpha = 0$) is a space-time harmonic function with respect to $(S_\alpha, \alpha \geq 0)$.

Proof

With the notation of Proposition 6.1, we have by formula (13) with $\Psi(\lambda) = \exp[-\psi(\lambda)]$:

$$(-i)^r \frac{\partial^r}{\partial \lambda^r} H_{(\lambda)}(x, \alpha) \Big|_{\lambda=0} = \alpha^r P_r^Y(\alpha^{-1} x).$$

The result then follows from Proposition 6.1. □

Corollary 6.3 For $0 \leq r \leq p$,

$$(\alpha^r P_r^Y(Y), \alpha \geq 0)$$

is a 1-martingale, an associated martingale being:

$$(\alpha^r P_r^Y(\alpha^{-1} S_\alpha), \alpha \geq 0).$$

Remarks

1. Barrieu and Schoutens studied in [3] space-time harmonic polynomials associated with homogeneous Markov processes. However, in our case, the Markov process involved in the definition of space-time harmonic polynomials, namely the Sato process (S_α) , generally is a non homogeneous Markov process. Nevertheless, in the particular case where the random variable Y is a stable variable, the Sato process (S_α) is a time changed Lévy process and the results of [3] could be used.
2. Suppose that $(\Lambda_t, t \geq 0)$ is a real valued Lévy process in L^p . Then, a consequence of formula (13) is that, for $0 \leq r \leq p$,

$$(P_r^{\Lambda_t}(\Lambda_t), t \geq 0)$$

is a martingale. In the case of examples given previously (Brownian motion or gamma process), this result is well-known.

3. In [14, p. 1239], the authors discuss a family of random pairs (X, Z) such that:

$$\forall \lambda \geq 0 \quad \mathbb{E}[\exp(i \lambda X)] \mathbb{E}[\exp(\lambda Z)] = 1.$$

In this set up, a simple additional hypothesis yields:

$$P_r^X(x) = \mathbb{E}[(x - i Z)^r].$$

7 Some examples of PCOC's without the self-decomposability assumption

In this section, we shall prove that real valued processes of the type of those introduced in Proposition 6.2 and in 2) of Proposition 5.5, are still PCOC's even though they do not satisfy the self-decomposability assumption. However, then we are not able to find associated martingales.

7.1 Preliminary results

The first result is quite elementary.

Lemma 7.1 *Let U be a real valued integrable random variable. Then the following properties are equivalent:*

- (i) For every $c \in \mathbb{R}$, $\mathbb{E}[1_{(U \geq c)} U] \geq 0$.
- (ii) $\mathbb{E}[U] \geq 0$.

Proof

(i) implies (ii), by letting $c \rightarrow -\infty$.

Conversely, assume (ii). Then, the result in (i) is obvious if $c \geq 0$. If $c < 0$,

$$\mathbb{E}[1_{(U \geq c)} U] = \mathbb{E}[U] + (-\mathbb{E}[1_{(U < c)} U])$$

and the RHS is nonnegative as it is the sum of two elements of \mathbb{R}_+ . □

The second result states sufficient conditions ensuring that a given process is a PCOC.

Proposition 7.1 *Let $V = (V_\alpha, \alpha \geq 0)$ be a real valued process satisfying the following hypotheses:*

- i) *The process V is almost surely continuous on $[0, +\infty)$ and derivable on $(0, +\infty)$, the derivative being denoted by $\frac{\partial V_\alpha}{\partial \alpha}$.*

- ii) *For every $a > 0$,*

$$\mathbb{E} \left[\sup_{\alpha \in [0, a]} |V_\alpha| \right] < \infty$$

and, for every $0 < a < b$,

$$\mathbb{E} \left[\sup_{\alpha \in [a, b]} \left| \frac{\partial V_\alpha}{\partial \alpha} \right| \right] < \infty .$$

Then, the process V is a PCOC if and only if the following properties hold:

- a) *The expectation $\mathbb{E}[V_\alpha]$ does not depend on $\alpha \geq 0$.*

b) For every $c \in \mathbb{R}$ and $\alpha > 0$,

$$\mathbb{E} \left[1_{(V_\alpha \geq c)} \frac{\partial V_\alpha}{\partial \alpha} \right] \geq 0 .$$

Proof

Suppose first that the process V satisfying Properties i) and ii), also satisfies Properties a) and b). To prove that V is a PCOC, it is enough to show that, for any convex C^1 -function φ which is affine on $(-\infty, A)$ and on $(B, +\infty)$ for some $A < B$, the function

$$f : \alpha \in [0, +\infty) \longrightarrow \mathbb{E}[\varphi(V_\alpha)]$$

is increasing. We have:

$$|\varphi(x)| \leq C + D|x| .$$

Therefore, by Properties i) and ii), the function f is continuous on $[0, +\infty)$ and derivable on $(0, +\infty)$ and

$$\forall \alpha > 0 \quad f'(\alpha) = \mathbb{E} \left[\varphi'(V_\alpha) \frac{\partial V_\alpha}{\partial \alpha} \right] .$$

Since φ' is a continuous increasing function, which is constant on $(-\infty, A)$ and on $(B, +\infty)$, $\varphi' - \varphi'(A)$ is an increasing limit of a sequence of functions which are finite sums of functions of the type: $d 1_{[c, +\infty)}$ with $d \geq 0$ and $c \in \mathbb{R}$. Taking into account Properties a) and b), this yields:

$$\forall \alpha > 0 \quad f'(\alpha) \geq 0 .$$

Suppose now that the process V satisfying Properties i) and ii) is a PCOC. Then, Property a) is obviously satisfied. We set, for $c \in \mathbb{R}$ and $\eta > 0$,

$$h_{c,\eta}(x) = \eta^{-1} \int_x^{x+\eta} (y - c)^+ dy .$$

Since $h_{c,\eta}$ is a convex C^1 -function,

$$\forall \alpha > 0 \quad \mathbb{E} \left[h'_{c,\eta}(V_\alpha) \frac{\partial V_\alpha}{\partial \alpha} \right] \geq 0 .$$

Letting then η tend to 0, we obtain that Property b) is satisfied.

□

The above results will be used in an essential way in the proof of Theorem 7.1 below. However, to illustrate the scope of these results, we first mention another interesting direct consequence of Lemma 7.1 and Proposition 7.1.

Corollary 7.1 *Let X be a centered real valued random variable. Then, the process*

$$(\alpha X, \alpha \geq 0)$$

is a PCOC.

7.2 A partial extension of Proposition 6.2

Theorem 7.1 *Let X be a real valued random variable such that*

$$\forall v \geq 0 \quad \mathbb{E}[\exp(v X)] < \infty$$

and set

$$\forall v \geq 0 \quad \phi_X(v) = \log(\mathbb{E}[\exp(v X)]) .$$

Let σ be a finite positive measure on \mathbb{R}_+ . Then

$$\left(\int \exp[\alpha v X - \phi_X(\alpha v)] d\sigma(v), \alpha \geq 0 \right)$$

is a PCOC.

Proof

1. Let $0 < C < D$. The following elementary inequalities hold:

$$\forall v \in [0, C] \quad \exp(v X) \leq 1 + \exp(C X) ,$$

$$\forall v \in [C, D] \quad \exp(v X) |X| \leq (e C)^{-1} + \exp[(D + 1) X] .$$

As a consequence, the function ϕ_X is continuous on $[0, \infty)$ and derivable on $(0, \infty)$.

2. Suppose now the measure σ has compact support, contained in $(0, \infty)$. We set, for $\alpha \geq 0$,

$$V_\alpha = \int \exp[\alpha v X - \phi_X(\alpha v)] d\sigma(v) .$$

By the above step 1, the process $(V_\alpha, \alpha \geq 0)$ satisfies Properties i) and ii) of Proposition 7.1, and since

$$\mathbb{E}(V_\alpha) = \int d\sigma(v) ,$$

Property a) also is satisfied. Moreover, for $c \in \mathbb{R}$ and $\alpha > 0$,

$$\mathbb{E} \left[1_{(V_\alpha \geq c)} \frac{\partial V_\alpha}{\partial \alpha} \right] =$$

$$\mathbb{E} \left[1_{(V_\alpha \geq c)} \int v \exp[\alpha v X - \phi_X(\alpha v)] (X - \phi'_X(\alpha v)) d\sigma(v) \right] .$$

For $r \geq 0$, we denote by $\mathbb{E}^{(r)}$ the expectation with respect to the probability measure:

$$\mathbb{P}^{(r)} = \exp[r X - \phi_X(r)] \mathbb{P} ,$$

and we set

$$U^{(r)} = X - \phi'_X(r) .$$

We have:

$$\mathbb{E}^{(r)}[U^{(r)}] = \frac{d}{dr} \mathbb{E}[\exp(r X - \phi_X(r))] = 0 .$$

Besides,

$$\mathbb{E} \left[1_{(V_\alpha \geq c)} \frac{\partial V_\alpha}{\partial \alpha} \right] = \int v \mathbb{E}^{(\alpha v)}[1_{(V_\alpha \geq c)} U^{(\alpha v)}] d\sigma(v) .$$

We set

$$\Gamma_\alpha(x) = \int \exp[\alpha v x - \phi_X(\alpha v)] d\sigma(v) .$$

The function Γ_α is clearly increasing and if we set

$$x(\alpha, c) = \inf\{x ; \Gamma_\alpha(x) \geq c\} ,$$

then,

$$(V_\alpha \geq c) = (\Gamma_\alpha(X) \geq c) = (X \geq x(\alpha, c)) .$$

Setting

$$\tilde{c} = x(\alpha, c) - \phi'_X(\alpha v) ,$$

we then obtain:

$$(V_\alpha \geq c) = (U^{(\alpha v)} \geq \tilde{c}) .$$

Since previously we noted that

$$\mathbb{E}^{(\alpha v)}[U^{(\alpha v)}] = 0 ,$$

we now deduce from Lemma 7.1:

$$\mathbb{E}^{(\alpha v)}[1_{(V_\alpha \geq c)} U^{(\alpha v)}] \geq 0 .$$

Therefore, Property b) in Proposition 7.1 is also satisfied. Thus, by Proposition 7.1, the process $(V_\alpha, \alpha \geq 0)$ is a PCOC.

3. Suppose finally that σ is any finite positive measure on \mathbb{R}_+ . We set again, for $\alpha \geq 0$,

$$V_\alpha = \int \exp[\alpha v X - \phi_X(\alpha v)] d\sigma(v) .$$

We still have:

$$\mathbb{E}(V_\alpha) = \int d\sigma(v) .$$

Therefore, to prove that $(V_\alpha, \alpha \geq 0)$ is a PCOC, it suffices to show that, for any *increasing* convex function φ ,

$$\alpha \longrightarrow \mathbb{E}[\varphi(V_\alpha)]$$

is increasing. Now, by step 2, the process V is the limit of an increasing sequence of PCOC's, which yields the desired result.

□

We note for the sequel the following obvious corollary, corresponding to σ equal to the Dirac measure at 1.

Corollary 7.2 *Let X be a real valued random variable such that*

$$\forall \alpha \geq 0 \quad \mathbb{E}[\exp(\alpha X)] < \infty .$$

Then

$$\left(\frac{\exp(\alpha X)}{\mathbb{E}[\exp(\alpha X)]}, \alpha \geq 0 \right)$$

is a PCOC.

We saw in Proposition 6.2 that, if X is self-decomposable, then Theorem 7.1 is valid with any *signed* measure σ . The following proposition shows that this is no longer true, in general, if X is not self-decomposable.

Proposition 7.2 *We assume that X is a symmetric Bernoulli random variable:*

$$\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2 .$$

We set, for $a > 1/2$ and $\alpha \geq 0$,

$$V_\alpha = \frac{\exp(\alpha X)}{\cosh \alpha} - a \frac{\exp(2\alpha X)}{\cosh 2\alpha}$$

Then, the function $\alpha \longrightarrow \mathbb{E}[V_\alpha^2]$ is not increasing.

Proof

An easy computation yields:

$$\frac{d}{d\alpha} \mathbb{E}[V_\alpha^2] = 2 \sinh \alpha \left[4a^2 \frac{\cosh \alpha}{(\cosh 2\alpha)^3} - 4a \frac{\cosh \alpha}{(\cosh 2\alpha)^2} + \frac{1}{(\cosh \alpha)^3} \right].$$

For α such that $\cosh 2\alpha = 2a$, we obtain:

$$\frac{d}{d\alpha} \mathbb{E}[V_\alpha^2] = -(2a)^{-1} (2a+1)^{-3/2} (2a-1)^{5/2} < 0.$$

□

7.3 A partial extension of Proposition 5.5, 2)

Theorem 7.2 *Let $(X_u, u \geq 0)$ be a real valued, right continuous process with independent increments. We assume that $X_0 = 0$ and*

$$\forall \alpha \geq 0, \forall u \geq 0, \quad \mathbb{E}[\exp(\alpha X_u)] < \infty.$$

We set

$$\phi_{X_u}(\alpha) = \log(\mathbb{E}[\exp(\alpha X_u)]).$$

Let σ be a signed measure on \mathbb{R}_+ . Then

$$\left(\int \exp[\alpha X_u - \phi_{X_u}(\alpha)] d\sigma(u), \alpha \geq 0 \right)$$

is a PCOC.

Proof

1. We begin the proof with some preliminaries.

(a) By the hypotheses on X , for every $\alpha \geq 0$, the process

$$(\exp[\alpha X_u - \phi_{X_u}(\alpha)], u \geq 0)$$

is a martingale with respect to a filtration satisfying the usual hypotheses. Consequently, this process admits a right continuous version.

This implies easily that the function

$$u \geq 0 \longrightarrow \phi_{X_u}(\alpha)$$

is right continuous, and therefore, for every $\alpha \geq 0$, the process $(\exp[\alpha X_u - \phi_{X_u}(\alpha)], u \geq 0)$ is a right continuous martingale.

(b) The above martingale is clearly in L^p for any $p \in (1, \infty)$. Therefore, the Doob L^p -inequality entails:

$$\forall \alpha \geq 0, \forall a > 0, \quad \mathbb{E} \left[\sup_{u \in [0, a]} \exp(\alpha X_u - \phi_{X_u}(\alpha)) \right] < \infty.$$

(c) By the independence of the increments of (X_u) , if $0 \leq u \leq v$ and $\alpha \geq 0$, then

$$\mathbb{E}[\exp(\alpha (X_v - X_u))] = \exp(\phi_{X_v}(\alpha) - \phi_{X_u}(\alpha)).$$

By Corollary 7.2, the process

$$(\exp[\alpha (X_v - X_u) - (\phi_{X_v}(\alpha) - \phi_{X_u}(\alpha))], \alpha \geq 0)$$

is a PCOC.

2. We shall now prove by induction, for $k \geq 1$, the following property:

(P_k) For every $0 \leq u_1 < u_2 < \dots < u_k$ and $d_1, d_2 \dots d_k \in \mathbb{R}$, the process

$$\left(\sum_{j=1}^k d_j \exp[\alpha X_{u_j} - \phi_{X_{u_j}}(\alpha)], \alpha \geq 0 \right)$$

is a PCOC.

Property (P_1) is a direct consequence of Corollary 7.2.

Suppose now that (P_k) is satisfied.

$$\sum_{j=1}^{k+1} d_j \exp[\alpha X_{u_j} - \phi_{X_{u_j}}(\alpha)] = \sum_{j=1}^k d_j \exp[\alpha X_{u_j} - \phi_{X_{u_j}}(\alpha)] +$$

$$d_{k+1} \exp[\alpha X_{u_k} - \phi_{X_{u_k}}(\alpha)] \exp[\alpha (X_{u_{k+1}} - X_{u_k}) - (\phi_{X_{u_{k+1}}}(\alpha) - \phi_{X_{u_k}}(\alpha))].$$

We set, for $\alpha, \beta \geq 0$,

$$V_{\alpha, \beta} = \sum_{j=1}^k d_j \exp[\alpha X_{u_j} - \phi_{X_{u_j}}(\alpha)] +$$

$$d_{k+1} \exp[\alpha X_{u_k} - \phi_{X_{u_k}}(\alpha)] \exp[\beta (X_{u_{k+1}} - X_{u_k}) - (\phi_{X_{u_{k+1}}}(\beta) - \phi_{X_{u_k}}(\beta))].$$

Let φ be a convex function. Using the independence of the increments of (X_u) , we obtain that $\mathbb{E}[\varphi(V_{\alpha, \beta})]$ is separately increasing with respect to α (from (P_k)), and with respect to β (from the above property (c)). Consequently, the function $\mathbb{E}[\varphi(V_{\alpha, \alpha})]$ is increasing, which is the property (P_{k+1}).

3. Reasoning by weak approximation of the measure σ by a sequence of linear combinations of Dirac measures, we deduce from step 2, using (a) and (b) of step 1, that, if the support of σ is compact, then

$$\left(\int \exp[\alpha X_u - \phi_{X_u}(\alpha)] d\sigma(u), \alpha \geq 0 \right)$$

is a PCOC. The general result follows again, approximating σ by measures with compact support, as in step 3 of the proof of Theorem 7.1.

□

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