A Model of Credit Events Based on Filtering Theory*

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Abstract

In this paper, we give a construction of default times admitting the same intensity, but having different conditional laws. This illustrates the well know fact that the intensity does not contain enough information to price derivative products. Our method is based on filtering theory.

This paper is friendly dedicated to Eckhard Platen's birthday. Even if it is not related with his exiting benchmark approach, we hope he will find some interest to our model.

1 Introduction

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space and η be a probability law on $\mathbb{R}_+$ which is assumed to have a density φ with respect to the Lebesgue measure. Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$, a density process is a family $(\alpha_t(u))_{t \geq 0}$, $u \geq 0$, of non-negative $\mathbb{F}$-martingales such that

$$\int_0^\infty \alpha_t(u)\, \eta(du) = 1 \tag{1}$$

for all $t \geq 0$. Without loss of generality, we shall restrict our attention to the case of $\alpha_0(u) = 1$, for $u \geq 0$. It is proved in [2] that one can associate with a density process a random time τ (on an appropriately extended probability space $(\Omega^*, \mathbb{P}^*)$ with $\mathbb{P}^*|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$) such that

$$\mathbb{P}^*(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty \alpha_t(v)\, \eta(dv)$$

for all $t, u \geq 0$. In particular, η is the law of τ (we shall give a new proof of this result later in this paper).
It follows that the Azéma supermartingale $G = (G_t)_{t \geq 0}$ (or conditional survival probability process) defined by $G_t = \mathbb{P}^*(\tau > t \,|\, \mathcal{F}_t)$ admits the integral representation

$$G_t = \int_t^\infty \alpha_t(u)\, \eta(du) \tag{2}$$

and the multiplicative decomposition

$$G_t = n_t\, e^{-\Lambda_t} \tag{3}$$

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for all $t \geq 0$. Here, $n = (n_t)_{t\geq 0}$ is an $\mathbb{F}$-local martingale and $\Lambda = (\Lambda_t)_{t\geq 0}$ is a continuous increasing process, called the intensity process, defined by

$$\Lambda_t = \int_0^t \lambda_s \, ds \tag{4}$$

where the non-negative $\mathbb{F}$-adapted process $\lambda = (\lambda_t)_{t\geq 0}$, called the intensity rate, is given by

$$\lambda_t = \frac{\alpha_t(t)}{G_t} \varphi(t)$$

for $t \geq 0$ (we assume that G does not vanish).

We denote by $\mathbb{H}$ the filtration generated by the default process $H = (H_t)_{t\geq 0}$, where $H_t = \mathbb{1}_{\tau \leq t}$ for all $t \geq 0$. As recalled in [2], the knowledge of the Azéma supermartingale allows to define the compensator of the default process: if the random time τ is such that (3) holds, then (whatever the local-martingale n is) the process

$$M_t := H_t - \int_0^t (1 - H_s)\lambda_s \, ds \tag{5}$$

is a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ (More precisely, $\mathcal{G}_t = \cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon)$ for $t \geq 0$). However, the knowledge of G is not sufficient for a complete characterization of the density process. Our aim is to construct various random times admitting the same intensity and corresponding to different local martingales n, and to study the impact of the process n on pricing. Our construction follows from filtering approach, another construction can be found in [3].

We recall (see [2]) that immersion property[1] between $\mathbb{F}$ and $\mathbb{G}$ holds if and only $\alpha_t(s) = \alpha_s(s)$ for all $0 \leq s \leq t$. In that case, the process G is decreasing, and $n \equiv 1$. Our study considers the case where the local martingale n is not trivial, and hence, the immersion property is not satisfied.

2 Filtering

The starting point of that research was a well known result on filtering that we recall now.

2.1 A one-dimensional model

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and τ be a random time, independent of W and such that $\mathbb{P}(\tau > t) = e^{-\lambda t}$, for all $t \geq 0$ and some $\lambda > 0$ fixed. We define $X = (X_t)_{t\geq 0}$ as the process

$$X_t = x \, \exp\left(\left(a + b - \frac{\sigma^2}{2}\right) t - b(t-\tau)^+ + \sigma \, W_t\right)$$

where a, and x, σ, b are some given strictly positive constants. It is easily shown that the process X solves the stochastic differential equation

$$dX_t = X_t \left(a + b \, \mathbb{1}_{\{\tau > t\}}\right) dt + X_t \, \sigma \, dW_t \, .$$

Let us take as $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the natural filtration of the process X, that is, $\mathcal{F}_t = \sigma(X_s \,|\, 0 \leq s \leq t)$ for $t \geq 0$. By means of standard arguments (see, e.g., [6, Chapter IV, Section 4] or [4, Chapter IX]), it can be shown that the process X admits the following representation in its own filtration

$$dX_t = X_t \left(a + b \, G_t\right) dt + X_t \, \sigma \, d\overline{W}_t \, .$$

[1] The immersion property is the fact that any $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale.

Here, $G = (G_t)_{t\geq 0}$ is the Azéma supermartingale given by $G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$ and the innovation process $\overline{W} = (\overline{W}_t)_{t\geq 0}$ defined by

$$\overline{W}_t = W_t + \frac{b}{\sigma} \int_0^t \big(\mathbb{1}_{\{\tau > s\}} - G_s \big) \, ds$$

is a standard $\mathbb{F}$-Brownian motion. It is easily shown using the arguments based on the notion of strong solutions of stochastic differential equations (see, e.g. [4, Chapter IV]) that the natural filtration of $\overline{W}$ coincides with $\mathbb{F}$. It follows from [4, Chapter IX] (see also [6, Chapter IV, Section 4]) that the process G solves the stochastic differential equation

$$dG_t = -\lambda G_t \, dt + \frac{b}{\sigma} \, G_t(1 - G_t) \, d\overline{W}_t \, . \tag{6}$$

Observe that the process $n = (n_t)_{t\geq 0}$ with $n_t = e^{\lambda t} \, G_t$ admits the representation

$$dn_t = d(e^{\lambda t} \, G_t) = \frac{b}{\sigma} \, e^{\lambda t} \, G_t(1 - G_t) \, d\overline{W}_t$$

and thus, n is an $\mathbb{F}$-martingale (to establish the true martingale property, note that the process $(G_t(1 - G_t))_{t\geq 0}$ is bounded). The equality (6) provides the (additive) Doob-Meyer decomposition of the supermartingale G, while $G_t = (G_t \, e^{\lambda t}) \, e^{-\lambda t}$ gives its multiplicative decomposition. It follows from these decompositions that the $\mathbb{F}$-intensity rate of τ is λ, so that, the process $M = (M_t)_{t\geq 0}$ with $M_t = H_t - \lambda(t \wedge \tau)$ is a $\mathbb{G}$-martingale.

2.2 Conditional laws and density process

It follows from the definition of the conditional survival probability process G and the fact that $(G_t \, e^{\lambda t})_{t\geq 0}$ is a martingale that the expression

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \mathbb{E}[\mathbb{P}(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t] = \mathbb{E}[G_u \, e^{\lambda u} \,|\, \mathcal{F}_t] \, e^{-\lambda u} = G_t \, e^{\lambda(t-u)}$$

holds for $0 \leq t < u$.

From the standard arguments in [5, Chapter IV, Section 4] (which are compressed in [6, Chapter IV, Section 4]), resulting from the application of Bayes' formula, we obtain that the conditional survival probability process can be expressed as

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{Y_{u\wedge t}}{Y_t} + \frac{Z_{u\wedge t}}{Y_t} \, e^{-\lambda u} \tag{7}$$

for all $t, u \geq 0$. Here, the process $Y = (Y_t)_{t\geq 0}$ is defined by

$$Y_t = \int_0^t Z_s \, \lambda e^{-\lambda s} \, ds + Z_t \, e^{-\lambda t} \tag{8}$$

and the process $Z = (Z_t)_{t\geq 0}$ is given by

$$Z_t = \exp\left(\frac{b}{\sigma^2} \Big(\ln \frac{X_t}{x} - \frac{2a + b - \sigma^2}{2} \, t \Big) \right) . \tag{9}$$

Moreover, by standard computations, we see that

$$dZ_t = \frac{b}{\sigma^2} \, Z_t \, (bG_t \, dt + \sigma \, dW_t)$$

and hence, the process $1/Y = (1/Y_t)_{t\geq 0}$, or its equivalent $(e^{\lambda t} \, G_t / Z_t)_{t\geq 0}$, admits the representation

$$d\Big(\frac{1}{Y_t} \Big) = d\Big(\frac{e^{\lambda t} \, G_t}{Z_t} \Big) = -\frac{b}{\sigma} \, \frac{G_t}{Y_t} \, d\overline{W}_t$$

and thus, it is an $\mathbb{F}$-local martingale (in fact, an $\mathbb{F}$-martingale since, for any $u \geq 0$, one has

$$\frac{1}{Y_t} = \frac{e^{\lambda u}}{Z_u} \mathbb{P}(\tau > u \,|\, \mathcal{F}_t)$$

for $u > t$). Hence, for each $u \geq 0$ fixed, it can be checked that the process $Z/Y = (Z_t/Y_t)_{t\geq 0}$ defines an $\mathbb{F}$-martingale, as it must be (this process being equal to $(e^{\lambda t}\, G_t)_{t\geq 0}$).

We also get the representation

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty \alpha_t(v)\, \eta(dv)$$

for $t, u \geq 0$, and hence, the density of τ is given by

$$\alpha_t(u) = \frac{Z_{u\wedge t}}{Y_t}$$

where $\eta(du) \equiv \mathbb{P}(\tau \in du) = \lambda e^{-\lambda u}\, du$. In particular, from (8), we have

$$\int_0^\infty \alpha_t(u)\, \eta(du) = 1$$

as expected. Thus, we have obtained a model with constant given intensity and with an explicit form of the density processes $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$.

It can be interesting to note that we are studying a model where

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Theta\}$$

with $\Lambda_t = \lambda t$ (so that $\lambda\tau = \Theta$) and the barrier Θ is an exponential random variable depending on $\mathcal{F}_\infty$: indeed

$$\mathbb{P}(\Theta > u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau > u/\lambda \,|\, \mathcal{F}_t) \neq e^{-u}$$

for $0 \leq t < u$.

3 A Family of Densities with Given Intensity Rate

Our aim here is to extend the above example obtained in the filtering case and to give non-trivial examples of density processes corresponding to a given $\mathbb{F}$-adapted intensity rate $\lambda = (\lambda_t)_{t\geq 0}$.

3.1 Constant Intensity Rate

Let us start with the case where the intensity rate λ is constant. We assume that an $\mathbb{F}$-martingale n is given so that $0 < n_t\, e^{-\lambda t} < 1$ for $t \geq 0$. Our aim is to construct τ and $\mathbb{Q}$ (on an extended probability space) so that $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$ and

$$\mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) = n_t\, e^{-\lambda t}$$

for all $t \geq 0$.

3.1.1 Construction of a density process

Let Y and Z be two processes and define

$$\alpha_t(u) = \frac{Z_{u\wedge t}}{Y_t} \tag{10}$$

for all $t, u \geq 0$. In order that $(\alpha_t(u))_{t\geq 0}$ is a density process associated with the law $\eta(du) = \lambda e^{-\lambda u}\, du$, for any $u \geq 0$ fixed, the two processes Y and Z must satisfy the following conditions:

(i) the processes $1/Y$ and Z/Y are strictly positive martingales, with initial value equal to 1, and

(ii) $\int_0^\infty \alpha_t(u)\,\eta(du) = 1$, i.e.

$$Y_t = \int_0^t Z_s\,\lambda e^{-\lambda s}\,ds + Z_t e^{-\lambda t} \tag{11}$$

for $t \geq 0$.

Let $X = 1/Y$ be a non-negative martingale solving the equation

$$dX_t = -X_t x_t\,dW_t$$

with $X_0 = 1$, where W is a Brownian motion and $x = (x_t)_{t\geq 0}$ is a square integrable $\mathbb{F}$-adapted process, or in a closed form

$$X_t = \mathcal{E}_t(-x \star W) \equiv \exp\left(-\int_0^t x_s\,dW_s - \frac{1}{2}\int_0^t x_s^2\,ds\right)$$

for $t \geq 0$. Let $Z = (Z_t)_{t\geq 0}$ be a diffusion satisfying the equation

$$dZ_t = Z_t(\widehat{z}_t\,dt + z_t\,dW_t)$$

with $Z_0 = 1$. Then, applying integration by parts formula, the process $ZX = (Z_t X_t)_{t\geq 0}$ is a non-negative local martingale if and only if

$$\widehat{z}_t = z_t x_t$$

i.e., if there exists an $\mathbb{F}$-adapted process $z = (z_t)_{t\geq 0}$ such that

$$dZ_t = Z_t z_t\,(dW_t + x_t\,dt)\,.$$

The condition in (11) is satisfied if

$$dY_t = (\lambda Z_t e^{-\lambda t} - \lambda Z_t e^{-\lambda t})\,dt + e^{-\lambda t}\,dZ_t = e^{-\lambda t}\,dZ_t$$

or if

$$Y_t x_t\,(dW_t + x_t\,dt) = e^{-\lambda t} Z_t z_t\,(dW_t + x_t\,dt)$$

holds implying that

$$z_t Z_t = x_t Y_t e^{\lambda t}$$

for all $t \geq 0$.

Therefore, one can construct a density process starting from a given x, with an explicit construction of Y and Z, setting

$$Y_t = \exp\left(\int_0^t x_u\,dW_u + \frac{1}{2}\int_0^t x_u^2\,du\right)$$

and using the fact that

$$dZ_t = e^{\lambda t}\,dY_t = e^{\lambda t} x_t Y_t\,(dW_t + x_t\,dt)$$

we get

$$Z_t = 1 + \int_0^t e^{-\lambda s} x_s \exp\left(\int_0^s x_u\,dW_u + \frac{1}{2}\int_0^s x_u^2\,du\right)(dW_s + x_s\,ds)$$

for $t \geq 0$. Hence, setting

$$\widehat{W}_t = W_t + \int_0^t x_s\,ds$$

we have

$$Z_t = 1 + \int_0^t e^{-\lambda s} x_s\,\mathcal{E}_s(x \star \widehat{W})\,d\widehat{W}_s$$

for $t \geq 0$.

We now assume that the process Z is strictly positive (we shall give examples below). Using (10), we can associate with the process x a density process $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$, and thus, a random time τ (see Subsection 3.1.3 for details) such that

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty \alpha_t(v)\, \eta(dv)$$

for all $t, u \geq 0$. In particular, we have

$$G_t \equiv \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = \frac{Z_t}{Y_t} e^{-\lambda t}$$

and the intensity rate of τ is λ. From integration by parts, one obtains

$$d\Big(\frac{Z_t}{Y_t}\Big) = -\frac{Z_t}{Y_t} x_t \left(1 - \frac{Y_t}{Z_t} e^{\lambda t}\right) dW_t \tag{12}$$

and hence, the Azéma supermartingale G satisfies

$$dG_t = -\lambda G_t\, dt + x_t (1 - G_t)\, dW_t \,.$$

In the previous filtering example, we have $x_t = b\, G_t/\sigma$ for $t \geq 0$ (so that, the processes G and thus Z are non-negative).

3.1.2 A family of density process with given Azéma's surpermartingale

Our aim is to construct a density process $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$, such that

$$G_t \equiv \int_t^\infty \alpha_t(u)\, \eta(du) = n_t\, e^{-\lambda t}$$

for $t \geq 0$. It is well known that the knowledge of the supermartingale $(n_t\, e^{-\lambda t})_{t\geq 0}$ does not allow for to reconstruct the density process in a unique way. We provide one way to do that.

From the predictable representation theorem, there exists a square integrable $\mathbb{F}$-predictable process $\nu = (\nu_t)_{t\geq 0}$ such that

$$dn_t = n_t \nu_t\, dW_t$$

holds. We construct two processes Y and Z as in Subsection 3.1.1 and such that $n_t = Z_t/Y_t$ for $t \geq 0$. By comparison with (12), we obtain

$$x_t = -\frac{\nu_t n_t}{n_t - e^{\lambda t}} \tag{13}$$

for all $t \geq 0$. The non-negativity of Z follows from the non-negativity of Y and of the ratio Z/Y, which is equal to n.

3.1.3 Construction of τ

We recall how one can construct a random time with a given intensity and with a given density. The traditional Cox approach, is to construct a random time with a given intensity. One sets

$$\tau = \inf\{t : \lambda t > -\ln U\} \equiv -\ln U/\lambda \tag{14}$$

where U is a random variable uniformly distributed on the interval $(0,1)$ and independent of $\mathcal{F}_\infty$. Under $\mathbb{P}$, we have

$$\begin{aligned} \alpha_t^{\mathbb{P}}(u) &= 1 \\ G_t \equiv \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) &= e^{-\lambda t} \\ \lambda^{\mathbb{P}} &= \lambda \\ \eta(du) &\equiv \mathbb{P}(\tau \in du) = \lambda e^{-\lambda u}\, du. \end{aligned}$$

Let us assume that a density process $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$, is given. Then, we set

$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t^{\mathbb{Q}}\, d\mathbb{P}|_{\mathcal{G}_t}$$

where $L^{\mathbb{Q}} = (L_t^{\mathbb{Q}})_{t\geq 0}$ is the $\mathbb{G}$-adapted process defined as

$$L_t^{\mathbb{Q}} = \mathbb{1}_{t<\tau}\, e^{\lambda t} \int_t^\infty \alpha_t(u)\, \lambda e^{-\lambda u}\, du + \mathbb{1}_{\tau \leq t}\, \alpha_t(\tau)$$

for $t \geq 0$. From the results obtained in [2] (also recalled in the appendix), the process $L^{\mathbb{Q}}$ is a $(\mathbb{P},\mathbb{G})$-martingale: indeed, the first condition (recalled in Subsection 5.1.1) is that $m = (m_t)_{t\geq 0}$ with

$$m_t := e^{\lambda t}\, G_t \int_t^\infty \alpha_t(u)\, \lambda e^{-\lambda u}\, du + \int_0^t \alpha_u(u)\, \lambda e^{-\lambda u}\, du = 1 - \int_0^t (\alpha_t(u) - \alpha_u(u))\, \lambda e^{-\lambda u}\, du$$

is a martingale (we have used (1)). From the martingale property of the density process, it is easy to see that

$$\begin{aligned} \mathbb{E}[m_t \,|\, \mathcal{F}_s] &= 1 - \int_0^t \mathbb{E}[\alpha_t(u) - \alpha_u(u) \,|\, \mathcal{F}_s]\, \lambda e^{-\lambda u}\, du \qquad (15) \\ &= 1 - \int_0^s (\alpha_s(u) - \alpha_u(u))\, \lambda e^{-\lambda u}\, du - \int_s^t (\alpha_s(u) - \alpha_s(u))\, \lambda e^{-\lambda u}\, du = m_s \end{aligned}$$

holds for $0 \leq s \leq t$. The second condition reads that, for any $s \geq 0$ fixed, the processes $(\alpha_t(s))_{t\geq s}$, are $\mathbb{F}$-local martingales.
In our case, when the density process is constructed as in Subsection 3.1.2 from the knowledge of the Azéma supermartingale (we emphasize again that this construction is not unique), from [2] (see also the Appendix), we obtain that under $\mathbb{Q}$:

$$\begin{aligned} \alpha_t^{\mathbb{Q}}(u) &= \alpha_t(u) = \frac{Z_{u\wedge t}}{Y_t} \\ G_t^{\mathbb{Q}} \equiv \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) &= \frac{Z_t}{Y_t}\, e^{-\lambda t} = n_t e^{-\lambda t} \\ \lambda^{\mathbb{Q}} &= \lambda \\ \mathbb{Q}(\tau \in du) &= \lambda e^{-\lambda u}\, du \end{aligned}$$

so that, we obtain a random time τ with deterministic intensity λ with another Azéma supermartingale and a different density process. Note that $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ holds for all $t \geq 0$ (indeed, $\mathbb{E}(L_t^{\mathbb{Q}}|\mathcal{F}_t) = 1$). The process n is a $(\mathbb{P},\mathbb{F})$ and a $(\mathbb{P},\mathbb{G})$-martingale, hence a $(\mathbb{Q},\mathbb{F})$-martingale, but is not a $(\mathbb{Q},\mathbb{G})$-martingale (see [2] for a characterization of $\mathbb{G}$-martingales).

3.2 General intensity

Let us now study the case with $\mathbb{F}$-adapted stochastic intensity $\Lambda = (\Lambda_t)_{t\geq 0}$ with

$$\Lambda_t = \int_0^t \lambda_s\, ds$$

where $(\lambda_t)_{t\geq 0}$ is a non-negative process.

In the usual Cox construction presented above, we have

$$\tau = \inf\{t \,:\, \Lambda_t \geq -\ln U\}$$

(see (14) for the case of constant intensity rate), where U is a uniform random variable on $(0,1)$ independent of $\mathbb{F}$. Hence, we get

$$\mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = e^{-\Lambda_t}$$

for $t \geq 0$, as well as

$$\eta(du) \equiv \mathbb{P}(\tau \in du) = \mathbb{E}[\lambda_u e^{-\Lambda_u}]\, du\,.$$

In a general setting, the multiplicative decomposition of the Azéma supermartingale associated with a random time τ with the given intensity rate process $(\lambda_t)_{t\geq 0}$ has the form of

$$\mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = n_t\, e^{-\Lambda_t}$$

where n is a martingale (satisfying $0 < n_t\, e^{-\Lambda_t} < 1$ for $t \geq 0$). In particular, the marginal law of τ is equal to

$$\mathbb{P}(\tau > u) = \mathbb{E}[n_u\, e^{-\Lambda_u}]$$

and is different from the case of the Cox model. Note that, for any $t > u$, the equality

$$\mathbb{E}[n_u\, e^{-\Lambda_u}] = \mathbb{E}[n_t\, e^{-\Lambda_u}]$$

holds, and the latter quantity can be written as

$$\mathbb{E}[n_t e^{-\Lambda_u}] = \mathbb{E}[n_t] - \int_0^u \mathbb{E}[n_t \lambda_s e^{-\Lambda_s}]\, ds$$

so that

$$d_u \mathbb{E}[n_u\, e^{-\Lambda_u}] = \mathbb{E}[n_t\, \lambda_u\, e^{-\Lambda_u}]\, du = \mathbb{E}[n_u\, \lambda_u\, e^{-\Lambda_u}]\, du\,. \tag{16}$$

Our aim is to find a density process $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$, such that the associated intensity rate is a given process $(\lambda_t)_{t\geq 0}$.

3.2.1 Construction of a density

An immediate generalization of the previous computations leads to the choice

$$\begin{aligned}
\alpha_t(u) &= \frac{1}{\varphi(u)}\, \mathbb{E}\left[\frac{Z_u}{Y_u}\, \lambda_u e^{-\Lambda_u} \,\Big|\, \mathcal{F}_t\right]\,,\ \text{for}\, t < u \\
&= \frac{1}{\varphi(u)}\, \frac{Z_t}{Y_t}\, \lambda_u e^{-\Lambda_u}\,,\ \text{for}\, t \geq u
\end{aligned}$$

where

$$\varphi(u) = \mathbb{E}\left[\frac{Z_u}{Y_u}\, \lambda_u e^{-\Lambda_u}\right]\,.$$

In order that $(\alpha_t(u))_{t\geq 0}, u \geq 0)$ is a density process with respect to $\eta(du) = \varphi(u)\, du$, for any $u \geq 0$ fixed, the two processes Y and Z must satisfy the following conditions:

(i) the process $1/Y$ is a non-negative martingale, with initial value equal to 1, and

(ii) $\int_0^\infty \alpha_t(u)\,\varphi(u)\,du = 1$ (or actually (1) holds), i.e.

$$\frac{1}{Y_t}\int_0^t Z_s\,\lambda_s e^{-\Lambda_s}\,ds + \frac{Z_t}{Y_t}\,\mathbb{E}\left[\int_t^\infty \lambda_u e^{-\Lambda_u}\,du \,\Big|\, \mathcal{F}_t\right] = \frac{1}{Y_t}\int_0^t Z_s\,\lambda_s e^{-\Lambda_s}\,ds + \frac{Z_t}{Y_t}\,e^{-\Lambda_t} = 1$$

or

$$Y_t = \int_0^t Z_s\,\lambda_s e^{-\Lambda_s}\,ds + Z_t\,e^{-\Lambda_t}$$

for $t \geq 0$.

One has

$$G_t \equiv \int_t^\infty \alpha_t(u)\,\lambda_u e^{-\Lambda_u}\,du = \frac{Z_t}{Y_t}\,e^{-\Lambda_t}$$

and hence, the intensity rate process is

$$\lambda_t = \frac{\alpha_t(t)}{G_t}\varphi(t)$$

for all $t \geq 0$.

Following the case of constant intensity rate, this leads to the choice of a process x such that the process $Z = (Z_t)_{t \geq 0}$ given by

$$Z_t = 1 + \int_0^t e^{-\Lambda_s} x_s\,\mathcal{E}_s(x \star \widehat{W})\,d\widehat{W}_s$$

with

$$\widehat{W}_t = W_t + \int_0^t x_s\,ds$$

is non-negative. Then, we have

$$Y_t = \exp\left(\int_0^t x_s\,dW_s + \frac{1}{2}\int_0^t x_s^2\,ds\right)$$

for $t \geq 0$, and

$$dG_t = -\lambda_t G_t\,dt + x_t(1 - G_t)\,dW_t$$

is satisfied.

3.2.2 A density with a given Azéma supermartingale

In full generality, the multiplicative decomposition of the Azéma supermartingale $G = (G_t)_{t \geq 0}$ associated with a random time τ is

$$\mathbb{P}(\tau > t \,|\, \mathcal{F}_t) = n_t\,e^{-\Lambda_t}$$

where n is a martingale (satisfying $0 < n_t\,e^{-\Lambda_t} < 1$ for $t \geq 0$) and Λ is an increasing process. We recall again that the knowledge of this supermartingale does not allow for to reconstruct the density process.

It suffices to construct two processes Y and Z satisfying the conditions above and such that $n_t = Z_t/Y_t$ for $t \geq 0$. Since

$$d\Big(\frac{Z_t}{Y_t}\Big) = -\frac{Z_t}{Y_t}x_t\left(1 - \frac{Y_t}{Z_t}e^{\Lambda_t}\right)dW_t$$

holds, writing the representation theorem for the positive martingale n on the form

$$dn_t = n_t \nu_t\,dW_t$$

with an appropriate square integrable $\mathbb{F}$-predictable process $\nu = (\nu_t)_{t \geq 0}$ leads to

$$x_t = -\frac{\nu_t n_t}{n_t - e^{\Lambda_t}}$$

for all $t \geq 0$.

3.2.3 Change of probabilities

Our aim is now to construct random times with a given intensity. As recalled above, the traditional approach is to set
$$\tau = \inf\{t \,:\, \Lambda_t > -\ln U\}$$
where U is an uniformly random variable on $(0,1)$ independent of $\mathcal{F}_\infty$. Then, we have

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty \alpha_t(v)\,\eta(dv)$$

where

$$\begin{aligned}
\eta(du) &\equiv \mathbb{P}(\tau \in du) = \mathbb{E}[\lambda_u e^{-\Lambda_u}]\,du = \varphi(u)\,du \\
\alpha_t^{\mathbb{P}}(u) &= \frac{\lambda_u\, e^{-\Lambda_u}}{\varphi(u)},\ \text{for}\, t \geq u \\
&= \frac{1}{\varphi(u)}\,\mathbb{E}[\lambda_u\, e^{-\Lambda_u} \,|\, \mathcal{F}_t]\,,\ \text{for}\, t < u \\
G_t \equiv \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) &= e^{-\Lambda_t} \\
\lambda_t^{\mathbb{P}} &= \lambda_t\,.
\end{aligned}$$

Let us now assume that n is an $\mathbb{F}$-martingale of the form

$$dn_t = n_t \nu_t\, dW_t$$

such that $0 < n_t\, e^{-\Lambda_t} < 1$ for $t \geq 0$, and start with τ as above. From the first part, one can construct a density process $(\alpha_t(u))_{t\geq 0}$, $u \geq 0$, with two processes Y and Z satisfying $n_t = Z_t/Y_t$, for all $t \geq 0$. Then, we set
$$d\mathbb{Q}|_{\mathcal{G}_t} = L_t^{\mathbb{Q}}\, d\mathbb{P}|_{\mathcal{G}_t}$$
where $L^{\mathbb{Q}}$ is the $\mathbb{G}$-martingale defined as

$$L_t^{\mathbb{Q}} = \mathbb{1}_{t<\tau}\, e^{\Lambda_t} \int_t^\infty \alpha_t(u)\,\eta(du) + \mathbb{1}_{\tau\leq t}\,\alpha_t(\tau)$$

for $t \geq 0$. From [2], we obtain that under $\mathbb{Q}$:

$$\begin{aligned}
\alpha_t^{\mathbb{Q}}(u) &= \alpha_t(u) \\
G_t^{\mathbb{Q}} \equiv \mathbb{Q}(\tau > t \,|\, \mathcal{F}_t) &= \frac{Z_t}{Y_t} e^{-\Lambda_t} = n_t\, e^{-\Lambda_t} \\
\lambda_t^{\mathbb{Q}} &= \lambda_t \\
\mathbb{Q}(\tau \in du) &= \mathbb{P}(\tau \in du)
\end{aligned}$$

so that, we obtain a random time τ with the given intensity $(\lambda_t)_{t\geq 0}$ with a given Azéma supermartingale G but, at the same time, with a different density process. Note that $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ holds for all $t \geq 0$. The process n is a $(\mathbb{P}, \mathbb{F})$ and a $(\mathbb{P}, \mathbb{G})$-local martingale, hence a $(\mathbb{Q}, \mathbb{F})$-local martingale, but is not a $(\mathbb{Q}, \mathbb{G})$-local martingale (see [2] for a characterization of $\mathbb{G}$-martingales).

4 Impact on the derivative pricing

4.1 Deterministic intensities

Let us first deal with the case of deterministic intensities, in a model where the interest rate is deterministic too (in what follows, for simplicity, the interest rate is null). More precisely, we consider the case where $G_t = n_t\, e^{-\Lambda_t}$ and $\Lambda = (\Lambda_t)_{t\geq 0}$ is a deterministic process. We shall see that, in that setting, the value of n has no influence on the prices of classical derivatives, as a defaultable zero-coupon bond (DZC) or the value of the spread of a credit default swap (CDS) with deterministic recovery. This means that calibration issues based on these assets will allow to recover only the intensity and give no information on the martingale n.

4.1.1 Defaultable zero-coupon bonds

It is known (see, e.g. [1]) that the price of a DZC is equal to

$$D(t,T) = \mathbb{1}_{t<\tau}\, \mathbb{Q}(T<\tau \,|\, \mathcal{G}_t) = \mathbb{1}_{t<\tau}\, \frac{\mathbb{Q}(T<\tau \,|\, \mathcal{F}_t)}{G_t} = \mathbb{1}_{t<\tau}\, \frac{\mathbb{E}_{\mathbb{Q}}[n_T e^{-\Lambda_T} \,|\, \mathcal{F}_t]}{G_t}$$

for all $0 \leq t \leq T$. Hence, assuming that $(\lambda_t)_{t\leq T}$ is deterministic and $(n_t)_{t\leq T}$ is a true martingale, we get

$$D(t,T) = \mathbb{1}_{t<\tau}\, e^{-(\Lambda_T - \Lambda_t)}$$

for $0 \leq t \leq T$.

4.1.2 Credit default swaps

It is also known (see, e.g. [1]) that the price of a CDS with recovery δ is equal to

$$S_t = \mathbb{1}_{t<\tau}\, \frac{1}{G_t}\, \mathbb{E}_{\mathbb{Q}}\left[\delta_\tau \mathbb{1}_{t<\tau\leq T} - \kappa((T\wedge\tau)-t)\mathbb{1}_{t<\tau} \,|\, \mathcal{F}_t\right] \tag{17}$$

for all $0 \leq t \leq T$. In terms of the Azéma's supermartingale and the intensity rate, the expression of (17) can be written as

$$S_t = \mathbb{1}_{\{t<\tau\}}\, \frac{1}{G_t}\, \mathbb{E}_{\mathbb{Q}}\left[\int_t^T G_u(\delta_u\lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t\right]$$

for $0 \leq t \leq T$ (see [1]). If the intensity rate and the recovery are deterministic, we have

$$S_t = \mathbb{1}_{\{t<\tau\}}\, \frac{1}{G_t} \int_t^T \mathbb{E}_{\mathbb{Q}}[G_u \,|\, \mathcal{F}_t](\delta_u\lambda_u - \kappa)\, du = \mathbb{1}_{\{t<\tau\}}\, e^{\Lambda_t} \int_t^T e^{-\Lambda_u}(\delta_u\lambda_u - \kappa)\, du$$

that does not depend on the particular choice of n.

As we have seen before, we have

$$S_t = \mathbb{1}_{\{t<\tau\}}\, \frac{1}{G_t}\, \mathbb{E}_{\mathbb{Q}}\left[\int_t^T G_u(\delta_u\lambda_u - \kappa)\, du \,\Big|\, \mathcal{F}_t\right]$$

and the spread is equal to

$$\kappa = \frac{\mathbb{E}_{\mathbb{Q}}\left[\int_0^T G_u\delta_u\lambda_u\, du\right]}{\mathbb{E}_{\mathbb{Q}}\left[\int_0^T G_u\, du\right]}.$$

If we consider the model where $G_t = n_t\, e^{-\lambda t}$, assuming that $(n_t)_{t\leq T}$ is a true martingale, the quantity

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T G_u\, du\right] = \int_0^T e^{-\lambda u}\, du$$

does not depend on n. However, the numerator, in the case of a stochastic recovery depends on n: indeed writing $n_t = Z_t/Y_t$ one has

$$\mathbb{E}_{\mathbb{Q}}\left[\int_0^T G_u\delta_u\lambda_u\, du\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \alpha_u(u)\delta_u\, du\right] = \mathbb{E}_{\mathbb{Q}}\left[\int_0^T \frac{Z_u}{Y_u}\, \delta_u\lambda e^{-\lambda u}\, du\right].$$

As an example, when $\delta_t = Y_t/Z_t$, the spread is equal to

$$\frac{1-e^{-\lambda T}}{\int_0^T e^{-\lambda u}\,du} = \lambda$$

whereas, in the model with the same intensity and $n \equiv 1$, the spread turns out to be

$$\frac{1}{1-e^{-\lambda T}} \int_0^T \mathbb{E}_{\mathbb{Q}}\left[\frac{Y_u}{Z_u}\right] \lambda e^{-\lambda u}\,du\,.$$

4.2 Stochastic Intensities

In the case of stochastic intensities (and/or stochastic interest rate), the problem is more difficult. Let $\beta = (\beta_t)_{t\geq 0}$ defined by

$$\beta_t = \exp\left(-\int_0^t r_s\,ds\right)$$

be the discounting factor, and assume that the process $r = (r_t)_{t\geq 0}$ is $\mathbb{F}$-adapted. The price of a DZC is then equal to

$$D(t,T) = \mathbb{1}_{t<\tau}\,\frac{1}{\beta_t}\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{T<\tau}\beta_T\,|\,\mathcal{G}_t) = \mathbb{1}_{t<\tau}\frac{1}{G_t\beta_t}\,\mathbb{E}_{\mathbb{Q}}(\mathbb{1}_{T<\tau}\beta_T\,|\,\mathcal{F}_t) = \mathbb{1}_{t<\tau}\frac{1}{G_t\beta_t}\,\frac{\mathbb{E}_{\mathbb{Q}}[n_T e^{-\Lambda_T}\beta_T\,|\,\mathcal{F}_t]}{G_t}$$

for all $0 \leq t \leq T$, and the role of n is important. The role of the density appears more in a case when the recovery is paid at maturity or in a multidefault setting. In the former case, the computation of $\mathbb{E}_{\mathbb{Q}}(R_\tau \mathbb{1}_{\tau<T}\beta_T|\mathcal{G}_t)$ requires the knowledge of $\mathbb{Q}(R_\tau \mathbb{1}_{\tau<T}\beta_T|\mathcal{F}_t)$, and thus, the knowledge of the density process.

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5 Appendix

5.1 Relation between various martingales

We recall the results obtained in [2].

5.1.1 Characterisation of $\mathbb{G}$-martingales

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a reference filtration $\mathbb{F}$, and τ a random time. We denote by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ and assume that τ admits a density $\alpha^{\mathbb{P}}$. We assume here that $G_t \equiv \mathbb{P}(\tau > t \,|\, \mathcal{F}_t) > 0$ for all $t \geq 0$ (see [2] for the general case).
A càdlàg $\mathbb{G}$-adapted process $Y_t^{\mathbb{G}} = (Y_t^{\mathbb{G}})_{t\geq 0}$ defined by

$$Y_t^{\mathbb{G}} = Y_t \mathbb{1}_{\{\tau > t\}} + Y_t(\tau)\mathbb{1}_{\{\tau \leq t\}}$$

is a $\mathbb{G}$-local martingale if and only the two following conditions are satisfied
(1) the process $(Y_t G_t + \int_0^t Y_s(s)\alpha_s^{\mathbb{P}}(s)\eta(ds))_{t\geq 0}$ is a $\mathbb{F}$-local martingale,
(2) for any fixed $s \geq 0$, the processes $(Y_t(s)\alpha_t^{\mathbb{P}}(s))_{t\geq s}$, are $\mathbb{F}$-local martingales.

5.1.2 Girsanov's Theorem

Let $Q^{\mathbb{G}} = (Q_t^{\mathbb{G}})_{t\geq 0}$ defined by

$$Q_t^{\mathbb{G}} = q_t \mathbb{1}_{\{\tau > t\}} + q_t(\tau)\mathbb{1}_{\{\tau \leq t\}}$$

be a càdlàg positive $\mathbb{G}$-martingale with $Q_0^{\mathbb{G}} = q_0 = 1$. Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by

$$d\mathbb{Q}|_{\mathcal{G}_t} = Q_t^{\mathbb{G}}\, d\mathbb{P}|_{\mathcal{G}_t}$$

for any $t \in \mathbb{R}_+$, and $\mathbb{Q}^{\mathbb{F}}$ be the restriction of $\mathbb{Q}$ to $\mathbb{F}$, which has the Radon-Nikodým density $Q^{\mathbb{F}}$, given by the projection of $Q^{\mathbb{G}}$ on $\mathbb{F}$, that is

$$Q_t^{\mathbb{F}} = q_t G_t + \int_0^t q_t(s)\alpha_t^{\mathbb{P}}(s)\,\eta(ds)$$

for $t \geq 0$. Then, under $\mathbb{Q}$, the random time τ admits a density with respect to $\mathbb{F}$, given by

$$\begin{aligned} \alpha_t^{\mathbb{Q}}(s) &= \alpha_t^{\mathbb{P}}(s)\,\frac{q_t(s)}{Q_t^{\mathbb{F}}}, \quad \text{for } t > s \\ &= \frac{1}{Q_t^{\mathbb{F}}}\,\mathbb{E}^{\mathbb{P}}[\alpha_s^{\mathbb{P}}(s) q_s(s) | \mathcal{F}_t], \quad \text{for } t \leq s\ . \end{aligned}$$

Furthermore:
1) the $\mathbb{Q}$-conditional survival process is defined by

$$S_t^{\mathbb{Q}} = S_t\,\frac{q_t}{Q_t^{\mathbb{F}}}\ ;$$

2) the $(\mathbb{F}, \mathbb{Q})$-intensity process is

$$\lambda_t^{\mathbb{F},\mathbb{Q}} = \lambda_t^{\mathbb{F}}\,\frac{q_t(t)}{q_t}, \qquad \eta(dt)\text{- a.s.}$$

5.2 Derivation of the density

The following arguments are presented in [5, Chapter IV, Section 4] but omitted in [6, Chapter IV, Section 4].

Suppose that the initial probability measure $\mathbb{P}$ has the following structure

$$\mathbb{P} = \int_0^\infty \mathbb{P}^s \, \eta(ds) \tag{18}$$

where P^s is a distribution law of the process X under condition that τ has happened at the time s, for $s \geq 0$. Thus, $\mathbb{P}(X \in \cdot \,|\, \tau = s) = \mathbb{P}^s(X \in \cdot\,)$ is the distribution law of a geometric Brownian motion with the diffusion coefficient $\sigma > 0$ and a local drift changing from a to $a + b$ at time $s \geq 0$.

By means of generalized Bayes' formula, we get

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \int_u^\infty \frac{d\mathbb{P}^s}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \eta(ds) \tag{19}$$

for all $t, u \geq 0$. Then, using the fact that the measures $\mathbb{P}^s$, $[0, \infty]$, are locally equivalent on the filtration $(\mathcal{F}_t)_{t \geq 0}$ by construction, we obtain

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \left(\int_u^\infty \frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) \right) \Big/ \left(\int_0^\infty \frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) \right) \tag{20}$$

for $t, u \geq 0$.

Note that, from construction of the probability measures $\mathbb{P}^s$, $[0, \infty]$, it follows that the property

$$\frac{d\mathbb{P}^s}{d\mathbb{P}^0}\Big|_{\mathcal{F}_t} = \frac{d\mathbb{P}^\infty}{d\mathbb{P}^0}\Big|_{\mathcal{F}_{t \wedge s}}$$

holds, for all $t, s \geq 0$. Hence, using the fact that $\mathbb{P}^s$ coincides with $\mathbb{P}^\infty$ on $\mathcal{F}_t$ under $s > t$, we have

$$\begin{aligned} \int_u^\infty \frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) &= \int_{u \wedge t}^t \frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) + \int_{u \vee t}^\infty \frac{d\mathbb{P}^s}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) \\ &= \int_{u \wedge t}^t \frac{d\mathbb{P}^s}{d\mathbb{P}^0}\Big|_{\mathcal{F}_t} \frac{d\mathbb{P}^0}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \eta(ds) + \eta(u \vee t, \infty) \\ &= \frac{d\mathbb{P}^0}{d\mathbb{P}^\infty}\Big|_{\mathcal{F}_t} \int_{u \wedge t}^t \frac{d\mathbb{P}^\infty}{d\mathbb{P}^0}\Big|_{\mathcal{F}_s} \eta(ds) + \eta(u \vee t, \infty) \end{aligned} \tag{21}$$

so that, the denominator in (20) can be obtained by setting $u = 0$ in (21).

We may therefore conclude that the conditional probability in (19) admits the representation from (7), where Y and Z are given by the formulas in (8) and (9), respectively. Note that we have

$$Z_t = \frac{d\mathbb{P}^\infty}{d\mathbb{P}^0}\Big|_{\mathcal{F}_t}$$

and $\eta(dt) = \lambda e^{-\lambda t} \, dt$, for any $t \geq 0$ and some $\lambda > 0$ fixed there.