

From an Itô type calculus for Gaussian processes to integrals of log-normal processes increasing in the convex order

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Abstract: In this paper, we present an Itô type formula for a Gaussian process, in which only the one-marginals of the Gaussian process are involved. Thus, this formula is well adapted to the study of processes increasing in the convex order, in a Gaussian framework. In particular, we give conditions ensuring that processes defined as integrals, with respect to one parameter, of exponentials of two-parameter Gaussian processes, are increasing in the convex order with respect to the other parameter. Finally, we construct Gaussian sheets allowing to exhibit martingales with the same one-marginals as the previously defined processes.

Key words: convex order; 1-martingale; Gaussian process; Gaussian sheet; log-normal process; Itô type formula; Itô's calculus.

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1 Introduction

The following notation will be used throughout our paper:

- If X and Y are two real valued random variables,

$$X \stackrel{d}{=} Y$$

means that these variables have the same law.

- If $(X_t, t \geq 0)$ and $(Y_t, t \geq 0)$ are two real valued processes,

$$(X_t, t \geq 0) \stackrel{(d)}{=} (Y_t, t \geq 0)$$

means that the two processes are identical in law.

- \mathbb{S}_n denotes the space of $n \times n$ symmetric matrices with real entries, and \mathbb{S}_n^+ denotes the convex cone in \mathbb{S}_n consisting of positive matrices. Thus, a matrix $(m_{j,k})_{1 \leq j,k \leq n} \in \mathbb{S}_n$ belongs to \mathbb{S}_n^+ if:

$$\forall \alpha_1, \dots, \alpha_n \in \mathbb{R} \quad \sum_{1 \leq j,k \leq n} \alpha_j \alpha_k m_{j,k} \geq 0 .$$

1.1 PCOC's and 1-martingales

An \mathbb{R} -valued process $(X_t, t \geq 0)$ is said to *increase in the convex order* if

$$\forall t \geq 0 \quad \mathbb{E}[|X_t|] < \infty ,$$

and for every convex function $\psi : \mathbb{R} \rightarrow \mathbb{R}$,

$$t \in \mathbb{R}_+ \rightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

We call such a process $(X_t, t \geq 0)$ a PCOC, this acronym being derived from the French name: Processus Croissant pour l'Ordre Convexe.

A process $(X_t, t \geq 0)$ is called a *1-martingale* if there exists (on a suitable filtered probability space) a martingale $(M_t, t \geq 0)$ which has the same one-dimensional marginals as $(X_t, t \geq 0)$, that is, for each $t \geq 0$,

$$X_t \stackrel{d}{=} M_t .$$

Such a martingale $(M_t, t \geq 0)$ is said to be *associated* with this process $(X_t, t \geq 0)$. Note that several different martingales may be associated with a given process.

It is an easy consequence of Jensen's inequality that an \mathbb{R} -valued process $(X_t, t \geq 0)$ which is a 1-martingale, is a PCOC. A remarkable result due to Kellerer [Ke] states that, conversely, any \mathbb{R} -valued process $(X_t, t \geq 0)$ which is a PCOC, is a 1-martingale. But generally, it is a difficult problem to give a concrete description of a martingale which is associated to a PCOC; this problem has been the aim of several recent papers ([BY, HY1, HY2, HY3, HRY]).

1.2 Our guiding example

Our interest in the study of PCOC's and associated martingales originated from the result by Carr, Ewald and Xiao [CEX] that the process:

$$A_t = \frac{1}{t} \int_0^t \exp\left(B_s - \frac{s}{2}\right) ds \quad , \quad t \geq 0 \quad ,$$

where $(B_s, s \geq 0)$ is a standard Brownian motion, is a PCOC. It has been shown later by Baker and Yor [BY] that a martingale associated with this process $(A_t, t \geq 0)$ is

$$M_t = \int_0^1 \exp\left(W_{s,t} - \frac{st}{2}\right) ds \quad , \quad t \geq 0$$

where W denotes the standard Brownian sheet.

1.3 Generalizations of our guiding example

The ubiquity of Brownian motion stems, for a large part, from the fact that it belongs to the intersection of important classes of stochastic processes, e.g: martingales, Lévy processes, Gaussian processes. Thus, the solution of a given problem involving Brownian motion often generalizes into one involving either of these classes of stochastic processes. The above result by Carr, Ewald and Xiao is no exception to this rule, as it has already been generalized as follows:

- (i) If $(N_t, t \geq 0)$ is a martingale, then the process

$$\frac{1}{t} \int_0^t N_s ds \quad , \quad t \geq 0$$

is a PCOC (see [HPRY]).

- (ii) An interesting particular case of (i) is: if $(L_t, t \geq 0)$ is a Lévy process such that $\mathbb{E}[\exp(L_1)] < \infty$, then the process

$$\frac{1}{t} \int_0^t \frac{\exp(L_s)}{\mathbb{E}[\exp(L_s)]} ds \quad , \quad t \geq 0$$

is a PCOC and an associated martingale may be expressed, using a Lévy sheet (see [BY, HY2, HRY]).

In this paper, we are concerned with generalizations of the result by Carr, Ewald and Xiao, in a Gaussian framework. Thus, we consider a family $(G_{\bullet,t}, t \geq 0)$ of real valued, centered, Gaussian processes:

$$G_{\bullet,t} := (G_{\lambda,t}, \lambda \in \Lambda) ,$$

where Λ denotes a measure space. For any signed finite measure σ on Λ , we set, for $t \geq 0$,

$$A_t^{(\sigma)} = \int_{\Lambda} \exp \left(G_{\lambda,t} - \frac{1}{2} \mathbb{E}[(G_{\lambda,t})^2] \right) \sigma(d\lambda)$$

and we give conditions ensuring that $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC. We also express, in some cases, an associated martingale in terms of a Gaussian sheet.

1.4 Organisation of the paper

The remainder of this paper is organised as follows:

- In Section 2, we develop an Itô type calculus for Gaussian processes.
- In Section 3, we use the previous calculus to prove that, under certain conditions, processes $(A_t^{(\sigma)}, t \geq 0)$ as defined in Subsection 1.3 are PCOC's.
- In Section 4, we construct Gaussian sheets allowing, for some processes $(A_t^{(\sigma)}, t \geq 0)$, to exhibit martingales having the same one-dimensional marginals. This yields, in these cases, another proof that they are PCOC's.

2 An Itô type calculus for centered Gaussian processes

2.1 An Itô type formula

In this subsection, we consider a family of \mathbb{R}^n -valued centered Gaussian variables:

$$(G_t, t \in [a, b])$$

where $[a, b]$ denotes a compact interval of \mathbb{R} . We denote, for $t \in [a, b]$, by $(G_t^{(1)}, \dots, G_t^{(n)})$ the components of the vector G_t , and by

$$C(t) = (c_{j,k}(t))_{1 \leq j, k \leq n}$$

the covariance matrix of G_t . The main result of this subsection is the following weak form of an Itô type formula.

Theorem 2.1 *Assume that*

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is a continuous function with finite variation. Let

$$F : (x, t) \in \mathbb{R}^n \times [a, b] \longrightarrow F(x, t) \in \mathbb{R}$$

be a $C^{2,1}$ -function whose derivatives of order 2 with respect to x : F''_{x_j, x_k} , $1 \leq j, k \leq n$, and whose derivative of order 1 with respect to t : F'_t , grow sub-exponentially at infinity with respect to x , uniformly with respect to $t \in [a, b]$. Then, for every s, t with $a \leq s \leq t \leq b$,

$$(1) \quad \mathbb{E}[F(G_t, t)] = \mathbb{E}[F(G_s, s)] + \int_s^t \mathbb{E}[F'_t(G_u, u)] du \\ + \frac{1}{2} \sum_{j,k} \int_s^t \mathbb{E}[F''_{x_j, x_k}(G_u, u)] dc_{j,k}(u).$$

Proof

The proof is based on the following lemma:

Lemma 2.1 *Assume that*

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is a continuous function with finite variation, and that, for any $t \in [a, b]$, the matrix $C(t)$ is invertible. We denote by $\mu(t, x)$ the density of the Gaussian variable G_t . Then, for every s, t with $a \leq s \leq t \leq b$ and for any $x \in \mathbb{R}^n$,

$$(2) \quad \mu(t, x) = \mu(s, x) + \frac{1}{2} \sum_{j,k} \int_s^t \mu''_{x_j, x_k}(u, x) \, dc_{j,k}(u) .$$

Proof Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{R}^n$, and denote by $\langle x, y \rangle$ the scalar product of $x, y \in \mathbb{R}^n$. Set, for $t \in [a, b]$,

$$\varphi_\gamma(t) = \int_{\mathbb{R}^n} e^{i \langle \gamma, x \rangle} \mu(t, x) \, dx .$$

We have, for $t \in [a, b]$,

$$\varphi_\gamma(t) = \mathbb{E}[\exp(i \langle \gamma, G_t \rangle)] = \exp \left(-\frac{1}{2} \sum_{j,k} \gamma_j \gamma_k c_{j,k}(t) \right) .$$

Then, for $a \leq s \leq t \leq b$,

$$\varphi_\gamma(t) = \varphi_\gamma(s) - \frac{1}{2} \sum_{j,k} \int_s^t \varphi_\gamma(u) \gamma_j \gamma_k \, dc_{j,k}(u) .$$

Now, an integration by parts yields:

$$\varphi_\gamma(u) \gamma_j \gamma_k = \int_{\mathbb{R}^n} e^{i \langle \gamma, x \rangle} \gamma_j \gamma_k \mu(u, x) \, dx = - \int_{\mathbb{R}^n} e^{i \langle \gamma, x \rangle} \mu''_{x_j, x_k}(u, x) \, dx .$$

Finally, using Fubini's theorem, we obtain that the Fourier transforms of both sides in (2) are equal, hence the desired result follows thanks to the injectivity of the Fourier transform. \square

We now prove Theorem 2.1. First, we may assume, by approximation, that F in (1) has compact support. Approximating G_t by $G_t + \varepsilon N$ where N denotes an \mathbb{R}^n -valued centered reduced Gaussian variable independent of $(G_t, t \in [a, b])$, we also may assume that the hypotheses of Lemma 2.1 are fulfilled. We still denote by $\mu(t, x)$ the density of G_t . Hence, for $t \in [a, b]$,

$$\mathbb{E}[F(G_t, t)] = \int F(x, t) \mu(t, x) \, dx .$$

Lemma 2.1 ensures that, for $a \leq s \leq t \leq b$ and $x \in \mathbb{R}^n$,

$$(3) \quad F(x, t) \mu(t, x) = F(x, s) \mu(s, x) + \int_s^t F'_t(x, u) \mu(u, x) \, du \\ + \frac{1}{2} \sum_{j,k} \int_s^t F(x, u) \mu''_{x_j, x_k}(u, x) \, dc_{j,k}(u) .$$

Now, by integration by parts,

$$\int F(x, u) \mu''_{x_j, x_k}(u, x) \, dx = \int F''_{x_j, x_k}(x, u) \mu(u, x) \, dx = \mathbb{E}[F''_{x_j, x_k}(G_u, u)] .$$

Therefore, integration with respect to x on both sides of (3) yields formula (1). □

Remarks

1. In formula (1), only the law of G_t for *each* $t \in [a, b]$, and consequently only the matrices $C(t)$, are involved. This explains why this formula is well adapted to the study of PCOC's.
2. Suppose that

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is an absolutely continuous function on $[a, b]$ and that the derivative $C'(t)$ is, for almost every $t \in [a, b]$, a positive symmetric matrix. Then, there exists a measurable function

$$t \in [a, b] \longrightarrow D(t) \in \mathbb{S}_n^+$$

such that, for almost every $t \in [a, b]$,

$$[D(t)]^2 = C'(t) .$$

In particular,

$$t \in [a, b] \longrightarrow D(t) \in \mathbb{S}_n$$

is a square integrable function. We set

$$M_t = G_a + \int_a^t D(s) \, dB_s$$

where B denotes a standard \mathbb{R}^n -valued Brownian motion starting from 0, independent of G_a . Then $(M_t, t \in [a, b])$ is a continuous Gaussian martingale, and, for any $t \in [a, b]$, the covariance matrix of M_t is $C(t)$. Consequently, to prove (1) in this case, we may replace $(G_t, t \in [a, b])$ by $(M_t, t \in [a, b])$, but then (1) is a direct consequence of the classical Itô formula.

3. As a Fourier argument easily shows, formula (1) characterizes, in some sense, families of \mathbb{R}^n -valued centered Gaussian variables.

2.2 Examples

We present, in this subsection, some examples in the scalar case $n = 1$. In particular, when $(G_t, t \in [a, b])$ has the same one-marginals as a semi-martingale, we compare our formula (1) with the one obtained by application of the classical Itô formula.

2.2.1 Time changed Brownian motion

We consider continuous functions u and v from an interval $[a, b]$ into \mathbb{R} and we suppose that u has a finite variation and v is increasing and nonnegative. Let (B_t) be the standard linear Brownian motion starting from 0 and set, for $t \in [a, b]$,

$$G_t = u(t) B_{v(t)} .$$

We have $c(t) = \mathbb{E}[G_t^2] = u^2(t)v(t)$. Let F be a C^2 -function on \mathbb{R} with compact support. Our formula (1) yields:

$$d_t \mathbb{E}[F(G_t)] = \frac{1}{2} \mathbb{E}[F''(G_t)] [2u(t)v(t) du(t) + u^2(t) dv(t)] ,$$

whereas the application of the classical Itô formula gives:

$$d_t \mathbb{E}[F(G_t)] = \mathbb{E}[F'(G_t) B_{v(t)}] du(t) + \frac{1}{2} \mathbb{E}[F''(G_t)] u^2(t) dv(t) .$$

Consequently, we obtain:

$$\mathbb{E}[F''(u(t) B_{v(t)})] u(t) v(t) = \mathbb{E}[F'(u(t) B_{v(t)}) B_{v(t)}] .$$

The above equality may be written down as:

$$(4) \quad \mathbb{E}[F''(\alpha B_1)] \alpha = \mathbb{E}[F'(\alpha B_1) B_1]$$

with $\alpha = u(t) \sqrt{v(t)}$. Obviously, α may be taken equal to 1, and (4) is a well-known characterization of the law of B_1 . In fact, formula (4) with g instead of F' , is the starting point of Stein's method (see [S]); see also [T, Appendix A4: Gaussian integration by parts].

The following examples may be considered as particular cases of the previous one.

2.2.2 Ornstein-Uhlenbeck process

For $\lambda \in \mathbb{R}$, let $(U_t^\lambda, t \geq 0)$ be the scalar Ornstein-Uhlenbeck process with parameter λ , starting from 0. Thus, U^λ is solution to the SDE:

$$(5) \quad U_t^\lambda = B_t + \lambda \int_0^t U_s^\lambda \, ds .$$

We have:

$$\forall t \geq 0 \quad U_t^\lambda = e^{\lambda t} \int_0^t e^{-\lambda s} \, dB_s .$$

Consequently, there exists a Brownian motion (β_u) such that:

$$\forall t \geq 0 \quad U_t^\lambda = e^{\lambda t} \beta_{v(t)}$$

with

$$v(t) = \int_0^t e^{-2\lambda s} \, ds .$$

We are therefore brought back to Example 2.2.1.

On the other hand, denoting by $\mu^{(\lambda)}(t, x)$ the density of $U_t^{(\lambda)}$, the Fokker-Planck equation corresponding to the SDE (5) yields:

$$\frac{\partial \mu^{(\lambda)}}{\partial t}(t, x) = -\lambda \frac{\partial}{\partial x}(x \mu^{(\lambda)}(t, x)) + \frac{1}{2} \frac{\partial^2 \mu^{(\lambda)}}{\partial x^2}(t, x) ,$$

whereas our Lemma 2.1 leads to:

$$\frac{\partial \mu^{(\lambda)}}{\partial t}(t, x) = \frac{1}{2} e^{2\lambda t} \frac{\partial^2 \mu^{(\lambda)}}{\partial x^2}(t, x) .$$

Consequently, we obtain:

$$\frac{\partial}{\partial x}(x \mu^{(\lambda)}(t, x)) = \frac{1 - e^{2\lambda t}}{2\lambda} \frac{\partial^2 \mu^{(\lambda)}}{\partial x^2}(t, x) ,$$

which is easy to verify directly.

2.2.3 Brownian bridge

Let $(b_t, 0 \leq t \leq 1)$ be the standard Brownian bridge satisfying $b_0 = b_1 = 0$. It can be obtained as solution to the SDE:

$$(6) \quad X_t = B_t - \int_0^t \frac{X_s}{1-s} ds$$

and one has: $\mathbb{E}[b_t^2] = t(1-t)$. Let F be a C^2 -function on \mathbb{R} with compact support. Our formula (1) yields:

$$\frac{d}{dt} \mathbb{E}[F(b_t)] = \frac{1-2t}{2} \mathbb{E}[F''(b_t)],$$

whereas the application of the classical Itô formula gives:

$$\frac{d}{dt} \mathbb{E}[F(b_t)] = -\frac{1}{1-t} \mathbb{E}[F'(b_t) b_t] + \frac{1}{2} \mathbb{E}[F''(b_t)].$$

Consequently, we obtain:

$$t(1-t) \mathbb{E}[F''(b_t)] = \mathbb{E}[F'(b_t) b_t],$$

which is equivalent to (4) with $\alpha = \sqrt{t(1-t)}$.

On the other hand, there exists a Brownian motion (β_u) such that:

$$\forall t \in [0, 1) \quad b_t = (1-t) \beta_{t/(1-t)}.$$

Thus, here again, we are brought back to Example 2.2.1.

Besides, denoting by $\mu(t, x)$ the density of b_t for $0 < t < 1$, the Fokker-Planck equation corresponding to the SDE (6) yields:

$$\frac{\partial \mu}{\partial t}(t, x) = \frac{1}{1-t} \frac{\partial}{\partial x} (x \mu(t, x)) + \frac{1}{2} \frac{\partial^2 \mu}{\partial x^2}(t, x),$$

whereas our Lemma 2.1 leads to:

$$\frac{\partial \mu}{\partial t}(t, x) = \frac{1-2t}{2} \frac{\partial^2 \mu}{\partial x^2}(t, x).$$

Consequently, we obtain:

$$\frac{\partial}{\partial x} (x \mu(t, x)) = -t(1-t) \frac{\partial^2 \mu}{\partial x^2}(t, x),$$

which is easy to verify directly.

2.2.4 Fractional Brownian motion

Let $(B_t^H, t \geq 0)$ be the fractional Brownian motion with Hurst index $H \in (0, 1)$. It is a continuous centered Gaussian process such that, for any $t \geq 0$, $\mathbb{E}[(B_t^H)^2] = t^{2H}$. Let F be a C^2 -function on \mathbb{R} with compact support. Our formula (1) yields, for $0 \leq s \leq t$:

$$(7) \quad \mathbb{E}[F(B_t^H)] = \mathbb{E}[F(B_s^H)] + H \int_s^t \mathbb{E}[F''(B_u^H)] u^{2H-1} du .$$

On the other hand, there exists a Brownian motion (β_u) such that, for each $t \geq 0$,

$$B_t^H \stackrel{d}{=} \beta_{t^{2H}} .$$

Formula (7) also follows from the classical Itô formula applied to the martingale $(\beta_{t^{2H}})$.

2.3 An application of Theorem 2.1

As an application of Theorem 2.1, we now present a simple proof of the Gordon-Slepian Lemma, the statement of which we recall below (see, for instance, [Ka] and [T, Proposition 1.3.2]).

Proposition 2.1 (Gordon-Slepian Lemma) *Let $X = (X_1, \dots, X_n)$ and $Y = (Y_1, \dots, Y_n)$ be two centered Gaussian vectors in \mathbb{R}^n , and let A and B be two subsets of $\{1, \dots, n\} \times \{1, \dots, n\}$. We assume:*

$$\begin{aligned} \mathbb{E}[X_j X_k] &\leq \mathbb{E}[Y_j Y_k] && \text{if } (j, k) \in A \\ \mathbb{E}[X_j X_k] &\geq \mathbb{E}[Y_j Y_k] && \text{if } (j, k) \in B \\ \mathbb{E}[X_j X_k] &= \mathbb{E}[Y_j Y_k] && \text{if } (j, k) \notin A \cup B \end{aligned}$$

Let $F : \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^2 -function whose derivatives of order 2 grow sub-exponentially at infinity. We assume:

$$F''_{x_j, x_k} \geq 0 \quad \text{if } (j, k) \in A \quad \text{and} \quad F''_{x_j, x_k} \leq 0 \quad \text{if } (j, k) \in B .$$

Then

$$\mathbb{E}[F(X)] \leq \mathbb{E}[F(Y)] .$$

Proof

As in Kahane's proof ([Ka]), we set, for $t \in [0, 1]$,

$$G_t = \sqrt{t} Y + \sqrt{1-t} X$$

where X and Y are assumed to be independent (a special case of the *smart path method* used, again and again, in [T]). Then, by Theorem 2.1,

$$\mathbb{E}[F(G_1)] = \mathbb{E}[F(G_0)] + \frac{1}{2} \sum_{j,k} \int_0^1 \mathbb{E}[F''_{x_j, x_k}(G_u)] (\mathbb{E}[Y_j Y_k] - \mathbb{E}[X_j X_k]) du .$$

Now, by hypothesis, for every (j, k) ,

$$(\mathbb{E}[Y_j Y_k] - \mathbb{E}[X_j X_k]) F''_{x_j, x_k} \geq 0 .$$

□

2.4 A variant of Theorem 2.1

In this subsection, we keep the framework and the notation of Subsection 2.1. A real valued function $h(x, t)$, defined on $\mathbb{R} \times \mathbb{R}_+$, will be called a *space-time harmonic function* if h is a $C^{2,1}$ -function satisfying, on $\mathbb{R} \times \mathbb{R}_+$ the equation:

$$\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} = 0 .$$

It is well-known that any nonnegative space-time harmonic function may be represented as:

$$h(x, t) = \int_{\mathbb{R}} \exp\left(yx - \frac{t y^2}{2}\right) d\nu(y)$$

where ν denotes a positive finite measure on \mathbb{R} . We refer to [Y, Theorem 1.3] for a probabilistic proof.

Theorem 2.2 *Assume that*

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is a continuous function with finite variation. Let $h^{(1)}, \dots, h^{(n)}$ be n space-time harmonic functions, and define the function $H = (H_1, \dots, H_n)$, from $\mathbb{R}^n \times [a, b]$ into \mathbb{R}^n , by

$$H_j(x, t) = h^{(j)}(x_j, c_{j,j}(t)) .$$

Let $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a C^2 -function such that the derivatives of order 2 with respect to x of the function F defined by

$$F(x, t) = \Phi[H(x, t)] ,$$

grow sub-exponentially at infinity with respect to x , uniformly with respect to $t \in [a, b]$. Then, for every s, t with $a \leq s \leq t \leq b$,

$$\begin{aligned} \mathbb{E}[\Phi[H(G_t, t)]] &= \mathbb{E}[\Phi[H(G_s, s)]] + \\ \frac{1}{2} \sum_{j,k} \int_s^t \mathbb{E} \left[\Phi''_{x_j, x_k} [H(G_u, u)] \frac{\partial h^{(j)}}{\partial x}(G_u^{(j)}, c_{j,j}(u)) \frac{\partial h^{(k)}}{\partial x}(G_u^{(k)}, c_{k,k}(u)) \right] &dc_{j,k}(u). \end{aligned}$$

Proof

If the covariance matrix C is a C^1 -function, the function F is of class $C^{1,2}$ and Theorem 2.2 follows by a direct application of Theorem 2.1, after simplifications which are consequences of the harmonicity property. Actually, the general case may be treated by a slight adaptation of the proof of Theorem 2.1. □

We now state two easy corollaries.

Corollary 2.1 *Assume that*

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is a continuous function with finite variation. Let $h^{(1)}, \dots, h^{(n)}$ be n space-time harmonic functions and $a_1, \dots, a_n \in \mathbb{R}$. Define the function k , from $\mathbb{R}^n \times [a, b]$ into \mathbb{R} , by

$$k(x, t) = \sum_j a_j h^{(j)}(x_j, c_{j,j}(t)).$$

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^2 -function such that the derivatives of order 2 with respect to x of the function F defined by

$$F(x, t) = \varphi[k(x, t)],$$

grow sub-exponentially at infinity with respect to x , uniformly with respect to $t \in [a, b]$. Then, for every s, t with $a \leq s \leq t \leq b$,

$$\begin{aligned} \mathbb{E}[\varphi[k(G_t, t)]] &= \mathbb{E}[\varphi[k(G_s, s)]] + \\ \frac{1}{2} \sum_{j,k} a_j a_k \int_s^t \mathbb{E} \left[\varphi''[k(G_u, u)] \frac{\partial h^{(j)}}{\partial x}(G_u^{(j)}, c_{j,j}(u)) \frac{\partial h^{(k)}}{\partial x}(G_u^{(k)}, c_{k,k}(u)) \right] &dc_{j,k}(u). \end{aligned}$$

Corollary 2.2 *Assume that*

$$t \in [a, b] \longrightarrow C(t) \in \mathbb{S}_n$$

is a continuous function with finite variation. Let $a_1, \dots, a_n \in \mathbb{R}$ and let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^2 -function whose second derivative grows at most polynomially at infinity. We set, for $1 \leq j \leq n$,

$$Y_u^j = \exp \left(G_u^{(j)} - \frac{c_{j,j}(u)}{2} \right)$$

and

$$K_u = \sum_j a_j Y_u^{(j)} .$$

Then, for every s, t with $a \leq s \leq t \leq b$,

$$\mathbb{E}[\varphi(K_t)] = \mathbb{E}[\varphi(K_s)] + \frac{1}{2} \sum_{j,k} a_j a_k \int_s^t \mathbb{E} [\varphi''(K_u) Y_u^{(j)} Y_u^{(k)}] \, dc_{j,k}(u) .$$

3 Application to PCOC's

3.1 Integrals of log-normal processes

Let Λ be a metric σ -compact space equipped with its Borel σ -field. We consider, for each $t \geq 0$, a real valued measurable centered Gaussian process

$$G_{\bullet,t} = (G_{\lambda,t} , \lambda \in \Lambda) .$$

For $\lambda, \mu \in \Lambda$ and $t \geq 0$, we set:

$$c_{\lambda,\mu}(t) = \mathbb{E}[G_{\lambda,t} G_{\mu,t}] .$$

We assume the following hypothesis:

(H_1) For all $t \geq 0$, the function:

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow c_{\lambda,\mu}(t) \in \mathbb{R}$$

is continuous.

For any signed finite measure σ on Λ , we set, for $t \geq 0$,

$$A_t^{(\sigma)} = \int_{\Lambda} \exp \left(G_{\lambda,t} - \frac{1}{2} c_{\lambda,\lambda}(t) \right) \sigma(d\lambda).$$

In the sequel, \mathbb{S}_n is assumed to be equipped with the following partial order, induced by the convex cone \mathbb{S}_n^+ :

$$\forall M, N \in \mathbb{S}_n \quad M \leq N \iff (N - M) \in \mathbb{S}_n^+.$$

The next theorem provides sufficient conditions for the process $(A_t^{(\sigma)}, t \geq 0)$ to be a PCOC.

Theorem 3.1 *We consider the following conditions:*

(C₁) *For every $\lambda, \mu \in \Lambda$, the function $c_{\lambda,\mu}$ is increasing on \mathbb{R}_+ .*

(C₂) *For every $n \geq 1$, for every $\lambda_1, \dots, \lambda_n \in \Lambda$, the matrix function*

$$t \geq 0 \longrightarrow (c_{\lambda_j, \lambda_k}(t))_{1 \leq j, k \leq n} \in \mathbb{S}_n$$

is increasing on \mathbb{R}_+ .

If either (C₁) is satisfied and σ is a positive finite measure, or (C₂) is satisfied and σ is a signed finite measure, then $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC.

Proof

We begin with a lemma for which we refer to [HPRY].

Lemma 3.1 *Let $(X_t, t \geq 0)$ be a real valued integrable process, i.e.:*

$$\forall t \geq 0 \quad \mathbb{E}[|X_t|] < \infty.$$

This process is a PCOC if (and only if):

i) $\mathbb{E}[X_t]$ *does not depend on $t \geq 0$;*

ii) *for any $\psi \in \mathcal{C}$, the function*

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)]$$

is increasing.

Here, \mathcal{C} denotes the set of all nonnegative, convex, C^2 -functions ψ such that ψ'' has a compact support.

Now, the proof of Theorem 3.1 proceeds in three steps.

1. We first assume that

$$\sigma = \sum_{j=1}^n a_j \delta_{\lambda_j}$$

where δ_λ denotes the Dirac measure at λ and $a_1, \dots, a_n \in \mathbb{R}$.

We have

$$A_t^{(\sigma)} = \sum_{j=1}^n a_j \exp \left(G_{\lambda_j, t} - \frac{1}{2} c_{\lambda_j, \lambda_j}(t) \right) .$$

Since for any t , $\mathbb{E}[|A_t^{(\sigma)}|] < \infty$, and $\mathbb{E}[A_t^{(\sigma)}] = \sum_{j=1}^n a_j$ does not depend on t , to prove that $(A_t^{(\sigma)}, t \geq 0)$ is a PCOC, it suffices to prove (Lemma 3.1) that, for any $\psi \in \mathcal{C}$, the function $t \geq 0 \rightarrow \mathbb{E}[\psi(A_t^{(\sigma)})]$ is increasing.

We fix $0 \leq s \leq t$. We set, for $1 \leq j \leq n$ and $u \in [0, 1]$,

$$G_u^{(j)} = \sqrt{u} G_{\lambda_j, t} + \sqrt{1-u} G_{\lambda_j, s}$$

where the Gaussian vectors $(G_{\lambda_1, t}, \dots, G_{\lambda_n, t})$ and $(G_{\lambda_1, s}, \dots, G_{\lambda_n, s})$ are supposed to be independent. This is yet another instance of the smart path method. Then, by Corollary 2.2, we have:

$$\begin{aligned} \mathbb{E}[\psi(K_1)] &= \mathbb{E}[\psi(K_0)] + \\ &\frac{1}{2} \sum_{j,k} a_j a_k \int_0^1 \mathbb{E} [\psi''(K_u) Y_u^{(j)} Y_u^{(k)}] (c_{\lambda_j, \lambda_k}(t) - c_{\lambda_j, \lambda_k}(s)) du \end{aligned}$$

where, for $1 \leq j \leq n$,

$$Y_u^{(j)} = \exp \left(G_u^{(j)} - \frac{1}{2} \mathbb{E}[G_u^{(j)} G_u^{(j)}] \right)$$

and

$$K_u = \sum_{j=1}^n a_j Y_u^{(j)} .$$

Since

$$K_1 = A_t^{(\sigma)}, \quad K_0 = A_s^{(\sigma)}, \quad \psi'' \geq 0 \quad \text{and} \quad Y_u^{(j)} \geq 0 ,$$

if either (C_1) is satisfied and σ is a positive measure, or (C_2) is satisfied, then, for $0 \leq s \leq t$,

$$\mathbb{E}[\psi(A_t^{(\sigma)})] \geq \mathbb{E}[\psi(A_s^{(\sigma)})] ,$$

which proves the result.

2. By hypothesis, there exists a sequence $(\Lambda_n)_{n \geq 0}$ of compact subsets of Λ with $\bigcup_{n \geq 0} \Lambda_n = \Lambda$. We now assume that the support of σ is contained in some compact set Λ_{n_0} . Then, there exists a sequence $(\sigma_n, n \geq 0)$, weakly converging to σ , such that, for each n , σ_n is as in step 1 a linear combination of Dirac measures supported by Λ_{n_0} . Besides, we may suppose

$$(8) \quad \forall n \quad \int |\sigma_n(d\lambda)| \leq \int |\sigma(d\lambda)|.$$

Moreover, if σ is a positive measure, all measures σ_n may be assumed to be positive.

Let $\psi \in \mathcal{C}$. By step 1, if either (C_1) is satisfied and σ is a positive measure, or (C_2) is satisfied, then for any $n \geq 0$ and $0 \leq s \leq t$,

$$(9) \quad \mathbb{E}[\psi(A_t^{(\sigma_n)})] \geq \mathbb{E}[\psi(A_s^{(\sigma_n)})].$$

On the other hand,

$$\mathbb{E}[(A_t^{(\sigma)} - A_t^{(\sigma_n)})^2] = \int \int_{\Lambda_{n_0}^2} e^{c\lambda, \mu(t)} d(\sigma - \sigma_n)(\lambda) d(\sigma - \sigma_n)(\mu).$$

Consequently, using (8) and (H_1) , we obtain the convergence, in L^2 , of the sequence $(A_t^{(\sigma_n)}, n \geq 0)$ to $A_t^{(\sigma)}$.

Since ψ is affine outside of a compact interval, then ψ is a Lipschitz continuous function, and the sequence $(\psi(A_t^{(\sigma_n)}), n \geq 0)$ converges in L^2 to $\psi(A_t^{(\sigma)})$. We may then pass to the limit in (9) and obtain, for $0 \leq s \leq t$,

$$\mathbb{E}[\psi(A_t^{(\sigma)})] \geq \mathbb{E}[\psi(A_s^{(\sigma)})].$$

As $\mathbb{E}[A_t^{(\sigma)}] = \int \sigma(d\lambda)$ does not depend on t , the desired result follows from Lemma 3.1.

3. In the general case, we set, for any $n \geq 0$

$$\sigma_n = 1_{\Lambda_n} \sigma.$$

We have, for $t \geq 0$,

$$\lim_{n \rightarrow \infty} A_t^{(\sigma_n)} = A_t^{(\sigma)} \text{ a.s. and } |A_t^{(\sigma_n)}| \leq A_t^{(|\sigma|)},$$

which allows to apply step 2 and to pass to the limit.

□

Remark Under the following hypothesis:

(H_2) For all $\lambda, \mu \in \Lambda$, the function $c_{\lambda, \mu}$ is absolutely continuous on \mathbb{R}_+ ,

Condition (C_2) may be written as:

For every $n \geq 1$, for every $\lambda_1, \dots, \lambda_n \in \Lambda$, the matrix

$$(c'_{\lambda_j, \lambda_k}(t))_{1 \leq j, k \leq n}$$

is a positive symmetric matrix for a.e. $t \geq 0$.

3.2 Examples

3.2.1 Processes ($t G_\lambda$)

Let Λ be a metric σ -compact space equipped with its Borel σ -field. We consider a real valued measurable centered Gaussian process $(G_\lambda, \lambda \in \Lambda)$. For $\lambda, \mu \in \Lambda$, we set:

$$c(\lambda, \mu) = \mathbb{E}[G_\lambda G_\mu].$$

We assume the following hypothesis:

(G_1) The function:

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow c(\lambda, \mu) \in \mathbb{R}$$

is continuous.

We set, for $\lambda \in \Lambda$ and $t \geq 0$,

$$G_{\lambda, t}^{(1)} = t G_\lambda.$$

Then, $G^{(1)}$ satisfies (H_1), (H_2) and (C_2).

3.2.2 Processes ($G_{\lambda t}$)

Here, we consider the particular case $\Lambda = \mathbb{R}_+$, and a measurable centered Gaussian process $(G_\lambda, \lambda \geq 0)$ satisfying the previous condition (G_1). We set, for $\lambda \geq 0$ and $t \geq 0$,

$$G_{\lambda, t}^{(2)} = G_{\lambda t}.$$

Furthermore, we assume that:

(G_2) For $\lambda, \mu \geq 0$, the function

$$t \geq 0 \longrightarrow c(t\lambda, t\mu)$$

is increasing.

Then the process $G^{(2)}$ satisfies (H_1) and (C_1) . In particular, Theorem 3.1 implies that the process

$$\left(\frac{1}{t} \int_0^t \exp \left(G_\lambda - \frac{1}{2} c(\lambda, \lambda) \right) d\lambda, t \geq 0 \right)$$

is a PCOC.

An example of a process $(G_\lambda, \lambda \geq 0)$ satisfying the above properties (G_1) and (G_2) , is the fractional Brownian motion B^H with Hurst index $H \in (0, 1)$. Indeed, then:

$$c(\lambda, \mu) = \frac{1}{2} (|\lambda|^{2H} + |\mu|^{2H} - |\lambda - \mu|^{2H}) \geq 0$$

and

$$c(t\lambda, t\mu) = t^{2H} c(\lambda, \mu).$$

Actually, for each $t \geq 0$, there is the equality in law:

$$B_{t\bullet}^H \stackrel{(d)}{=} t^H B_\bullet^H.$$

Therefore, we may as well apply the previous paragraph. Consequently, for any *signed* finite measure σ on \mathbb{R}_+ ,

$$\int_{\mathbb{R}_+} \exp \left(B_{t\lambda}^H - \frac{(t\lambda)^{2H}}{2} \right) d\sigma(\lambda), \quad t \geq 0$$

is a PCOC.

We now introduce another example. Let

$$a : (\lambda, s) \in \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow a(\lambda, s) \in \mathbb{R}_+$$

be a nonnegative measurable function such that:

- i) For every $\lambda \geq 0$, $a(\lambda, \bullet) \in L^2(\mathbb{R}_+)$.
- ii) The function

$$(\lambda, \mu) \in \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \int_0^\infty a(\lambda, s) a(\mu, s) ds$$

is continuous.

- iii) For any $s \geq 0$, the function

$$\lambda \in \mathbb{R}_+ \longrightarrow a(\lambda, s)$$

is increasing.

Setting

$$G_\lambda = \int_0^\infty a(\lambda, s) dB_s \quad , \quad \lambda \geq 0$$

where (B_s) is a standard Brownian motion, we see that properties (G_1) and (G_2) are satisfied.

Finally, we consider, for $\varepsilon \in \mathbb{R}$, the process:

$$G_\lambda = B_{\lambda \wedge 1} - \varepsilon (\lambda \wedge 1) B_1$$

where B denotes the standard Brownian motion. Obviously, (G_1) is satisfied. An easy computation shows that (G_2) is satisfied if and only if

$$|1 - \varepsilon| \geq \frac{1}{\sqrt{2}} .$$

Consider now the case $\varepsilon = 1$. Then, $(G_t = B_t - tB_1, 0 \leq t \leq 1)$ is a representation of the standard Brownian bridge $(b_t, 0 \leq t \leq 1)$. The following proposition shows that, in this case, the conclusion of Theorem 3.1 fails.

Proposition 3.1 *Let $(b_t, 0 \leq t \leq 1)$ be the standard Brownian bridge. We set, for $t \in [0, 1]$,*

$$A_t^{(1)} = \int_0^1 \exp\left(b_{ut} - \frac{ut(1-ut)}{2}\right) du \quad \text{and} \quad A_t^{(2)} = \exp\left(b_{at} - \frac{at(1-at)}{2}\right)$$

with $a \in (1/2, 1]$. Then, neither $(A_t^{(1)}, 0 \leq t \leq 1)$ nor $(A_t^{(2)}, 0 \leq t \leq 1)$ are PCOC's.

Proof

It is not difficult to see that the left derivative at $t = 1$ of $\mathbb{E}[(A_t^{(j)})^2]$ is < 0 for $j = 1, 2$.

□

3.2.3 Brownian sheet

Let, for $\lambda, t \geq 0$,

$$G_{\lambda,t} = W_{\lambda,t}$$

where W denotes the standard Brownian sheet. We have:

$$c_{\lambda,\mu}(t) = t(\lambda \wedge \mu) .$$

Then G satisfies hypotheses (H_1) , (H_2) , (C_1) and (C_2) . In fact, for any $t \geq 0$,

$$G_{\bullet,t} \stackrel{(d)}{=} \sqrt{t} B_{\bullet}$$

where B denotes the standard Brownian motion. We may then consider this case as a particular case of Example 3.2.1 (replacing t by \sqrt{t}). On the other hand, for any $\lambda \geq 0$,

$$\left(\exp \left(W_{\lambda,t} - \frac{t\lambda}{2} \right), t \geq 0 \right)$$

is a (\mathcal{W}_t) -martingale, with

$$\mathcal{W}_t = \sigma\{W_{\lambda,s} ; \lambda \geq 0, 0 \leq s \leq t\}.$$

Therefore, Theorem 3.1 is obvious in this case since $(A_t^{(\sigma)})$ is a (\mathcal{W}_t) -martingale.

3.2.4 Stochastic integrals

Let Λ be a metric σ -compact space and let

$$h : (\lambda, s) \in \Lambda \times \mathbb{R}_+ \longrightarrow h(\lambda, s) \in \mathbb{R}$$

be a measurable function such that:

- i) For every $\lambda \in \Lambda$, $h(\lambda, \bullet) \in L_{loc}^2(\mathbb{R}_+)$.
- ii) For any $t \geq 0$, the function

$$(\lambda, \mu) \in \Lambda \times \Lambda \longrightarrow \int_0^t h(\lambda, s) h(\mu, s) ds$$

is continuous.

We note that, for

$$G_{\lambda,t} = \int_0^t h(\lambda, s) dB_s \quad ; \quad \lambda \in \Lambda, t \geq 0$$

where (B_s) is a standard Brownian motion, then:

$$c_{\lambda,\mu}(t) = \int_0^t h(\lambda, s) h(\mu, s) ds.$$

Therefore, G satisfies hypotheses (H_1) , (H_2) and (C_2) .

3.2.5 On a theorem of Kahane

Let Λ be a σ -compact metric space and let

$$X = (X_\lambda, \lambda \in \Lambda) \quad , \quad Y = (Y_\lambda, \lambda \in \Lambda)$$

be two real valued measurable centered Gaussian processes. We set, for $\lambda, \mu \in \Lambda$,

$$c_X(\lambda, \mu) = \mathbb{E}[X_\lambda X_\mu] \quad , \quad c_Y(\lambda, \mu) = \mathbb{E}[Y_\lambda Y_\mu]$$

and we assume that c_X and c_Y are continuous functions on $\Lambda \times \Lambda$. The following proposition is stated in [Ka] with the additional assumption that the convex function ψ below is increasing.

Proposition 3.2 *We assume:*

$$\forall \lambda, \mu \in \Lambda \quad c_X(\lambda, \mu) \leq c_Y(\lambda, \mu) .$$

Then, for any positive finite measure σ on Λ and for any convex function ψ on \mathbb{R} ,

$$\begin{aligned} & \mathbb{E} \left[\psi \left\{ \int_\Lambda \exp \left(X_\lambda - \frac{c_X(\lambda, \lambda)}{2} \right) \sigma(d\lambda) \right\} \right] \\ & \leq \mathbb{E} \left[\psi \left\{ \int_\Lambda \exp \left(Y_\lambda - \frac{c_Y(\lambda, \lambda)}{2} \right) \sigma(d\lambda) \right\} \right] . \end{aligned}$$

Proof

We shall use again the smart path method. We set, for $\lambda \in \Lambda$ and $t \in [0, 1]$,

$$G_{\lambda,t} = \sqrt{t} Y_\lambda + \sqrt{1-t} X_\lambda$$

where the processes X and Y are assumed to be independent. Then properties (H_1) and (C_1) are satisfied for $t \in [0, 1]$. Therefore, by Theorem 3.1, the process

$$\int_\Lambda \exp \left(G_{\lambda,t} - \frac{t c_Y(\lambda, \lambda) + (1-t) c_X(\lambda, \lambda)}{2} \right) \sigma(d\lambda) \quad , \quad 0 \leq t \leq 1$$

is a PCOC, which leads to the desired result. □

4 PCOC's and Gaussian sheets

In this section, our aim is to associate, to certain process $(A_t^{(\sigma)}, t \geq 0)$ as defined in Section 3, a martingale having the same one-dimensional marginals as this process. This will produce another proof that they are PCOC's.

Our main tool is the construction of Gaussian sheets.

Throughout this section, Λ denotes a measure space, and we consider, for each $t \geq 0$, a real valued measurable centered Gaussian process

$$G_{\bullet,t} = (G_{\lambda,t}, \lambda \in \Lambda).$$

For $\lambda, \mu \in \Lambda$ and $t \geq 0$, we set:

$$c_{\lambda,\mu}(t) = \mathbb{E}[G_{\lambda,t} G_{\mu,t}].$$

4.1 Gaussian sheets

We introduce the following hypotheses (H_0) , (H'_1) , and we recall hypotheses (H_2) and (C_2) :

(H_0) Λ is a separable metric space equipped with its Borel σ -field.

(H'_1) The function:

$$(\lambda, \mu, t) \in \Lambda \times \Lambda \times \mathbb{R}_+ \longrightarrow c_{\lambda,\mu}(t) \in \mathbb{R}$$

is continuous.

(H_2) For all $\lambda, \mu \in \Lambda$, the function $c_{\lambda,\mu}$ is absolutely continuous on \mathbb{R}_+ .

(C_2) For every $n \geq 1$, for every $\lambda_1, \dots, \lambda_n \in \Lambda$, the matrix function

$$t \geq 0 \longrightarrow (c_{\lambda_j, \lambda_k}(t))_{1 \leq j, k \leq n} \in \mathbb{S}_n$$

is increasing.

Theorem 4.1 *Under (H_0) , (H'_1) , (H_2) and (C_2) , there exists a measurable centered Gaussian process:*

$$(\Gamma_{\lambda,t} ; \lambda \in \Lambda, t \geq 0),$$

such that

$$(10) \quad \forall (\lambda, s), (\mu, t) \in \Lambda \times \mathbb{R}_+ \quad \mathbb{E}[\Gamma_{\lambda,s} \Gamma_{\mu,t}] = c_{\lambda,\mu}(s \wedge t).$$

Proof

We first prove that

$$[(\lambda, s), (\mu, t)] \longrightarrow c_{\lambda, \mu}(s \wedge t)$$

is a covariance on $\Lambda \times \mathbb{R}_+$.

Let $\lambda_1, \dots, \lambda_n \in \Lambda$. Conditions (H_2) and (C_2) ensure the existence of a measurable function

$$t \geq 0 \longrightarrow D^{(\lambda_1, \dots, \lambda_n)}(t) \in \mathbb{S}_n^+$$

such that, for almost every $t \geq 0$,

$$[D^{(\lambda_1, \dots, \lambda_n)}(t)]^2 = (c'_{\lambda_j, \lambda_k}(t))_{1 \leq j, k \leq n} .$$

Let $(B_t = (B_t^1, \dots, B_t^n), t \geq 0)$ be a standard \mathbb{R}^n -valued Brownian motion, independent of $G_{\bullet, 0}$. Let $s_1, \dots, s_n \in \mathbb{R}_+$. We set, for $1 \leq j \leq n$,

$$Z_j = G_{\lambda_j, 0} + \sum_{l=1}^n \int_0^{s_j} d_{j,l}^{(\lambda_1, \dots, \lambda_n)}(u) dB_u^l$$

where

$$d_{j,k}^{(\lambda_1, \dots, \lambda_n)}(u) \quad , \quad 1 \leq j, k \leq n$$

denote the entries of the matrix $D^{(\lambda_1, \dots, \lambda_n)}(u)$. Then, the formula:

$$\mathbb{E}[Z_j Z_k] = c_{\lambda_j, \lambda_k}(0) + \int_0^{s_j \wedge s_k} c'_{\lambda_j, \lambda_k}(u) du = c_{\lambda_j, \lambda_k}(s_j \wedge s_k) ,$$

ensures the covariance property.

From the preceding, there exists a centered Gaussian process:

$$(\Gamma_{\lambda, t} \quad ; \quad \lambda \in \Lambda, t \geq 0) ,$$

such that

$$\mathbb{E}[\Gamma_{\lambda, s} \Gamma_{\mu, t}] = c_{\lambda, \mu}(s \wedge t) .$$

Moreover, hypotheses (H_0) and (H'_1) easily entail that the Gaussian space generated by this process Γ is separable. Therefore, by [N, Corollaire 3.8, p. 44], the process admits a measurable version. □

In some particular cases, we can give more explicit constructions, without assuming hypotheses (H_0) , (H'_1) , (H_2) .

Proposition 4.1 Assume that there exists an increasing function φ on \mathbb{R}_+ such that, for every $\lambda, \mu \in \Lambda$ and $t \geq 0$,

$$c_{\lambda, \mu}(t) = \varphi(t) c_{\lambda, \mu}(1) .$$

Let $(G_{\bullet}^{(n)}, n \geq 0)$ be a sequence of independent copies of $G_{\bullet, 1}$, and let $(e_n, n \geq 0)$ be a Hilbert basis of $L^2(\mathbb{R}_+)$. We set

$$\Gamma_{\lambda, t} = \sum_{n=0}^{\infty} \left(\int_0^{\varphi(t)} e_n(s) ds \right) G_{\lambda}^{(n)} .$$

Then,

$$(\Gamma_{\lambda, t} ; \lambda \in \Lambda, t \geq 0) ,$$

is a measurable centered Gaussian process such that (10) is satisfied.

Proof

Since the function φ is increasing, the result follows from Parseval's identity. □

Proposition 4.2 Let $g : \Lambda \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a measurable function such that, for every $\lambda \in \Lambda$, $g(\lambda, \bullet) \in L^2(\mathbb{R}_+)$. We suppose that

$$G_{\lambda, t} = t \int_0^{\infty} g(\lambda, s) dB_s$$

where (B_s) is a standard Brownian motion. We denote by $(W_{s, t}; s, t \geq 0)$ the Brownian sheet and we set:

$$\Gamma_{\lambda, t} = \int_0^{\infty} g(\lambda, u) d_u W_{u, t^2} .$$

Then,

$$(\Gamma_{\lambda, t} ; \lambda \in \Lambda, t \geq 0) ,$$

is a measurable centered Gaussian process such that (10) is satisfied.

Proof

We have

$$c_{\lambda, \mu}(t \wedge s) = (t^2 \wedge s^2) \int_0^{\infty} g(\lambda, u) g(\mu, u) du$$

and the result follows. □

Proposition 4.3 *Let $h : \Lambda \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be a measurable function such that, for every $\lambda \in \Lambda$, $h(\lambda, \bullet) \in L_{loc}^2(\mathbb{R}_+)$. We suppose that*

$$G_{\lambda,t} = \int_0^t h(\lambda, s) dB_s$$

where (B_s) is a standard Brownian motion. We set:

$$\Gamma_{\lambda,t} = G_{\lambda,t} .$$

Then,

$$(\Gamma_{\lambda,t} \ ; \ \lambda \in \Lambda , t \geq 0) ,$$

is a measurable centered Gaussian process such that (10) is satisfied.

The proof is straightforward.

The following proposition states the properties of Γ which are essential in the sequel.

Proposition 4.4 *Let $(\Gamma_{\lambda,t} ; \lambda \in \Lambda, t \geq 0)$ be a measurable centered Gaussian process such that (10) holds. We set, for $t \geq 0$,*

$$\mathcal{G}_t = \sigma\{\Gamma_{\lambda,s} ; \lambda \in \Lambda, 0 \leq s \leq t\} .$$

Then,

1) For $0 \leq s \leq t$, the process $(\Gamma_{\lambda,t} - \Gamma_{\lambda,s} , \lambda \in \Lambda)$ is independent of the σ -field \mathcal{G}_s .

2) For any $t \geq 0$,

$$\Gamma_{\bullet,t} \stackrel{(d)}{=} G_{\bullet,t} .$$

The proof is straightforward.

4.2 Application to PCOC's

For any signed finite measure σ on Λ , we set as in Subsection 3.1, for $t \geq 0$,

$$A_t^{(\sigma)} = \int_{\Lambda} \exp\left(G_{\lambda,t} - \frac{1}{2} c_{\lambda,\lambda}(t)\right) \sigma(d\lambda) .$$

Proposition 4.5 *Assume there exists a measurable centered Gaussian process:*

$$(\Gamma_{\lambda,t} \ ; \ \lambda \in \Lambda \ , \ t \geq 0) \ ,$$

such that (10) is satisfied. We set, for $t \geq 0$,

$$\mathcal{G}_t = \sigma\{\Gamma_{\lambda,s} \ ; \ \lambda \in \Lambda \ , \ 0 \leq s \leq t\} \ .$$

Let σ be a signed finite measure on Λ . We set, for $t \geq 0$,

$$M_t^{(\sigma)} = \int_{\Lambda} \exp\left(\Gamma_{\lambda,t} - \frac{1}{2} c_{\lambda,\lambda}(t)\right) \sigma(d\lambda) \ .$$

Then $(M_t^{(\sigma)} \ , \ t \geq 0)$ is a (\mathcal{G}_t) -martingale and, for each $t \geq 0$,

$$M_t^{(\sigma)} \stackrel{d}{=} A_t^{(\sigma)} \ .$$

In particular, $(A_t^{(\sigma)} \ , \ t \geq 0)$ is a PCOC.

Proof

This is a direct consequence of Proposition 4.4, using the following consequence of (10):

$$\forall 0 \leq s \leq t, \forall \lambda \in \Lambda, \quad \mathbb{E}[(\Gamma_{\lambda,t} - \Gamma_{\lambda,s})^2] = c_{\lambda,\lambda}(t) - c_{\lambda,\lambda}(s) \ .$$

□

Theorem 4.1, Proposition 4.1, Proposition 4.2 and Proposition 4.3 give conditions entailing the hypothesis of the above proposition. In particular, Theorem 4.1 and Proposition 4.5 yield another proof (with slightly different hypotheses) of Theorem 3.1 under Condition (C_2) .

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