Constructing Random Times with Given Survival Processes and Applications to Valuation of Credit Derivatives

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Abstract

We provide an explicit construction of a random time when the associated Azéma semimartingale (also known as the survival process) is given in advance. Our approach hinges on the use of a variant of Girsanov's theorem combined with a judicious choice of the Radon-Nikodým density process. The proposed solution is also partially motivated by the classic example arising in the filtering theory.

This paper is dedicated to Professor Eckhard Platen on the occasion of his 60th birthday. Even though its topic is not related to his exciting benchmark approach, we hope he will find some interest in this research.

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1 Introduction

The goal of this work is to address the following problem:

Problem (P). Let $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ be a probability space endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Assume that we are given a strictly positive, càdlàg, $(\mathbb{P}, \mathbb{F})$-local martingale N with $N_0 = 1$ and an $\mathbb{F}$-adapted, continuous, increasing process Λ, with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$, such that $G_t := N_t e^{-\Lambda_t} \leq 1$ for every $t \in \mathbb{R}_+$. The goal is to construct a random time τ on an extended probability space and a probability measure $\mathbb{Q}$ on an extended space such that:
(i) $\mathbb{Q}$ is equal to $\mathbb{P}$ when restricted to $\mathbb{F}$, that is, $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$ for every $t \in \mathbb{R}_+$,
(ii) the Azéma supermartingale $G^{\mathbb{Q}} := \mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t)$ of τ under $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$ satisfies

$$G_t^{\mathbb{Q}} = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{1}$$

In that case, the pair $(\tau, \mathbb{Q})$ is called a solution to Problem (P).

We will sometimes refer to the Azéma supermartingale $G^{\mathbb{Q}}$ as the *survival process* of τ under $\mathbb{Q}$ with respect to $\mathbb{F}$. The solution to this problem is well known if $N_t = 1$ for every $t \in \mathbb{R}_+$ (see Section 3) and thus we will focus in what follows on the case where N is not equal identically to 1.

Condition (i) implies that the postulated inequality $G_t := N_t e^{-\Lambda_t} \leq 1$ is necessary for the existence of a solution $(\tau, \mathbb{Q})$. Note also that in view of (i), the joint distribution of (N, Λ) is set to be identical under $\mathbb{P}$ and $\mathbb{Q}$ for any solution $(\tau, \mathbb{Q})$ to Problem (P). In particular, N is not only a $(\mathbb{P}, \mathbb{F})$-local martingale, but also a $(\mathbb{Q}, \mathbb{F})$-local martingale. However, in the construction of a solution to Problem (P) provided in this work, the so-called H-hypothesis is not satisfied under $\mathbb{Q}$ by the filtration $\mathbb{F}$ and the enlarged filtration $\mathbb{G}$ generated by $\mathbb{F}$ and the observations of τ. Hence the process N is not necessarily a $(\mathbb{Q}, \mathbb{G})$-local martingale.

In the approach proposed in this work, in the first step we construct a finite random time τ on an extended probability space using the *canonical construction* in such a way that

$$G_t^{\mathbb{P}} := \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) = e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+.$$

To avoid the need for an extension of Ω, it suffices to postulate, without loss of generality, that there exists a random variable Θ defined on Ω such that Θ is exponentially distributed under $\mathbb{P}$ and it is independent of $\mathcal{F}_\infty$. In the second step, we propose a change of the probability measure by making use of a suitable version of Girsanov's theorem. Since we purportedly identify the extended space with Ω, it make sense to compare the probability measures $\mathbb{P}$ and $\mathbb{Q}$. Let us mention in this regard that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are not necessarily equivalent. However, for any solution $(\tau, \mathbb{Q})$ to Problem (P), the equality $\mathbb{Q}(\tau < \infty) = 1$ is satisfied in the present set-up (see Lemma 4.1) and thus τ is necessarily a finite random time under $\mathbb{Q}$.

In the existing literature, one can find easily examples where the Doob-Meyer decomposition of the Azéma supermartingale is given, namely, $G_t = M_t - A_t$ (see, e.g., Mansuy and Yor [6]). It is then straightforward to deduce the multiplicative decomposition by setting $N_t = \int_0^t e^{\Lambda_s}\, dM_s$ and $\Lambda_t = \int_0^t \frac{dA_s}{G_{s-}}$. However, to the best of our knowledge, a complete solution to the problem stated above is not yet available, though some partial results were obtained. Nikeghbali and Yor [7] study a similar problem for a particular process Λ, namely, $\Lambda_t = \sup_{s \leq t} N_s$ for a local martingale N which converges to 0 as t goes to infinity. It is worth stressing that in [7] the process G can take the value one for some $t > 0$. We will conduct the first part of our study under the standing assumption that $G_t \leq 1$. However, to provide an explicit construction of a probability measure $\mathbb{Q}$, we will work in Section 5 under the stronger assumption that the inequality $G_t < 1$ holds for every $t > 0$

The paper is organized as follows. We start by presenting in Section 2 an example of a random time τ, which is not a stopping time with respect to the filtration $\mathbb{F}$, such that the Azéma supermartingale of τ with respect to $\mathbb{F}$ can be computed explicitly. In fact, we revisit here a classic example arising in the non-linear filtering theory. In the present context, it can be seen as a motivation for the problem stated at the beginning. In addition, some typical features of the Azéma supermartingale, which are apparent in the filtering example, are later rediscovered in a more general set-up, which is examined in the subsequent sections.

The goal of Section 3 is to furnish some preliminary results on Girsanov's change of a probability measure in the general set-up. In Section 4, the original problem is first reformulated and then reduced to a more tractable analytical problem (see Problems (P.1)–(P.3) therein). In Section 5, we analyze in some detail the case of a Brownian filtration. Under the assumption that $G_t < 1$ for every $t \in \mathbb{R}_+$, we identify a solution to the original problem in terms of the Radon-Nikodým density process.

Section 6 discusses the relevance of the multiplicative decomposition of survival process of a default time τ for the risk-neutral valuation of credit derivatives. From this perspective, it is important to observe that a random time τ constructed in this work has the same intensity under $\mathbb{P}$ and $\mathbb{Q}$, but it has different conditional probability distributions with respect to $\mathbb{F}$ under $\mathbb{P}$ and $\mathbb{Q}$. This illustrates the important fact that the default intensity does not contain enough information to price credit derivatives (in this regard, we refer to El Karoui et al. [2]). In several papers in the financial literature, the modeling of credit risk is based on the postulate that the process

$$M_t := \mathbb{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \lambda_u \, du$$

is a martingale with respect to a filtration $\mathbb{G}$ such that τ is a (totally inaccessible) $\mathbb{G}$-stopping time. However, we will argue in Section 6 that this information is insufficient for the computation of prices of credit derivatives. Indeed, it appears that, except for the most simple examples of credit derivatives, the martingale component in the multiplicative decomposition of the Azéma supermartingale of a default time τ has a non-negligible impact on risk-neutral values of credit derivatives.

2 Filtering Example

The starting point for this research was a well known problem arising in the filtering theory. The goal of this section is to recall this example and to examine some interesting features of the conditional distributions of a random time, which will be later rediscovered in a different set-up.

2.1 Azéma Supermartingale

Let $W = (W_t, t \in \mathbb{R}_+)$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and τ be a random time, independent of W and such that $\mathbb{P}(\tau > t) = e^{-\lambda t}$ for every $t \in \mathbb{R}_+$ and some fixed $\lambda > 0$. We define the process $U = (U_t, t \in \mathbb{R}_+)$ by setting

$$U_t = \exp\left(\left(a + b - \frac{\sigma^2}{2}\right) t - b(t-\tau)^+ + \sigma\, W_t\right),$$

where a, b and σ are some given strictly positive constants. One can check that the process U solves the stochastic differential equation

$$dU_t = U_t \left(a + b\, \mathbb{1}_{\{\tau > t\}}\right) dt + U_t\, \sigma\, dW_t. \tag{2}$$

In the filtering problem, the goal is to assess the conditional probability that the moment τ has already occurred by a given date t, using the observations of the process U driven by (2).

Let us take as $\mathbb{F}$ the natural filtration of the process U, that is, $\mathcal{F}_t = \sigma(U_s \,|\, 0 \leq s \leq t)$ for $t \in \mathbb{R}_+$. By means of standard arguments (see, e.g., [9, Chapter IV, Section 4] or [5, Chapter IX, Section 4]), it can be shown that the process U admits the following semimartingale decomposition in its own filtration

$$dU_t = U_t(a + bG_t)\, dt + U_t \sigma\, d\overline{W}_t,$$

where $G = (G_t, t \in \mathbb{R}_+)$ is the Azéma supermartingale, given by $G_t = \mathbb{P}(\tau > t \,|\, \mathcal{F}_t)$, and the innovation process $\overline{W} = (\overline{W}_t, t \in \mathbb{R}_+)$, defined by

$$\overline{W}_t = W_t + \frac{b}{\sigma} \int_0^t \left(\mathbb{1}_{\{\tau > u\}} - G_u\right) du,$$

is the standard Brownian motion with respect to $\mathbb{F}$. It is easy to show, using the arguments based on the notion of strong solutions of stochastic differential equations (see, e.g. [5, Chapter IV,Section 4]), that the natural filtration of $\overline{W}$ coincides with $\mathbb{F}$. It follows from [5, Chapter IX,Section 4] (see also [9, Chapter IV, Section 4]) that the process G solves the following stochastic differential equation

$$dG_t = -\lambda G_t\, dt + \frac{b}{\sigma}\, G_t(1 - G_t)\, d\overline{W}_t, \tag{3}$$

so that the process $N = (N_t, t \in \mathbb{R}_+)$, given by $N_t = e^{\lambda t} G_t$, satisfies

$$dN_t = \frac{b}{\sigma}\, e^{\lambda t}\, G_t(1 - G_t)\, d\overline{W}_t. \tag{4}$$

Since $G(1 - G)$ is bounded, it is clear that N is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $N_0 = 1$. We conclude that the Azéma supermartingale G of τ with respect to the filtration $\mathbb{F}$ admits the following representation

$$G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+, \tag{5}$$

where the $(\mathbb{P}, \mathbb{F})$-martingale N is given by (4).

Let us finally observe that equality (3) provides the (additive) Doob-Meyer decomposition of the bounded $(\mathbb{P}, \mathbb{F})$-supermartingale G, whereas equality (5) yields its multiplicative decomposition.

2.2 Conditional Distributions

From the definition of the Azéma supermartingale G and the fact that $(G_t\, e^{\lambda t}, t \in \mathbb{R}_+)$ is a $(\mathbb{P}, \mathbb{F})$-martingale it follows that, for every fixed $u > 0$ and every $t \in [0, u]$,

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\big(\mathbb{P}(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t\big) = e^{-\lambda u}\, \mathbb{E}_{\mathbb{P}}(N_u \,|\, \mathcal{F}_t) \, = e^{-\lambda u}\, N_t. \tag{6}$$

Standard arguments given in [8, Chapter IV, Section 4] (which are also summarized in [9, Chapter IV, Section 4]), based on an application of the Bayes formula, yield the following result, which extends formula (6) to $t \in [u, \infty)$.

Proposition 2.1. *The conditional survival probability process equals, for every $t, u \in \mathbb{R}_+$,*

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{X_t}{X_{u \wedge t}} + X_t Y_{u \wedge t}\, e^{-\lambda u}, \tag{7}$$

where the process Y is given by

$$Y_t = \exp\left(\frac{b}{\sigma^2}\Big(\ln U_t - \frac{2a + b - \sigma^2}{2}\, t\Big)\right) \tag{8}$$

and the process X satisfies

$$\frac{1}{X_t} = 1 + \int_0^t e^{-\lambda u}\, dY_u. \tag{9}$$

It follows immediately from (7) that

$$G_t = X_t Y_t\, e^{-\lambda t}$$

so that the equality $N = XY$ is valid, and thus (7) coincides with (6) when $t \in [0, u]$. Moreover, by standard computations, we see that

$$dY_t = \frac{b}{\sigma}\, Y_t\, dW_t + \frac{b^2}{\sigma^2}\, G_t Y_t\, dt. \tag{10}$$

Using (4) and (10), we obtain

$$dX_t = d\Big(\frac{N_t}{Y_t}\Big) = -\frac{b}{\sigma}\, G_t X_t\, d\overline{W}_t,$$

and thus X is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale with $X_0 = 1$.

Finally, it is interesting to note that we deal here with the model where

$$\tau = \inf \{t \in \mathbb{R}_+ : \Lambda_t \geq \Theta\}$$

with $\Lambda_t = \lambda t$ (so that $\lambda\tau = \Theta$) and the barrier Θ is an exponentially distributed random variable, which is not independent of the σ-field $\mathcal{F}_\infty$. Indeed, we have that, for every $u > 0$ and $0 \leq t < u/\lambda$,

$$\mathbb{P}(\Theta > u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau > u/\lambda \,|\, \mathcal{F}_t) = N_t e^{-u} \neq e^{-u}.$$

3 Preliminary Results

We start by introducing notation. Let τ be a random time, defined the probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F}$, and such that $\mathbb{P}(\tau > 0) = 1$. We denote by $\mathbb{G} = (\mathcal{G}_t)_{t \in \mathbb{R}_+}$ the $\mathbb{P}$-completed and right-continuous version of the progressive enlargement of the filtration $\mathbb{F}$ by the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in \mathbb{R}_+}$ generated by the process $H_t = \mathbb{1}_{\{\tau \leq t\}}$. It is assumed throughout that the H-*hypothesis* (see, for instance, Elliott et al. [2]) is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$ so that, for every $u \in \mathbb{R}_+$,

$$\mathbb{P}\left(\tau > u \,|\, \mathcal{F}_t\right) = \mathbb{P}\left(\tau > u \,|\, \mathcal{F}_u\right), \quad \forall\, t \in [u, \infty).$$

The main tool in a construction of a random time with a given Azéma supermartingale will be a locally equivalent change of a probability measure. For this reason, we first present some results related to Girsanov's theorem in the present set-up.

3.1 Properties of $(\mathbb{P}, \mathbb{G})$-Martingales

It order to define the Radon-Nikodým density process, we first analyze the properties of $(\mathbb{P}, \mathbb{G})$-martingales. The following auxiliary result is based on El Karoui et al. [3], in the sense that it can be seen as a consequence of Theorem 5.7 therein. For the sake of completeness, we provide a simple proof of Proposition 3.1. In what follows, Z stands for a càdlàg, $\mathbb{F}$-adapted, $\mathbb{P}$-integrable process, whereas $Z_t(u)$ denotes an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map, where $\mathcal{O}(\mathbb{F})$ stands for the $\mathbb{F}$-optional σ-field in $\Omega \times \mathbb{R}_+$ (for details, see [3]).

Proposition 3.1. *Assume that the H-hypothesis is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$. Let the $\mathbb{G}$-adapted, $\mathbb{P}$-integrable process $Z^{\mathbb{G}}$ be given by the formula*

$$Z^{\mathbb{G}}_t = Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau) \mathbb{1}_{\{\tau \leq t\}}, \quad \forall\, t \in \mathbb{R}_+, \tag{11}$$

where:
(i) the projection of $Z^{\mathbb{G}}$ onto $\mathbb{F}$, which is defined by

$$Z^{\mathbb{F}}_t := \mathbb{E}_{\mathbb{P}}\left(Z^{\mathbb{G}}_t \,\middle|\, \mathcal{F}_t\right) = Z_t\, \mathbb{P}\left(\tau > t \,|\, \mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau) \mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t\right),$$

is a $(\mathbb{P}, \mathbb{F})$-martingale,
(ii) for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u), t \in [u, \infty))$ is a $(\mathbb{P}, \mathbb{F})$-martingale.
Then the process $Z^{\mathbb{G}}$ is a $(\mathbb{P}, \mathbb{G})$-martingale.

Proof. Let us take $s < t$. Then

$$\mathbb{E}_{\mathbb{P}}\left(Z^{\mathbb{G}}_t \,\middle|\, \mathcal{G}_s\right) = \mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau > t\}} \,\middle|\, \mathcal{G}_s\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau) \mathbb{1}_{\{s < \tau \leq t\}} \,\middle|\, \mathcal{G}_s\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau) \mathbb{1}_{\{\tau \leq s\}} \,\middle|\, \mathcal{G}_s\right) = I_1 + I_2 + I_3.$$

For I_1 and I_2, we apply the standard formula

$$I_1 + I_2 = \mathbb{1}_{\{\tau > s\}} \frac{1}{G^{\mathbb{P}}_s} \mathbb{E}_{\mathbb{P}}\left(Z_t G^{\mathbb{P}}_t \,\middle|\, \mathcal{F}_s\right) + \mathbb{1}_{\{\tau > s\}} \frac{1}{G^{\mathbb{P}}_s} \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau) \mathbb{1}_{\{s < \tau \leq t\}} \,\middle|\, \mathcal{F}_s\right),$$

whereas for I_3, we obtain

$$I_3 = \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{G}_s\right) = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left(Z_t(u)\,|\,\mathcal{F}_s\right)_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left(Z_s(u)\,|\,\mathcal{F}_s\right)_{u=\tau} = \mathbb{1}_{\{\tau\le s\}}Z_s(\tau),$$

where the first equality holds under the H-hypothesis[1] (see Section 3.2 in El Karoui et al. [3]) and the second follows from (ii). It thus suffices to show that $I_1 + I_2 = Z_s\mathbb{1}_{\{\tau>s\}}$. Condition (i) yields

$$\mathbb{E}_{\mathbb{P}}\left(Z_tG_t^{\mathbb{P}}\,\middle|\,\mathcal{F}_s\right)+\mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_s\right)-\mathbb{E}_{\mathbb{P}}\left(Z_s(\tau)\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right)=Z_sG_s^{\mathbb{P}}.$$

Therefore,

$$I_1 + I_3 = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s^{\mathbb{P}}}\Big(Z_sG_s^{\mathbb{P}} + \mathbb{E}_{\mathbb{P}}\left((Z_s(\tau)-Z_t(\tau))\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right)\Big) = Z_s\mathbb{1}_{\{\tau>s\}},$$

where the last equality holds since

$$\mathbb{E}_{\mathbb{P}}\left((Z_s(\tau)-Z_t(\tau))\mathbb{1}_{\{\tau\le s\}}\,\middle|\,\mathcal{F}_s\right) = \mathbb{1}_{\{\tau\le s\}}\mathbb{E}_{\mathbb{P}}\left((Z_s(u)-Z_t(u))\,|\,\mathcal{F}_s\right)_{u=\tau} = 0.$$

For the last equality in the formula above, we have again used condition (ii) in Proposition 3.1. □

In order to define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ under which (1) holds, we will search for a process $Z^{\mathbb{G}}$ satisfying the following set of assumptions.

Assumption 3.1. The process $Z^{\mathbb{G}}$ is a $\mathbb{G}$-adapted and $\mathbb{P}$-integrable process given by

$$Z_t^{\mathbb{G}} = Z_t\mathbb{1}_{\{\tau>t\}} + Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}, \quad \forall\, t\in\mathbb{R}_+, \tag{12}$$

such that the following properties are valid:
(A.1) the projection of $Z^{\mathbb{G}}$ onto $\mathbb{F}$ is equal to one, that is, $Z_t^{\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}}\,\middle|\,\mathcal{F}_t\right) = 1$ for every $t\in\mathbb{R}_+$,
(A.2) $Z_t(\tau)$ is such that the process $Z^{\mathbb{G}}$ is a strictly positive $(\mathbb{P},\mathbb{G})$-martingale.

Remarks 3.1. Since $\mathbb{P}(\tau > 0) = 1$ is clear that $Z_0^{\mathbb{G}} = Z_0 = 1$, so that $\mathbb{E}_{\mathbb{P}}(Z_t^{\mathbb{G}}) = 1$ for every $t\in\mathbb{R}_+$. We will later define a probability measure $\mathbb{Q}$ using the process $Z^{\mathbb{G}}$ as the Radon-Nikodým density. Then condition (A.1) will imply that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ equals $\mathbb{P}$ and, together with the equality $Z = N$, will give us control over the Azéma supermartingale of τ under $\mathbb{Q}$. Let us also note that assumption (A.1) implies that condition (i) of Proposition 3.1 is trivially satisfied.

The following lemma provides a simple condition, which is equivalent to property (A.1).

Lemma 3.1. *The projection of $Z^{\mathbb{G}}$ on $\mathbb{F}$ equals $Z_t^{\mathbb{F}} := \mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}}\,\middle|\,\mathcal{F}_t\right) = 1$ if and only if the processes Z and $Z_t(\tau)$ satisfy the following relationship*

$$Z_t = \frac{1-\mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_t\right)}{\mathbb{P}\left(\tau>t\,|\,\mathcal{F}_t\right)}. \tag{13}$$

Proof. Straightforward calculations yield

$$\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}}\,\middle|\,\mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_t\mathbb{1}_{\{\tau>t\}} + Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_t\right) = Z_t\,\mathbb{P}\left(\tau>t\,|\,\mathcal{F}_t\right) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\,\middle|\,\mathcal{F}_t\right) = 1.$$

The last equality is equivalent to formula (13). □

We find it convenient to work with the following assumption, which is more explicit and slightly stronger than Assumption 3.1.

Assumption 3.2. We postulate that the processes Z and $Z_t(u)$ are such that:
(B.1) equality (13) is satisfied,
(B.2) for every $u\in\mathbb{R}_+$, the process $(Z_t(u),\, t\in[u,\infty))$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale.

Lemma 3.2. *Assumption 3.2 implies Assumption 3.1.*

[1] Essentially, this equality holds, since under the H-hypothesis the σ-fields $\mathcal{F}_t$ and $\mathcal{G}_s$ are conditionally independent given $\mathcal{F}_s$.

Proof. In view of Lemma 3.1, the conditions (A.1) and (B.1) are equivalent. In view of Proposition 3.1, conditions (B.1) and (B.2) imply (A.2). □

Remarks 3.2. It is not true that Assumption 3.1 implies Assumption 3.2, since it is not true that Assumption 3.1 implies condition (B.2), in general. However, if the intensity $(\lambda_u, u \in \mathbb{R}_+)$ of τ under $\mathbb{P}$ exists then one can show that for any $u \in \mathbb{R}_+$ the process $(Z_t(u)\lambda_u G_u,\, t \in [u,\infty))$ is a $(\mathbb{P},\mathbb{F})$-martingale (see El Karoui et al. [2]). This property implies in turn condition (B.2) provided that the intensity process λ does not vanish.

3.2 Girsanov's Theorem

To establish a suitable version of Girsanov's theorem, we need to specify a set-up in which the H-hypothesis is satisfied. Let Λ be an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. We define a random time τ using the *canonical construction*, that is, by setting

$$\tau = \inf \{ t \in \mathbb{R}_+ : \Lambda_t \geq \Theta \}, \tag{14}$$

where Θ is an exponentially distributed random variable with parameter 1, independent of $\mathcal{F}_\infty$, and defined on a suitable extension of the space Ω. In fact, we identify the probability space Ω with its extension so that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ will be defined on the same space.

It is easy to check that in the case of the canonical construction, the H-hypothesis is satisfied under $\mathbb{P}$ by the filtrations $\mathbb{F}$ and $\mathbb{G}$. Moreover, the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{P}$ equals

$$G_t^{\mathbb{P}} := \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) = e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+.$$

Under Assumption 3.2, the strictly positive $(\mathbb{P},\mathbb{G})$-martingale $Z^{\mathbb{G}}$ given by (cf. (12))

$$Z_t^{\mathbb{G}} = Z_t \mathbb{1}_{\{\tau > t\}} + Z_t(\tau)\mathbb{1}_{\{\tau \leq t\}}, \quad \forall\, t \in \mathbb{R}_+, \tag{15}$$

defines a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{G}_\infty)$, locally equivalent to $\mathbb{P}$, by setting $\frac{d\mathbb{Q}}{d\mathbb{P}} \,|\, \mathcal{G}_t = Z_t^{\mathbb{G}}$ for every $t \in \mathbb{R}_+$.

The next result describes the conditional distributions of τ under a locally equivalent probability measure $\mathbb{Q}$.

Proposition 3.2. *Under Assumption 3.2, let the probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ be given by (15). Then the following properties hold:*
(i) the restriction of $\mathbb{Q}$ to the filtration $\mathbb{F}$ is equal to $\mathbb{P}$,
(ii) the Azéma supermartingale of τ under $\mathbb{Q}$ satisfies

$$G_t^{\mathbb{Q}} := \mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = Z_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+, \tag{16}$$

(iii) for every $u \in \mathbb{R}_+$,

$$\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t \right), & t \leq u, \\ Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau) \mathbb{1}_{\{u < \tau \leq t\}} \,\middle|\, \mathcal{F}_t \right), & t \geq u. \end{cases} \tag{17}$$

Proof. We will now check that (16) is satisfied. By the abstract Bayes formula

$$\mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \mathbb{1}_{\{\tau > t\}} \,\middle|\, \mathcal{F}_t \right)}{\mathbb{E}_{\mathbb{P}}\left(Z_t^{\mathbb{G}} \,\middle|\, \mathcal{F}_t \right)} = \frac{\mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau > t\}} \,\middle|\, \mathcal{F}_t \right)}{Z_t^{\mathbb{F}}} = Z_t\, \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) = Z_t e^{-\Lambda_t},$$

as expected. Using the abstract Bayes formula, we can also find expressions for the conditional probabilities $\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t)$ for every $u, t \in \mathbb{R}_+$. For a fixed $u \in \mathbb{R}_+$ and every $t \in [0, u]$, we simply have that

$$\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) = \mathbb{Q}\,(\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t \right).$$

For a fixed $u \in \mathbb{R}_+$ and every $t \in [u, \infty)$, we obtain

$$\begin{aligned}\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}\left(Z^{\mathbb{G}}_t \mathbb{1}_{\{\tau>u\}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}_{\mathbb{P}}\left(Z^{\mathbb{G}}_t \,\middle|\, \mathcal{F}_t\right)} = \mathbb{E}_{\mathbb{P}}\left(\left(Z_t \mathbb{1}_{\{\tau>t\}} + Z_t(\tau)\mathbb{1}_{\{\tau\le t\}}\right)\mathbb{1}_{\{\tau>u\}} \,\middle|\, \mathcal{F}_t\right)\\ &= \mathbb{E}_{\mathbb{P}}\left(Z_t \mathbb{1}_{\{\tau>t\}} + Z_t(\tau)\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right) = Z_t\,\mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right)\\ &= Z_t G^{\mathbb{P}}_t + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right) = Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right).\end{aligned}$$

This completes the proof. □

Remarks 3.3. (i) It is worth stressing that the H-hypothesis does not hold under $\mathbb{Q}$. This property follows immediately from (16), since under the H-hypothesis the Azéma supermartingale is necessarily a decreasing process.
(ii) It is not claimed that τ is a finite random time under $\mathbb{Q}$. Indeed, this property holds if and only if

$$\lim_{t\to\infty} \mathbb{Q}(\tau > t) = \lim_{t\to\infty} \mathbb{E}_{\mathbb{Q}}(Z_t e^{-\Lambda_t}) =: c = 0,$$

otherwise, we have that $\mathbb{Q}(\tau = \infty) = c$. Of course, if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ then necessarily $\mathbb{Q}(\tau < \infty) = 1$ since from (14) we deduce that $\mathbb{P}(\tau < \infty) = 1$ (this is a consequence of the assumption that $\Lambda_\infty = \infty$).
(iii) We observe that the formula

$$G^{\mathbb{Q}}_t := \mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = Z_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+, \tag{18}$$

represents the multiplicative decomposition of the Azéma supermartingale $G^{\mathbb{Q}}$ if and only if the process Z is a $(\mathbb{Q}, \mathbb{F})$-local martingale or, equivalently, if Z is a $(\mathbb{P}, \mathbb{F})$-local martingale (it is worth stressing that Assumption 3.2 does not imply that Z is a $(\mathbb{P}, \mathbb{F})$-martingale; see Example 3.1). In other words, an equivalent change of a probability measure may result in a change of the decreasing component in the multiplicative decomposition as well. The interested reader is referred to Section 6 in El Karoui et al. [2] for a more detailed analysis of the change of a probability measure in the framework of the so-called *density approach* to the modelling of a random time.

Example 3.1. To illustrate the above remark, let us set $Z_t(u) = 1/2$ for every $u \in \mathbb{R}_+$ and $t \in [0, u]$. Then the process $Z^{\mathbb{G}}$, which is given by the formula

$$Z^{\mathbb{G}}_t = \frac{1 - (1/2)\mathbb{P}\,(\tau \le t \,|\, \mathcal{F}_t)}{\mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t)} \mathbb{1}_{\{\tau>t\}} + (1/2)\mathbb{1}_{\{\tau\le t\}},$$

satisfies Assumption 3.2. The process Z is not a $(\mathbb{P}, \mathbb{F})$-local martingale, however, since

$$Z_t = \frac{1 + \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t)}{2\mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t)} = \frac{1 + e^{-\Lambda_t}}{2e^{-\Lambda_t}}.$$

Moreover, the Azéma supermartingale of τ under $\mathbb{Q}$ equals, for every $t \in \mathbb{R}_+$,

$$\mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = Z_t e^{-\Lambda_t} = \frac{1}{2}(1 + e^{-\Lambda_t}) = e^{-\widehat{\Lambda}_t},$$

where $\widehat{\Lambda}$ is an $\mathbb{F}$-adapted, continuous, increasing process different from Λ. Note, however, that $Z^{\mathbb{G}}$ is not a uniformly integrable $(\mathbb{P}, \mathbb{G})$-martingale and $\mathbb{Q}$ is not equivalent to $\mathbb{P}$ on $\mathcal{G}_\infty$ since $\widehat{\Lambda}_\infty = \ln 2 < \infty$, so that $\mathbb{Q}(\tau < \infty) < 1$. Note that the martingale $Z^{\mathbb{G}}_t$ has a jump at time τ, an this in fact implies the processes Λ and $\widehat{\Lambda}$ do not coincide. We will see in the sequel (cf. Lemma 4.2) that, under mild technical assumptions, it is neccesary to set $Z_t(t) = Z_t$ when solving Problem (P), so that the density process $Z^{\mathbb{G}}$ is continuous at τ. This is by no means surprising, since this equality was identified in El Karoui et al. [2] within the density approach as the crucial condition for the preservation of the $\mathbb{F}$-intensity of a random time τ under an equivalent change of a probability measure.

4 Construction Through a Change of Measure

We are in a position to address the issue of finding a solution $(\tau, \mathbb{Q})$ to Problem (P). Let N be a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale with $N_0 = 1$ and let Λ be an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. We postulate, in addition, that $G_t = Ne^{-\Lambda} \leq 1$. Recall that a strictly positive local martingale is a supermartingale; this implies, in particular, that the process N is $\mathbb{P}$-integrable.

Before we proceed to an explicit construction of a random time τ and a probability measure $\mathbb{Q}$, let us show that for any solution $(\tau, \mathbb{Q})$ to Problem (P), we necessarily have that $\mathbb{Q}(\tau < \infty) = 1$.

Lemma 4.1. *For any solution $(\tau, \mathbb{Q})$ to Problem (P), we have that $\mathbb{Q}(\tau = \infty) = 0$.*

Proof. Note first that

$$\mathbb{Q}(\tau = \infty) = \lim_{t \to \infty} \mathbb{Q}(\tau > t) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{Q}} \left(\mathbb{Q} \left(\tau > t \,|\, \mathcal{F}_t \right) \right) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}} \left(\mathbb{Q} \left(\tau > t \,|\, \mathcal{F}_t \right) \right) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}} \left(N_t e^{-\Lambda_t} \right).$$

Since N is a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale, and thus a positive supermartingale, we have that $\lim_{t \to \infty} N_t =: N_\infty < \infty$, $\mathbb{P}$-a.s. By assumption $0 \leq N_t e^{-\Lambda_t} \leq 1$, and thus the dominated convergence theorem yields

$$\mathbb{Q}(\tau = \infty) = \lim_{t \to \infty} \mathbb{E}_{\mathbb{P}} \left(N_t e^{-\Lambda_t} \right) = \mathbb{E}_{\mathbb{P}} \left(\lim_{t \to \infty} N_t e^{-\Lambda_t} \right) = \mathbb{E}_{\mathbb{P}} \left(N_\infty \lim_{t \to \infty} e^{-\Lambda_t} \right) = 0,$$

where the last equality follows from the assumption that $\Lambda_\infty = \infty$. □

In the first step, using the canonical construction, we define a random time τ by formula (14). In the second step, we propose a suitable change of a probability measure.

In order to define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ under which (1) holds, we wish to employ Proposition 3.2 with $Z = N$. To this end, we postulate that Assumption 3.2 is satisfied by $Z = N$ and a judiciously selected $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$. The choice of a map $Z_t(u)$, for a given in advance process N, is studied in what follows.

Let $\mathbb{Q}$ be defined by (15) with Z replaced by N. Then, by Proposition 3.2, we conclude that the Azéma supermartingale of τ under $\mathbb{Q}$ equals

$$G_t^{\mathbb{Q}} := \mathbb{Q} \left(\tau > t \,|\, \mathcal{F}_t \right) = N_t e^{-\Lambda_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{19}$$

By the same token, formula (17) remains valid when Z is replaced by N.

Our next goal is to investigate Assumption 3.2, which was crucial in the proof of Proposition 3.2 and thus also in obtaining equality (19). For this purpose, let us formulate the following auxiliary problem, which combines Assumption 3.2 with the assumption that $N = Z$.

Problem (P.1) Let a strictly positive $(\mathbb{P}, \mathbb{F})$-local martingale N with $N_0 = 1$ be given. Find an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ such that the following conditions are satisfied:
(i) for every $t \in \mathbb{R}_+$,

$$1 - N_t \, \mathbb{P} \left(\tau > t \,|\, \mathcal{F}_t \right) = \mathbb{E}_{\mathbb{P}} \left(Z_t(\tau) \mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t \right), \tag{20}$$

(ii) for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u, \infty))$ is a strictly positive $(\mathbb{P}, \mathbb{F})$-martingale.

Of course, equality (20) is obtained by combining (B.1) with the equality $Z = N$. If, for a given process N, we can find a solution $(Z_t(u),\, t \in [u, \infty))$ to Problem (P.1) then the pair $(Z_t, Z_t(u)) = (N_t, Z_t(u))$ will satisfy Assumption 3.2.

To examine the existence of a solution to Problem (P.1), we note first that formula (20) can be represented as follows

$$1 - G_t^{\mathbb{P}} N_t = \mathbb{E}_{\mathbb{P}} \left(Z_t(\tau) \mathbb{1}_{\{\tau \leq t\}} \,\middle|\, \mathcal{F}_t \right) = \int_0^t Z_t(u) \, d\mathbb{P} \left(\tau \leq u \,|\, \mathcal{F}_t \right).$$

Since the H-hypothesis is satisfied under $\mathbb{P}$, the last formula is equivalent to

$$1 - G^{\mathbb{P}}_t N_t = \int_0^t Z_t(u)\, d\mathbb{P}\,(\tau \le u \,|\, \mathcal{F}_u) = \int_0^t Z_t(u)\, d\big(1 - G^{\mathbb{P}}_u\big) = -\int_0^t Z_t(u)\, dG^{\mathbb{P}}_u.$$

Since we work under the standing assumption that $G^{\mathbb{P}}_t = e^{-\Lambda_t}$, we thus obtain the following equation, which is equivalent to (20)

$$N_t e^{-\Lambda_t} = 1 + \int_0^t Z_t(u)\, de^{-\Lambda_u}. \tag{21}$$

We conclude that, within the present set-up, Problem (P.1) is equivalent to the following one.

Problem (P.2) Let a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N with $N_0 = 1$ be given. Find an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ such that the following conditions hold:
(i) for every $t \in \mathbb{R}_+$

$$N_t e^{-\Lambda_t} = 1 + \int_0^t Z_t(u)\, de^{-\Lambda_u}, \tag{22}$$

(ii) for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u,\infty))$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale.

Assume that $\widehat{Z}_t := Z_t(t)$ is an $\mathbb{F}$-optional process. We will now show that the equality $N = \widehat{Z}$ necessarily holds for any solution to (22), in the sense made precise in Lemma 4.2. In particular, it follows immediately from this result that the processes N and $\widehat{Z}$ are indistinguishable when $\widehat{Z}$ is an $\mathbb{F}$-adapted, càdlàg process and the process Λ has strictly increasing sample paths (for instance, when $\Lambda_t = \int_0^t \lambda_u\, du$ for some strictly positive *intensity process* λ).

Lemma 4.2. *Suppose that $Z_t(u)$ solves Problem (P.2) and the process $(\widehat{Z}_t,\, t \in \mathbb{R}_+)$ given by $\widehat{Z}_t = Z_t(t)$ is $\mathbb{F}$-optional. Then $N = \widehat{Z}$, ν-a.e., where the measure ν on $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathbb{F}))$ is generated by the increasing process Λ, that is, for every $s < t$ and any bounded, $\mathbb{F}$-optional process V*

$$\nu(\mathbb{1}_{[s,t[}V) = \mathbb{E}_{\mathbb{P}}\Big(\int_s^t V_u\, d\Lambda_u\Big).$$

Proof. The left-hand side in (22) has the following Doob-Meyer decomposition

$$N_t e^{-\Lambda_t} = 1 + \int_0^t e^{-\Lambda_u}\, dN_u + \int_0^t N_u\, de^{-\Lambda_u}, \tag{23}$$

whereas the right-hand side in (22) can be represented as follows

$$1 + \int_0^t Z_t(u)\, de^{-\Lambda_u} = 1 + \int_0^t (Z_t(u) - \widehat{Z}_u)\, de^{-\Lambda_u} + \int_0^t \widehat{Z}_u\, de^{-\Lambda_u} = 1 + I_1(t) + I_2(t), \tag{24}$$

where I_2 is an $\mathbb{F}$-adapted, continuous process of finite variation. We will show that I_1 is a $(\mathbb{P},\mathbb{F})$-martingale, so that right-hand side in (24) yields the Doob-Meyer decomposition as well. To this end, we need to show that the equality $\mathbb{E}_{\mathbb{P}}\,(I_1(t)\,|\,\mathcal{F}_s) = I_1(s)$ holds for every $s < t$ or, equivalently,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^t Z_t(u)\, de^{-\Lambda_u} - \int_0^s Z_s(u)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \widehat{Z}_u\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right). \tag{25}$$

We first observe that, for every $s < t$,

$$\mathbb{E}_{\mathbb{P}}\left(\int_0^s Z_s(u)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^s \mathbb{E}_{\mathbb{P}}\,(Z_t(u)\,|\,\mathcal{F}_s)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_0^s Z_t(u)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right),$$

and thus the right-hand side in (25) satisfies

$$\mathbb{E}_{\mathbb{P}}\left(\int_s^t Z_t(u)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \mathbb{E}_{\mathbb{P}}\,(Z_t(u)\,|\,\mathcal{F}_u)\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \widehat{Z}_u\, de^{-\Lambda_u} \,\middle|\, \mathcal{F}_s\right),$$

where we have used the following equality, which holds for every $\mathbb{F}$-adapted, continuous process A of finite variation and every càdlàg process V (not necessarily $\mathbb{F}$-adapted)

$$\mathbb{E}_{\mathbb{P}}\left(\int_s^t V_u\,dA_u \,\Big|\, \mathcal{F}_s\right) = \mathbb{E}_{\mathbb{P}}\left(\int_s^t \mathbb{E}_{\mathbb{P}}\left(V_u \,|\, \mathcal{F}_u\right) dA_u \,\Big|\, \mathcal{F}_s\right).$$

We thus see that (25) holds, so that I_1 is a $(\mathbb{P},\mathbb{F})$-martingale. By comparing the right-hand sides in (23) and (24) and using the uniqueness of the Doob-Meyer decomposition, we conclude that

$$\int_0^t (N_u - \widehat{Z}_u)\,de^{-\Lambda_u} = 0, \quad \forall\, t \in \mathbb{R}_+.$$

The formula above implies that $N = \widehat{Z}$, $\widetilde{\nu}$-a.e., where the measure $\widetilde{\nu}$ on $(\Omega \times \mathbb{R}_+, \mathcal{O}(\mathbb{F}))$ is generated by the decreasing process $e^{-\Lambda_u}$. It is easily see that the measures $\widetilde{\nu}$ and ν are equivalent and thus $N = \widehat{Z}$, ν-a.e. □

To address the issue of existence of a solution to Problem (P.2) (note that it is not claimed that a solution $Z_t(u)$ to Problem (P.2) is unique), we start by postulating that, as in the filtering case described in Section 2, an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable map $Z_t(u)$ satisfies: $Z_t(u) = X_t Y_{t\wedge u}$ for some $\mathbb{F}$-adapted, continuous, strictly positive processes X and Y. It is then easy to check that condition (ii) implies that the process X is necessarily a $(\mathbb{P},\mathbb{F})$-martingale. Moreover, equation (22) becomes

$$N_t e^{-\Lambda_t} = 1 + X_t \int_0^t Y_u\,de^{-\Lambda_u}. \tag{26}$$

Note that at this stage we are searching for a pair (X,Y) of strictly positive, $\mathbb{F}$-adapted processes such that X is a $(\mathbb{P},\mathbb{F})$-martingale and equality (26) holds for every $t \in \mathbb{R}_+$. In view of Lemma 4.2, it is also natural to postulate that $N = XY$. We will then be able to find a simple relation between processes X and Y (see formula (27) below) and thus to reduce the dimensionality of the problem.

Lemma 4.3. *(i) Assume that a pair (X,Y) of strictly positive processes is such that the process $Z_t(u) = X_t Y_{t\wedge u}$ solves Problem (P.2) and the equality $N = XY$ holds. Then the process X is a $(\mathbb{P},\mathbb{F})$-martingale and the process Y equals*

$$Y_t = Y_0 + \int_0^t e^{\Lambda_u}\,d\left(\frac{1}{X_u}\right). \tag{27}$$

(ii) Conversely, if X is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale and the process Y given by (27) is strictly positive, then the process $Z_t(u) = X_t Y_{t\wedge u}$ solves Problem (P.2) for the process $N = XY$.

Proof. For part (i), we observe that under the assumption that $N = XY$, equation (26) reduces to

$$X_t Y_t e^{-\Lambda_t} = 1 + X_t \int_0^t Y_u\,de^{-\Lambda_u}, \tag{28}$$

which in turn is equivalent to

$$Y_t e^{-\Lambda_t} = \frac{1}{X_t} + \int_0^t Y_u\,de^{-\Lambda_u}. \tag{29}$$

The integration by parts formula yields

$$\frac{1}{X_t} = Y_0 + \int_0^t e^{-\Lambda_u}\,dY_u$$

and this in turn is equivalent to (27). To establish part (ii), we first note that (27) implies (28), which means that (22) is satisfied by the processes $N = XY$ and $Z_t(u) = X_t Y_{t\wedge u}$. It is also clear that for any fixed $u \in \mathbb{R}_+$, the process $(Z_t(u),\, t \in [u,\infty))$ is a strictly positive $(\mathbb{P},\mathbb{F})$-martingale. Let us also note that, by the Itô formula, the process XY satisfies

$$d(X_t Y_t) = Y_t\,dX_t - e^{\Lambda_t}(1/X_t)\,dX_t, \tag{30}$$

and thus it is a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale. □

We conclude that in order to find a solution $Z_t(u) = X_t Y_{t\wedge u}$ to Problem (P.2), it suffices to solve the following problem.

Problem (P.3) Assume that we are given a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N with $N_0 = 1$ and an $\mathbb{F}$-adapted, continuous, increasing process Λ with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$. Find a strictly positive $(\mathbb{P},\mathbb{F})$-martingale X such that for the process Y given by (27) we have that $N = XY$.

The following corollary is an easy consequence of part (ii) in Lemma 4.3.

Corollary 4.1. *Assume that a process X solves Problem (P.3) and let Y be given by (27). Then the processes $Z = N$ and $Z_t(u) = X_t Y_{t\wedge u}$ solve Problem (P.2) and thus they satisfy Assumption 3.2.*

5 Case of a Brownian Filtration

The aim of this section is to examine the existence of a solution to Problem (P.3) under the following standing assumptions:
(i) the filtration $\mathbb{F}$ is generated by a Brownian motion W,
(ii) we are given an $\mathbb{F}$-adapted, continuous, increasing process Λ with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$ and a strictly positive $(\mathbb{P},\mathbb{F})$-local martingale N satisfying

$$N_t = 1 + \int_0^t \nu_u N_u \, dW_u, \quad \forall\, t \in \mathbb{R}_+, \tag{31}$$

for some $\mathbb{F}$-predictable process ν,
(iii) the inequality $G_t := N_t e^{-\Lambda_t} < 1$ holds for every $t > 0$, so that $N_t < e^{\Lambda_t}$ for every $t > 0$.

We start by noting that X is postulated to be a strictly positive $(\mathbb{P},\mathbb{F})$-martingale and thus it is necessarily given by

$$X_t = \exp\left(\int_0^t x_s \, dW_s - \frac{1}{2}\int_0^t x_s^2 \, ds\right), \quad \forall\, t \in \mathbb{R}_+, \tag{32}$$

where the process x is yet unknown. The goal is to specify x in terms N and Λ in such a way that the equality $N = XY$ will hold for Y given by (27).

Lemma 5.1. *Let X be given by (32) with the process x satisfying*

$$x_t = \frac{\nu_t N_t}{N_t - e^{\Lambda_t}}, \quad \forall\, t \in \mathbb{R}_+. \tag{33}$$

Assume that x is a square-integrable process. Then the equality $N = XY$ holds, where the process Y given by (27) with $Y_0 = 1$.

Proof. Using (30) and (32), we obtain

$$d(X_tY_t) = Y_t \, dX_t - e^{\Lambda_t}(1/X_t)\, dX_t = Y_t x_t X_t \, dW_t - e^{\Lambda_t}(1/X_t)\, x_t X_t \, dW_t,$$

and thus

$$d(X_tY_t) = x_t\big(X_tY_t - e^{\Lambda_t}\big)\, dW_t. \tag{34}$$

Let us denote $V = XY$. Then V satisfies the following SDE

$$dV_t = x_t\,(V_t - e^{\Lambda_t})\, dW_t. \tag{35}$$

In view of (33), it is clear that the process N solves this equation as well. Hence to show that the equality $N = XY$ holds, it suffices to show that a solution to the SDE (35) is unique. We note that we deal here with the integral equation of the form

$$V_t = H_t + \int_0^t x_u V_u \, dW_u. \tag{36}$$

We will show that a solution to (36) is unique. To this end, we argue by contradiction. Suppose that V^i, $i = 1, 2$ are any two solutions to (36). Then the process $U = V^1 - V^2$ satisfies

$$dU_t = x_t U_t \, dW_t, \quad U_0 = 0, \tag{37}$$

which admits the obvious solution $U = 0$. Suppose that $\widehat{U}$ is a non-null solution to (37). Then the Doléans-Dade equation $dX_t = x_t X_t \, dW_t$, $X_0 = 1$ would admit the usual solution X given by (32) and another solution $X + \widehat{U} \neq X$, and this well known to be false. We conclude that (37) admits a unique solution, and this in turn implies the uniqueness of a solution to (36). This shows that $N = XY$, as was stated. □

The following result is an immediate consequence of Corollary 4.1 and Lemma 5.1.

Corollary 5.1. *Let the filtration $\mathbb{F}$ be generated by a Brownian motion W on $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. Assume that we are given an $\mathbb{F}$-adapted, continuous supermartingale G such that $G_t < 1$ for every $t > 0$ and*

$$G_t = N_t e^{-\Lambda_t}, \quad \forall \, t \in \mathbb{R}_+, \tag{38}$$

where Λ is an $\mathbb{F}$-adapted, continuous, increasing process with $\Lambda_0 = 0$ and $\Lambda_\infty = \infty$, and N is a strictly positive $\mathbb{F}$-local martingale, so that there exists an $\mathbb{F}$-predictable process ν such that

$$N_t = 1 + \int_0^t \nu_u N_u \, dW_u, \quad \forall \, t \in \mathbb{R}_+. \tag{39}$$

Let X be given by (32) with the process x satisfying

$$x_t = \frac{\nu_t G_t}{G_t - 1}, \quad \forall \, t \in \mathbb{R}_+. \tag{40}$$

Then:
(i) the equality $N = XY$ holds for the process Y given by (27),
(ii) the processes $Z = N$ and $Z_t(u) = X_t Y_{t \wedge u}$ satisfy Assumption 3.2.

In the next result, we denote by τ the random time defined by the canonical construction on a (possibly extended) probability space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$. By construction, the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{P}$ equals

$$G_t^{\mathbb{P}} := \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) = e^{-\Lambda_t}, \quad \forall \, t \in \mathbb{R}_+.$$

Let us note that, since the H-hypothesis is satisfied, the Brownian motion W remains a Brownian motion with respect to the enlarged filtration $\mathbb{G}$ under $\mathbb{P}$. It is still a Brownian motion under $\mathbb{Q}$ with respect to the filtration $\mathbb{F}$, since the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$. However, the process W is not necessarily a Brownian motion under $\mathbb{Q}$ with respect to the enlarged filtration $\mathbb{G}$.

The following result furnishes a solution to Problem (P) within the set-up described at the beginning of this section.

Proposition 5.1. *Under the assumptions of Corollary 5.1, we define a probability measure $\mathbb{Q}$ locally equivalent to $\mathbb{P}$ by the Radon-Nikodým density process $Z^{\mathbb{G}}$ given by formula (12) with $Z_t = X_t Y_t = N_t$ and $Z_t(u) = X_t Y_{t \wedge u}$ or, more explicitly,*

$$Z_t^{\mathbb{G}} = N_t \mathbb{1}_{\{\tau > t\}} + X_t Y_{t \wedge \tau} \mathbb{1}_{\{\tau \le t\}}, \quad \forall \, t \in \mathbb{R}_+. \tag{41}$$

Then the Azéma supermartingale of τ with respect to $\mathbb{F}$ under $\mathbb{Q}$ satisfies

$$\mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = X_t Y_t e^{-\Lambda_t} = N_t e^{-\Lambda_t}, \quad \forall \, t \in \mathbb{R}_+. \tag{42}$$

Moreover, the conditional distribution of τ given $\mathcal{F}_t$ satisfies

$$\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t \right), & t < u, \\ N_t e^{-\Lambda_t} + X_t \, \mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u < \tau \le t\}} \,\middle|\, \mathcal{F}_t \right), & t \ge u. \end{cases}$$

Proof. In view of Corollary 5.1, Assumption 3.2 is satisfied and thus the probability measure $\mathbb{Q}$ is well defined by the Radon-Nikodým density process $Z^{\mathbb{G}}$ given by (15), which is now equivalent to (41). Therefore, equality (42) is an immediate consequence of Proposition 3.2. Using (17), for every $u \in \mathbb{R}_+$, we obtain, for every $t \in [0, u]$

$$\mathbb{Q}\left(\tau > u \,|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(Z_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right) = \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\Lambda_u} \,\middle|\, \mathcal{F}_t\right),$$

whereas for every $t \in [u, \infty)$, we get

$$\begin{aligned}
\mathbb{Q}\left(\tau > u \,|\, \mathcal{F}_t\right) &= Z_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(Z_t(\tau)\mathbb{1}_{\{u<\tau\leq t\}} \,\middle|\, \mathcal{F}_t\right) \\
&= X_t Y_t e^{-\Lambda_t} + \mathbb{E}_{\mathbb{P}}\left(X_t Y_{t\wedge\tau}\mathbb{1}_{\{u<\tau\leq t\}} \,\middle|\, \mathcal{F}_t\right) \\
&= N_t e^{-\Lambda_t} + X_t\, \mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u<\tau\leq t\}} \,\middle|\, \mathcal{F}_t\right),
\end{aligned}$$

as required. □

Example 5.1. This example is related to the filtering problem examined in Section 2. Let $W = (W_t, t \in \mathbb{R}_+)$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and let $\mathbb{F}$ be its natural filtration. We wish to model a random time with the Azéma semimartingale with respect to the filtration $\mathbb{F}$ given by the solution to the following SDE (cf. (3))

$$dG_t = -\lambda G_t\, dt + \frac{b}{\sigma}\, G_t(1 - G_t)\, dW_t, \quad G_0 = 1. \tag{43}$$

A comparison theorem for SDEs implies that $0 < G_t < 1$ for every $t > 0$. Moreover, by an application of the Itô formula, we obtain

$$G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+,$$

where the martingale N satisfies

$$dN_t = \frac{b}{\sigma}(1 - G_t) N_t\, dW_t. \tag{44}$$

As in Section 3, we start the construction of τ by first defining a random variable Θ with exponential distribution with parameter 1 and independent of $\mathcal{F}_\infty$ under $\mathbb{P}$ and by setting

$$\tau = \inf\{t \in \mathbb{R}_+ : \lambda t \geq \Theta\}.$$

In the second step, we propose an equivalent change of a probability measure. For this purpose, we note that the process x is here given by (cf. (33))

$$x_t = \frac{\nu_t N_t}{N_t - e^{\lambda t}} = -\frac{b}{\sigma} G_t,$$

and thus x is a bounded process. Next, in view of (32) and (44), the process X solves the SDE

$$dX_t = -\frac{b}{\sigma} G_t X_t\, dW_t,$$

and the process Y satisfies (cf. (27))

$$dY_t = e^{\lambda t}\, d\Big(\frac{1}{X_t}\Big) = \frac{N_t}{X_t}\Big(\frac{b}{\sigma}\, dW_t + \frac{b^2}{\sigma^2}\, G_t\, dt\Big).$$

The integration by parts formula yields

$$\begin{aligned}
d\left(\frac{N_t}{X_t}\right) &= \frac{1}{X_t}\, dN_t + N_t\, d\left(\frac{1}{X_t}\right) + d\left[\frac{1}{X}, N\right]_t \\
&= \frac{N_t}{X_t}\frac{b}{\sigma}\,(1 - G_t)\, dW_t + \frac{N_t}{X_t}\left(\frac{b}{\sigma}\, G_t\, dW_t + \frac{b^2}{\sigma^2}\, G_t^2\, dt\right) + \frac{N_t}{X_t}\frac{b^2}{\sigma^2}\, G_t(1 - G_t)\, dt \\
&= \frac{N_t}{X_t}\Big(\frac{b}{\sigma}\, dW_t + \frac{b^2}{\sigma^2}\, G_t\, dt\Big).
\end{aligned}$$

It is now easy to conclude that $N = XY$, as was expected. Under the probability measure $\mathbb{Q}$ introduced in Proposition 5.1, we have that

$$G_t^{\mathbb{Q}} := \mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = G_t = N_t e^{-\lambda t}, \quad \forall\, t \in \mathbb{R}_+.$$

Moreover, the conditional distribution of τ given $\mathcal{F}_t$ satisfies

$$\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) = \begin{cases} \mathbb{E}_{\mathbb{P}}\left(N_u e^{-\lambda u} \,\middle|\, \mathcal{F}_t\right) = N_t e^{-\lambda u}, & t < u, \\ N_t e^{-\lambda t} + X_t\, \mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right), & t \ge u. \end{cases}$$

Since τ is here independent of $\mathcal{F}_\infty$, we obtain, for $t \ge u$,

$$\mathbb{E}_{\mathbb{P}}\left(Y_\tau \mathbb{1}_{\{u<\tau\le t\}} \,\middle|\, \mathcal{F}_t\right) = \int_u^t Y_v \lambda e^{-\lambda v}\, dv = -\int_u^t Y_v \, de^{-\lambda v} = Y_u e^{-\lambda u} - Y_t e^{-\lambda t} + \left(\frac{1}{X_t} - \frac{1}{X_u}\right),$$

where the last equality can be deduced, for instance, from (29). Therefore, for every $t \ge u$

$$\mathbb{Q}\,(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{X_t}{X_u} + X_t Y_u e^{-\lambda u}.$$

We conclude that, for every $t, u \in \mathbb{R}_+$,

$$\mathbb{Q}(\tau > u \,|\, \mathcal{F}_t) = 1 - \frac{X_t}{X_{u\wedge t}} + X_t Y_{u\wedge t} e^{-\lambda u}.$$

It is interesting to note that this equality agrees with the formula (7), which was established in Proposition 2.1 in the context of filtering problem using a different technique.

6 Applications to Valuation of Credit Derivatives

We will examine very succinctly the importance of the multiplicative decomposition of the Azéma supermartingale (i.e., the survival process) of a random time for the risk-neutral valuation of credit derivatives. Unless explicitly stated otherwise, we assume that the interest rate is null. This assumption is made for simplicity of presentation and, obviously, it can be easily relaxed.

As a risk-neutral probability, we will select either the probability measure $\mathbb{P}$ or the equivalent probability measure $\mathbb{Q}$ defined here on $(\Omega, \mathcal{G}_T)$, where T stands for the maturity date of a credit derivative. Recall that within the framework considered in this paper the survival process of a random time τ is given under $\mathbb{P}$ and $\mathbb{Q}$ by the following formulae

$$G_t^{\mathbb{P}} := \mathbb{P}\,(\tau > t \,|\, \mathcal{F}_t) = e^{-\Lambda_t},$$

and

$$G_t^{\mathbb{Q}} := \mathbb{Q}\,(\tau > t \,|\, \mathcal{F}_t) = N_t e^{-\Lambda_t},$$

respectively. The random time τ is here interpreted as the *default time* of a reference entity of a credit derivative.

We assume from now on that the increasing process Λ satisfies $\Lambda_t = \int_0^t \lambda_u \, du$ for some non-negative, $\mathbb{F}$-progressively measurable process λ. Then we have the following well known result.

Lemma 6.1. *The process M, given by the formula*

$$M_t = \mathbb{1}_{\{\tau \le t\}} - \int_0^{t\wedge\tau} \lambda_u \, du,$$

is a $\mathbb{G}$-martingale under the probability measures $\mathbb{P}$ and $\mathbb{Q}$.

The property established in Lemma 6.1 is frequently adopted in the financial literature as the definition of the *default intensity* λ. In the present set-up, Lemma 6.1 implies that the default intensity is the same under the equivalent probability measures $\mathbb{P}$ and $\mathbb{Q}$, despite the fact that the corresponding survival processes $G^{\mathbb{P}}$ and $G^{\mathbb{Q}}$ are different (recall that we postulate that N is a non-trivial local martingale). Hence the following question arises: is the specification of the default intensity λ sufficient for the risk-neutral valuation of credit derivatives related to a reference entity? Similarly as in El Karoui et al. [2], we will argue that the answer to this question is negative. To support our claim, we will show that the risk-neutral valuation of credit derivatives requires the full knowledge of the survival process, and thus the knowledge of the decreasing component Λ of the survival process in not sufficient for this purpose, in general.

To illustrate the importance of martingale component N for the valuation of credit derivatives, we first suppose $\Lambda = (\Lambda_t, t \in \mathbb{R}_+)$ is deterministic. We will argue that in that setting the process N has no influence on the prices of some simple credit derivatives, such as: a defaultable zero-coupon bond with zero recovery or a stylized credit default swap (CDS) with a deterministic protection payment. This means that the model calibration based on these assets will only allow us to recover the function Λ, but will provide no information regarding the local martingale component N of the survival process $G^{\mathbb{Q}}$. However, if the assumptions of the deterministic character of Λ and/or protection payment of a CDS are relaxed, then the corresponding prices will depend on the choice of N as well, and thus the explicit knowledge of N becomes important (for an example, see Corollary 6.1). As expected, this feature becomes even more important when we deal with a credit risk model in which the default intensity λ is stochastic, as is typically assumed in the financial literature.

6.1 Defaultable Zero-Coupon Bonds

By definition, the risk-neutral price under $\mathbb{Q}$ of the T-maturity defaultable zero-coupon bond with zero recovery equals, for every $t \in [0, T]$,

$$D^{\mathbb{Q}}(t,T) := \mathbb{Q}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{Q}(\tau > T \,|\, \mathcal{F}_t)}{G_t^{\mathbb{Q}}} = \mathbb{1}_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{Q}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)}{G_t^{\mathbb{Q}}}. \tag{45}$$

Assuming that Λ is deterministic, we obtain the pricing formulae independent of N. Indeed, the risk-neutral price of the bond under $\mathbb{P}$ equals, for every $t \in [0, T]$,

$$D^{\mathbb{P}}(t,T) := \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{-(\Lambda_T - \Lambda_t)}.$$

On the other hand, using (45) and the fact that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$, we obtain

$$\begin{aligned} D^{\mathbb{Q}}(t,T) \quad &:= \quad \mathbb{Q}(\tau > T \,|\, \mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{Q}}(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t) \\ &= \quad \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}}\, e^{-\Lambda_T}\, \mathbb{E}_{\mathbb{P}}(N_T \,|\, \mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}}\, e^{-(\Lambda_T - \Lambda_t)}, \end{aligned}$$

where we have assumed that N is a (true) $(\mathbb{P}, \mathbb{F})$-martingale.

If we allow for a stochastic process Λ, then the role of N in the valuation of defaultable zero-coupon bonds becomes important, as can be seen from the following expressions

$$D^{\mathbb{P}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{P}}\big(e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)$$

and

$$D^{\mathbb{Q}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{Q}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}}\, \mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big),$$

where the last equality follows from the standing assumption that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$. We thus see that the inequality $D^{\mathbb{P}}(t,T) \neq D^{\mathbb{Q}}(t,T)$ is likely to hold when Λ is stochastic.

In the case of a stochastic interest rate, the bond valuation problem is more difficult. Let the discount factor $\beta = (\beta_t, t \in \mathbb{R}_+)$ be defined by

$$\beta_t = \exp\left(-\int_0^t r_s\, ds\right),$$

where the short-term interest rate process $r = (r_t, t \in \mathbb{R}_+)$ is assumed to be $\mathbb{F}$-adapted. Then the risk-neutral prices under $\mathbb{P}$ and $\mathbb{Q}$ of the T-maturity defaultable zero-coupon bond with zero recovery are given by the following expressions

$$D^{\mathbb{P}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{e^{-\Lambda_t}\beta_t}\, \mathbb{E}_{\mathbb{P}}\big(e^{-\Lambda_T}\beta_T \,|\, \mathcal{F}_t\big)$$

and

$$D^{\mathbb{Q}}(t,T) = \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}\beta_t}\, \mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T}\beta_T \,|\, \mathcal{F}_t\big).$$

Once again, it is clear that the role of the martingale component of the survival process is non-trivial, even in the case when the default intensity is assumed to be deterministic.

6.2 Credit Default Swaps

Let us now consider a stylized CDS with the protection payment process R and fixed spread κ, which gives protection over the period $[0,T]$. It is known that the risk-neutral price under $\mathbb{P}$ of this contract is given by the formula, for every $t \in [0,T]$,

$$S_t^{\mathbb{P}} := \mathbb{E}_{\mathbb{P}}\Big(\mathbb{1}_{\{t<\tau\le T\}} R_\tau - \kappa\big((T\wedge\tau) - (t\vee\tau)\big) \,\Big|\, \mathcal{G}_t\Big). \tag{46}$$

In the case where $\Lambda_t = \int_0^t \lambda_u\, du$, one can also show that (see Bielecki et al. [1])

$$S_t^{\mathbb{P}} = \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{P}}}\, \mathbb{E}_{\mathbb{P}}\bigg(\int_t^T G_u^{\mathbb{P}}(R_u\lambda_u - \kappa)\, du \,\bigg|\, \mathcal{F}_t\bigg). \tag{47}$$

Analogous formulae are valid under $\mathbb{Q}$, if we decide to choose $\mathbb{Q}$ as a risk-neutral probability.

To analyze the impact of N on the value of the CDS, let us first consider the special case when the default intensity λ and the protection payment R are assumed to be deterministic. In that case, the risk-neutral price of the CDS under $\mathbb{P}$ can be represented as follows

$$S_t^{\mathbb{P}} = \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t} \int_t^T e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du.$$

For the risk-neutral price under $\mathbb{Q}$, we obtain

$$\begin{aligned}
S_t^{\mathbb{Q}} &= \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{Q}}}\, \mathbb{E}_{\mathbb{Q}}\bigg(\int_t^T G_u^{\mathbb{Q}}(R_u\lambda_u - \kappa)\, du \,\bigg|\, \mathcal{F}_t\bigg) \\
&= \mathbb{1}_{\{\tau>t\}} \frac{1}{G_t^{\mathbb{Q}}} \int_t^T \mathbb{E}_{\mathbb{P}}\big(G_u^{\mathbb{Q}} \,|\, \mathcal{F}_t\big)(R_u\lambda_u - \kappa)\, du \\
&= \mathbb{1}_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}} \int_t^T \mathbb{E}_{\mathbb{P}}\big(N_u \,|\, \mathcal{F}_t\big) e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du \\
&= \mathbb{1}_{\{\tau>t\}}\, e^{\Lambda_t} \int_t^T e^{-\Lambda_u}(R_u\lambda_u - \kappa)\, du,
\end{aligned}$$

where in the second equality we have used once again the standing assumption that the restriction of $\mathbb{Q}$ to $\mathbb{F}$ is equal to $\mathbb{P}$ and the last one holds provided that N is a $(\mathbb{P},\mathbb{F})$-martingale. It is thus clear that the equality $S_t^{\mathbb{P}} = S_t^{\mathbb{Q}}$ holds for every $t \in [0,T]$, so that the price of the CDS does not depend on the particular choice of N when λ and R are deterministic.

Let us now consider the case when the intensity λ is assumed to be deterministic, but the protection payment R is allowed to be stochastic. Since our goal is to provide an explicit example, we postulate that $R = 1/N$, though we do not pretend that this is a natural choice of the protection payment process R.

Corollary 6.1. *Let us set $R = 1/N$ and let us assume that the default intensity λ is deterministic. If the process N is a $(\mathbb{P}, \mathbb{F})$-martingale then then the fair spread $\kappa_0^{\mathbb{P}}$ of the CDS under $\mathbb{P}$ satisfies*

$$\kappa_0^{\mathbb{P}} = \frac{\int_0^T \mathbb{E}_{\mathbb{P}}\big((N_u)^{-1}\big)\lambda_u e^{-\Lambda_u}\,du}{\int_0^T e^{-\Lambda_u}\,du} \tag{48}$$

and the fair spread $\kappa_0^{\mathbb{Q}}$ under $\mathbb{Q}$ equals

$$\kappa_0^{\mathbb{Q}} = \frac{1 - e^{-\Lambda_T}}{\int_0^T e^{-\Lambda_u}\,du}. \tag{49}$$

Proof. Recall that the fair spread at time 0 is defined as the level of κ for which the value of the CDS at time 0 equals zero, that is, $S_0(\kappa) = 0$. By applying formula (47) with $t = 0$ to risk-neutral probabilities $\mathbb{P}$ and $\mathbb{Q}$, we thus obtain

$$\kappa_0^{\mathbb{P}} = \frac{\mathbb{E}_{\mathbb{P}}\left(\int_0^T G_u^{\mathbb{P}} R_u \lambda_u\,du\right)}{\mathbb{E}_{\mathbb{P}}\left(\int_0^T G_u^{\mathbb{P}}\,du\right)}$$

and

$$\kappa_0^{\mathbb{Q}} = \frac{\mathbb{E}_{\mathbb{Q}}\left(\int_0^T G_u^{\mathbb{Q}} R_u \lambda_u\,du\right)}{\mathbb{E}_{\mathbb{Q}}\left(\int_0^T G_u^{\mathbb{Q}}\,du\right)}.$$

Let us first observe that the denominators in the two formulae above are in fact equal since

$$\mathbb{E}_{\mathbb{P}}\bigg(\int_0^T G_u^{\mathbb{P}}\,du\bigg) = \int_0^T e^{-\Lambda_u}\,du$$

and

$$\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T G_u^{\mathbb{Q}}\,du\bigg) = \mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T N_t e^{-\Lambda_u}\,du\bigg) = \int_0^T \mathbb{E}_{\mathbb{Q}}(N_t) e^{-\Lambda_u}\,du = \int_0^T \mathbb{E}_{\mathbb{P}}(N_t) e^{-\Lambda_u}\,du = \int_0^T e^{-\Lambda_u}\,du$$

since the restriction of $\mathbb{Q}$ to $\mathbb{F}$ equals $\mathbb{P}$ and N is a $(\mathbb{P}, \mathbb{F})$-martingale, so that $\mathbb{E}_{\mathbb{P}}(N_t) = 1$ for every $t \in [0, T]$. For the numerators, using the postulated equality $R = 1/N$, we obtain

$$\mathbb{E}_{\mathbb{P}}\bigg(\int_0^T G_u^{\mathbb{P}} R_u \lambda_u\,du\bigg) = \mathbb{E}_{\mathbb{P}}\bigg(\int_0^T (N_u)^{-1}\lambda_u e^{-\Lambda_u}\,du\bigg) = \int_0^T \mathbb{E}_{\mathbb{P}}\big((N_u)^{-1}\big)\lambda_u e^{-\Lambda_u}\,du$$

and

$$\mathbb{E}_{\mathbb{Q}}\bigg(\int_0^T G_u^{\mathbb{Q}} R_u \lambda_u\,du\bigg) = \int_0^T \lambda_u e^{-\Lambda_u}\,du = 1 - e^{-\Lambda_T}.$$

This proves equalities (48) and (49). □

Formulae (48) and (49) make it clear that the fair spreads $\kappa_0^{\mathbb{P}}$ and $\kappa_0^{\mathbb{Q}}$ are not equal, in general. This example supports our claim that the knowledge of the default intensity λ, even in the case when λ is deterministic, is not sufficient for the determination of the risk-neutral price of a credit derivative, in general. To conclude, a specific way in which the default time of the underlying entity is modeled should always be scrutinized in detail, and, when feasible, the multiplicative decomposition of the associated survival process should be computed explicitly.

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