

# PORTFOLIO OPTIMIZATION UNDER A PARTIALLY OBSERVED JUMP-DIFFUSION MODEL

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**ABSTRACT.** This paper studies the question of maximizing terminal wealth from expected utility in a multidimensional jump-diffusion model. The special feature of our approach is that the investor only observes the vector of stock prices, therefore leading to a partial information framework. Using non-linear filtering and change of measure techniques, we show that the optimization problem can be rewritten such that parameters depend only on the past history of observed prices. Through duality approach, we derive the optimal value function. As examples, special attention is given to three standard utility functions for which we exhibit the optimal value functions.

**Mathematics Subject Classification:** 60G48, 60G35, 90E11.

**JEL Classification:** G11.

**Key words:** portfolio optimization, partial information, discontinuous processes.

## 1. INTRODUCTION

In this paper, we study the question of maximizing terminal wealth from expected utility in a partial information framework. The situation appears when investors only observe the vector of stock prices and when they cannot disentangle the drift terms from the other sources of uncertainty. More specifically, in the economy we consider, growth rates are altered by infrequent large shocks and continuous small shocks. Investors observe changes in returns but they cannot perfectly distinguish their dynamics. Instead, they solve a signal extraction problem.

Motivated by recent findings which indicate the importance of jumps in returns to fully capture the empirical features of equity index returns (Eraker et al. 2003 [9]), we consider a market model where the stock price process solves a mixed jump-diffusion stochastic differential equation where the average growth rate and the jump time intensity are unobservable. Such a model seems reasonable, since jumps are often generated by external sources whose impact cannot be completely analyzed. The random feature of the drift term and the jump intensity are modeled via a strong Markov process (see Duffie et al. (2003) [8]). This Markov unobservable process may be interpreted as an environment process which represents relevant factors affecting stock price dynamics, like economical news, political situations, technical progress.

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The optimization setting with full information goes back to Merton (1971) [22] who solved the problem in a Black-Scholes environment via the Hamilton-Jacobi-Bellman equation and dynamic programming. For the case of complete markets, we refer to Karatzas et al. (1987) [16] or Cox and Huang (1989) [5]. Models with incomplete information have been investigated by Detemple (1986) [6], Dothan and Feldman (1986) [7] and Gennotte (1986) [11]. The authors deal with an unobserved average growth rate, using dynamic programming methods within the linear Gaussian filtering framework. Lakner (1995, 1998) [18], [19] solved the partial optimization problem via a martingale approach : he characterized the optimal strategy via Malliavin calculus and he worked out the special case of linear Gaussian models. Pham and Quenez (2001) [24] treated the case of an incomplete stochastic volatility model and Sass and Haussmann (2004) [30] solved the case of hidden Markov filtering. None of these papers have considered the case of a jump-diffusion model, which would be more realistic. References on dynamic optimization with jumps include Jeanblanc-Picqué and Pontier (1990) [14], Shirakawa (1990) [31], Bellamy (1999) [2] or Liu et al. (2003) [21]. Unlike the present setup, market models are generally supposed to be *completely* observable, in the sense that both the drift term and the jump intensity are known. Callegaro, Di Masi and Runggaldier (2006) [4] recently studied the case of a partially observed market driven by purely discontinuous asset prices. In our paper, and similarly with Bauerle and Rieder (2007) [1], who studied the one-dimensional case, we consider the situation where both drift terms and intensities are unknown. We intend to solve Merton's problem through maximizing the expected utility of terminal wealth.

The common way to solve the problem is to use filtering theory, so as to reduce the stochastic control problem with partial information to one with complete observation. It is then possible to derive the solution either using the martingale approach, cf. Kramkov and Schachermayer (1999) [17], or via stochastic control methods, cf. Framstad et al (1999) [10]. In this paper, we combine stochastic filtering techniques and a martingale duality approach to characterize the value function. Nevertheless, as the reduced market model is incomplete, we use the theory of stochastic control to solve the problem explicitly.

The paper is organized as follows: Section 2 states the framework and recalls some known results on portfolio optimization. Section 3 shows that conditioning arguments can be used to replace the original partial information problem by a full information one which depends only on the past history of observed prices. Section 4 derives the value function within partial information. The special cases of power, logarithmic and exponential utility functions are studied.

## 2. FORMULATION OF THE PROBLEM

**2.1. The Economy.** We consider an economy defined on the complete probability space  $(\Omega, \mathcal{A}, \mathbb{P})$  for a finite time span  $[0, T]$  with  $T \in (0, \infty)$ , equipped with a filtration  $\mathbb{A} = (\mathcal{A}_t)_{t \in [0, T]}$  satisfying the usual conditions and on which all stochastic processes are defined. In this market,  $1 + m$  assets are traded. The first one is a non-risky asset (savings account) which pays no dividend and satisfies :

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 = 1,$$

where  $(r_t, t \in [0, T])$  is uniformly bounded. The  $m$  other assets are risky and we refer to them as the stocks. Letting  $S_t^i$  be the positive price at time  $t$  of the  $i^{\text{th}}$  asset, and  $S_t$  the  $m$ -vector of asset prices, we assume that its evolution is modeled, for  $i \in \llbracket 1; m \rrbracket$ , through the following equation:

$$dS_t^i = S_{t-}^i \left( \mu_t^i(\theta_t) dt + \sum_{j=1}^m \left\{ \sigma^{ij}(t, S_t) dW_t^j + \omega^{ij}(t, S_{t-}) \left( dM_t^j + \lambda_t^j(\theta_t) dt \right) \right\} \right),$$

where the real-valued process  $(\theta_t, t \in [0, T])$  stands for an economic factor process. Here,  $W$  is a  $m$ -dimensional Brownian motion,  $M$  is the compensated martingale of a  $m$ -dimensional inhomogeneous Poisson process  $N$ , whose components have no common jumps: each  $N^j$  is such that  $M_t^j = N_t^j - \int_0^t \lambda_s^j(\theta_s) ds$ , and  $\lambda^j(\theta)$  is the  $\mathbb{R}^+$ -valued intensity. Processes  $W$  and  $N$  are independent of each other and  $\mathbb{A}$ -adapted. We denote by  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$  the *model* information, which is the filtration generated by the processes  $W$ ,  $N$  and  $\theta$ , so:

$$\mathcal{F}_t = \sigma(W_s, s \in [0, t]) \vee \sigma(N_s, s \in [0, t]) \vee \sigma(\theta_s, s \in [0, t]), \quad \mathcal{F}_t \subset \mathcal{A}_t.$$

In general, the economic factor process  $\theta$  is not directly observable, which is equivalent to say that  $\mathbb{F}$  is not available. This situation therefore entails that the *instantaneous means* of the continuous and counting processes are not observed.

Thereafter, we assume the following:

**Assumption 1.** For  $(i, j) \in \llbracket 1; m \rrbracket$ ,  $x \in \mathbb{R}_+^m$ ,  $y \in \mathbb{R}$  and  $t \in [0, T]$ :

- (1)  $r_t > 0$ ,  $\mu_t^i(y) \in \mathbb{R}$ , a.s.,
- (2)  $\mu_t^i(y)$  and  $\lambda_t^i(y)$  as well as  $(\lambda_t^i(y))^{-1}$  are  $\mathbb{F}$ -predictable and uniformly bounded processes,
- (3)  $x^i \sigma^{ij}(t, x)$  and  $x^i \omega^{ij}(t, x)$  are Borel functions satisfying Lipschitz and growth conditions, such that  $\sigma^{ij}(t, x) > 0$  and  $\omega^{ij}(t, x) > -1$ ,
- (4)  $\theta$  is an  $\mathbb{R}$ -valued càdlàg homogeneous and  $\mathbb{A}$ -adapted Markov process,
- (5)  $W$  and  $N$  are assumed to be independent of  $\theta$  and the possible jumps of  $\theta$  are disjoint from those of  $N$ .

Using vector notation, the process  $(S_t, t \in [0, T])$  satisfies :

$$(2.1) \quad dS_t = S_{t-} (b_t(\theta_t) dt + \sigma_t dW_t + \omega_t dM_t)$$

where:

$$b_t(\theta_t) = (b_t^1(\theta_t), \dots, b_t^m(\theta_t))^{\top},$$

with  $b_t^i(\theta_t) \equiv b^i(t, \theta_t, S_t) = \mu_t^i(\theta_t) + \omega^i(t, S_t) \lambda^{\top}(t, \theta_t)$  and:

$$\lambda_t(\theta_t) = (\lambda_t^1(\theta_t), \dots, \lambda_t^m(\theta_t)), \quad \sigma_t^i = (\sigma_t^{i,1}, \dots, \sigma_t^{i,m}), \quad \omega_t^i = (\omega_t^{i,1}, \dots, \omega_t^{i,m})$$

with:

$$\sigma_t^{i,j} \equiv \sigma^{i,j}(t, S_t), \quad \omega_t^{i,j} \equiv \omega^{i,j}(t, S_{t-}).$$

Under Assumption 1, there exists a unique  $\mathbb{A}$ -adapted solution that does not explode until time  $T$ , cf. Protter (1990) [25]. At this stage, we also assume that the  $m \times m$ -matrix  $\sigma_t$

is *invertible* for all  $t \in [0, T]$  and that  $\mu(\theta)$  and  $\sigma$  satisfy:

$$(2.2) \quad \int_0^T E[|\mu_t(\theta_t)|] dt < \infty, \quad \int_0^T E[|\sigma_t \sigma_t^\top|] dt < \infty, \quad \mathbb{P} - a.s.$$

Finally, the  $m \times m$ -matrix  $\omega_t$  is *invertible*, for all  $t \in [0, T]$ . This implies that, since there are no common jumps among the components of the counting process  $N$ , there exists a one-to-one correspondence between the observation of the size of a jump of  $S$  and the knowledge of which of the process  $N^i$ ,  $i \in \llbracket 1; m \rrbracket$ , has jumped.

**2.1.1. Partial Information Setup.** We now consider, as in Detemple (1986) [6] or Lakner (1995, 1998) [18], [19], that agents do not have access to filtration  $\mathbb{F}$ . They only observe the asset prices. Thus observations are given by the sequence  $(S_t^0, S_t^i)_{t \geq 0}^{i \in \llbracket 1; m \rrbracket}$ . We denote by  $\mathbb{G} = (\mathcal{G}_t, t \in [0, T])$ , with  $\mathbb{G} \subsetneq \mathbb{F}$ , the  $\mathbb{P}$ -augmentation of the *market* filtration generated by the  $1 + m$  assets:

$$\mathcal{G}_t = \sigma((S_s^0, S_s^i), i \in \llbracket 1; m \rrbracket, s \leq t).$$

We assume that  $\theta_0$  is independent of  $\mathcal{G}_\infty$ . From a filtering perspective,  $\theta_0$  is a random variable (which is a-priori fixed) and so  $\mathcal{G}_0$  is not trivial.

**Remark 1.** *By definition, the processes  $\sigma$  and  $\omega$  are  $\mathbb{G}$ -adapted, the interest rate  $r$  is  $\mathbb{G}$ -adapted.*

## 2.2. The Optimization Problem.

**2.2.1. Trading Strategies.** By denoting  $\phi_t = (\phi_t^1, \dots, \phi_t^m)^\top$  the vector of fractions of positive wealth invested in the  $m$  risky assets at time  $t$ , a self-financing trading strategy is a pair  $(x_0, \phi)$  where  $x_0 \geq 0$  is the initial investment and  $\phi$  is an  $\mathbb{R}^m$ -valued and  $\mathbb{G}$ -predictable process such that the wealth process:

$$(2.3) \quad dX_t^\phi = X_{t-}^\phi ((r_t + \phi_t^\top (b_t(\theta_t) - r_t \mathbf{1}_m)) dt + \phi_t^\top \sigma_t dW_t + \phi_t^\top \omega_t dM_t)$$

with  $X_0^\phi = x_0$ , is  $\mathbb{P}$ -a.s. well defined. The class of admissible strategies reads:

$$\mathcal{S}(t) = \{\phi : [t, T] \times \Omega \rightarrow \mathbb{R}^m, \mathbb{G} - \text{predictable}\}$$

The numéraire investment is given by  $\phi_t^0 = 1 - \sum_{i=1}^m \phi_t^i$ .

**2.2.2. Optimizing Terminal Wealth.** A function  $U : \mathbb{R} \rightarrow \mathbb{R}$  is called a utility function if it is strictly increasing, strictly concave and of class  $\mathcal{C}^2$ . The optimization problem the investor faces is to maximize the expected utility of his/her terminal wealth over the class of admissible policies.

**Definition 1.** *Let  $U$  be a utility function. We then define<sup>1</sup>, for  $\phi \in \mathcal{S}(t)$ :*

$$(2.4) \quad \begin{aligned} J_\phi^\mathbb{G}(t, x) &\triangleq E^\mathbb{P} \left[ U(X_T^{x,t,\phi}) | \mathcal{G}_t \right], \\ u^\mathbb{G}(t, x) &= \sup_{\phi \in \mathcal{S}(t)} J_\phi^\mathbb{G}(t, x). \end{aligned}$$

<sup>1</sup>The notation  $X^{x,t,\phi}$  is a shorthand for the process  $X^\phi$  satisfying  $X_s = x + \int_t^s \phi_u^\top X_{u-} \frac{dS_u}{S_{u-}}$ ,  $s \geq t$ .

Then, for  $x_0 > 0$ , a portfolio strategy  $\phi^* \in \mathcal{S}(0)$  is optimal if and only if:

$$u^{\mathbb{G}}(0, x_0) = J_{\phi^*}^{\mathbb{G}}(0, x_0).$$

The control problem (2.4) is stated under partial information. In order to solve it, we reduce it to a control problem with complete observation.

### 3. MECHANICS OF THE LEARNING PROCESS

**3.1. Setup.** We introduce the  $\mathbb{G}$ -conditional law of the random variable  $\theta_t$ , say:

$$(3.1) \quad \pi_t(f) = E^{\mathbb{P}}[f(\theta_t) | \mathcal{G}_t]$$

for any  $\mathbb{R}$ -valued measurable function  $f$  such that  $E^{\mathbb{P}}[|f(\theta_t)|] < \infty$ .

**3.1.1. Reference Measure.** Following Kallianpur (1980) [15], we introduce a new probability measure  $\mathbb{P}^0$ , termed the *reference measure*. To this end, let:

$$(3.2) \quad H_t = 1 - \int_0^t H_{s-} [a_s^{\top}(\theta_s) dW_s + (\mathbf{1}_m^{\top} - c_s^{\top}(\theta_s)) dM_s]$$

where<sup>2</sup>:

$$(3.3) \quad a_s(\theta_s) = \sigma_s^{-1}(b_s(\theta_s) - r_s \mathbf{1}_m), \quad c_s(\theta_s) = \lambda_s^{-1}(\theta_s).$$

Note that, since  $\theta$  and  $N$  do not have common jumps, one has:

$$\xi_t := \int_0^t (\mathbf{1}_m^{\top} - c_s^{\top}(\theta_s)) dM_s = \int_0^t (\mathbf{1}_m^{\top} - c_s^{\top}(\theta_{s-})) dM_s$$

and the process  $\xi$  is a local martingale.

**Proposition 1.** *Let us assume that  $\ln c_s(\theta_s)$  is (component-wise) bounded on  $[0, T]$ . Then  $H$  is a strictly positive  $(\mathbb{P}, \mathbb{F})$ -martingale.*

*Proof.* Let  $H^W$  and  $H^M$  be the unique solutions of:

$$dH_t^W = -H_t^W a_t dW_t, \quad dH_t^M = -H_{t-}^M (1 - c_t) dM_t, \quad H_0^W = H_0^M = 1,$$

(in the one-dimensional case). By integration by parts formula, we have  $H_t = H_t^W H_t^M$ . From Assumption 1,  $H^W$  and  $H^M$  are  $(\mathbb{P}, \mathbb{F})$ -martingales. The strict positivity of  $H^M$  follows from the strict positivity of  $c$ . As  $H^W$  and  $H^M$  are orthogonal, the conclusion follows.  $\square$

From Proposition 1, we then have:

**Proposition 2.** *On  $(\Omega, \mathbb{F})$ , we define the measure  $\mathbb{P}^0$  equivalent to  $\mathbb{P}$  by:*

$$(3.4) \quad \frac{d\mathbb{P}^0}{d\mathbb{P}} \Big|_{\mathcal{F}_T} = H_T.$$

*Girsanov's transformation ensures that:*

$$(3.5) \quad W_t^0 = W_t + \int_0^t a_s(\theta_s) ds \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-Brownian motion,}$$

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<sup>2</sup>We note  $a_s(\theta_s)$  and  $c_s(\theta_s)$  to recall the dependence on the unknown process  $\theta_s$ .

$$(3.6) \quad M_t^0 = M_t + \int_0^t D_s^\theta [\mathbf{1}_m - c_s(\theta_s)] ds \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-martingale,}$$

with  $D_s^\theta = \text{Diag}(\lambda_s(\theta_s))$ . Thus,  $N$  is an  $\mathbb{F}$ -standard Poisson process with  $\mathbb{P}^0$ -intensity  $\text{Diag}(\mathbf{1}_m)$ , so that  $M_t^0 = N_t - \mathbf{1}_m^\top t$  is the  $\mathbb{P}^0$ -compensated martingale of  $N$ .

We begin by proving a lemma which will be highly relevant in the following. It extends a result by Pham and Quenez (2001) [24] :

**Lemma 1.** *The filtration  $\mathbb{G}$  is the augmented filtration of  $(W^0, M^0)$ .*

*Proof.* From  $dS_t = S_{t-} (r_t dt + \sigma_t dW_t^0 + \omega_t dM_t^0)$  one obtains:

$$(3.7) \quad \omega_t^{-1} \left( \frac{dS_t}{S_{t-}} - r_t dt + \omega_t dt \right) = \sigma_t \omega_t^{-1} dW_t^0 + dN_t$$

from which we get, using the fact that the predictable covariation process is  $\mathbb{G}$ -adapted, that  $\int_0^t (\sigma_s \omega_s^{-1})^2 ds + t$  is  $\mathbb{G}$ -adapted. Besides, as covariation processes are adapted, it follows that  $\int_0^t (\sigma_s \omega_s^{-1})^2 \frac{dS_s}{S_{s-}} + N_t$  is  $\mathbb{G}$ -adapted which leads to  $N$  is  $\mathbb{G}$ -adapted. Then, by (3.7),  $W^0$  is  $\mathbb{G}$ -adapted. Noting  $\mathbb{F}^0$  the augmented filtration of the processes  $(W^0, N)$ , this implies that  $\mathbb{F}^0 \subseteq \mathbb{G}$ . Conversely, following Protter (1990) [25], as  $\sigma$  and  $\omega$  are  $\mathbb{G}$ -adapted processes and under Assumption 1, we get that the unique solution of (3.7), say  $Y^0$ , is  $\mathbb{F}^0$ -adapted, so that  $\mathbb{G} \subseteq \mathbb{F}^0$ , hence yielding  $\mathbb{G} = \mathbb{F}^0$ .  $\square$

This lemma allows to extend results of Lipster and Shiryaev (2001) [20] or Brémaud (1981) [3] to the case of jump-diffusions. By Itô's calculus applied to (3.2), the  $(\mathbb{P}^0, \mathbb{F})$ -martingale  $H^{-1}$  reads:

$$(3.8) \quad H_t^{-1} = 1 + \int_0^t H_{s-}^{-1} \left[ a_s^\top(\theta_s) dW_s^0 + \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) dM_s^0 \right].$$

**Theorem 1.** *For all  $t \in [0, T]$ , we have:*

$$(3.9) \quad \begin{aligned} \mathbb{E}^{\mathbb{P}^0} [H_t^{-1} | \mathcal{G}_t] &= 1 + \int_0^t \mathbb{E}^{\mathbb{P}^0} [H_{s-}^{-1} a_s^\top(\theta_s) | \mathcal{G}_s] dW_s^0 \\ &\quad + \int_0^t \mathbb{E}^{\mathbb{P}^0} \left[ H_{s-}^{-1} \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) | \mathcal{G}_s \right] dM_s^0, \end{aligned}$$

*Proof.* The equality (3.9) is equivalent to:

$$(3.10) \quad \mathbb{E}^{\mathbb{P}^0} [H_t^{-1} A] = \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t A]$$

for any bounded  $\mathcal{G}_t$ -measurable random variable  $A$ , where  $\mathcal{R}_t$  denotes the right-hand side of (3.9). Via the Predictable Representation Theorem (PRT) for  $\mathbb{G}$ -martingales w.r.t. Brownian motion  $W^0$  and compensated Poisson process  $M^0$ , as quoted in Runggaldier (2003) [29], the random variable  $A$  is, for  $t \geq 0$ , of the form:

$$(3.11) \quad A = \mathcal{M}_0 + \int_0^t U_s dW_s^0 + \int_0^t V_s dM_s^0 \triangleq \mathcal{M}_0 + \mathcal{M}_t^0$$

where  $U$  and  $V$  are  $\mathbb{G}$ -predictable processes and  $\mathcal{M}_0 \in \mathcal{G}_0$  is not necessarily a constant.

Hence, (3.10) will follow from:

$$\mathbb{E}^{\mathbb{P}^0} [H_t^{-1} \mathcal{M}_0] + \mathbb{E}^{\mathbb{P}^0} [H_t^{-1} \mathcal{M}_t^0] = \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t \mathcal{M}_0] + \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t \mathcal{M}_t^0],$$

or equivalently:

$$(3.12) \quad \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t H_s^{-1} d\nu_s^0 \cdot \mathcal{M}_0 \right] = \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t \mathcal{M}_0],$$

$$(3.13) \quad \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t H_s^{-1} d\nu_s^0 \cdot \mathcal{M}_t^0 \right] = \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t \mathcal{M}_t^0],$$

with:

$$d\nu_s^0 = a_s^\top(\theta_s) dW_s^0 + \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) dM_s^0,$$

and:

$$\mathcal{R}_t^0 = \int_0^t \mathbb{E}^{\mathbb{P}^0} [H_{s-}^{-1} a_s^\top(\theta_s) | \mathcal{G}_s] dW_s^0 + \int_0^t \mathbb{E}^{\mathbb{P}^0} \left[ H_{s-}^{-1} \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) | \mathcal{G}_s \right] dM_s^0.$$

Using the fact that  $\mathcal{M}^0$  and  $\nu^0$  are martingales, an integration by parts shows that (3.13) reads:

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t \left( H_{s-}^{-1} a_s^\top U_s + H_{s-}^{-1} \left( (c_s^\top)^{-1} - \mathbf{1}_m^\top \right) V_s \right) ds \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t \left( \mathbb{E}^{\mathbb{P}^0} [H_{s-}^{-1} U_s a_s^\top | \mathcal{G}_s] + \mathbb{E}^{\mathbb{P}^0} [H_{s-}^{-1} V_s \left( (c_s^\top)^{-1} - \mathbf{1}_m^\top \right) | \mathcal{G}_s] \right) ds \right], \end{aligned}$$

which is obviously satisfied. Also, in (3.12), we note that  $\mathcal{M}_0 \in \mathcal{G}_0$  and similar arguments yield:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^0} [\mathcal{R}_t^0 \mathcal{M}_0] &= \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t \mathbb{E}^{\mathbb{P}^0} [H_{s-}^{-1} a_s^\top(\theta_s) \mathcal{M}_0 | \mathcal{G}_s] dW_s^0 \right. \\ &\quad \left. + \int_0^t \mathbb{E}^{\mathbb{P}^0} \left[ H_{s-}^{-1} \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) \mathcal{M}_0 | \mathcal{G}_s \right] dM_s^0 \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \int_0^t H_{s-}^{-1} a_s^\top(\theta_s) \mathcal{M}_0 dW_s^0 + \int_0^t H_{s-}^{-1} \left( (c_s^\top(\theta_s))^{-1} - \mathbf{1}_m^\top \right) \mathcal{M}_0 dM_s^0 \right], \end{aligned}$$

which concludes the proof.  $\square$

**3.2. Resolution of Uncertainty.** We now consider the restriction of  $\mathbb{P}$  to  $\mathbb{G}$ . From the Bayes formula:

$$(3.14) \quad \mathbb{E}^{\mathbb{P}^0} [X | \mathcal{G}_t] = \frac{\mathbb{E}^{\mathbb{P}} [H_t X | \mathcal{G}_t]}{\mathbb{E}^{\mathbb{P}} [H_t | \mathcal{G}_t]}$$

for any  $\mathcal{F}_t$ -measurable and  $\mathbb{P}^0$ -integrable random variable  $X$ . Hence, letting  $X = 1/H_t$  in (3.14), we get:

$$(3.15) \quad Z_t \triangleq \mathbb{E}^{\mathbb{P}^0} \left[ \frac{1}{H_t} | \mathcal{G}_t \right] = \frac{1}{\mathbb{E}^{\mathbb{P}} [H_t | \mathcal{G}_t]}.$$

As  $(1/H)$  is a  $(\mathbb{P}^0, \mathbb{F})$ -martingale,  $Z$  is a  $(\mathbb{P}^0, \mathbb{G})$ -martingale whose martingale decomposition, given by Theorem 1, is:

$$(3.16) \quad Z_t = 1 + \int_0^t Z_{s-} [\pi_s^\top(a) dW_s^0 + (\pi_s^\top(c^{-1}) - \mathbf{1}_m) dM_s^0]$$

with processes  $(a, c)$  given by (3.3). Then, from (3.4) and (3.15):

$$(3.17) \quad \frac{d\mathbb{P}}{d\mathbb{P}^0} \Big|_{\mathcal{G}_T} = Z_T.$$

As a consequence of Girsanov's theorem, we have:

**Lemma 2.** *For all  $t \in [0, T]$ :*

$$(3.18) \quad \overline{W}_t = W_t + \int_0^t \sigma_s^{-1} [\mu_s(b_s) - \pi_s(b)] ds \text{ is a } (\mathbb{P}, \mathbb{G})\text{-Brownian motion,}$$

$$\overline{M}_t = M_t + \int_0^t [\lambda_s(\theta_s) - \pi_s(\lambda)] ds \text{ is a } (\mathbb{P}, \mathbb{G})\text{-martingale.}$$

Processes  $\overline{W}$  and  $\overline{M}$  are called *innovation* processes in filtering theory. Importantly enough, they include the distances between the true values of  $\mu$  and  $\lambda$  and their estimates.

3.2.1. *Complete Observation Problem.* Using these *innovation* processes, we can now describe the dynamics of the prices within the framework of a complete observation model:

$$(3.19) \quad dS_t = S_{t-} [\pi_t(b) dt + \sigma_t d\overline{W}_t + \omega_t d\overline{M}_t]$$

with the  $\mathcal{G}_t$ -conditional counterpart of  $b_t(\theta_t)$  given by  $\pi_t(b) = \pi_t(\mu) + \omega_t \pi_t(\lambda)$ . In the same way, the reduced wealth process (2.3) satisfies:

$$(3.20) \quad dX_t^\phi = X_{t-}^\phi ((r_t + \phi_t^\top(\pi_t(b) - r_t \mathbf{1}_m)) dt + \phi_t^\top \sigma_t d\overline{W}_t + \phi_t^\top \omega_t d\overline{M}_t)$$

We can therefore restate the partially observable stochastic control problem (2.4) as follows:

**Definition 2.** *Let  $U$  be a utility function. We then define:*

$$(3.21) \quad J_\phi(t, x) \triangleq E^{\mathbb{P}|\mathcal{G}_t} [U(X_T^{x,t,\phi})]$$

$$u(t, x) = \sup_{\phi \in \mathcal{S}(t)} J_\phi(t, x)$$

*Then, for an initial endowment  $x_0 > 0$ , a portfolio strategy  $\phi^* \in \mathcal{S}(0)$  is optimal if and only if:*

$$u(0, x_0) = J_{\phi^*}(0, x_0)$$

The reduced problem (3.21) solves the original one (2.4). The main point here is that functions  $u$  and  $J$  depend on the whole history  $\mathcal{G}$  only through  $\pi_0(b)$ , i.e.: the filter contains the necessary information to solve the control problem.

## 4. OPTIMAL VALUE FUNCTION

**4.1. Duality Theory.** As shown by Kramkov and Schachermayer (1999) [17], resolution of problem (3.21) relies upon solving the dual optimization problem:

$$(4.1) \quad v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}} E \left[ V \left( y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right], \quad y > 0,$$

where  $\mathcal{Q}$  is the set of equivalent martingale measures given by:

$$\mathcal{Q} = \{ \mathbb{Q} \sim \mathbb{P}|_{\mathcal{G}} \mid Y \text{ is a local } (\mathbb{Q}, \mathbb{G})\text{-martingale,} \}$$

and where the conjugate version of the utility function  $U(x)$  is defined by:

$$V(y) = \sup_{x \in \mathbb{R}} [U(x) - xy], \quad y > 0.$$

**4.2. Martingale Measures.** Thereafter, we will use the following definition of equivalent martingale measures:

**Proposition 3.** *Let  $(\gamma, \psi)$  be predictable processes satisfying the integrability conditions:*

$$E \int_0^T |\gamma_t \gamma_t^\top| dt < \infty, \quad E \int_0^T |\psi_t| \pi_t(\lambda) dt < \infty, \quad \psi_t > 0, \quad a.e. \ t \in [0, T]$$

*Then, the process:*

$$(4.2) \quad \Lambda_t = 1 - \int_0^t \Lambda_{s-} [\gamma_s^\top d\overline{W}_s + (\mathbf{1}_m^\top - \psi_s^\top) d\overline{M}_s], \quad \Lambda_0 = 1$$

*is a strictly positive  $(\mathbb{P}, \mathcal{G})$ -local martingale.*

When  $\mathbb{E}^\mathbb{P}[\Lambda_T] = 1$ , it is a martingale and then the measure  $\mathbb{Q}$  defined as:

$$(4.3) \quad \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{G}_T} = \Lambda_T$$

is a probability equivalent to  $\mathbb{P}$ . The process  $S/S^0$  is a local-martingale if and only if<sup>3</sup>:

$$(4.4) \quad \pi_t(b) - r_t \mathbf{1}_m = \sigma_t \gamma_t^\pi + \omega_t D_t^\pi (\mathbf{1}_m - \psi_t^\pi).$$

This equation will be important when choosing the optimal premiums (see Proposition 5 below).

Girsanov's theorem ensures that:

$$\widehat{W}_t = \overline{W}_t + \int_0^t \gamma_s^\pi ds \text{ is a } (\mathbb{Q}, \mathbb{G})\text{-Brownian motion,}$$

$$\widehat{M}_t = \overline{M}_t + \int_0^t D_s^\pi [\mathbf{1}_m - \psi_s^\pi] ds \text{ is a } (\mathbb{Q}, \mathbb{G})\text{-martingale,}$$

with  $D_s^\pi = \text{Diag}(\pi_s(\lambda))$  and  $N$  is a  $(\mathbb{Q}, \mathbb{G})$ -Poisson process with  $\mathbb{G}$ -intensity  $\lambda^\pi = D^\pi \psi^\pi$ .

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<sup>3</sup>We note  $\gamma_s^\pi$  and  $\psi_s^\pi$  to make apparent the dependence on both time and the filter.

**4.3. Value Functions.** Thereafter, we note  $I(\cdot) = (U'(\cdot))^{-1}$ ,  $\beta_\cdot = \exp(-\int_0^\cdot r_s ds)$  and  $\mathbb{Q}^y \equiv \mathbb{Q}$ . The following theorem is adapted from Owen (2002) [23].

**Theorem 2.** *Let  $U$  be a utility function. Then:*

- (1) *For  $y > 0$ , there exists a unique measure  $\mathbb{Q}^y$  solution of (4.1),*
- (2) *There exists a unique number  $\hat{y}$  s.t.  $\mathbb{E}^{\mathbb{Q}^{\hat{y}}} \left[ \beta_T I \left( \hat{y} \beta_T \Lambda_T^{\mathbb{Q}^{\hat{y}}} \right) \right] = x_0$ ,*
- (3) *The optimal terminal wealth is given by  $\hat{X}_T^\phi = I \left( \hat{y} \beta_T \Lambda_T^{\mathbb{Q}^{\hat{y}}} \right)$ .*

**4.4. Special Cases.** Problem (3.21) will now be solved in the case of the three standard utility functions: power, logarithmic and exponential, defined by:

$$(4.5) \quad U(x) = \begin{cases} x^p/p & x \in \mathbb{R}^+, p \in (0, 1) \\ \log x & , x \in \mathbb{R}^+ \\ -e^{-x} & x \in \mathbb{R} \end{cases}$$

These utility functions have the special feature of solving equation (4.1) independently of  $y$ . Indeed:

$$V(y) = \begin{cases} -y^q/q & y \in \mathbb{R}, q = \frac{p}{p-1} \\ -(1 + \ln y) & , y \in \mathbb{R}^+ \\ y(\ln y - 1) & y \in \mathbb{R}^+ \end{cases}$$

which implies:

$$(4.6) \quad v(y) = \begin{cases} y^q/q \inf_{\mathbb{Q} \in \mathcal{Q}} E \left[ - \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)^q \right] \\ -1 - \ln y + \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[ - \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \\ y \ln y + y \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E} \left[ \frac{d\mathbb{Q}}{d\mathbb{P}} \ln \frac{d\mathbb{Q}}{d\mathbb{P}} \right] \end{cases}$$

The quantity appearing in the right hand side of (4.6) under the expectation term can be called martingale distance measures, cf. Goll and Ruschendorf (2001) [12].

We then have the useful martingale distance decomposition results:

**Proposition 4.** *Case of power utility:*

$$(4.7) \quad \mathbb{E}[-\Lambda_T^q] = \mathbb{E} \left[ \frac{q(1-q)}{2} \int_0^T \gamma_s^\top \gamma_s ds + ((\psi_s^q)^\top - \mathbf{1}_m - q(\psi_s^\top - \mathbf{1}_m)) \lambda_s^\pi ds \right].$$

*Case of logarithmic utility:*

$$(4.8) \quad \mathbb{E}[-\ln \Lambda_T] = \mathbb{E} \left[ \frac{1}{2} \int_0^T \gamma_s^\top \gamma_s ds + (\psi_s^\top - \ln \psi_s^\top - \mathbf{1}_m) \lambda_s^\pi ds \right].$$

*Case of exponential utility:*

$$(4.9) \quad \mathbb{E}[\Lambda_T \ln \Lambda_T] = \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{2} \int_0^T \Lambda_s \gamma_s^\top \gamma_s ds + (\Lambda_s \psi_s^\top \ln \psi_s^\top - \psi_s^\top + \mathbf{1}_m) \lambda_s^\pi ds \right],$$

where  $\gamma \equiv \gamma^\pi$ ,  $\psi \equiv \psi^\pi$  and  $\lambda_s^\pi \equiv \pi_s(\lambda)$ .

*Proof.* **Proof of equation (4.7)**

The  $q$ -Kakutani-Hellinger distance is related to the Hellinger process of order  $q$  as shown by Jacod and Shiryaev (2003) [13]. The result is then a direct consequence of their Corollary IV.1.37.

**Proof of equation (4.8)**

An application of Itô's formula for jump-diffusion processes leads to:

$$d \ln \Lambda_t = dU_t - \frac{1}{2} \gamma_t^\top \gamma_t dt + (\ln \psi_t^\top - \psi_t^\top + \mathbf{1}_m^\top) \lambda_t dt,$$

where  $U$  is a martingale. Therefore,

$$\mathbb{E} [-\ln \Lambda_T] = \mathbb{E} \left[ \frac{1}{2} \int_0^T \gamma_s^\top \gamma_s ds + (\psi^\top - \ln \psi^\top - \mathbf{1}_n^\top) \lambda_s ds \right],$$

which concludes the proof.

**Proof of equation (4.9)**

Proof in one dimensional case. Let:

$$d\Lambda_t = -\Lambda_t(\gamma_t dW_t + (1 - \psi_t) dM_t),$$

and  $\lambda$  be the intensity of  $M$ . One has, using Itô's formula, or the closed form for the solution of the Doléans-Dade exponential:

$$d(\ln \Lambda_t) = -\gamma_t dW_t - \frac{1}{2} \gamma_t^2 dt - (1 - \psi_t) \lambda_t dt + \ln(\psi_t) (dM_t + \lambda_t dt).$$

Hence, using integration by parts formula,

$$d(\Lambda_t \ln \Lambda_t) = d\mu_t + \Lambda_t \left( -\frac{1}{2} \gamma_t^2 - (1 - \psi_t) \lambda_t + \lambda_t \ln(\psi_t) + \gamma_t^2 - (1 - \psi_t) \lambda_t \ln(\psi_t) \right) dt,$$

where  $\mu$  is a martingale. With obvious simplifications:

$$d(\Lambda_t \ln \Lambda_t) = d\mu_t + \Lambda_t \left( \frac{1}{2} \gamma_t^2 + \lambda_t (\psi_t - 1 + \psi_t \ln(\psi_t)) \right) dt,$$

and

$$E(\Lambda_T \ln \Lambda_T) = E \int_0^T \Lambda_t \left( \frac{1}{2} \gamma_t^2 + \lambda_t (\psi_t - 1 + \psi_t \ln(\psi_t)) \right) dt.$$

□

We can now determine the minimal distance premiums, i.e., Girsanov parameters  $(\gamma, \psi)$  which solve problem (4.6) subject to the martingale condition (4.4).

**Proposition 5.** *Case of power utility:*

$$\gamma_s = \frac{\sigma_s \Upsilon_s}{q(1-q)}, \quad \psi_s = \left( \mathbf{1}_m - \frac{\Upsilon_s}{q} \right)^{\frac{1}{q-1}}$$

*Case of logarithmic utility:*

$$\gamma_s = \sigma_s \Upsilon_s, \quad \psi_s = \frac{1}{\mathbf{1}_m - \Upsilon_s}$$

*Case of exponential utility:*

$$\gamma_s = \sigma_s \Upsilon_s, \quad \psi_s = \exp(\Upsilon_s)$$

where  $\Upsilon_s$  is such that the pair  $\gamma, \psi$  solves (4.4).

*Proof.* Solving (4.1) reduces to optimize the canonical decomposition of a constrained utility-distance based functional. Relying on Proposition 4, we need to minimize a concave function subject to a convex constraint. Following Rockafellar (1970) [26], it is enough to consider the Lagrangian function and resort to the saddle point theorem where we let  $\Upsilon$  to be a  $\mathbb{R}^m$ -valued Lagrange multiplier. From the first order conditions, we obtain the optimal  $(\gamma, \psi)$  in terms of the Lagrange multiplier  $\Upsilon$  which in turn satisfies the martingale condition (4.4), say  $\mathcal{L}(\Upsilon)$ . Furthermore, it can be verified that in each case, the function  $\mathcal{L}(\Upsilon)$  is continuous and strictly increasing, thus admitting a unique solution yielding the explicit risk premiums.  $\square$

We can now determine the optimal value functions in the case of our utility functions (4.5) by a straight application of Theorem 2:

**Proposition 6.** *For specific choices of  $(\gamma, \psi)$  given in proposition 5: The "power" optimal value function is given by:*

$$u(0, x) = \frac{x^p}{p} \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( r_t + \frac{1}{2} p(p-1) (\gamma_t^\pi)^\top \gamma_t^\pi \right) dt - \int_0^T \left( (\psi_t^{-p})^\top - \mathbf{1}_m^\top + p(\psi_t^\top - \mathbf{1}_m^\top) \right) \pi_t(\lambda) dt \right].$$

The "logarithmic" optimal value function reads:

$$u(0, x) = \ln x + \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( r_t + \frac{1}{2} (\gamma_t^\pi)^\top \gamma_t^\pi \right) dt + \int_0^T (\psi_t^\top - \ln \psi_t^\top - \mathbf{1}_m^\top) \pi_t(\lambda) dt \right].$$

The "exponential" optimal value function writes:

$$u(0, x) = -e^{-x} \exp \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T \left( r_t + \frac{1}{2} (\gamma_t^\pi)^\top \gamma_t^\pi \right) dt + \int_0^T (\psi_t^\top \ln \psi_t^\top - \psi_t^\top + \mathbf{1}_m^\top) \pi_t(\lambda) dt \right].$$

From this proposition, we conclude that in the partial observation case, the optimal value functions write similarly with the completely observable case. The filtering equivalence principle holds: the unknown drift  $b_t$  and jump intensity  $\lambda_t$  are replaced by the estimates  $\pi_t(b)$  and  $\pi_t(\lambda)$ .

## 5. CONCLUSION

In this article, we have investigated the question of optimal policies in a multi-dimensional jump-diffusion model of incomplete market under the setup of partial information. When the model is only partially observable, we have extended the framework of Lakner (1998) [19] by allowing learning in the intensity of the Poisson process. We ended proving that the optimal value functions write similarly in both complete and partial observations, with the

difference that the unknown parameters are replaced by their estimates when the model is partially observed.

## 6. APPENDIX

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