# Applying Itô's motto: "look at the infinite dimensional picture" by constructing sheets to obtain processes increasing in the convex order

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**Abstract:** Strongly inspired by the result due to Carr-Ewald-Xiao that the arithmetic average of geometric Brownian motion is an increasing process in the convex order, we extend this result to integrals of Lévy processes and Gaussian processes. Our method consists in finding an appropriate sheet associated to the original Lévy or Gaussian process, from which the one-dimensional marginals of the integrals will appear to be those of a martingale, thus proving the increase in the convex order property.

**Key words:** convex order; 1-martingale; PCOC; Lévy sheet; Sato sheet; Gaussian sheet.
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# 1 Introduction: Itô's motto, PCOC's and 1-martingales

## 1.1

Although this paper is dedicated to Professors E. Csáki and P. Révész, who have, at times, been interested in the Wiener sheet, we would also like to closely associate to our work Professor K. Itô, who departed in November 2008, one year before the celebration of E. Csáki and P. Révész 75th birthdays.

In a Foreword to his Selected Papers [I], K. Itô writes: "After some time, it became my habit, even for finite dimensional probabilistic phenomena, to look at an infinite dimensional related set-up, the properties of which may illuminate / explain / those of the finite dimensional set-up considered previously."

This sentence is, intentionally, not exactly that of Itô, but it expresses precisely the contents and the spirit of his sentence, which is intended there to explain how he came to think of excursions of, say, Brownian motion away from 0, as a Poisson Point Process. See K. Itô [I] and P.A. Meyer [M]. The Tribute for K. Itô in preparation [VY] gives, in particular, a number of developments of excursion theory.

## 1.2

In the present paper, we shall show how Itô's motto translates, in our study of PCOC's, which we now define. PCOC is the acronym, coming from the French translation: <u>P</u>rocessus <u>C</u>roissant pour l'<u>O</u>rdre <u>C</u>onvexe, of the English term: Process Increasing in the Convex Order. We prefer, as a pun, the French acronym to the less evocative English one, which would be: PICO.

- A real valued process $(X_t\ ,\ t \geq 0)$ is a PCOC, i.e: a process increasing in the convex order, if

  $$\forall t \geq 0 \quad \mathbb{E}[|X_t|] < \infty\ ,$$

  and for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$,

  $$t \in \mathbb{R}_+ \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

  is increasing.

- Here is now, apparently, a totally different notion: a real valued process $(X_t\ ,\ t \geq 0)$ is said to be a *1-martingale* if there exists, on some filtered probability space, a martingale $(M_t\ ,\ t \geq 0)$ such that

  $$X_t \overset{(1.d)}{=} M_t\ ,$$

  this notation meaning that the processes $(X_t\ ,\ t \geq 0)$ and $(M_t\ ,\ t \geq 0)$ have the same one-dimensional marginals, i.e:

  $$\forall t \geq 0, \text{ fixed}, \quad X_t \overset{(law)}{=} M_t\ .$$

  Jensen's inequality entails that a martingale is a PCOC. Consequently, *a 1-martingale is a PCOC.*

- The converse result, i.e: *a PCOC is a 1-martingale*, also holds true (Kellerer [K], 1972), but Kellerer's proof does not allow to construct effectively a martingale *associated* to a given PCOC, i.e: a martingale with the same 1-dimensional marginals as this PCOC. Note that several different martingales may be associated with a given PCOC.

The aim of this paper is to describe certain processes $(X_t\ ,\ t \geq 0)$ as 1-martingales, via the intervention of related sheets, thus showing at the same time that those processes $(X_t\ ,\ t \geq 0)$ are PCOC's.

## 1.3

Here is the organisation of the remainder of this paper.

- In Section 2, we present the example of a PCOC which motivated our study of this subject, i.e: the arithmetic mean of geometric Brownian motion. Two equivalent phrasings of this example shall guide us throughout the sequel of our paper.

- In Section 3, we present a general candidate to be a PCOC, and we show it is indeed one, once we have been able to associate with the two-parameter process of reference $(Y_{\lambda,t})$, an adequate sheet $(Z_{\lambda,t})$, from which we shall construct martingales indexed by $t$.

- Section 4 is an illustration of Section 3, for

  $$Y_{\lambda,t} = L_{\lambda t}\ ,$$

  where $(L_u)$ is a Lévy process.

- Section 5 is an illustration of Section 3, for

  $$Y_{\lambda,t} = t\, L_\lambda \ ,$$

  where $(L_u)$ is a self-decomposable Lévy process, i.e: $L_1$ is a self-decomposable random variable.

- Section 6 is an illustration of Section 3, for $(Y_{\lambda,t})$ such that, for each $t \geq 0$, $Y_{\bullet,t}$ is a Gaussian process whose covariance admits a dependence of a special kind with respect to $t$.

- Finally, Section 7 concludes with an opening to a yet wider, but promising, framework of study.

# 2 Our guiding example

## 2.1

Let $\nu \in \mathbb{R}$, and $(B_s \ , \ s \geq 0)$ denote standard Brownian motion. Carr-Ewald-Xiao [CEX] have shown "directly" that the process:

$$A_t = \frac{1}{t} \int_0^t \exp\left(\nu\, B_s - \frac{\nu^2\, s}{2}\right) \mathrm{d}s \quad , \quad t \geq 0$$

(we take, by continuity, $A_0 = 1$), is a PCOC.

## 2.2

Note that the above PCOC property may be presented in at least two different ways. Indeed, set:

$$A_t^{[1]} = \int_0^1 \exp\left(\nu\, B_{tu} - \frac{\nu^2\, t\, u}{2}\right) \mathrm{d}u \quad , \quad t \geq 0 \tag{1}$$

and

$$A_t^{[2]} = \int_0^1 \exp\left(\nu\, t\, B_u - \frac{\nu^2\, t^2\, u}{2}\right) \mathrm{d}u \quad , \quad t \geq 0 \ . \tag{2}$$

Then, one has:

$$A_t = A_t^{[1]} \overset{(1.\mathrm{d})}{=} A_{\sqrt{t}}^{[2]} \ .$$

Hence, it is equivalent to prove that $(A_t = A_t^{[1]} \ , \ t \geq 0)$ is a PCOC, or that $(A_t^{[2]} \ , \ t \geq 0)$ is a PCOC. In fact, in [CEX], it is shown that $(A_t^{[2]} \ , \ t \geq 0)$ is a PCOC.

We note that, in Mathematical Finance terms, $t$ appears in (1) as a *maturity*, whereas in (2), $t$ appears as a *volatility*. These two kinds of parameters shall guide us throughout our discussion in this paper.

## 2.3

Let $(W_{u,v}\ ;\ u,v \geq 0)$ denote the standard Brownian sheet indexed by $\mathbb{R}_+^2$, and

$$(\mathcal{W}_t = \sigma\{W_{u,v}\ ;\ u \geq 0\,,\ 0 \leq v \leq t\})$$

be the natural filtration of the path-valued process $(W_{\bullet,t}\ ,\ t \geq 0)$.

Baker-Yor ([BY]) gave an alternative proof of the Carr-Ewald-Xiao result by showing the following result.

### 2.3.1

Set:

$$M_t^{[1]} = \int_0^1 \exp\left(\nu\, W_{u,t} - \frac{\nu^2\, t\, u}{2}\right)\, \mathrm{d}u\ .$$

Then $A_t^{[1]} \overset{(1.d)}{=} M_t^{[1]}$ , and $(M_t^{[1]}\ ,\ t \geq 0)$ is a $(\mathcal{W}_t)$-martingale.

Obviously, this result also entails the following one:

### 2.3.2

Set:

$$M_t^{[2]} = \int_0^1 \exp\left(\nu\, W_{u,t^2} - \frac{\nu^2\, t^2\, u}{2}\right)\, \mathrm{d}u\ .$$

Then $A_t^{[2]} \overset{(1.d)}{=} M_t^{[2]}$ , and $(M_t^{[2]}\ ,\ t \geq 0)$ is a $(\mathcal{W}_{t^2})$-martingale.

In the following, we shall extend these results 2.3.1 and 2.3.2 by replacing, in (1) and (2), the process $(\nu\, B_t)$ by a Lévy process or a Gaussian process.

# 3 A general framework involving sheets

## 3.1

We consider a measurable space $\Lambda$ and, for every $t \geq 0$, a real valued measurable process

$$Y_{\bullet,t} = (Y_{\lambda,t}\ ,\ \lambda \in \Lambda)$$

such that

$$\forall \lambda \in \Lambda,\ \forall t \geq 0, \qquad \mathbb{E}[\exp(Y_{\lambda,t})] < \infty\ .$$

(For example (1), we take $\Lambda = \mathbb{R}_+$ and $Y^{[1]}_{\lambda,t} = \nu\, B_{\lambda t}$ , whereas for example (2), we take $\Lambda = \mathbb{R}_+$ and $Y^{[2]}_{\lambda,t} = \nu\, t\, B_{\lambda}$ .)
Now, for any finite signed measure $\sigma$ on $\Lambda$, we set:

$$A^{(\sigma)}_t = \int_\Lambda \frac{\exp(Y_{\lambda,t})}{\mathbb{E}[\exp(Y_{\lambda,t})]}\, \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0\ ,$$

and we look for conditions which ensure that $(A^{(\sigma)}_t\ ,\ t \geq 0)$ is a 1-martingale, hence a PCOC. (For examples (1) and (2) we take, as a measure $\sigma$, the Lebesgue measure on $[0,1]$ .)

## 3.2

Here is our general presentation of the *sheets method.*

**Proposition 3.1** *Assume the existence of a measurable sheet:*

$$(Z_{\lambda,t} \quad ; \quad \lambda \in \Lambda\ ,\ t \geq 0)$$

*such that:*

$(H_1)$ *For every* $t \geq 0$,

$$Y_{\bullet,t} \overset{(law)}{=} Z_{\bullet,t}\ .$$

$(H_2)$ *For every* $0 \leq s \leq t$, $Z_{\bullet,t} - Z_{\bullet,s}$ *is independent from*

$$\mathcal{Z}_s = \sigma\{Z_{\lambda,u}\ ;\ \lambda \in \Lambda\,,\, 0 \leq u \leq s\}\ .$$

*Then the process*

$$M^{(\sigma)}_t := \int_\Lambda \frac{\exp(Z_{\lambda,t})}{\mathbb{E}[\exp(Z_{\lambda,t})]}\, \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0$$

*is a* $(\mathcal{Z}_t)$*-martingale and*

$$A^{(\sigma)}_t \overset{(1.\mathrm{d})}{=} M^{(\sigma)}_t\ .$$

**Proof**

For the first point, we obtain, as a consequence of $(H_2)$:

$$\mathbb{E}[M_t^{(\sigma)} \mid \mathcal{Z}_s] = \int_\Lambda \exp(Z_{\lambda,s}) \, \frac{\mathbb{E}[\exp(Z_{\lambda,t} - Z_{\lambda,s})]}{\mathbb{E}[\exp(Z_{\lambda,t})]} \, \sigma(\mathrm{d}\lambda) \; ,$$

and the right-hand side is equal to $M_s^{(\sigma)}$ since, by $(H_2)$:

$$\mathbb{E}[\exp(Z_{\lambda,t})] = \mathbb{E}[\exp(Z_{\lambda,s})] \, \mathbb{E}[\exp(Z_{\lambda,t} - Z_{\lambda,s})] \; .$$

The second point follows immediately from $(H_1)$.

$\square$

Note that, in example (1), we may set: $Z_{\lambda,t}^{[1]} = \nu \, W_{\lambda,t}$. Then, the result 2.3.1 of Baker-Yor is a particular case of Proposition 3.1. Likewise, in example (2), we may set: $Z_{\lambda,t}^{[2]} = \nu \, W_{\lambda,t^2}$, which leads to 2.3.2.

### 3.3

In the sequel, we shall develop systematically three classes of examples, which involve, in the proof of the PCOC property, Lévy sheets, Sato sheets and Gaussian sheets.

## 4 Lévy sheets

### 4.1

Consider $(L_t \; , \; t \geq 0)$ a real valued Lévy process starting from 0. In connection with $L$, we introduce below, as in Hirsch-Yor [HY2, Subsection 2.4] and Hirsch-Roynette-Yor [HRY1, Subsection 2.2], the *Lévy sheet associated with* $L$, which we denote by $\widehat{L}$. In the case $L = B$, Brownian motion, then $\widehat{L} = W$, the standard Wiener sheet.

**Theorem 4.1** *There exists a sheet*

$$\widehat{L} = (\widehat{L}_{s,t} \; ; \; s \geq 0 \, , \, t \geq 0)$$

*such that:*

*i)*

$$\forall s, t \geq 0, \quad \widehat{L}_{s,0} = \widehat{L}_{0,t} = 0 \; .$$

*ii) Almost surely, for every* $s, t \geq 0$, $\widehat{L}_{s,\bullet}$ *and* $\widehat{L}_{\bullet,t}$ *are càdlàg processes.*

*iii) Denote as*

$$(\mathcal{L}_t \equiv \sigma\{\widehat{L}_{u,v}\ ;\ u \geq 0\ ,\ 0 \leq v \leq t\} \quad , \quad t \geq 0)\ ,$$

*the natural filtration of* $(\widehat{L}_{\bullet,t}\ ,\ t \geq 0)$. *Then, for* $0 \leq t_1 \leq t_2$, *the process* $(\widehat{L}_{s,t_2} - \widehat{L}_{s,t_1}\ ,\ s \geq 0)$ *is a Lévy process starting from 0, which is independent of* $\mathcal{L}_{t_1}$, *and is distributed as* $(L_{(t_2-t_1)\,s}\ ,\ s \geq 0)$.

*iv) The sheets* $(\widehat{L}_{s,t}\ ;\ s, t \geq 0)$ *and* $(\widehat{L}_{t,s}\ ;\ s, t \geq 0)$ *have the same law.*

A proof of this theorem may be found in Dalang-Walsh [DW], who, themselves, refer Adler et al [AMSW]. It is analogous to the construction of a Lévy process $(L_t)$, via the Lévy-Itô method starting from the characteristic function $\exp(-\psi(\lambda))$ of an infinitely divisible law to arrive to the Lévy-Khintchine formula:

$$\mathbb{E}[\exp(\mathrm{i}\,\lambda\,L_t)] = \exp(-t\,\psi(\lambda))\ .$$

## 4.2

Coming back to our set-up in Section 3, let us show how it translates when

$$\Lambda = \mathbb{R}_+ \qquad \text{and} \qquad Y_{\lambda,t} = L_{\lambda\,t}\ ,$$

assuming furthermore the following exponential integrability assumption:

$$(EI) \qquad\qquad \forall \alpha \geq 0, \quad \mathbb{E}[\exp(\alpha\,L_1)] < \infty\ .$$

In the sequel, if property $(EI)$ is satisfied, we set, for $\alpha \geq 0$,

$$\phi_L(\alpha) = \log(\mathbb{E}[\exp(\alpha\,L_1)])\ ,$$

and $\phi_L = \phi_L(1)$. Then, taking

$$Z_{\lambda,t} = \widehat{L}_{\lambda,t}\ ,$$

properties $(H_1)$ and $(H_2)$ in Proposition 3.1 are satisfied from *iii*) in Theorem 4.1.

As a particular application of our discussion in Section 3, we obtain that, if $\sigma$ is a signed finite measure on $\mathbb{R}_+$,

$$A_t^{(\sigma)} = \int_{\mathbb{R}_+} \exp(L_{\lambda\,t} - \lambda\,t\,\phi_L)\ \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is:

$$M_t^{(\sigma)} = \int_{\mathbb{R}_+} \exp(\widehat{L}_{\lambda,t} - \lambda\, t\, \phi_L)\, \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0\ .$$

This result, as well as various extensions, were first proved in Hirsch-Yor [HY2].
As a particular example, taking for $\sigma$ the Lebesgue measure on $[0,1]$, we obtain the following extension of 2.3.1:

$$\frac{1}{t}\int_0^t \exp(L_s - s\,\phi_L)\, \mathrm{d}s \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$\int_0^1 \exp(\widehat{L}_{s,t} - s\, t\, \phi_L)\, \mathrm{d}s \quad , \quad t \geq 0\ .$$

### 4.3

The reader will have noticed that we have only discussed in the previous subsection the case $Y_{\lambda,t} = L_{\lambda t}$, and not $\widetilde{Y}_{\lambda,t} = t\, L_\lambda$. The latter study shall be undertaken in the next section, under the additional assumption of self-decomposability of the Lévy process $(L_t\ ,\ t \geq 0)$, or equivalently, of the random variable $L_1$. In fact, we know (see [HRY1, Theorem 7.2]), even without the hypothesis of self decomposability assumed on $L_1$, that if $\sigma$ is a finite *positive* measure, then

$$\widetilde{A}_t^{(\sigma)} := \int_{\mathbb{R}_+} \exp[t\, L_\lambda - \lambda\, \phi_L(t)]\, \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0$$

is a PCOC but, we do not know how to associate a martingale to this PCOC. We shall be able to do this, in the next section, for any signed finite measure $\sigma$, under the restrictive hypothesis that $L_1$ is self-decomposable.

## 5 Sato sheets

### 5.1

We first recall some general facts concerning the notion of self-decomposability. We refer for background, complements and references, to Sato [S, Chapter 3].

A real valued random variable $R$ is said to be *self-decomposable* if, for each $u$ with $0 < u < 1$, there is the equality in law:

$$R \overset{(law)}{=} u\,R + R_u$$

for some variable $R_u$ independent of $R$.
Such variables $R$ are particular infinitely divisible variables.
In Jeanblanc-Pitman-Yor [JPY], a number of equivalent properties and representations of such variables are presented, and ways to go from one representation to another are exhibited.

For our purpose, the important representation of (the law of) $R$ is the following:
$R$ is self-decomposable if and only if, there exists a Lévy process $(C_s\ ,\ s \geq 0)$ such that:

$$R \overset{(law)}{=} \int_0^\infty \mathrm{e}^{-s}\, \mathrm{d}C_s\ .$$

The if part is straightforward. On the other hand, the law of the process $C$ above is uniquely determined from that of $R$. More precisely, assume that:

$$\mathbb{E}[\exp(\mathrm{i}\,\lambda\,R)] = \exp(-\psi(\lambda)) \quad \text{and} \quad \mathbb{E}[\exp(\mathrm{i}\,\lambda\,C_s)] = \exp(-s\,\widetilde{\psi}(\lambda))\ ,$$

then:

$$\exp(-\psi(\lambda)) = \exp\left(-\int_0^\infty \widetilde{\psi}(\lambda\,\mathrm{e}^{-s})\ \mathrm{d}s\right)$$

so that:

$$\psi(\lambda) = \int_0^\lambda \frac{\widetilde{\psi}(v)}{v}\ \mathrm{d}v$$

and $\widetilde{\psi}(\lambda) = \lambda\,\psi'(\lambda)$ , for $\lambda \neq 0$ .
Thus, we shall call $(C_s)$ the Lévy process *associated* to the self-decomposable random variable $R$ . As an example, the Lévy process associated to $\nu\,B_1$ is $(\nu\,B_{2s}\ ,\ s \geq 0)$ .

## 5.2

In this subsection, we shall associate with any self-decomposable random variable $R$ , a sheet $(\widetilde{R}_{\lambda,t}\ ;\ \lambda \geq 0\,,\,t \geq 0)$ . The construction of this sheet, inspired from Jeanblanc-Pitman-Yor [JPY], follows Hirsch-Roynette-Yor [HRY1, Subsection 2.2].

Let $(C_s\ ,\ s \geq 0)$ be the Lévy process associated with the self-decomposable random variable $R$. Let us denote by $(\widehat{C}_{s,t}\ ;\ s \geq 0\,,\,t \geq 0)$ the Lévy sheet associated with $(C_t)$ by Theorem 4.1. We then extend the definition of $(\widehat{C}_{s,t})$ to $s \leq 0\,, t \geq 0$, by taking, for $(\widehat{C}_{s,t}\ ;\ s \leq 0\,, t \geq 0)$, an independent copy of $(-\widehat{C}_{-s,t}\ ;\ s \leq 0\,, t \geq 0)$. Thus, $(\widehat{C}_{s,t}\ ;\ s \in \mathbb{R}\,, t \geq 0)$ still retains the properties of a Lévy sheet. We define, for $\lambda \geq 0$, and $t \geq 0$:

$$\widetilde{R}_{\lambda,t} = \int_{-\log t}^{+\infty} \mathrm{e}^{-s}\, \mathrm{d}_s \widehat{C}_{s,\lambda}\ .$$

This is, by definition, the *Sato sheet associated with* $R$.

We gather, in the next statement, several important properties of the sheet $(\widetilde{R}_{\lambda,t})$. The proof is similar to that of Theorem 2.3 in Hirsch-Roynette-Yor [HRY1].

**Theorem 5.1** *The sheet* $\widetilde{R}$ *satisfies the following:*

*i) The process* $(\widetilde{R}_{\bullet,t}\ ,\ t \geq 0)$ *has independent increments.*

*ii) For any fixed* $t \geq 0$*,* $\widetilde{R}_{\bullet,t} \overset{(law)}{=} t\,\widetilde{R}_{\bullet,1}\ .$

*iii) For any fixed* $t \geq 0$*,* $\widetilde{R}_{\bullet,t}$ *is a Lévy process.*

*iv)* $\widetilde{R}_{1,1} \overset{(law)}{=} R\ .$

## 5.3

Coming back to our set-up in Section 3, let us show how it translates when

$$\Lambda = \mathbb{R}_+ \qquad \text{and} \qquad Y_{\lambda,t} = t\,\text{\L}_\lambda\ ,$$

where $(L_t\ ,\ t \geq 0)$ is a Lévy process starting from 0 and satisfying property $(EI)$. We assume furthermore the following self-decomposability assumption:

$(SD)$ The random variable $L_1$ is self-decomposable.

We set $R = L_1$ and, for $\lambda \geq 0$ and $t \geq 0$,

$$Z_{\lambda,t} = \widetilde{R}_{\lambda,t}\ .$$

By *iii*) and *iv*) in Theorem 5.1,

$$L_\bullet \overset{(law)}{=} Z_{\bullet,1}\ .$$

Therefore, by *ii*), property $(H_1)$ in Proposition 3.1 is satisfied. Moreover, property *i*) in Theorem 5.1 says exactly that $(H_2)$ is satisfied.

As a consequence, if $\sigma$ is a signed finite measure on $\mathbb{R}_+$, then

$$\int_{\mathbb{R}_+} \exp[t\, L_\lambda - \lambda\, \phi_L(t)]\; \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0$$

is a PCOC, and an associated martingale is

$$\int_{\mathbb{R}_+} \exp[\widetilde{R}_{\lambda,t} - \lambda\, \phi_L(t)]\; \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0\ .$$

This generalizes 2.3.2.

# 6 Gaussian sheets

## 6.1

Let $\Lambda$ denote a measure space and assume that for every $t \geq 0$, fixed, $(Y_{\lambda,t}\ ,\ \lambda \in \Lambda)$ is a real valued, centered, measurable Gaussian process. Define:

$$c_{\lambda,\mu}(t) = \mathbb{E}[Y_{\lambda,t}\, Y_{\mu,t}]\ .$$

Denote by $\mathbb{S}_n^+$ the set of symmetric positive, $n \times n$ real matrices. Thus, a symmetric matrix $A = (a_{j,k})_{1 \leq j,k \leq n}$ belongs to $\mathbb{S}_n^+$ if

$$\forall \alpha_1, \cdots, \alpha_n \in \mathbb{R} \qquad \sum_{1 \leq j,k \leq n} \alpha_j\, \alpha_k\, a_{j,k} \geq 0\ .$$

In the sequel, $\mathbb{S}_n^+$ is assumed to be equipped with the following partial order:

$$\forall A, B \in \mathbb{S}_n^+ \qquad A \leq B \Longleftrightarrow (B - A) \in \mathbb{S}_n^+\ .$$

The following theorem and its consequences are taken from [HRY2, Section 4].

**Theorem 6.1** *Assume that:*

$(C)$ *For every $n \geq 1$, for every $\lambda_1, \cdots, \lambda_n \in \Lambda$, the matrix function*

$$t \in \mathbb{R}_+ \longrightarrow (c_{\lambda_j,\lambda_k}(t))_{1 \leq j,k \leq n} \in \mathbb{S}_n^+$$

*is increasing with respect to the previously defined order on $\mathbb{S}_n^+$.*

*Then, the function:*

$$[(\lambda, s), (\mu, t)] \longrightarrow c_{\lambda,\mu}(s \wedge t)$$

*is a covariance on $\Lambda \times \mathbb{R}_+$.*

**Proof**

We only give a sketch of the proof, assuming to simplify that, for every $\lambda, \mu \in \Lambda$, the function:

$$t \longrightarrow c_{\lambda,\mu}(t)$$

is absolutely continuous.
Let $\lambda_1, \cdots, \lambda_n \in \Lambda$. The hypothesis $(C)$ translates then as:

$$C'(t) := (c'_{\lambda_j,\lambda_k}(t))_{1\leq j,k\leq n}$$

belongs to $\mathbb{S}_n^+$ for almost every $t$.
Let

$$t \longrightarrow D(t) = (d_{j,k}(t))_{1\leq j,k\leq n} \in \mathbb{S}_n^+$$

denote a measurable application such that

$$D^2(t) = C'(t) \qquad \text{for almost every } t \ .$$

For $t_1, \cdots, t_n \in \mathbb{R}_+$, we denote, for $1 \leq j \leq n$,

$$V_j = Y_{\lambda_j,0} + \sum_{l=1}^{n} \int_0^{t_j} d_{j,l}(s) \, \mathrm{d}B_s^l$$

where $((B_s^1, \cdots, B_s^n) \ , \ s \geq 0)$ is a standard $\mathbb{R}^n$-valued Brownian motion, independent of $Y_{\bullet,0}$. Then, the formula:

$$\mathbb{E}[V_j \, V_k] = c_{\lambda_j,\lambda_k}(0) + \int_0^{t_j\wedge t_k} c'_{\lambda_j,\lambda_k}(s) \, \mathrm{d}s = c_{\lambda_j,\lambda_k}(t_j \wedge t_k) \ ,$$

ensures the covariance property.

□

## 6.2

We now consider an interesting particular case of the previous set-up.
Let $\Lambda$ denote a measure space and assume that $(G_\lambda \ , \ \lambda \in \Lambda)$ is a real valued, centered, measurable Gaussian process.
Define:

$$c(\lambda, \mu) = \mathbb{E}[G_\lambda \, G_\mu] \ .$$

We assume that for every $t \geq 0$, fixed, the processes $(Y_{\lambda,t} \ , \ \lambda \in \Lambda)$ and $(t \, G_\lambda \ , \ \lambda \in \Lambda)$ have the same law. Then,

$$c_{\lambda,\mu}(t) = t^2 \, c(\lambda, \mu) \ ,$$

and property $(C)$ is obviously satisfied. In this case, Theorem 6.1 admits a more precise formulation as well as an alternative proof.

**Proposition 6.1** *There exists a real valued, centered, measurable Gaussian process:*

$$(Z_{\lambda,t} \quad ; \quad \lambda \in \Lambda \ , \ t \geq 0)$$

*such that*

$$(3) \qquad \forall (\lambda, s), (\mu, t) \in \Lambda \times \mathbb{R}_+ \qquad \mathbb{E}[Z_{\lambda,s} \, Z_{\mu,t}] = c_{\lambda,\mu}(s \wedge t) \ ,$$

*and this process satisfies* $(H_1)$ *and* $(H_2)$.

**Proof**

Let $(G^{(n)} \ , \ n \geq 0)$ be a sequence of independent copies of $G$, and let $(e_n \ , \ n \geq 0)$ be a Hilbert basis of $L^2(\mathbb{R}_+)$. We set

$$Z_{\lambda,t} = \sum_{n=0}^{\infty} \left( \int_0^{t^2} e_n(s) \, \mathrm{d}s \right) \, G_\lambda^{(n)} \ .$$

Then, (3) is clearly satisfied. Moreover, as $Z$ is a centered Gaussian process, properties $(H_1)$ and $(H_2)$ easily follow from (3).

$\square$

An interesting particular example is:

$$\Lambda = \mathbb{R}_+ \qquad \text{and} \qquad Y_{\lambda,t} = B^{(H)}_{\lambda t} \ ,$$

where $B^{(H)}$ is the fractional motion with Hurst index $H \in [0, 1]$.
Then, for every $t \geq 0$, fixed,

$$(B^{(H)}_{\lambda t} \ , \ \lambda \in \Lambda) \overset{(law)}{=} (t^H \ B^{(H)}_{\lambda} \ , \ \lambda \in \Lambda) \ ,$$

and therefore, the above result applies, up to some changes of $t$ into $t^H$ .
Thus by Proposition 3.1, for any finite signed measure $\sigma$ on $\mathbb{R}_+$,

$$\int_{\mathbb{R}_+} \exp\left( \nu \, B^{(H)}_{\lambda t} - \frac{\nu^2 \, (\lambda \, t)^{2H}}{2} \right) \, \sigma(\mathrm{d}\lambda) \quad , \quad t \geq 0$$

is a PCOC. In particular,

$$\frac{1}{t} \int_0^t \exp\left( \nu \, B^{(H)}_s - \frac{\nu^2 \, s^{2H}}{2} \right) \, \mathrm{d}s \quad , \quad t \geq 0$$

is a PCOC, a result already noted in Baker-Yor [BY].

In the next subsection, we shall extend, under some additional hypotheses, Proposition 6.1 to the general set-up of Subsection 6.1.

### 6.3

We now come back to the framework of Subsection 6.1, of which we keep the notation.

**Proposition 6.2** *Suppose that* $\Lambda$ *is a separable metric space equipped with its Borel* $\sigma$*-field, and that the function*

$$(\lambda, \mu, t) \in \Lambda \times \Lambda \times \mathbb{R}_+ \longrightarrow c_{\lambda,\mu}(t) \in \mathbb{R}$$

*is continuous. We moreover assume that property* $(C)$ *is satisfied.*
*Then, there exists a real valued, centered, measurable Gaussian process:*

$$(Z_{\lambda,t} \quad ; \quad \lambda \in \Lambda \ , \ t \geq 0)$$

*such that*

$$\forall (\lambda, s), (\mu, t) \in \Lambda \times \mathbb{R}_+ \qquad \mathbb{E}[Z_{\lambda,s}\, Z_{\mu,t}] = c_{\lambda,\mu}(s \wedge t) \ ,$$

*and this process satisfies* $(H_1)$ *and* $(H_2)$*.*

**Proof**

By Theorem 6.1, there exists a centered Gaussian process:

$$(Z_{\lambda,t} \quad ; \quad \lambda \in \Lambda \ , \ t \geq 0) \ ,$$

such that

$$\mathbb{E}[Z_{\lambda,s}\, Z_{\mu,t}] = c_{\lambda,\mu}(s \wedge t) \ .$$

Moreover, the hypotheses easily entail that the Gaussian space generated by this process $Z$ is separable. Therefore, by Neveu [Ne, Corollaire 3.8, p. 44] (see also, for instance, Janson [J, Chapter VIII]), the process $Z$ admits a measurable version. Properties $(H_1)$ and $(H_2)$ follow as previously.

□

# 7 A wider panorama, as a temporary conclusion

### 7.1

We hope that the reader who went through our discussion is now convinced about some advantage of associating sheets to certain classes of one-parameter processes: those sheets provide room to create martingales, whose one-dimensional marginals are those of e.g., the processes

$$\frac{1}{t}\int_0^t \exp(L_s - s\,\phi_L)\,\mathrm{d}s \quad , \quad t \geq 0$$

seen in Subsection 4.2, thus proving that these latter processes are PCOC's.

## 7.2

Let us recall that the Wiener sheet plays a key role in Malliavin calculus (see, e.g: Nualart [Nu]), a formidable machinery to study regularity properties of Brownian functionals.

The third author of this paper also noticed in [Y], that the Wiener sheet arises as limit in law of Brownian local times $(L_t^x\ ;\ x \in \mathbb{R}\,,\, t \geq 0)$ . Precisely:

$$\left(\frac{1}{\sqrt{\varepsilon}}\,(L_t^{\varepsilon a} - L_t^0)\ ;\ a \in \mathbb{R}\,,\, t \geq 0\right)$$

converges in law to

$$(W_{a,t}\ ;\ a \in \mathbb{R}\,,\, t \geq 0)$$

as $\varepsilon$ tends to 0, a result which got Professors Csáki and Révész interested (see [CFK, CCFR]).

Thus, the role of sheets in obtaining PCOC's is yet another instance of their potential for studying properties of one parameter processes.

## 7.3

Right at the time of writing this paper, we also noticed the interest of another enlargement of our original parameter set, namely $\mathbb{R}_+(\ni t)$; indeed, we have just proven that, if $\sigma(\mathrm{d}u)$ and $\tau(\mathrm{d}u)$ are two probability measures on $\mathbb{R}_+$, then:

$$\sigma \overset{(stoch)}{\leq} \tau, \quad \text{i.e: } \forall u \geq 0, \quad \sigma([u,\infty)) \leq \tau([u,\infty))$$

implies

$$A_\sigma \overset{(convex)}{\leq} A_\tau, \quad \text{i.e: } \forall \psi \text{ convex}, \quad \mathbb{E}[\psi(A_\sigma)] \leq \mathbb{E}[\psi(A_\tau)]$$

where:

$$A_\sigma = \int \exp\left(\nu\, B_u - \frac{\nu^2\, u}{2}\right)\, \sigma(\mathrm{d}u)\ .$$

Note that in Subsection 2.1, where we presented the guiding example $(A_t)$ of [CEX], we may write: $A_t = A_{\sigma_t}$ , where

$$\sigma_t(\mathrm{d}u) = \frac{1}{t}\, 1_{[0,t]}(u)\ \mathrm{d}u$$

is the law of $t\,U$, with $U$ uniform on $[0,1]$; hence, if $s \leq t$, then $\sigma_s \overset{(stoch)}{\leq} \sigma_t$, and we recover that $(A_t)$ is a PCOC.

We note that we might take up the whole discussion of the present paper replacing systematically $t \in \mathbb{R}_+$ by $\sigma_t(\mathrm{d}u)$ , a positive probability on $\mathbb{R}_+$, a study we intend to pursue in the near future.

### 7.4

Finally, other methods of deciding whether a given process is a PCOC are detailed at length in the monograph in preparation [HPRY], to which we refer the interested reader.

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