

# Robust utility maximization in a discontinuous filtration

Monique Jeanblanc <sup>\*</sup>   Anis Matoussi <sup>†</sup>   Armand Noguey  
University of Evry   University of Maine   University of Evry  
Institut Europlace de Finance

9th February 2010

## Abstract

We study a problem of utility maximization under model uncertainty. The problem is formulated as a *sup/inf*, the supremum being over a terminal value and an intermediate control and the infimum over a set of models (measures)  $\mathcal{Q}$ . We extend the work of Bordogoni, Matoussi and Schweizer [3] to the case of a discontinuous filtration and prove that the solution of the robust problem is a solution of a quadratic-exponential backward stochastic differential equation. Moreover, we prove a dynamic maximum principle for the maximization problem which generalizes the results of Duffie and Skiadas [6] and El Karoui et al. [9] to the robust case and including model with jumps.

**Keywords:** robust maximization, model uncertainty, stochastic control, stochastic differential utility, Backward Stochastic Differential Equations, maximum principle, jump model.

## 1 Introduction

This paper deals with the problem of utility maximization from a terminal value and an intermediate control under model uncertainty. In the standard problem of utility maximization, one assumes that the investor knows the "historical" probability  $\mathbb{P}$  that describes the dynamics of the state process. In reality, the investor has some uncertainty on this probability. This has led Bordogoni, Matoussi and Schweizer [3], denoted hereafter [BMS], to introduce the set  $\mathcal{Q}$  of probability measures absolutely

---

<sup>\*</sup>This research benefited from the support of the "Chaire Risque de Cr dit", F d ration Bancaire Fran aise.

<sup>†</sup>The second author was partially supported by "Chaire Risques Financiers de la Fondation du Risque", CMAP  cole Polytechnique.

continuous with respect to the reference model measure  $\mathbb{P}$ , and to choose "a worst case" criteria in the optimization problem. More precisely, the goal is to solve

$$\sup_{\pi, c} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbf{U}((\psi, c), \mathbb{Q}) \quad (1)$$

where  $\psi$  runs through a set of random variables,  $c$  through a set of control processes and  $\mathbb{Q}$  through a set of models (measures), and where the criteria  $\mathbf{U}((\psi, c), \mathbb{Q})$  is the sum of a  $\mathbb{Q}$ -expected utility and a term relative to the relative entropy (*a penalization term*). This approach has been suggested by [1] and [12].

Some results in the robust maximization problem have been obtained in Gundel [11], Quenez [22], Schied and Wu [24], Skiadas [28] in the case of continuous filtration. Schied [23] has been working on the problem (1) with a fairly general penalization term for  $\mathbb{Q}$ . However, his (static) results do not contain ours; they only cover the simple case  $\delta \equiv 0$ . Bordigoni [2] has used classical optimization technics to study the same problem in the continuous case.

Our first motivation is to extend the results of [BMS] to a discontinuous filtration concerning the *inf* problem and then to study the maximization problem. We also extend some results obtained by the second author in [7] where the utility maximization part of the problem is studied in the case of a continuous filtration, and in a complete market.

As in [BMS], the primary inspiration clearly comes from the papers [28, 25, 26, 27]. In [28], Skiadas studies essentially the optimization problem (1), and proves that the dynamic value process  $V$  can be described by some quadratic BSDE. Skiadas points out that the BSDE coincides with the one describing a stochastic differential utility; hence working with a standard expected utility under (a particular form of) model uncertainty is equivalent to working with a corresponding stochastic differential utility under a fixed model.

Our second source of inspiration is the work of Anderson with coauthors in [1, 12] in which more references can be found. These authors introduce and discuss the basic problem of robust utility maximization when model uncertainty is penalized by a relative entropy term. Both papers are cast in a Markovian setting and use mainly formal manipulations of Hamilton-Jacobi-Bellman (HJB) equations to provide insights about the optimal investment behaviour in these situations.

By using BSDE technics, we generalize the characterization of optimality obtained by El Karoui, Quenez and Peng [9] in the framework of robust case and including model with jumps. Indeed, we derive a maximum principle which gives a necessary and sufficient condition of optimality. Our results may also be considered as a generalization of the works of ([6, 25, 26, 27]).

The paper is structured as follows. Section 2 presents the model and the form of the criteria  $\mathbf{U}(\cdot, \mathbb{Q})$ . In Section 3, we show that the optimal model measure exists. Section 4 provides the optimal control using a BSDE approach. For a specific choice of utility functions, the value function is given in Section 5 in terms of the optimal plan. The final Section 6 contains a technical proof concerning a regularity result of our generalized quadratic-exponential backward stochastic differential equation.

## 2 The Model and the Robust Optimization Problem

In this section, we present the optimization problem relative to the choice of an optimal probability measure.

### 2.1 The Model

We consider a filtered probability space  $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ . All the processes are  $\mathbb{G}$ -adapted, and defined on the time interval  $[0, T]$  where  $T$  is the finite horizon. We recall that any special  $\mathbb{G}$ -semimartingale  $Y$  admits a canonical decomposition  $Y = Y_0 + A + M^{Y,c} + M^{Y,d}$  where  $A$  is a predictable finite variation process,  $M^{Y,c}$  is a continuous martingale and  $M^{Y,d}$  is a discontinuous martingale.

**Assumption A 1.** *We make the following assumptions:*

- 1) *For each  $i = 1, \dots, d$ ,  $H^i$  is a counting process and there exists a positive adapted process  $\lambda^i$ , called the  $\mathbb{P}$  intensity of  $H^i$ , such that the process  $N^i$  with  $N_t^i := H_t^i - \int_0^t \lambda_s^i ds$  is a martingale. We assume that the processes  $H^i, i = 1, \dots, d$  have no common jumps.*
- 2) *Any discontinuous martingale admits a representation of the form  $dM_t^{Y,d} = \sum_{i=1}^d y_t^i dN_t^i$  where  $y^i, i = 1, \dots, d$  are predictable processes.*

This hypothesis is satisfied in the case where the filtration is generated by a Brownian motion and an inhomogeneous Poisson process and in the case of credit risk, under immersion property (see Kusuoka [15] for details).

**Definition 1.**

$L^{\text{exp}}$  is the space of all  $\mathcal{G}_T$ -measurable random variables  $X$  with

$$\mathbb{E}^{\mathbb{P}} [\exp(\gamma |X|)] < \infty \quad \text{for all } \gamma > 0$$

$D_0^{\text{exp}}$  is the space of all progressively measurable processes  $X = (X_t)_{0 \leq t \leq T}$  with

$$\mathbb{E}^{\mathbb{P}} [\exp(\gamma \text{ess sup}_{0 \leq t \leq T} |X_t|)] < \infty \quad \text{for all } \gamma > 0$$

$D_1^{\text{exp}}$  is the space of all progressively measurable processes  $X = (X_t)_{0 \leq t \leq T}$  such that

$$\mathbb{E}^{\mathbb{P}} \left[ \exp \left( \gamma \int_0^T |X_s| ds \right) \right] < \infty \quad \text{for all } \gamma > 0$$

$\mathcal{M}_0^p(\mathbb{P})$  is the space of all  $\mathbb{P}$ -martingales  $M = (M_t)_{0 \leq t \leq T}$  with  $M_0 = 0$  and  $\mathbb{E}^{\mathbb{P}} [\sup_{0 \leq t \leq T} |M_t|^p] < \infty$

$\mathcal{L}^2(\lambda, \mathbb{P})$  is the space of all  $\mathbb{R}^d$ -valued predictable processes  $X$  such that  $\sum_{i=1}^d \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (X_s^i)^2 \lambda_s^i ds \right] < \infty$

We denote by  $\mathcal{H}^2(\mathbb{P})$  the space  $\mathcal{L}^2(\lambda, \mathbb{P})$  for  $\lambda = 1$

$\mathcal{S}^2(\mathbb{P})$  is the space of all  $\mathbb{R}$ -valued predictable processes  $X$  such that  $\mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |X_s|^2 \right] < \infty$

**Definition 2.**

For any probability measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{G}_T)$ ,

$$H(\mathbb{Q}|\mathbb{P}) := \begin{cases} \mathbb{E}^{\mathbb{Q}} \left[ \ln \frac{d\mathbb{Q}}{d\mathbb{P}} |_{\mathcal{G}_T} \right] & \text{if } \mathbb{Q} \ll \mathbb{P} \text{ on } \mathcal{G}_T \\ +\infty & \text{otherwise} \end{cases}$$

is the relative entropy of  $\mathbb{Q}$  with respect to  $\mathbb{P}$ . We denote by  $\mathcal{Q}_f$  the space of all probability measures  $\mathbb{Q}$  on  $(\Omega, \mathcal{G}_T)$  with  $\mathbb{Q} \ll \mathbb{P}$  on  $\mathcal{G}_T$  and  $H(\mathbb{Q}|\mathbb{P}) < +\infty$ . Note that the reference probability measure  $\mathbb{P}$  belongs to  $\mathcal{Q}_f$ .

**2.2 The robust optimization problem**

We define the discounting process  $S_t^\delta := e^{-\int_0^t \delta_s ds}$  for all  $t \in [0, T]$  where  $\delta$  is a non-negative adapted process. For  $\mathbb{Q} \in \mathcal{Q}_f$ , we denote by  $Z^\mathbb{Q} = (Z_t^\mathbb{Q})_{0 \leq t \leq T}$  (a càdlàg  $\mathbb{P}$ -martingale) its Radon-Nikodym density with respect to  $\mathbb{P}$ .

Let  $U$  be a given process (the cost process) and  $\bar{U}_T$  a given random variable (the terminal target). The robust utility maximization problem  $\mathcal{P}(U, \bar{U}_T, \beta)$  is to find the infimum of  $\Gamma(\mathbb{Q})$  over the set  $\mathcal{Q}_f$  where

$$\Gamma(\mathbb{Q}) = \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T S_s^\delta U_s ds + S_T^\delta \bar{U}_T \right] + \beta \mathbb{E}^{\mathbb{Q}} \left[ \int_0^T \delta_s S_s^\delta \ln Z_s^\mathbb{Q} ds + S_T^\delta \ln Z_T^\mathbb{Q} \right]. \quad (2)$$

The first term in the right-hand side of (2) will be linked, in the following section, to the  $\mathbb{Q}$ -expected discounted utility from target and cost process. The second term is a discounted relative entropy term and  $\beta > 0$  is a given positive constant which determines the strength of this penalty term. Note that the optimal probability  $\mathbb{Q}$  for the problem  $\mathcal{P}(U, \bar{U}_T, \beta)$  is optimal for the minimization problem  $\mathcal{P}(U^\beta, \bar{U}_T^\beta, 1)$  where  $U^\beta = U/\beta$ ,  $\bar{U}_T^\beta = \bar{U}_T/\beta$ , therefore, we shall restrict our attention to the problem  $\mathcal{P}(U, \bar{U}_T) := \mathcal{P}(U, \bar{U}_T, 1)$ .

**Assumption A 2.** For a more precise formulation of our problem, we make the following further assumptions:

- i) the discount rate  $\delta$  is a non-negative bounded process, more precisely there exists  $\epsilon > 0$  such that for any  $t \geq 0$ ,  $0 < \epsilon \leq \delta_t \leq \|\delta\|_\infty$ , a.s.
- ii) the cost process  $U$  belongs to  $D_1^{\text{exp}}$  and the terminal target  $\bar{U}_T$  is in  $L^{\text{exp}}$ .
- iii) the process  $\Lambda_t^i := \int_0^t \lambda_s^i ds$  is assumed to be uniformly bounded, i.e.,  $\Lambda_T^i \leq C$ , a.s..

**Remark 1.** The assumption iii) is a technical hypothesis needed only in the proof of Theorem 4.

### 3 The Optimal Model Measure

In this section, we study the characterization of the optimal probability measure for the minimization problem  $\inf_{\mathbb{Q} \in \mathcal{Q}} \Gamma(\mathbb{Q})$ . We recall the general result on existence and uniqueness given in [BMS]:

**Proposition 1.** *Under Assumptions A1-A2, there exists a unique  $\mathbb{Q}^*$  which minimizes  $\Gamma(\mathbb{Q})$  over all  $\mathbb{Q} \in \mathcal{Q}_f$ :*

$$\Gamma(\mathbb{Q}^*) = \inf_{\mathbb{Q} \in \mathcal{Q}_f} \Gamma(\mathbb{Q}) \quad (3)$$

Furthermore,  $\mathbb{Q}^*$  is equivalent to  $\mathbb{P}$ .

We use stochastic control techniques to study the value process  $V$  associated with this optimization problem. We show that  $V$  is the unique solution of a backward stochastic differential equation (BSDE) with a quadratic-exponential driver:

$$\begin{cases} dY_t = \left[ \sum_{i=1}^d g(y_t^i) \lambda_t^i - U_t + \delta_t Y_t \right] dt + \frac{1}{2} d\langle M^{Y,c} \rangle_t + dM_t^{Y,c} + \sum_{i=1}^d y_t^i dN_t^i \\ Y_T = \bar{U}_T \end{cases} \quad (4)$$

where  $g$  is the convex function  $g(x) = e^{-x} + x - 1$

A solution of (4) is a triple  $(Y, M^{Y,c}, y)$  where  $Y$  is a  $\mathbb{P}$ -semimartingale,  $M^{Y,c}$  is a locally square-integrable continuous local  $\mathbb{P}$ -martingale null at 0 and  $y = (y^1, \dots, y^d)$  an  $\mathbb{R}^d$ -valued predictable locally bounded process. Note that  $Y$  is a special  $\mathbb{P}$ -semimartingale.

#### 3.1 Some properties of solutions of the BSDE

We establish some auxiliary results about the existence and uniqueness of the solutions of (4).

**Proposition 2.** *Let  $(Y, M^{Y,c}, y) \in D_0^{\text{exp}} \times \mathcal{M}_{0,loc}^c(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$  be a solution of the BSDE (4). Then,  $Y$  satisfies the following recursion equality: for any stopping time  $\tau$  valued in  $[t, T]$ ,*

$$Y_t = -\ln \mathbb{E}^{\mathbb{P}} \left[ \exp \left( -Y_\tau + \int_t^\tau (\delta_s Y_s - U_s) ds \right) \middle| \mathcal{G}_t \right]. \quad (5)$$

**Proof:** Assume that  $(Y, M^{Y,c}, y)$  is a solution of (4), and define

$$X_t = Y_t - Y_0 - \int_0^t (\delta_s Y_s - U_s) ds$$

and  $Z_t = e^{-X_t}$ . An application of Itô's formula leads to

$$dZ_t = Z_{t-} \left( -dM_t^{Y,c} + \sum_{i=1}^d \left( e^{-y_t^i} - 1 \right) dN_t^i \right)$$

hence,  $Z$  is a non-negative local martingale. Assuming that  $Z$  is a martingale, one obtains, for  $t < \tau < T$ :

$$e^{-Y_t} = \mathbb{E}^{\mathbb{P}} \left[ \exp \left( -Y_\tau + \int_t^\tau (\delta_s Y_s - U_s) ds \right) \middle| \mathcal{G}_t \right]. \quad (6)$$

In general, we use a localizing sequence  $\tau_n$  in order to have the  $\mathbb{P}$ -martingale property and thus obtain (6) and (5) with  $\tau_n \wedge \tau$  instead of  $\tau$ . Then by the integrability Assumption 2 and the assumption that  $Y \in D_0^{\text{exp}}$ , we obtain a  $\mathbb{P}$ -integrable upper bound for the right-hand side of (6) and letting  $n$  go to infinity, by dominated convergence we obtain (5) for  $\tau$ .  $\square$

In the case  $\delta = 0$ , the process  $Y$ , part of the solution of (4) is given in a closed form as

$$Y_t = -\ln \mathbb{E}^{\mathbb{P}} \left[ \exp \left( -\bar{U}_T - \int_t^T U_s ds \right) \middle| \mathcal{G}_t \right].$$

We give now the main result of this section which extends earlier works by [16, 28, 25] and [BMS]:

**Theorem 1.** *There exists a unique triple  $(Y, M^{Y,c}, y) \in D_0^{\text{exp}} \times \mathcal{M}_0^p(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$  solution of (4). Furthermore, the optimal measure  $\mathbb{Q}^*$  solution of (3) admits the Radon-Nikodym density  $Z^{\mathbb{Q}^*} = \mathcal{E}(L)$  w.r.t.  $\mathbb{P}$  where*

$$dL_t = -dM_t^{Y,c} + \sum_{i=1}^d \left( e^{-y_t^i} - 1 \right) dN_t^i, \quad L_0 = 0. \quad (7)$$

**Proof:**

**Step 1:** In this step, we follow closely [BMS]. For  $\mathbb{Q} \in \mathcal{Q}_f^e$ , the space of probability measures equivalent to  $\mathbb{P}$  with finite entropy, we denote by  $L^{\mathbb{Q}}$  the stochastic logarithm of  $Z^{\mathbb{Q}}$ , i.e., the  $\mathbb{P}$ -local martingale such that  $dZ_t^{\mathbb{Q}} = Z_{t-}^{\mathbb{Q}} dL_t^{\mathbb{Q}}$ . From Assumption 1, the local martingale  $L^{\mathbb{Q}}$  admits the decomposition

$$dL_t^{\mathbb{Q}} = dL_t^{\mathbb{Q},c} + \sum_{i=1}^d \ell_t^i dN_t^i,$$

where  $L^{\mathbb{Q},c}$  is a continuous  $\mathbb{P}$ -local martingale, and  $\ell^i$  are predictable processes, and one has

$$d \ln Z_t^{\mathbb{Q}} = dL_t^{\mathbb{Q},c} - \frac{1}{2} d\langle L^{\mathbb{Q},c} \rangle_t + \sum_{i=1}^d \ln(1 + \ell_t^i) dN_t^i + \sum_{i=1}^d (\ln(1 + \ell_t^i) - \ell_t^i) \lambda_t^i dt. \quad (8)$$

Following [BMS], we establish that there exists a special semi-martingale  $V$  (the value process) such that the process  $J^{\mathbb{Q}}$  defined as

$$J_t^{\mathbb{Q}} = S_t^{\delta} V_t + \int_0^t S_s^{\delta} U_s ds + \int_0^t \delta_s S_s^{\delta} \ln Z_s^{\mathbb{Q}} ds + S_t^{\delta} \ln Z_t^{\mathbb{Q}} \quad \text{for all } t \in [0, T]$$

is a  $\mathbb{Q}$ -submartingale for each  $\mathbb{Q} \in \mathcal{Q}_f^e$  and a martingale for a particular  $\mathbb{Q}^*$ , hence  $\mathbb{Q}^*$  is an optimal probability measure (see [BMS] for details, in particular for the fact that  $\mathcal{E}(L)$  is a true martingale). We denote by  $V = A^V + M^V$  the canonical decomposition of the special semi-martingale  $V$ . The local martingale  $M^V$  admits a decomposition  $dM^V = dM^{V,c} + \sum_{i=1}^d v^i dN^i$  where  $M^{V,c}$  is a continuous  $\mathbb{P}$ -martingale. Using integration by parts formula, we obtain after some simple computations and using (8):

$$\begin{aligned} dJ_t^{\mathbb{Q}} &= S_t^{\delta} \left( (-\delta_t V_t + U_t) dt + (dV_t + d \ln Z_t^{\mathbb{Q}}) \right) \\ &= S_t^{\delta} \left[ (-\delta_t V_t + U_t) dt + dM_t^{V,c} + dA_t^V + dL_t^{\mathbb{Q},c} - \frac{1}{2} d\langle L^{\mathbb{Q},c} \rangle_t \right. \\ &\quad \left. + \sum_{i=1}^d (v_t^i + \ln(1 + \ell_t^i)) dN_t^i + \sum_{i=1}^d (\ln(1 + \ell_t^i) - \ell_t^i) \lambda_t^i dt \right] \end{aligned}$$

From Girsanov's theorem, the processes  $(N_t^i)_{t \geq 0}$  and  $(\widetilde{M}_t^c)_{t \geq 0}$  defined as:

$$\begin{aligned} d\widetilde{N}_t^i &= dN_t^i - \ell_t^i \lambda_t^i dt \\ d\widetilde{M}_t^c &= d(M_t^{V,c} + L_t^{\mathbb{Q},c}) - d\langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c} \rangle_t \end{aligned}$$

are  $\mathbb{Q}$ -local martingales, hence:

$$\begin{aligned} dJ_t^{\mathbb{Q}} &= S_t^{\delta} \left[ (-\delta_t V_t + U_t) dt + d\widetilde{M}_t^c + dA_t^V + d\langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c} \rangle_t - \frac{1}{2} d\langle L^{\mathbb{Q},c} \rangle_t \right. \\ &\quad \left. + \sum_{i=1}^d (v_t^i + \ln(1 + \ell_t^i)) d\widetilde{N}_t^i + \sum_{i=1}^d (\ell_t^i (v_t^i - 1) + (1 + \ell_t^i) \ln(1 + \ell_t^i)) \lambda_t^i dt \right]. \end{aligned}$$

In order that the process  $J^{\mathbb{Q}}$  is a  $\mathbb{Q}$ -submartingale for each  $\mathbb{Q} \in \mathcal{Q}_f^e$ , we impose that its finite variation part is a non-decreasing process.

$$\begin{aligned} A_t^V &= -\text{ess inf}_{\mathcal{Q}_f^e} \int_0^t (U_s - \delta_s V_s) ds + \langle M^{V,c} + L^{\mathbb{Q},c}, L^{\mathbb{Q},c} \rangle_t - \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle_t \\ &\quad + \sum_{i=1}^d \int_0^t (\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i)) \lambda_s^i ds. \end{aligned} \tag{9}$$

To find the *ess inf*, we divide (9) in two parts, the continuous part and the discontinuous part; hence we have two optimization problems:

$$\begin{aligned} A_t^V &= \int_0^t (\delta_s V_s - U_s) ds - \operatorname{ess\,inf}_{\mathbb{Q}_f^e} \{ \langle M^{V,c}, L^{\mathbb{Q},c} \rangle_t + \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle_t \} \\ &\quad - \operatorname{ess\,inf}_{\mathbb{Q}_f^e} \sum_{i=1}^d \int_0^t (\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i)) \lambda_s^i ds. \end{aligned}$$

In [BMS], it is proved that the first infimum is obtained for  $L^{\mathbb{Q},c} = -M^{V,c}$  and

$$-\operatorname{ess\,inf}_{\mathbb{Q}_f^e} \{ \langle M^{V,c}, L^{\mathbb{Q},c} \rangle + \frac{1}{2} \langle L^{\mathbb{Q},c} \rangle \} = \frac{1}{2} \langle M^{V,c} \rangle.$$

The second part of the optimisation problem reduces to find the optimal  $\ell^i$ , solution of:

$$\operatorname{ess\,inf} (\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i))$$

which is an easy task, the solution being  $\ell_s^{*,i} = e^{-v_s^i} - 1$ , which leads to

$$-\operatorname{ess\,inf} (\ell_s^i (v_s^i - 1) + (1 + \ell_s^i) \ln(1 + \ell_s^i)) = e^{-v_s^i} + v_s^i - 1 = g(v_s^i)$$

where  $g(x) = e^{-x} + x - 1$ . Therefore,

$$A_t^V = \int_0^t (\delta_s V_s - U_s) ds + \frac{1}{2} \langle M^{V,c} \rangle_t + \int_0^t \sum_{i=1}^d g(v_s^i) \lambda_s^i ds.$$

It follows that  $(V, M^{V,c}, v)$  is a solution of

$$\begin{cases} dV_t = \left( \delta_t V_t - U_t + \sum_{i=1}^d g(v_t^i) \lambda_t^i \right) dt + \frac{1}{2} d \langle M^{V,c} \rangle_t + dM_t^{V,c} + \sum_{i=1}^d v_t^i dN_t^i \\ V_T = \bar{U}_T \end{cases}$$

hence, the optimal probability measure  $\mathbb{Q}^*$  is characterized by its Radon-Nikodym density

$$dZ_t^{\mathbb{Q}^*} = Z_t^{\mathbb{Q}^*} dL_t, \quad dL_t = -dM_t^{V,c} + \sum_{i=1}^d (e^{-v_t^i} - 1) dN_t^i$$

The value process  $V$  is a solution of (4), then the solution exists.

**Step 2:** We now study the uniqueness of the BSDE solution. Assume that  $(Y, M^{Y,c}, y)$  and  $(\bar{Y}, M^{\bar{Y},c}, \bar{y})$  are two solutions of (4) in  $D_0^{\text{exp}} \times \mathcal{M}^p(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$ . Suppose that, for some  $t \in [0, T]$ ,

the set  $A = \{Y_t > \bar{Y}_t\} \in \mathcal{F}_t$  satisfies  $\mathbb{P}(A) > 0$  and define  $\tau = \inf\{s \geq t | \bar{Y}_s \geq Y_s\}$ , so that  $\bar{Y}_\tau \geq Y_\tau$ . Since  $Y_T = \bar{Y}_T$ , one has  $\tau \leq T$ , and:

$$\int_t^\tau (\delta_s Y_s - U_s) ds - Y_\tau > \int_t^\tau (\delta_s \bar{Y}_s - U_s) ds - \bar{Y}_\tau \quad \text{on } A$$

Then, from Proposition 2, it follows that:

$$\exp(-Y_t) = \mathbb{E}^\mathbb{P} \left[ \exp \left( \int_t^\tau \delta_s Y_s - U_s ds - Y_\tau \right) \middle| \mathcal{G}_t \right] > \exp(-\bar{Y}_t) \quad \text{on } A$$

which implies that  $Y_t < \bar{Y}_t$  on  $A$  in contradiction with the definition of  $A$ ; therefore  $Y$  and  $\bar{Y}$  are indistinguishable.

**Step 3:** In this step we prove that the solution  $(Y, M^{Y,c}, y)$  of the BSDE (4) belongs to the required spaces.

- As in [BMS], the recursive property implies that  $Y \in D_0^{exp}$ .
- We now study the process  $M^{Y,c}$ . Let us consider the  $\mathbb{P}$ -martingale:

$$K_t := \mathbb{E}^\mathbb{P} \left[ \exp \left( \int_0^T (\delta_s Y_s - U_s) ds - \bar{U}_T \right) \middle| \mathcal{G}_t \right]$$

Using the fact that  $Y \in D_0^{exp}$ , we obtain that the process  $K$  belongs to  $\mathcal{M}^p(\mathbb{P})$ . Now, the recursive property leads to

$$K_t = \exp \left( -Y_t + \int_0^t (\delta_s Y_s - U_s) ds \right)$$

and it is not difficult to show that, from Itô's formula and the canonical decomposition of  $Y$ ,

$$dM_t^{Y,c} = -\frac{dK_t^c}{K_{t-}}. \quad (10)$$

From Assumption 1, there exists  $k^i$  and  $M^{K,c}$  such that the martingale  $K$  equals

$$K_t = K_0 + M_t^{K,c} + \sum_{i=1}^d \int_0^t k_s^i dN_s^i$$

Hence, from (10)

$$\langle M^{Y,c} \rangle_T \leq \int_0^T \frac{1}{K_t^2} d\langle K^c \rangle_t \leq \langle K^c \rangle_T \sup_{0 \leq t \leq T} \frac{1}{K_t^2} \leq \langle K^c \rangle_T \exp \left( 2 \sup_{0 \leq t \leq T} |Y_t| (1 + \|\delta\|_\infty T) + 2 \int_0^T |U_s| ds \right)$$

By BDG's inequalities, there exists a constant  $C$  such that for every  $p \in [1, +\infty)$ :

$$\mathbb{E}^{\mathbb{P}} \left[ \langle K^c \rangle_T + \int_0^T (k_t^i)^2 dH_t^i \right]^{\frac{p}{2}} \leq C \mathbb{E}^{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |K_t|^p \right) \quad (11)$$

Since  $K \in \mathcal{M}^p(\mathbb{P})$ , we conclude that  $M^{Y,c}$  lies in the space  $\mathcal{M}_0^p(\mathbb{P})$  for every  $p \in [1, +\infty)$ . We conclude, using again BDG's inequalities.

• Space of  $y$ : Using the recursive relation and the decomposition of the process  $K$  we get:

$$\ln(K_{t^-} + k_t^i) - \ln(K_{t^-}) = -y_t^i$$

hence,

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-y_t^i} - 1)^2 dH_t^i \right]^{\frac{p}{2}} = \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( \frac{k_t^i}{K_{t^-}} \right)^2 dH_t^i \right]^{\frac{p}{2}} \leq \mathbb{E}^{\mathbb{P}} \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{K_t^p} \right) \left( \int_0^T (k_t^i)^2 dH_t^i \right)^{\frac{p}{2}} \right]$$

Since  $\sup_{0 \leq t \leq T} \left( \frac{1}{K_t} \right) \in \mathbf{L}^p(\mathbb{P})$  for any  $p \in [1, +\infty]$ , using (11) and Cauchy inequalities, we conclude:

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-y_t^i} - 1)^2 dH_t^i \right]^{\frac{p}{2}} < \infty. \quad (12)$$

In particular

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-y_t^i} - 1)^2 \lambda_t^i dt \right] = \mathbb{E}^{\mathbb{P}} \left[ \int_0^T \left( \frac{k_t^i}{K_{t^-}} \right)^2 dH_t^i \right] \leq \mathbb{E}^{\mathbb{P}} \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{K_t^2} \right) \int_0^T (k_t^i)^2 dH_t^i \right] < \infty. \quad (13)$$

By using similar arguments, one prove that:

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{y_t^i} - 1)^2 \lambda_t^i dt \right] < \infty. \quad (14)$$

Moreover, by using the inequality

$$|y|^2 \leq 2(|e^{-y} - 1|^2 + |e^y - 1|^2), \quad \forall y \in \mathbb{R}$$

and (13)-(14) we conclude that the process  $y$  is in  $\mathcal{L}^2(\lambda, \mathbb{P})$ .

It remains to prove that the martingale part of the BSDE solution, i.e.,

$$M = -M^{Y,c} + \sum_{i=1}^d \int_0^\cdot (e^{-y_t^i} - 1) dN_t^i$$

belongs to  $\mathcal{M}_0^p(\mathbb{P})$  for any  $p \in [1, +\infty)$ .

Since  $M^{Y,c} \in \mathcal{M}_0^p(\mathbb{P})$ , and (12)

$$\mathbb{E}^{\mathbb{P}} \left[ \langle M^{Y,c} \rangle_T + \sum_{i=1}^d \int_0^T (e^{-y_t^i} - 1)^2 dH_t^i \right]^{\frac{p}{2}} < \infty,$$

and using BDG inequality, we obtain

$$\mathbb{E}^{\mathbb{P}} \left( \sup_{0 \leq t \leq T} |M_t|^p \right) < \infty.$$

□

### 3.2 Comparison theorem and properties of the value process

In this part, we establish a comparison theorem and we study the properties of the value process for a given pair  $(U, \bar{U}_T)$ .

**Definition 3.** For two random variables  $X$  and  $Y$ , we write  $X \leq Y$  for  $X \leq Y$  a.s.

For two processes  $A$  and  $B$ , we write  $A \leq B$  for  $A_t \leq B_t, \forall t \in [0, T]$ , a.s.. We write  $(X, A) \leq (Y, B)$  if  $X \leq Y$  and  $A \leq B$ .

**Theorem 2.** Assume that for  $k = 1, 2$ ,  $(Y^k, M^{k,c}, y^k)$  is the solution of the BSDE (4) associated with  $(U^k, \bar{U}_T^k)$ . We denote  $Y^{12} := Y^1 - Y^2$ ,  $\tilde{U}^{12} := U^1 - U^2$  and  $\bar{U}_T^{12} := \bar{U}_T^1 - \bar{U}_T^2$ . Then,

$$S_t^\delta Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ \int_t^T S_s^\delta \tilde{U}_s^{12} ds + S_T^\delta \bar{U}_T^{12} \middle| \mathcal{G}_t \right] \quad (15)$$

where  $\mathbb{Q}^{*,2}$  is the solution of  $\mathcal{P}(U^2, \bar{U}_T^2)$ , i.e., the probability measure equivalent to  $\mathbb{P}$  with Radon Nikodym density  $Z^{\mathbb{Q}^{*,2}}$  given by

$$dZ_t^{\mathbb{Q}^{*,2}} = Z_t^{\mathbb{Q}^{*,2}} \left( -dM_t^{2,c} + \sum_{i=1}^d (e^{-y_t^{i,2}} - 1) dN_t^i \right). \quad (16)$$

In particular, if  $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$ , one obtains

$$Y_t^1 \leq Y_t^2, \quad dP \otimes dt\text{-a.e.}$$

**Proof:** We denote  $y^{i,12} := y^{i,1} - y^{i,2}$  and  $M^{12,c} = M^{1,c} - M^{2,c}$ . Then:

$$\begin{aligned} Y_t^{12} &= \bar{U}_T^{12} + \int_t^T \left( \tilde{U}_s^{12} - \delta_s Y_s^{12} \right) ds - \sum_{i=1}^d \int_t^T y_s^{i,12} dN_s^i - \sum_{i=1}^d \int_t^T [g(y_s^{i,1}) - g(y_s^{i,2})] \lambda_s^i ds \\ &\quad + \frac{1}{2} \int_t^T (d\langle M^{2,c} \rangle_s - d\langle M^{1,c} \rangle_s) - \int_t^T dM_s^{12,c} \end{aligned} \quad (17)$$

Note that, since  $M^{k,c}$  are continuous martingales,

$$-\langle M^{2,c}, M^{12,c} \rangle - \frac{1}{2} \langle M^{2,c} \rangle + \frac{1}{2} \langle M^{1,c} \rangle = \frac{1}{2} \langle M^{12,c} \rangle \quad (18)$$

Using the fact that the process  $\langle M^{12,c} \rangle$  is increasing and that the function  $g$  is convex we get:

$$\begin{aligned} Y_t^{12} &\leq \bar{U}_T^{12} + \int_t^T \left( \tilde{U}_s^{12} - \delta_s Y_s^{12} \right) ds + \sum_{i=1}^d \int_t^T (e^{-y_s^{i,2}} - 1) y_s^{i,2} \lambda_s^i ds + \int_t^T d\langle M^{2,c}, M^{12,c} \rangle_s \\ &\quad - \int_t^T dM_s^{12,c} - \sum_{i=1}^d \int_t^T y_s^{i,2} dN_s^i. \end{aligned}$$

Let  $N^*$  and  $M^{*,c}$  be the  $\mathbb{Q}^{*,2}$ -martingales obtained by Girsanov's transformation from  $N$  and  $M^{12,c}$ , where  $d\mathbb{Q}^{*,2} = Z^{\mathbb{Q}^{*,2}} d\mathbb{P}$  and where  $Z^{\mathbb{Q}^{*,2}}$  is given by (16). Then,

$$Y_t^{12} \leq \bar{U}_T^{12} + \int_t^T \left( \tilde{U}_s^{12} - \delta_s Y_s^{12} \right) ds - \sum_{i=1}^d \int_t^T y_s^{i,2} dN_s^{i*} - \int_t^T dM_s^{*,c}$$

which implies that

$$Y_t^{12} \leq \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ \int_t^T e^{-\int_t^s \delta_r dr} \tilde{U}_s^{12} ds + e^{-\int_t^T \delta_r dr} \bar{U}_T^{12} \middle| \mathcal{G}_t \right]$$

In particular, if  $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$ , then  $Y_t^1 \leq Y_t^2$   $d\mathbb{P} \otimes dt$ -a.e..  $\square$

We have also the following standard a priori estimates.

**Proposition 3.** *Let  $(Y^k, M^{k,c}, y^k)$  be the solution associated with  $(U^k, \bar{U}_T^k)$  for  $k = 1, 2$  where we assume that  $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$ . Then there exists a constant  $C > 0$  such that:*

$$\mathbb{E}^{\mathbb{Q}^{*,2}} \left[ \sup_{0 \leq t \leq T} |Y_t^{12}|^2 + \langle M^{12,c} \rangle_T + \sum_{i=1}^d \int_0^T |y_t^{i,12}|^2 \lambda_t^{i,*} dt \right] \leq C \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ |\bar{U}_T^{12}|^2 + \int_0^T |U_t^{12}|^2 dt \right] \quad (19)$$

where  $\lambda^{i,*}$  is the intensity process of  $H^i$  under the probability  $\mathbb{Q}^{*,2}$ .

In the case  $(U^2, \bar{U}_T^2) \leq (U^1, \bar{U}_T^1)$ , the same inequality holds with  $\mathbb{Q}^{*,1}$ .

**Proof:** Using Itô's formula:

$$\begin{aligned} d(Y_t^{12})^2 &= 2Y_t^{12} \left[ \left( \delta_t Y_t^{12} - \tilde{U}_t^{12} \right) dt + \frac{1}{2} d\langle M^{1,c} \rangle_t - \frac{1}{2} d\langle M^{2,c} \rangle_t \right] + d\langle M^{12,c} \rangle_t \\ &\quad + 2Y_t^{12} \left[ \sum_{i=1}^d (g(y_t^{i,1}) - g(y_t^{i,2})) \lambda_t^i \right] dt + \sum_{i=1}^d (y_t^{i,12})^2 \lambda_t^i dt + d\text{mart}_t \end{aligned}$$

where  $d\text{mart}_t = 2Y_t^{12} \left[ dM_t^{12,c} + \sum_{i=1}^d y_t^{i,12} dN_t^i \right] + \sum_{i=1}^d (y_t^{i,12})^2 dN_t^i$  corresponds to a martingale.

Assuming  $(U^1, \bar{U}_T^1) \leq (U^2, \bar{U}_T^2)$ , it follows from the comparison Theorem 2 that  $Y^1 \leq Y^2$ . Using the relation (18) and the convexity property of the function  $g$ , we get :

$$\begin{aligned} (Y_t^{12})^2 &+ \int_t^T d\langle M^{12,c} \rangle_s \leq (\bar{U}_T^{12})^2 + 2 \int_t^T Y_s^{12} \left[ -\delta_s Y_s^{12} + \tilde{U}_s^{12} \right] ds + 2 \sum_{i=1}^d \int_t^T Y_s^{12} (e^{-y_s^{i,2}} - 1) y_s^{i,12} \lambda_s^i ds \\ &- 2 \int_t^T Y_s^{12} d\langle M^{1,c}, M^{2,c} \rangle_s + \int_t^T Y_s^{12} d\langle M^{2,c} \rangle_s - \sum_{i=1}^d \int_t^T (y_s^{i,12})^2 \lambda_s^i ds + \int_t^T d\text{mart}_s \end{aligned}$$

hence

$$(Y_t^{12})^2 + \int_t^T d\langle M^{12,c} \rangle_s \leq (\bar{U}_T^{12})^2 + 2 \int_t^T Y_s^{12} \left[ -\delta_s Y_s^{12} + \tilde{U}_s^{12} \right] ds - \sum_{i=1}^d \int_t^T (y_s^{i,12})^2 \lambda_s^{i*} ds + \int_t^T d\text{mart}_s^* \quad (20)$$

where  $\text{mart}^*$  is a  $\mathbb{Q}^{*,2}$  martingale and  $\lambda_s^{i*} := \lambda_s^i e^{-y_s^{i,2}}$  is the intensity of  $H^i$  under  $\mathbb{Q}^{*,2}$ . From the obvious inequality

$$(Y_t^{12})^2 - \frac{1}{\delta_t} (\tilde{U}_t^{12}) Y_t^{12} \geq \frac{-1}{4\delta_t^2} (\tilde{U}_t^{12})^2$$

and the non-negativity of  $\delta$ , we deduce easily that

$$-Y_t^{12} \left( \delta_t Y_t^{12} - \tilde{U}_t^{12} \right) \leq \frac{1}{4\delta_t} (\tilde{U}_t^{12})^2 \quad (21)$$

Plotting relation (21) in (20) and using the fact that the process  $\delta$  is bounded below, there exists a constant  $C > 0$  such that:

$$\mathbb{E}^{\mathbb{Q}^{*,2}} \left[ \sup_{t \in [0, T]} |Y_t^{12}|^2 + \langle M^{12,c} \rangle_T + \sum_{i=1}^d \int_0^T |y_t^{12,i}|^2 \lambda_t^{i,*} dt \right] \leq C \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ |\bar{U}_T^{12}|^2 + \int_0^T |\tilde{U}_t^{12}|^2 dt \right].$$

Permuting  $Y^1$  and  $Y^2$  and assuming  $(U^1, \bar{U}_T^1) \geq (U^2, \bar{U}_T^2)$  leads to the kind of inequality.  $\square$

**Theorem 3.** (Concavity property) Define the map  $F : D_1^{\text{exp}} \times L^{\text{exp}} \longrightarrow D_0^{\text{exp}}$  as

$$F(U, \bar{U}) = V$$

where  $(V, M^{V,c}, v)$  is the solution associated with  $(U, \bar{U})$ . Then  $F$  is concave, namely, for all  $\theta \in (0, 1)$  and  $(U^1, \bar{U}_T^1), (U^2, \bar{U}_T^2) \in D_1^{\text{exp}} \times L^{\text{exp}}$  :

$$F(\theta U^1 + (1 - \theta)U^2, \theta \bar{U}_T^1 + (1 - \theta)\bar{U}_T^2) \geq \theta F(U^1, \bar{U}_T^1) + (1 - \theta)F(U^2, \bar{U}_T^2).$$

**Proof:** Let  $(V^k, M^{k,c}, v^k)$  be the solution of BSDE (4) associated with  $(U^k, \bar{U}_T^k) \in D_1^{\text{exp}} \times L^{\text{exp}}$ . Then for any  $\theta \in (0, 1)$ :

$$\begin{aligned} d(\theta V_t^1 - (1 - \theta)V_t^2) &= [\delta_t(\theta V_t^1 + (1 - \theta)V_t^2) - (\theta U_t^1 + (1 - \theta)U_t^2)] dt \\ &\quad + \theta d\langle M^{1,c} \rangle_t + (1 - \theta)d\langle M^{2,c} \rangle_t + d(\theta M_t^{1,c} + (1 - \theta)M_t^{2,c}) \\ &\quad + \sum_{i=1}^d [\theta v_t^{1,i} + (1 - \theta)v_t^{2,i}] dN_t^i + \sum_{i=1}^d [\theta g(v_t^{1,i}) + (1 - \theta)g(v_t^{2,i})] \lambda_t^i dt \end{aligned} \quad (22)$$

We recall the following general result: Let  $X$  and  $Y$  be two continuous martingales. Then, for all  $\theta \in (0, 1)$ ,  $\theta\langle X \rangle + (1 - \theta)\langle Y \rangle - \langle \theta X + (1 - \theta)Y \rangle$  is an increasing process. Indeed, we have:

$$\begin{aligned} &\langle \theta X + (1 - \theta)Y \rangle - \theta\langle X \rangle - (1 - \theta)\langle Y \rangle \\ &= (\theta^2 - \theta)\langle X \rangle + ((1 - \theta)^2 - (1 - \theta))\langle Y \rangle + 2\theta(1 - \theta)\langle X, Y \rangle \\ &= \theta(\theta - 1)[\langle X \rangle + \langle Y \rangle - 2\langle X, Y \rangle] = \theta(\theta - 1)\langle X - Y \rangle \end{aligned}$$

Therefore using the convexity property of the function  $g$  we get:

$$\begin{aligned} \theta V_t^1 + (1 - \theta)V_t^2 &\leq (\theta \bar{U}_T^1 + (1 - \theta)\bar{U}_T^2) - \int_t^T [\delta_s(\theta V_s^1 + (1 - \theta)V_s^2) - (\theta U_s^1 + (1 - \theta)U_s^2)] ds \\ &\quad - \int_t^T d\langle \theta M^{1,c} + (1 - \theta)M^{2,c} \rangle_s - \int_t^T d(\theta M_s^{1,c} + (1 - \theta)M_s^{2,c}) \\ &\quad - \sum_{i=1}^d \int_t^T (\theta v_s^{1,i} + (1 - \theta)v_s^{2,i}) dN_s^i - \sum_{i=1}^d \int_t^T g(\theta v_s^{1,i} + (1 - \theta)v_s^{2,i}) \lambda_s^i ds \end{aligned} \quad (23)$$

Let  $(V^\theta, M^{\theta,c}, v^\theta)$  be the solution of the BSDE associated with  $(\theta U^1 + (1 - \theta)U^2, \theta \bar{U}^1 + (1 - \theta)\bar{U}^2)$  and set  $M^{V,c,\theta} = \theta M^{1,c} + (1 - \theta)M^{2,c}$  and for  $i = 1, \dots, d$ ,  $\widehat{v}^{\theta,i} = \theta v^{1,i} + (1 - \theta)v^{2,i}$ . Then, using (23):

$$\begin{aligned} \theta V_t^1 + (1 - \theta)V_t^2 - V_t^\theta &\leq \int_t^T \delta_s(V_s^\theta - (\theta V_s^1 + (1 - \theta)V_s^2)) ds - \int_t^T d\langle M^{V,c,\theta} \rangle_s + \int_t^T d\langle M^{\theta,c} \rangle_s \\ &\quad - \int_t^T d(M_s^{V,c,\theta} - M_s^{\theta,c}) + \sum_{i=1}^d \int_t^T (g(v_s^{\theta,i}) - g(\widehat{v}_s^{\theta,i})) \lambda_s^i ds - \sum_{i=1}^d \int_t^T (\widehat{v}_s^{\theta,i} - v_s^{\theta,i}) dN_s^i \end{aligned}$$

Using (18) and the convexity property of the function  $g$  we get:

$$\begin{aligned}
\theta V_t^1 + (1-\theta)V_t^2 - V_t^\theta &\leq \int_t^T [\delta_s(V_s^\theta - (\theta V_s^1 + (1-\theta)V_s^2))] ds + \sum_{i=1}^d \int_t^T (e^{-v_s^{\theta,i}} - 1) (\widehat{v}_s^{\theta,i} - v_s^{\theta,i}) \lambda_s^i ds \\
&\quad - \int_t^T d(\langle M^{\theta,c}, M^{V,c,\theta} \rangle_s + \langle M^{\theta,c} \rangle_s) - \int_t^T d(M_s^{V,c,\theta} - M_s^{\theta,c}) - \sum_{i=1}^d \int_t^T (\widehat{v}_s^{\theta,i} - v_s^{\theta,i}) dN_s^i \\
&\leq \int_t^T [\delta_s(V_s^\theta - (\theta V_s^1 + (1-\theta)V_s^2))] ds - \sum_{i=1}^d \int_t^T (\widehat{v}_s^{\theta,i} - v_s^{\theta,i}) (dN_s^i - (e^{-v_s^{\theta,i}} - 1) \lambda_s^i ds) \\
&\quad - \int_t^T d((M_s^{V,c,\theta} - M_s^{\theta,c}) + \langle M^{V,c,\theta} - M^{\theta,c}, M^{\theta,c} \rangle_s).
\end{aligned}$$

Let  $\mathbb{Q}^{*,\theta}$  be the probability measure equivalent to  $\mathbb{P}$  with Radon-Nikodym density

$$dZ_t^{\mathbb{Q}^{*,\theta}} = Z_{t-}^{\mathbb{Q}^{*,\theta}} \left( -dM_t^{\theta,c} + \sum_{i=1}^d (e^{-v_t^{\theta,i}} - 1) dN_t^i \right).$$

Then, using integration by parts and Girsanov's theorem, taking  $\mathbb{Q}^{*,\theta}$ -conditional expectations, we have

$$S_t^\delta (\theta V_t^1 + (1-\theta)V_t^2 - V_t^\theta) \leq 0$$

which gives the result.  $\square$

## 4 The second optimization problem

In this section, we assume that  $U_s = U(c_s)$  and  $\bar{U}_T = \bar{U}(\psi)$  where  $U$  and  $\bar{U}$  are given functions,  $c$  is a non-negative  $\mathbb{G}$ -adapted process and  $\psi$  a  $\mathcal{G}_T$ -measurable non-negative random variable. We fix a probability  $\tilde{\mathbb{P}}$  equivalent to  $\mathbb{P}$  with a Radon-Nikodym density  $\tilde{Z}$  with respect to  $\mathbb{P}$  given by:

$$d\tilde{Z}_t = \tilde{Z}_{t-} (\theta_t dM_t^c + \sum_{i=1}^n (e^{-z_t^i} - 1) dN_t^i), \quad \tilde{Z}_0 = 1. \quad (24)$$

### 4.1 The optimal plan

**Definition 4.**

$\mathcal{A}(x)$  is the closed convex set of controls parameters  $(c, \psi) \in \mathcal{H}^2([0, T]) \times \mathbf{L}^2(\Omega, \mathcal{G}_T)$  such that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_0^T c_t dt + \psi \right] \leq x,$$

and  $(U(c), \bar{U}(\psi)) \in D_1^{\text{exp}} \times L^{\text{exp}}$  and  $(\bar{c}U'(c), \bar{\psi}\bar{U}'(\psi)) \in D_1^{\text{exp}} \times L^{\text{exp}}$  for any pair  $(\bar{c}, \bar{\psi}) \in \mathcal{H}^2([0, T]) \times \mathbf{L}^2(\Omega, \mathcal{G}_T)$ , as well as the process  $\exp(\gamma \int_0^\cdot |U(c_t)| dt)$  (respectively  $\exp(\gamma \int_0^\cdot |c_t| |U'(c_t)| dt)$ ) belongs to the class  $\mathbf{[D]}$  (see Dellacherie and Meyer [5] for definition).

We study the following optimization problem:

$$\sup_{(c, \psi) \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{Q}^*} \left[ \int_0^T S_s^\delta U(c_s) ds + S_T^\delta \bar{U}(\psi) \right] + \mathbb{E}^{\mathbb{Q}^*} \left[ \int_0^T \delta_s S_s^\delta \ln Z_s^{\mathbb{Q}^*} ds + S_T^\delta \ln Z_T^{\mathbb{Q}^*} \right] = \sup_{(c, \psi) \in \mathcal{A}(x)} V_0^{x, \psi, c}$$

where  $V_0$  is the value at initial time of the value process  $V$ , part of the solution  $(V, M^{V, c}, v)$  of the BSDE (4), in the case  $U_s = U(c_s)$  and  $\bar{U}_T = \bar{U}(\psi)$ . Here,  $\mathbb{Q}^*$  is the optimal model measure for  $\mathcal{P}(U(c), \bar{U}(\psi))$ , and depends on  $c, \psi$ .

In a complete market setting, denoting by  $\tilde{\mathbb{P}}$  the unique risk neutral probability, the process  $c$  can be interpreted as a consumption and  $\psi$  as a terminal wealth.

**Assumption A 3.** *The utility functions  $U$  and  $\bar{U}$  satisfy the usual conditions:*

- i) *Strictly increasing and concave.*
- ii) *Continuous differentiable on the set  $\{U > -\infty\}$  and  $\{\bar{U} > -\infty\}$ , respectively,*
- iii)  *$U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0$  and  $\bar{U}'(\infty) := \lim_{x \rightarrow \infty} \bar{U}'(x) = 0$ ,*
- iv)  *$U'(0) := \lim_{x \rightarrow 0} U'(x) = +\infty$  and  $\bar{U}'(0) := \lim_{x \rightarrow 0} \bar{U}'(x) = +\infty$ ,*
- v) *Asymptotic elasticity  $AE(U) := \limsup_{x \rightarrow +\infty} \frac{xU'(x)}{U(x)} < 1$ .*

## 4.2 Properties of the value process

**Proposition 4.** *Define the map  $G : \mathcal{A}(x) \rightarrow D_0^{\text{exp}}$  as  $G(c, \psi) = V$ , where  $(V, M^{V, c}, v)$  is the solution of the BSDE (4) associated with  $(U(c), \bar{U}(\psi))$ . Then*

- (i)  *$G$  is concave, i.e., for all  $\theta \in (0, 1)$  and  $(c^1, \psi^1), (c^2, \psi^2) \in \mathcal{A}(x)$ :*

$$G(\theta c^1 + (1 - \theta)c^2, \theta \psi^1 + (1 - \theta)\psi^2) \geq \theta G(c^1, \psi^1) + (1 - \theta)G(c^2, \psi^2).$$

- (ii) *Let  $G_0(c, \psi)$  be the value at initial time of  $G(c, \psi)$ , i.e.,  $G_0(c, \psi) = V_0$ . If  $(c^n, \psi^n) \in \mathcal{A}(x)$  converges decreasingly to  $(c, \psi) \in \mathcal{A}(x)$ , then  $G_0(c^n, \psi^n)$  converges decreasingly to  $G_0(c, \psi)$ . Moreover  $G_0$  is upper continuous with respect to the control parameters.*

**Proof:** Let  $(V^k, M^{k, c}, v^k)$  be the solution of the BSDE (4) associated with  $(U(c^k), \bar{U}(\psi^k))$  for  $k = 1, 2$ . For any  $\theta \in (0, 1)$ , let

- $(\tilde{V}^\theta, \tilde{M}^{\theta, c}, \tilde{v}^\theta)$  be the solution of (4) associated with  $(U(\theta c^1 + (1 - \theta)c^2), \bar{U}(\theta \psi^1 + (1 - \theta)\psi^2))$
- $(V^\theta, M^{\theta, c}, v^\theta)$  be the solution of (4) associated with  $(\theta U(c^1) + (1 - \theta)U(c^2), \theta \bar{U}(\psi^1) + (1 - \theta)\bar{U}(\psi^2))$

and set  $\bar{V}^\theta = \theta V^1 + (1-\theta)V^2$ . Then, by using both the concavity properties of  $(U, \bar{U})$  and Theorem 2, we get  $\tilde{V}^\theta \geq V^\theta$ . Moreover, as consequence of Theorem 3, we obtain  $V^\theta \geq \bar{V}^\theta$ , which gives the assertion (i).

Let us now consider  $(c^n, \psi^n)$  a decreasing sequence of control parameters in  $\mathcal{A}(x)$  which converges to  $(c, \psi)$ ,  $c^n \rightarrow c$  a.s and  $\psi^n \rightarrow \psi$  a.s; then, by using inequality (15), and the fact that the functions  $U$  and  $\bar{U}$  are non-decreasing, we get

$$|V_0^{c^n, \psi^n} - V_0^{c, \psi}| \leq \mathbb{E}^{\mathbb{Q}^*} \left[ \int_0^T (U(c_s^n) - U(c_s)) ds + (\bar{U}(\psi^n) - \bar{U}(\psi)) \right] \quad (25)$$

where  $\mathbb{Q}^*$  is the optimal density associated with  $(U(c), \bar{U}(\psi))$ . Thus, by using the convergence monotone theorem and the a priori estimate (19),  $V^{c^n, \psi^n}$  converges decreasingly to  $V^{c, \psi}$ .

Let  $(c^n, \psi^n) \in \mathcal{A}(x)$  be a sequence of control parameters such that  $c^n \rightarrow c$  a.s and  $\psi^n \rightarrow \psi$  a.s where  $(c, \psi) \in \mathcal{A}(x)$  and denote  $\tilde{c}^n = \sup_{m \geq n} c^m$ ,  $\tilde{\psi}^n = \sup_{m \geq n} \psi^m$ . Then,  $\tilde{c}^n \rightarrow c$  a.s decreasingly and  $\tilde{\psi}^n \rightarrow \psi$  a.s decreasingly. It follows that  $V_0^{\tilde{c}^n, \tilde{\psi}^n}$  converges to  $V^{c, \psi}$  decreasingly and therefore:

$$\limsup_n V_0^{c^n, \psi^n} \leq \lim_n V_0^{\tilde{c}^n, \tilde{\psi}^n} = V_0^{c, \psi}$$

Hence,  $G_0$  is upper semicontinuous with respect to the control parameters. □

**Definition 5.** The pairs  $(c^1, \psi^1), (c^2, \psi^2) \in \mathcal{A}(x)$  are comparable if either  $(c^1, \psi^1) \geq (c^2, \psi^2)$  or  $(c^1, \psi^1) \leq (c^2, \psi^2)$  with the order introduced in Definition 3.

**Proposition 5.** Assume that Assumption A. 3 holds and let  $(c^1, \psi^1), (c^2, \psi^2)$  be two comparable plans in  $\mathcal{A}(x)$ . Then the function  $\Psi$  defined on  $(0, 1)$  and valued in  $D_0^{\text{exp}}$

$$\Psi(\epsilon) = G(c^1 + \epsilon(c^2 - c^1), \psi^1 + \epsilon(\psi^2 - \psi^1))$$

is right continuous at 0.

**Proof:**

• Assume first that  $(c^1, \psi^1) \leq (c^2, \psi^2)$ . Let, for  $\epsilon \in ]0, 1[$ ,  $V^\epsilon = G(c^1 + \epsilon(c^2 - c^1), \psi^1 + \epsilon(\psi^2 - \psi^1))$  and  $V = G(c^1, \psi^1)$ . From Proposition 3 and the obvious inequalities  $U(c^1 + \epsilon(c^2 - c^1)) \geq U(c^1)$  and  $\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) \geq \bar{U}(\psi^1)$ , we obtain

$$\mathbb{E}^{\mathbb{Q}^{*,2}} \left( \sup_{0 \leq t \leq T} |V_t - V_t^\epsilon|^2 \right) \leq C \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ |\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1)|^2 + \int_0^T |U(c_s^1 + \epsilon(c_s^2 - c_s^1)) - U(c_s^1)|^2 ds \right].$$

Using now the concavity properties of  $U$  and  $\bar{U}$ , we obtain

$$\begin{aligned} 0 &\leq U(c_t^1 + \epsilon(c_t^2 - c_t^1)) - U(c_t^1) \leq \epsilon U'(c_t^1)(c_t^2 - c_t^1) \\ 0 &\leq \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1) \leq \epsilon \bar{U}'(\psi^1)(\psi^2 - \psi^1). \end{aligned}$$

Thus, we have

$$\mathbb{E}^{\mathbb{Q}^{*,2}} \left( \sup_{0 \leq t \leq T} \left| \frac{V_t - V_t^\epsilon}{\epsilon} \right|^2 \right) \leq C \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ (\bar{U}'(\psi^1))^2 (\psi^2 - \psi^1)^2 + \int_0^T (U'(c_s^1))^2 (c_s^2 - c_s^1)^2 ds \right].$$

• Assume now that  $(c^1, \psi^1) \geq (c^2, \psi^2)$ . Then, using the fact that  $G$  is concave with respect to the control parameters, one has

$$V^\epsilon \geq (1 - \epsilon)V^1 + \epsilon V^2$$

where  $V^k$  are associated with  $(c^k, \psi^k)$ . Moreover, since  $c^1 + \epsilon(c^2 - c^1) \leq c^1$  and  $\psi^1 + \epsilon(\psi^2 - \psi^1) \leq \psi^1$ , we have by Theorem 2 that

$$0 \geq \frac{V^\epsilon - V^1}{\epsilon} \geq V^2 - V^1.$$

Therefore:

$$\left| \frac{V_t^\epsilon - V_t^1}{\epsilon} \right| \leq |V_t^2 - V_t^1|, \quad t \in [0, T].$$

Using now Proposition 3, we get

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left( \sup_{0 \leq t \leq T} \left| \frac{V_t^1 - V_t^\epsilon}{\epsilon} \right|^2 \right) \leq c \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ (\bar{U}'(\psi^2))^2 (\psi^2 - \psi^1)^2 + \int_0^T (U'(c_s^2))^2 (c_s^2 - c_s^1)^2 ds \right].$$

• Finally, we conclude there exists a constant  $C > 0$  such that:

$$\mathbb{E}^{\mathbb{Q}^*} \left[ \sup_{0 \leq t \leq T} \left| \frac{V_t^1 - V_t^\epsilon}{\epsilon} \right|^2 \right] \leq C$$

where  $\mathbb{Q}^* = \mathbb{Q}^{*,1}$  if  $(c^1, \psi^1) \geq (c^2, \psi^2)$  and  $\mathbb{Q}^* = \mathbb{Q}^{*,2}$  if  $(c^1, \psi^1) \leq (c^2, \psi^2)$ . Then, by Kolmogorov's criteria, we deduce that  $\Psi$  is right-continuous at 0.  $\square$

We now give a regularity result that will be useful in the next section. The proof is postponed to the Appendix

**Theorem 4.** *Let  $(c^1, \psi^1)$  and  $(c^2, \psi^2)$  be two comparable plans in  $\mathcal{A}(x)$ . Let*

$$\begin{aligned} (V^\epsilon, M^{\epsilon,c}, v^\epsilon) & \text{ the solution of (4) associated with } (U(c^1 + \epsilon(c^2 - c^1)), \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1))) \\ (V^1, M^{1,c}, v^1) & \text{ the solution of (4) associated with } (U(c^1), \bar{U}(\psi^1)) \end{aligned}$$

*Then,  $V^\epsilon$  is right-differentiable with respect to  $\epsilon$  at 0. Moreover, if we denote by  $\partial_\epsilon V := \lim_{\epsilon \rightarrow 0} \frac{V^\epsilon - V^1}{\epsilon}$ , then there exists  $\partial_\epsilon \widetilde{M}^{V,c}, \partial_\epsilon v \in \mathbf{L}^2(\mathbb{Q}^{1,*}) \times \mathcal{L}^2(\widetilde{\lambda}, \mathbb{Q}^{1,*})$  such that the triple  $(\partial_\epsilon V, \partial_\epsilon \widetilde{M}^{V,c}, \partial_\epsilon v)$  is the solution of the following BSDE:*

$$\begin{cases} d\partial_\epsilon V_t = (\delta_t \partial_\epsilon V_t - U'(c_t^1)(c_t^2 - c_t^1)) dt + d\partial_\epsilon \widetilde{M}_t^{V,c} + \sum_{i=1}^d \partial_\epsilon v_t^i d\widetilde{N}_t^i, & \mathbb{Q}^{*,1}\text{-a.s.} \\ \partial_\epsilon V_T = \bar{U}'(\psi^1)(\psi^2 - \psi^1), \end{cases} \quad (26)$$

where  $\tilde{\lambda}^i := \lambda^i e^{-v^{1,i}}$  and  $\tilde{N}^i := N^i - \int_0^{\cdot} (e^{-v_t^{1,i}} - 1) \lambda_t^i dt$  is a  $\mathbb{Q}^{1,*}$ -martingale.

Moreover, we obtain

$$\partial_\epsilon V_t = \mathbb{E}^{\mathbb{P}} \left[ \frac{Z_T^{\mathbb{Q}^{*,1}} S_T^\delta}{Z_t^{\mathbb{Q}^{*,1}} S_t^\delta} \bar{U}'(\psi^1) (\psi^2 - \psi^1) + \int_t^T \frac{Z_s^{\mathbb{Q}^{*,1}} S_s^\delta}{Z_t^{\mathbb{Q}^{*,1}} S_t^\delta} U'(c_s^1) (c_s^2 - c_s^1) ds \middle| \mathcal{G}_t \right] \quad \forall t \in [0, T]. \quad (27)$$

### 4.3 The optimization problem

In this section, we solve the following optimization problem: we associate with a pair  $(c, \psi) \in \mathcal{A}(x)$  the quantity

$$X_0^{c,\psi} = \mathbb{E}^{\tilde{\mathbb{P}}} \left( \int_0^T c_s ds + \psi \right)$$

and we study

$$u(x) = \sup_{X_0^{c,\psi} \leq x} V_0^{(c,\psi)}. \quad (28)$$

Here  $V_0^{(c,\psi)} = V_0$ , where  $(V, M^{V,c}, v)$  is the solution of the BSDE (4) associated with  $(U(c), \bar{U}(\psi))$ . Note that, in a complete market setting with zero interest rate, when  $\tilde{\mathbb{P}}$  is the unique equivalent martingale measure,  $X_0$  is the initial wealth associated with the consumption  $c$  and terminal wealth  $\psi$ .

**Proposition 6.** *There exists an optimal pair  $(c^0, \psi^0)$  which solves (28).*

**Proof:** The uniqueness is a consequence of the strictly concavity property of  $V_0$ . We shall prove the existence by using Komlós theorem.

**First step:** Let us first prove that  $\sup_{(c,\psi) \in \mathcal{A}(x)} V_0^{c,\psi} < +\infty$ . Because  $\mathbb{P} \in \mathcal{Q}_f^e$ , we have:

$$\sup_{(c,\psi) \in \mathcal{A}(x)} V_0^{c,\psi} \leq \sup_{(c,\psi) \in \mathcal{A}(x)} \mathbb{E}^{\mathbb{P}} \left[ \bar{U}(\psi) + \int_0^T U(c_s) ds \right] := \tilde{u}(x)$$

Using the elasticity assumption on  $U$  and  $\bar{U}$ , we can find  $\gamma \in (0, 1)$  and  $x_0 \in \mathbb{R}$  such that, for any  $\theta > 1$ , one has:

$$\begin{aligned} U(\theta x) &< \theta^\gamma U(x) & \forall x > x_0, \\ \bar{U}(\theta x) &< \theta^\gamma \bar{U}(x) & \forall x > x_0, \end{aligned}$$

hence, for any  $x > x_0$ :

$$\tilde{u}(\theta x) = \mathbb{E}^{\mathbb{P}} \left[ \bar{U}(\theta \frac{\psi^{\theta x}}{\theta}) + \int_0^T U(\theta \frac{c_s^{\theta x}}{\theta}) ds \right] \leq \theta^\gamma \tilde{u}(x).$$

Then,  $AE(\tilde{u}) < 1$ , which permits to conclude that, for any  $x > 0$ ,  $\tilde{u}(x) < +\infty$  (see [14] and [21] chap. 3, Lemma 3).

**Second step:** Let  $(c^n, \psi^n) \in \mathcal{A}(x)$  be a maximizing sequence such that:

$$\nearrow \lim_{n \rightarrow +\infty} V_0^{c^n, \psi^n} = \sup_{(c, \psi) \in \mathcal{A}(x)} V_0^{c, \psi} < +\infty,$$

where the RHS is finite thanks to step 1. Using Komlòs criterion, we can find a convex combination  $(\bar{c}^n, \bar{\psi}^n) \in \text{conv}\left((c^n, \psi^n), (c^{n+1}, \psi^{n+1}), \dots\right)$  which converges  $\mathbb{P}$ -a.s. We denote by  $(c^*, \psi^*)$  this limit, which belongs to  $\mathcal{A}(x)$  since it is a closed convex set. Moreover, there exists  $N_n \geq n$  and a positive sequence  $(\theta^m)_{m \in \mathbb{N}}$  satisfying  $\sum_{m=n}^{N_n} \theta^m = 1$  such that  $(\bar{c}_n, \bar{\psi}_n) = (\sum_{m=n}^{N_n} \theta^m c^m, \sum_{m=n}^{N_n} \theta^m \psi^m)$ . Therefore, by using the concavity and the increasing properties of the functional  $V_0$  which respect to the control plan we get:

$$V_0^{\bar{c}^n, \bar{\psi}^n} = V_0^{\sum_{m=n}^{N_n} \theta^m c^m, \sum_{m=n}^{N_n} \theta^m \psi^m} \geq \sum_{m=n}^{N_n} \theta^m V_0^{c^m, \psi^m} \geq V_0^{c^n, \psi^n}.$$

Moreover, using the upper semi-continuous property of the functional  $V_0$  which respect to the control plan we get:

$$\sup_{(c, \psi) \in \mathcal{A}(x)} V_0^{c, \psi} = \limsup_n V_0^{c^n, \psi^n} \leq \limsup_n V_0^{\bar{c}^n, \bar{\psi}^n} = V_0^{c^*, \psi^*}.$$

□

In order to characterize the optimal solution, we recall the classical convex analysis result.

**Proposition 7.** *There exists a constant  $\nu^* > 0$  such that :*

$$u(x) = \sup_{(c, \psi)} \left\{ V_0^{(c, \psi)} + \nu^* \left( x - X_0^{(c, \psi)} \right) \right\} \quad (29)$$

and if the maximum is attained in (28) by  $(c^*, \psi^*)$ , then it is attained in (29) by  $(c^*, \psi^*)$  with  $X_0^{(c^*, \psi^*)} = x$ . Conversely, if there exists  $\nu^0 > 0$  and  $(c^0, \psi^0)$  such that the maximum is attained in

$$\sup_{(c, \psi)} \left\{ V_0^{(c, \psi)} + \nu^0 \left( x - X_0^{(c, \psi)} \right) \right\}$$

with  $X_0^{(c^0, \psi^0)} = x$ , then the maximum is attained in (29) by  $(c^0, \psi^0)$ .

Let  $\nu > 0$  be fixed and  $L$  be the map given by  $L(c, \psi) = V_0^{(c, \psi)} - \nu X_0^{(c, \psi)}$ . We now study the following optimization problem:

$$\sup_{(c, \psi)} L(c, \psi). \quad (30)$$

**Proposition 8.** *The optimal plan  $(c^0, \psi^0)$  which solves (30) satisfies the following (implicit) equations:*

$$U'(c_t^0) = \frac{Z_t^{\tilde{\mathbb{P}}}}{Z_t^0} \nu \quad dt \otimes d\mathbb{P} \text{ a.s.}, \quad \bar{U}'(\psi^0) = \frac{Z_T^{\tilde{\mathbb{P}}}}{Z_T^0} \nu, \quad d\mathbb{P} \text{ a.s.} \quad (31)$$

where  $Z^0$  is the Radon-Nikodym density of the probability measure  $\mathbb{Q}^0$  associated with the optimal plan  $(c^0, \psi^0)$ .

**Proof:** Consider the optimal plan  $(c^0, \psi^0)$  which solves (30) and another plan  $(c, \psi)$ . For  $\epsilon \in (0, 1)$ , one has:

$$L(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0)) \leq L(c^0, \psi^0)$$

Then

$$\frac{1}{\epsilon} \left( V_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - V_0^{(c^0, \psi^0)} \right) - \nu \frac{1}{\epsilon} \left( X_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - X_0^{(c^0, \psi^0)} \right) \leq 0 \quad (32)$$

From the definition, we obtain that

$$\partial_\epsilon X_0^{(c^0, \psi^0)} := \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (X_0^{(c^0 + \epsilon(c - c^0), \psi^0 + \epsilon(\psi - \psi^0))} - X_0^{(c^0, \psi^0)}) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_0^T (c_s - c_s^0) ds + (\psi - \psi^0) \right].$$

Taking the limit when  $\epsilon$  goes to 0 in (32), we obtain:

$$\partial_\epsilon V_0^{(c^0, \psi^0)} - \nu \partial_\epsilon X_0^{(c^0, \psi^0)} \leq 0 \quad (33)$$

where  $\partial_\epsilon V_0^{(c^0, \psi^0)}$  exists and is given explicitly by Theorem 4. From the explicit expression of  $\partial_\epsilon X_0^{(c^0, \psi^0)}$  we get:

$$\begin{aligned} \partial_\epsilon V_0^{(c^0, \psi^0)} - \nu \partial_\epsilon X_0^{(c^0, \psi^0)} &= \mathbb{E}^{\mathbb{P}} \left[ S_T^\delta Z_T^0 \bar{U}'(\psi^0) (\psi - \psi^0) + \int_0^T S_s^\delta Z_s^0 U'(c_s^0) (c_s - c_s^0) ds \right] \\ &\quad - \nu \mathbb{E}^{\mathbb{P}} \left[ Z_T^{\tilde{\mathbb{P}}} (\psi - \psi^0) + \int_0^T Z_s^{\tilde{\mathbb{P}}} (c_s - c_s^0) ds \right] \end{aligned}$$

It follows from equality (33) that

$$\mathbb{E}^{\mathbb{P}} \left[ \left( S_T^\delta Z_T^0 \bar{U}'(\psi^0) - \nu Z_T^{\tilde{\mathbb{P}}} \right) (\psi - \psi^0) + \int_0^T \left( S_s^\delta Z_s^0 U'(c_s^0) - \nu Z_s^{\tilde{\mathbb{P}}} \right) (c_s - c_s^0) ds \right] \leq 0$$

The rest of the proof is the same as in El Karoui et al. [9] (proof of Theorem 4.2, p. 677). In particular, for any  $\psi$ ,  $\mathbb{E}^{\mathbb{P}} \left[ \left( S_T^\delta Z_T^0 \bar{U}'(\psi^0) - \nu Z_T^{\tilde{\mathbb{P}}} \right) (\psi - \psi^0) \right] \leq 0$ , hence

$$S_T^\delta Z_T^0 \bar{U}'(\psi^0) - \nu Z_T^{\tilde{\mathbb{P}}} = 0 \quad \text{a.s.}$$

We find the optimal  $c$  with similar arguments. □

**Theorem 5.** Let  $I$  and  $\bar{I}$  be the inverse of the functions  $U'$  and  $\bar{U}'$ . The optimal plan  $(c^0, \psi^0)$  which solve the problem (29) is given by:

$$c_t^0 = I \left( \frac{\nu^0 Z_t^{\tilde{\mathbb{P}}}}{S_t^\delta Z_t^0} \right) dt \otimes d\mathbb{P} \text{ a.s.}, \quad \psi^0 = \bar{I} \left( \frac{\nu^0 Z_T^{\tilde{\mathbb{P}}}}{S_T^\delta Z_T^0} \right) \text{ a.s. .}$$

where  $\nu^0 > 0$  satisfies:

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_0^T I \left( \frac{\nu^0 Z_t^{\tilde{\mathbb{P}}}}{S_t^\delta Z_t^0} \right) dt + \bar{I} \left( \frac{\nu^0 Z_T^{\tilde{\mathbb{P}}}}{S_T^\delta Z_T^0} \right) \right] = x.$$

**Proof:** Define the map:  $f : (0, +\infty) \rightarrow (0, +\infty)$  as

$$f(\nu) = \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \int_0^T I \left( \frac{\nu Z_t^{\tilde{\mathbb{P}}}}{S_t^\delta Z_t^0} \right) dt + \bar{I} \left( \frac{\nu Z_T^{\tilde{\mathbb{P}}}}{S_T^\delta Z_T^0} \right) \right].$$

Then, using assumption A.3,  $f$  is monotone and satisfies  $\lim_{\nu \rightarrow 0} f(\nu) = +\infty$  and  $\lim_{\nu \rightarrow +\infty} f(\nu) = 0$ . For any initial wealth  $x \in (0, +\infty)$ , there exists a unique  $\nu^0$  such that  $f(\nu^0) = x$ .

Let  $(c, \psi) \in \mathcal{A}(x)$  and  $(V^{(c, \psi)}, M^{V, c}, v)$  (resp.  $(V^{(c^0, \psi^0)}, M^{V^0, c^0}, v^0)$ ) the solution of the BSDE (4) associated with  $(U(c^0), \bar{U}(\psi^0))$  (resp.  $(U(c), \bar{U}(\psi))$ ) then from the inequality (15) (see the comparison theorem), we get:

$$\begin{aligned} V_0^{(c, \psi)} - V_0^{(c^0, \psi^0)} &\leq \mathbb{E}^{\mathbb{Q}^0} \left[ S_T^\delta (\bar{U}(\psi) - \bar{U}(\psi^0)) + \int_0^T S_s^\delta (U(c_s) - U(c_s^0)) ds \right] \\ &\leq \mathbb{E}^{\mathbb{Q}^0} \left[ S_T^\delta \bar{U}'(\psi^0) (\psi - \psi^0) + \int_0^T S_s^\delta U'(c_s^0) (c_s - c_s^0) ds \right]. \end{aligned}$$

It follows that:

$$\begin{aligned} V_0^{(c, \psi)} - V_0^{(c^0, \psi^0)} &\leq \nu^0 \mathbb{E}^{\mathbb{Q}^0} \left( \frac{Z_T^{\tilde{\mathbb{P}}}}{Z_T^0} (\psi - \psi^0) + \int_0^T \frac{Z_s^{\tilde{\mathbb{P}}}}{Z_s^0} (c_s - c_s^0) ds \right) \\ &\leq \nu^0 \left( \mathbb{E}^{\tilde{\mathbb{P}}} \left( \psi + \int_0^T c_s ds \right) - \mathbb{E}^{\tilde{\mathbb{P}}} \left( \psi^0 + \int_0^T c_s^0 ds \right) \right) \end{aligned}$$

Since  $(c, \psi) \in \mathcal{A}(x)$ , then  $\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \psi + \int_0^T c_s ds \right] \leq x$ . Using that  $\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \psi^0 + \int_0^T c_s^0 ds \right] = x$ , we conclude:

$$V_0^{(c, \psi)} \leq V_0^{(c^0, \psi^0)}.$$

□

## 5 Logarithm Case

In this section, we assume that the process  $\delta$  is deterministic and that  $U(x) = \ln(x)$  and  $\bar{U}(x) = 0$  (hence  $I(x) = \frac{1}{x}$  for all  $x \in (0, +\infty)$ ). We introduce, as in Theorem 5, the optimal process  $c_t^* = I\left(\frac{\nu}{S_t^\delta} \frac{\tilde{Z}_t}{Z_t^*}\right) = \frac{S_t^\delta Z_t^*}{\nu \tilde{Z}_t}$ . Recall that the Radon-Nikodym density  $\tilde{Z}$ , and the Radon-Nikodym density of the optimal probability measure  $Z^*$  (given in (7)) satisfy

$$d\tilde{Z}_t = \tilde{Z}_{t-}(\theta_t dM_t^c + \sum_{i=1}^n (e^{-z^i} - 1) dN_t^i), \tilde{Z}_0 = 1 \quad (34)$$

$$dZ_t^* = Z_{t-}^*(-dM_t^{Y,c} + \sum_{i=1}^n (e^{-y^i} - 1) dN_t^i), Z_0^* = 1 \quad (35)$$

For any deterministic function  $\alpha$  such that  $\alpha(T) = 0$ ,  $V$  admits a decomposition as

$$V_t = \alpha(t) \ln(c_t^*) + \beta_t$$

where  $\beta$  is a process such that  $\beta_T = 0$ . Our goal is to characterize the process  $\beta$ . As in [2], we introduce  $J_t = \frac{1}{1+\alpha(t)}\beta_t$  in order to obtain a simple BSDE. Note that, even if  $Z^*$  is implicit (the coefficients depend on the solution  $c^*$ ), the BSDE for  $J$  is explicitly determined in terms of the given parameters  $\lambda^i$  and of the given probability  $\bar{\mathbb{P}}$ .

**Proposition 9.** *The value function  $V$  has the form*

$$V_t = \alpha(t) \ln(c_t^*) + (1 + \alpha(t))J_t$$

where

$$\alpha(t) = - \int_t^T e^{\int_t^s \delta(u) du} ds$$

and  $(J, \bar{M}^{J,c}, j)$  is the unique solution of the following Backward Stochastic Differential Equation, where  $k(t) = -\frac{\alpha(t)}{1+\alpha(t)}$ :

$$\begin{aligned} dJ_t &= \left( (1 + \delta(t))(1 + k(t))J_t - k(t)\delta(t) \right) dt + d\bar{M}_t^{J,c} + \frac{1}{2} d\langle \bar{M}^{J,c} \rangle_t + \frac{1}{2} k(t)(1 + k(t))\theta_t^2 d\langle M^c \rangle_t \\ &\quad + \sum_{i=1}^d j_t^i d\bar{N}_t^i + \sum_{i=1}^d \left( g(j_t^i) \bar{\lambda}_t^i + \left( k(t)(e^{-z^i} - 1) + e^{k(t)z^i} - 1 \right) \lambda_t^i \right) dt \end{aligned} \quad (36)$$

$$J_T = 0$$

Here, the processes  $\bar{M}^{J,c}$  and  $d\bar{N}_t^i = dH_t^i - \bar{\lambda}_t^i dt$  are  $\bar{\mathbb{P}}$ -martingales where  $d\bar{\mathbb{P}}|_{\mathcal{G}_t} = \bar{Z}_t d\mathbb{P}|_{\mathcal{G}_t}$  and  $\bar{\lambda}_t^i = e^{k(t)z^i} \lambda_t^i$  where

$$d\bar{Z}_t = -\bar{Z}_{t-} \left( k(t)\theta_t dM_t^c - \sum_{i=1}^d (e^{k(t)z^i} - 1) dN_t^i \right) \quad (37)$$

Note that, in a complete market, one obtains a forward backward system for the pair  $J$ -optimal wealth.

**Proof:** Using the fact that  $V$  satisfies the BSDE (4) and the assumed form of  $V$  in terms of  $(\alpha, \beta)$ , one obtains

$$dV_t = (\delta(t)V_t - \ln(c_t^*)) dt - d(\ln Z_t^*) = \alpha(t)d(\ln c_t^*) + (\ln c_t^*)\alpha'(t)dt + d\beta_t$$

Therefore

$$\begin{aligned} d\beta_t &= \delta(t)(V_t + \alpha(t))dt - (1 + \alpha'(t)) \ln(c_t^*)dt + \alpha(t)d \ln \tilde{Z}_t + (\alpha(t) + 1)d \ln Z_t^* \\ &= ((\delta(t)\alpha(t) - 1 - \alpha'(t)) \ln c_t^*(t) + \delta(t)\beta_t + \alpha(t)\delta(t))dt + \alpha(t)d \ln \tilde{Z}_t + (\alpha(t) + 1)d \ln Z_t^* \end{aligned}$$

We choose  $\alpha$  so that  $\delta(t)\alpha(t) = 1 + \alpha'(t)$ . It follows that

$$d\beta_t = \delta(t)(\beta_t + \alpha(t))dt + \alpha(t)d \ln \tilde{Z}_t + (\alpha(t) + 1)d \ln Z_t^*$$

After some obvious computations taking into account the form of  $\tilde{Z}$  and  $Z^*$ , one obtains

$$\begin{aligned} d\beta_t &= \delta(t)(\beta_t + \alpha(t))dt + \sum_{i=1}^d \left( (\alpha(t) + 1)(e^{-y_t^i} - 1) - \alpha(t)(e^{-z_t^i} - 1) \right) \lambda_t^i dt \\ &\quad + \alpha(t)\theta_t dM_t^c + (\alpha(t) + 1)dM_t^{V,c} - \frac{1}{2} \left( \alpha(t)\theta_t^2 d\langle M^c \rangle_t - (\alpha(t) + 1)d\langle M^{V,c} \rangle_t \right) \\ &\quad + \sum_{i=1}^d \left( (\alpha(t) + 1)y_t^i - \alpha(t)z_t^i \right) dH_t^i \end{aligned}$$

We now define  $J = \frac{1}{1+\alpha(t)}\beta_t$  and set  $k(t) = -\frac{\alpha(t)}{1+\alpha(t)}$ . Then

$$\begin{aligned} dJ_t &= \left( \frac{1 + \delta(t)}{1 + \alpha(t)} J_t - \delta(t)k(t) \right) dt + \sum_{i=1}^d \left( g(y_t^i) + k(t)g(z_t^i) \right) \lambda_t^i dt \\ &\quad + dM_t^{V,c} - k(t)\theta_t dM_t^c + \frac{1}{2} \left( k(t)\theta_t^2 d\langle M^c \rangle_t + d\langle M^{V,c} \rangle_t \right) + \sum_{i=1}^d \left( y_t^i + k(t)z_t^i \right) dN_t^i \end{aligned}$$

We introduce the martingale  $M^{J,c}$  as  $dM_t^{J,c} = dM_t^{V,c} - k(t)\theta_t dM_t^c$ . It is easy to check that

$$d\langle M^{J,c} \rangle_t = d\langle M^{V,c} \rangle_t - k^2(t)\theta_t^2 d\langle M^c \rangle_t - 2k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t$$

and we denote  $j_t^i = y_t^i + k(t)z_t^i$ . Using the fact that, due to the form of  $g$ , for any  $x, k, z, \lambda$ ,

$$xdN + \lambda(g(x - kz) + kg(z))dt = x(dN - (e^{kz} - 1)\lambda dt) + (g(x)e^{kz} + (e^{-z} - 1)k + e^{kz} - 1) \lambda dt$$

one obtains

$$\begin{aligned} dJ_t &= \left( (1 + \delta(t))(1 + k(t))J_t - \delta(t)k(t) \right) dt + \sum_{i=1}^d \left( g(j_t^i) e^{k(t)z_t^i} + k(t)(e^{-z_t^i} - 1) + e^{k(t)z_t^i} - 1 \right) \lambda_t^i dt \\ &\quad + dM_t^{J,c} + \frac{1}{2} d\langle M^{J,c} \rangle_t + k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t + \frac{1}{2} k(t)(k(t) + 1)\theta_t^2 d\langle M^c \rangle_t \\ &\quad + \sum_{i=1}^d j_t^i (dN_t^i - (e^{kz_t^i} - 1)\lambda_t^i dt) \end{aligned}$$

We define  $\bar{\mathbb{P}}$  as  $d\bar{\mathbb{P}} = \bar{Z}d\mathbb{P}$ , where

$$d\bar{Z}_t = -\bar{Z}_{t-} \left( k(t)\theta_t dM_t^c - \sum_{i=1}^d (e^{k(t)z_t^i} - 1) dN_t^i \right)$$

The processes  $\bar{M}^{J,c}$  and  $\bar{N}^i$  defined as

$$\begin{aligned} d\bar{M}_t^{J,c} &= dM_t^{J,c} + k(t)\theta_t d\langle M^{J,c}, M^c \rangle_t \\ d\bar{N}_t^i &= dN_t^i - (e^{k(t)z_t^i} - 1)\lambda_t^i dt = dH_t^i - \bar{\lambda}_t^i dt \end{aligned}$$

are  $\bar{\mathbb{P}}$  martingales. The result follows.

## 6 Appendix

In this Appendix, we give the proof of Theorem 4. Let

$$\begin{aligned} (V^\epsilon, M^{\epsilon,c}, v^\epsilon) &\text{ be the solution of (4) associated with } (U(c^1 + \epsilon(c^2 - c^1)), \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1))) \\ (V^1, M^{1,c}, v^1) &\text{ be the solution of (4) associated with } (U(c^1), \bar{U}(\psi^1)) \end{aligned}$$

and denote

$$\begin{aligned} \Delta_\epsilon V &:= \frac{V^\epsilon - V^1}{\epsilon}, & \Delta_\epsilon M^c &:= \frac{M^{\epsilon,c} - M^{1,c}}{\epsilon}, & \Delta_\epsilon v^i &:= \frac{v^{\epsilon,i} - v^{1,i}}{\epsilon}, \\ \Delta_\epsilon U &:= \frac{U(c^1 + \epsilon(c^2 - c^1)) - U(c^1)}{\epsilon}, & \Delta_\epsilon \bar{U}_T &:= \frac{\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1)}{\epsilon}. \end{aligned} \quad (38)$$

Then,  $(\Delta_\epsilon V, \Delta_\epsilon M^c, \Delta_\epsilon v)$  satisfies the following equation:

$$\begin{aligned} \Delta_\epsilon V_t - \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds &= \frac{1}{2\epsilon} (\langle M^{\epsilon,c} \rangle_t - \langle M^{1,c} \rangle_t) + \frac{1}{\epsilon} \sum_{i=1}^d \int_0^t (g(v_s^{\epsilon,i}) - g(v_s^{1,i})) \lambda_s^i ds + \Delta_\epsilon M_t^c \\ &\quad + \sum_{i=1}^d \int_0^t \Delta_\epsilon v_s^i dN_s^i, \end{aligned} \quad (39)$$

with final condition  $\Delta_\epsilon V_T = \Delta_\epsilon \bar{U}_T$ .

We start first to give the following a priori estimates:

**Lemma 1.** *Assume the same conditions as in Theorem 4. Then, there exists a constant  $C > 0$  such that:  $\forall i = 1, \dots, d, \forall p \in \mathbb{N}^*, \forall \epsilon > 0$ ,*

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} |\Delta_\epsilon V_t|^2 + \left\langle \Delta_\epsilon \widetilde{M}^c \right\rangle_T + \sum_{i=1}^d \int_0^T \frac{|\Delta_\epsilon v_s^i|^p}{p!} \widetilde{\lambda}_s^i ds \right] \leq C, \quad (40)$$

where  $\Delta_\epsilon \widetilde{M}^c$  is the  $\mathbb{Q}^{*,1}$  martingale part of the  $\mathbb{Q}^{*,1}$  semimartingale  $\Delta_\epsilon M^c$ , and  $\widetilde{\lambda}^i := \lambda^i e^{-v^{1,i}}$  is the intensity process of the process  $H^i$  under the probability measure  $\mathbb{Q}^{*,1}$ .

**Proof:** Let  $(c^1, \psi^1)$  and  $(c^2, \psi^2)$  be two comparable plans. We introduce the processes

$$\begin{aligned} K_t^\epsilon &:= \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \int_0^T (\delta_s V_s^\epsilon - U(c_s^1 + \epsilon(c_s^2 - c_s^1))) ds - \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) \right) \middle| \mathcal{G}_t \right] \\ K_t^1 &:= \mathbb{E}^{\mathbb{P}} \left[ \exp \left( \int_0^T (\delta_s V_s^1 - U(c_s^1)) ds - \bar{U}(\psi^1) \right) \middle| \mathcal{G}_t \right]. \end{aligned}$$

Obviously, for all  $t \in [0, T]$ , one has:

$$\begin{aligned} V_t^\epsilon &= -\ln(K_t^\epsilon) + \int_0^t (\delta_s V_s^\epsilon - U(c_s^1 + \epsilon(c_s^2 - c_s^1))) ds \\ V_t^1 &= -\ln(K_t^1) + \int_0^t (\delta_s V_s^1 - U(c_s^1)) ds, \end{aligned}$$

hence,

$$\frac{V_t^\epsilon - V_t^1}{\epsilon} = -\ln \left[ \left( \frac{K_t^\epsilon}{K_t^1} \right)^{\frac{1}{\epsilon}} \right] + \int_0^t \left[ \delta_s \frac{1}{\epsilon} (V_s^\epsilon - V_s^1) - \frac{1}{\epsilon} (U(c_s^1 + \epsilon(c_s^2 - c_s^1)) - U(c_s^1)) \right] ds. \quad (41)$$

For  $t \in [0, T]$ , we define  $\widetilde{K}_t^\epsilon := \frac{K_t^\epsilon}{K_t^1}$  and  $\bar{K}_t^\epsilon = (\widetilde{K}_t^\epsilon)^{1/\epsilon}$ . The processes  $\bar{K}^\epsilon$  and  $(\bar{K}^\epsilon)^{-1}$  are positive semi-martingales which belong to  $\mathbf{L}^p(\mathbb{P})$  since:

$$(K_t^\epsilon)^{-1/p} = \exp \left( p \Delta_\epsilon V_t + \int_0^t p (\Delta_\epsilon U(c_s^1) - \delta_s \Delta_\epsilon V_s) ds \right).$$

In the other hand, by using the dynamics of  $K^\epsilon$  and  $K^1$  under the probability measure  $\mathbb{P}$ :

$$\begin{aligned} dK_t^\epsilon &= K_{t-}^\epsilon \left( -dM_t^{\epsilon,c} + \sum_{i=1}^d (e^{-v_t^{\epsilon,i}} - 1) dN_t^i \right) \\ dK_t^1 &= K_{t-}^\epsilon \left( -dM_t^{1,c} + \sum_{i=1}^d (e^{-v_t^{1,i}} - 1) dN_t^i \right) \end{aligned}$$

and applying integration by parts formula, we get the dynamics of  $\widetilde{K}^\epsilon$  given by:

$$d\widetilde{K}_t^\epsilon = \widetilde{K}_{t-}^\epsilon \left[ -d(M_t^{\epsilon,c} - M_t^{1,c} - \langle M^{\epsilon,c} - M^{1,c}, M^{1,c} \rangle_t) + \sum_{i=1}^d (e^{-(v_t^{\epsilon,i} - v_t^{1,i})} - 1) [dH_t^i - e^{-v_t^{1,i}} \lambda_t^i dt] \right] \quad (42)$$

Clearly,  $\widetilde{K}^\epsilon$  is  $\mathbb{Q}^{*,1}$ -local martingale. Then, the processes  $\bar{K}^\epsilon$  and  $(\bar{K}^\epsilon)^{-1}$  are positive  $\mathbb{Q}^{*,1}$ -submartingales. We now split the study into two cases.

*First case:*  $(c^1, \psi^1) \leq (c^2, \psi^2)$ . Using the inequality (15), for all  $t \in [0, T]$ :

$$\begin{aligned} |\Delta_\epsilon V_t| &\leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \frac{S_T^\delta}{S_t^\delta} \bar{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T \frac{S_s^\delta}{S_t^\delta} U'(c_s^1)(c_s^2 - c_s^1) ds \middle| \mathcal{G}_t \right] \\ \sup_{0 \leq t \leq T} \left( \frac{1}{\bar{K}_t^\epsilon} \right)^p &\leq \exp \left( p(\|\delta\|_\infty + 1) \sup_{0 \leq t \leq T} |\Delta_\epsilon V_t| + \int_0^T pU'(c_s^1)(c_s^2 - c_s^1) ds \right). \end{aligned}$$

Setting  $\kappa = p(\|\delta\|_\infty + 1)$ , we obtain from (6),

$$\sup_{0 \leq t \leq T} \left( \frac{1}{\bar{K}_t^\epsilon} \right)^p \leq \exp \left( \kappa \sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{Q}^{*,1}} (\bar{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T U'(c_s^1)(c_s^2 - c_s^1) ds) \middle| \mathcal{G}_t \right) + \int_0^T pU'(c_s^1)(c_s^2 - c_s^1) ds.$$

Using Jensen inequality, we have:

$$\begin{aligned} \sup_{0 \leq t \leq T} \left( \frac{1}{\bar{K}_t^\epsilon} \right)^p &\leq \sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \exp \left( \bar{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T U'(c_s^1)(c_s^2 - c_s^1) ds \right) \middle| \mathcal{G}_t \right]^\kappa \\ &\quad \times \exp \left( \int_0^T pU'(c_s^1)(c_s^2 - c_s^1) ds \right). \end{aligned} \quad (43)$$

Thanks to the assumption  $(c^i, \psi^i) \in \mathcal{A}(x)$ , we conclude that  $\sup_{0 \leq t \leq T} \left( \frac{1}{\bar{K}_t^\epsilon} \right) \in \mathbf{L}^p(\mathbb{P})$ .

*Second case:*  $(c^2, \psi^2) \geq (c^1, \psi^1)$ . Then, using concavity property, we obtain for all  $t \in [0, T]$ :

$$\left| \frac{V_t^\epsilon - V_t^1}{\epsilon} \right| \leq |V_t^1 - V_t^2|, \quad |\Delta_\epsilon U(c_t^1)| \leq U'(c_t^2)(c_t^1 - c_t^2)$$

Now, using the same arguments as in the first step, we get that:

$$\begin{aligned} \sup_{0 \leq t \leq T} \left( \frac{1}{\bar{K}_t^\epsilon} \right)^p &\leq \sup_{0 \leq t \leq T} \mathbb{E}^{\mathbb{Q}^{*,2}} \left[ \exp \left( \bar{U}'(\psi^2)(\psi^1 - \psi^2) + \int_t^T U'(c_s^2)(c_s^1 - c_s^2) ds \right) \middle| \mathcal{G}_t \right]^\kappa \\ &\quad \times \exp \left( \int_0^T p U'(c_s^2)(c_s^1 - c_s^2) ds \right). \end{aligned}$$

We use the same arguments to prove  $\sup_{0 \leq t \leq T} |\bar{K}_t^\epsilon| \in \mathbf{L}^p(\mathbb{P})$ .

From the representation theorem, there exist two continuous martingales  $\bar{M}^{\epsilon,c}, \widetilde{M}^{\epsilon,c}$  and  $d$  predictable processes  $\bar{k}^\epsilon, \widetilde{k}^\epsilon$  such that:

$$\begin{aligned} \bar{K}_t^\epsilon &= K_0^\epsilon + \bar{M}_t^{\epsilon,c} + \sum_{i=1}^d \int_0^t \bar{k}_s^\epsilon dN_s^i \\ \frac{1}{\bar{K}_t^\epsilon} &= \frac{1}{K_0^\epsilon} + \widetilde{M}_t^{\epsilon,c} + \sum_{i=1}^d \int_0^t \widetilde{k}_s^\epsilon dN_s^i. \end{aligned}$$

These processes being positive  $\mathbb{Q}^{*,1}$ -submartingales, using (43) there exist two constants  $C_K$  and  $\widetilde{C}_K$  such that:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (\bar{k}_s^\epsilon)^2 \widetilde{\lambda}_s^i ds \right] &\leq \mathbb{E}^{\mathbb{Q}^{*,1}} [(\bar{K}_T^\epsilon)^2] \leq C_K \\ \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (\widetilde{k}_s^\epsilon)^2 \widetilde{\lambda}_s^i ds \right] &\leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \left( \frac{1}{\bar{K}_T^\epsilon} \right)^2 \right] \leq \widetilde{C}_K \end{aligned} \tag{44}$$

>From the uniqueness of the representation theorem and equation (41), we get, for  $1 \leq i \leq d$ , :

$$-\Delta_\epsilon v_t^i = \ln \left( 1 + \frac{\bar{k}_t^{\epsilon,i}}{\bar{K}_{t^-}^\epsilon} \right) \quad \text{and} \quad \Delta_\epsilon v_t^i = \ln \left( 1 + \widetilde{k}_t^i K_{t^-}^\epsilon \right).$$

Therefore

$$\exp(|\Delta_\epsilon v_t^i|) - 1 \leq \frac{|\bar{k}_t^{\epsilon,i}|}{\bar{K}_{t^-}^\epsilon} + |\widetilde{k}_t^{\epsilon,i}| \bar{K}_{t^-}^\epsilon$$

and

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (e^{|\Delta_\epsilon v_s^i|} - 1) \widetilde{\lambda}_s^i ds \right] &\leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T \frac{|\bar{k}_s^{\epsilon,i}|}{\bar{K}_{s^-}^\epsilon} \widetilde{\lambda}_s^i ds + \int_0^T |\widetilde{k}_s^{\epsilon,i}| \bar{K}_{s^-}^\epsilon \widetilde{\lambda}_s^i ds \right] \\ &\leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \left( \sup_{0 \leq t \leq T} \frac{1}{\bar{K}_t^\epsilon} \right) \int_0^T |\bar{k}_s^{\epsilon,i}| \widetilde{\lambda}_s^i ds + \left( \sup_{0 \leq t \leq T} \bar{K}_t^\epsilon \right) \int_0^T |\widetilde{k}_s^{\epsilon,i}| \widetilde{\lambda}_s^i ds \right] \end{aligned}$$

$$\leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \left( \int_0^T \tilde{\lambda}_s^i ds \right)^{1/2} \left( \sup_{0 \leq t \leq T} \frac{1}{\bar{K}_t^\epsilon} \right) \left( \int_0^T |\bar{k}_s^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right)^{1/2} + \left( \int_0^T \tilde{\lambda}_s^i ds \right)^{1/2} \left( \sup_{0 \leq t \leq T} \bar{K}_t^\epsilon \right) \left( \int_0^T |\tilde{k}_s^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right)^{1/2} \right].$$

Then, by Schwarz inequality one get:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (e^{|\Delta_\epsilon v_s^i|} - 1) \tilde{\lambda}_s^i ds \right] &\leq \left[ \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \sup_{0 \leq t \leq T} \frac{1}{(\bar{K}_t^\epsilon)^2} \int_0^T \tilde{\lambda}_s^i ds \right) \right]^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \int_0^T |\tilde{k}_t^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right)^{\frac{1}{2}} \\ &\quad + \left[ \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \sup_{0 \leq t \leq T} (\bar{K}_t^\epsilon)^2 \int_0^T \tilde{\lambda}_s^i ds \right) \right]^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \int_0^T |\tilde{k}_s^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right)^{\frac{1}{2}}. \end{aligned}$$

Using Cauchy-Schwartz inequalities, we get:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T \tilde{\lambda}_t^i dt \right]^2 &= \mathbb{E}^{\mathbb{P}} \left[ Z_T^{\mathbb{Q}^{*,1}} \int_0^T e^{-v_s^{1,i}} \lambda_s^i ds \right]^2 \leq \mathbb{E}^{\mathbb{P}} \left[ (Z_T^{\mathbb{Q}^{*,1}})^2 \int_0^T \lambda_s^i ds \int_0^T e^{-2v_s^{1,i}} \lambda_s^i ds \right] \\ &\leq c \left( \mathbb{E}^{\mathbb{P}} (Z_T^{\mathbb{Q}^{*,1}})^4 \right)^{\frac{1}{2}} \left[ \mathbb{E}^{\mathbb{P}} \left( \int_0^T e^{-2v_s^{1,i}} \lambda_s^i ds \right)^2 \right]^{\frac{1}{2}} \\ &\leq c_1 \left( \mathbb{E}^{\mathbb{P}} (Z_T^{\mathbb{Q}^{*,1}})^4 \right)^{\frac{1}{2}} \left[ \mathbb{E}^{\mathbb{P}} \left( \int_0^T e^{-4v_s^{1,i}} \lambda_s^i ds \right) \right]^{\frac{1}{2}} \end{aligned}$$

where we make use several times of Assumption A2-iii). Moreover, we can see that

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T e^{-4v_s^{1,i}} \lambda_s^i ds \right] = \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-v_s^{1,i}} - 1 + 1)^4 \lambda_s^i ds \right] \leq 16 \mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-v_s^{1,i}} - 1)^4 \lambda_s^i ds + \int_0^T \lambda_s^i ds \right].$$

Therefore, since the martingale  $-M^{1,c} + \int \sum_{i=1}^d (e^{-v_t^{1,i}} - 1) dN_t^i$  belongs to  $\mathbf{L}^p(\mathbb{P})$ , and by assumption A2-iii) again, we conclude that  $\mathbb{E}^{\mathbb{P}} \left[ \int_0^T (e^{-v_s^{1,i}} - 1)^p \lambda_s^i ds \right] < +\infty$  for any  $p \geq 1$ . Moreover since  $Z^{\mathbb{Q}^{*,1}} \in \mathbf{L}^p(\mathbb{P})$ , we get that  $\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T \tilde{\lambda}_s^i ds \right] < \infty$ .

Then, using again Cauchy inequality:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (e^{|\Delta_\epsilon v_s^i|} - 1) \tilde{\lambda}_s^i ds \right] &\leq C \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} \frac{1}{(\bar{K}_t^\epsilon)^4} \right] \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T |\bar{k}_s^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right] \right)^{\frac{1}{2}} \\ &\quad + C \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} (\bar{K}_t^\epsilon)^4 \right] \right)^{\frac{1}{2}} \left( \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T |\tilde{k}_s^{\epsilon,i}|^2 \tilde{\lambda}_s^i ds \right] \right)^{\frac{1}{2}} \end{aligned}$$

>From (43) and (44), we deduce that there exists a constant  $C_2 > 0$  such that:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (e^{|\Delta_\epsilon v_s^i|} - 1) \tilde{\lambda}_s^i ds \right] \leq C_2$$

and then using the expansion of the functional  $x \rightarrow e^x$  we get:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T |\Delta_\epsilon v_s^i|^p \tilde{\lambda}_s^i ds \right] \leq C_2 p!$$

In order to conclude the proof of the lemma, it remains to establish that there exists a constant  $C_1$  satisfying:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} [\langle \Delta_\epsilon \tilde{M}^c \rangle_T] \leq C_1.$$

*First case:*  $(c^2, \psi^2) \geq (c^1, \psi^1)$ , then  $U(c^1 + \epsilon(c^2 - c^1)) \geq U(c^1)$  and  $\bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) \geq \bar{U}(\psi^1)$ . From Proposition 3, it follows that:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} |V_t^\epsilon - V_t^1|^2 + \langle \tilde{M}^{\epsilon,c} - \tilde{M}^{1,c} \rangle_T + \sum_{i=1}^d \int_0^T (v_t^{\epsilon,i} - v_t^{1,i})^2 \tilde{\lambda}_t^i dt \right] \\ & \leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ [\bar{U}(\psi + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1)]^2 + \int_0^T [U(c_s^1 + \epsilon(c_s^2 - c_s^1)) - U(c_s^1)]^2 ds \right] \end{aligned}$$

Since

$$0 \leq U(c_t^1 + \epsilon(c_t^2 - c_t^1)) - U(c_t^1) \leq \epsilon U'(c_t^1)(c_t^2 - c_t^1)$$

and

$$0 \leq \bar{U}(\psi^1 + \epsilon(\psi^2 - \psi^1)) - \bar{U}(\psi^1) \leq \epsilon \bar{U}'(\psi^1)(\psi^2 - \psi^1).$$

we get:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} |\Delta_\epsilon V_t|^2 + \langle \Delta_\epsilon \tilde{M}^c \rangle_T + \int_0^T (\Delta_\epsilon v_s^i)^2 \tilde{\lambda}_s^i ds \right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ (\bar{U}'(\psi^1))^2 (\psi^2 - \psi^1)^2 + \int_0^T (U'(c_s^1))^2 (c_s^2 - c_s^1)^2 ds \right]$$

The process  $Z^{\mathbb{Q}^{*,1}}$  belongs to  $\mathbf{L}^p(\mathbb{P})$ ; moreover  $U'(\psi^1)(\psi^2 - \psi^1) \in \mathbf{L}^{\text{exp}}$  and  $U'(c_s^1)(c_s^2 - c_s^1) \in D_1^{\text{exp}}$  since  $(c^1, \psi^1), (c^2, \psi^2) \in \mathcal{A}(x)$ . It follows that there exists a constant  $C > 0$  such that:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{0 \leq t \leq T} |\Delta_\epsilon V_t|^2 + \langle \Delta_\epsilon \tilde{M}^c \rangle_T + \sum_{i=1}^d \int_0^T (\Delta_\epsilon v_s^i)^2 \tilde{\lambda}_s^i ds \right] \leq C$$

*Second case:*  $(c^2, \psi^2) \leq (c^1, \psi^1)$ . We first prove that for all  $t \in [0, T]$ ,  $\bar{K}_t^\epsilon \geq 1$ . Let us recall that:

$$\bar{K}_t^\epsilon = \exp \left( -\Delta_\epsilon V_t + \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds \right)$$

Define the process  $X$  as

$$X_t = -\Delta_\epsilon V_t + \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds, \quad 0 \leq t \leq T.$$

>From integration by part formula, we get:

$$S_t^\delta X_t = -\Delta_\epsilon V_0 - \int_0^t S_s^\delta d\Delta_\epsilon V_s - \int_0^t S_s^\delta \Delta_\epsilon U_s ds$$

Since the process  $\delta$  is positive and bounded, there exists a constant  $L > 0$  such that  $S^\delta < L < 1$ . It follows that:

$$S_t^\delta X_t \geq (-1 + L)\Delta_\epsilon V_0 - L\Delta_\epsilon V_t - \int_0^t S_s^\delta \Delta_\epsilon U_s ds$$

Note that, for all  $t \in [0, T]$ ,  $\Delta_\epsilon U_t \leq 0$  since  $(c^2, \psi^2) \leq (c^1, \psi^1)$  and using comparison theorem  $\Delta_\epsilon V_t \leq 0$ .

Therefore, for all  $t \in [0, T]$ ,  $X_t \geq 0$ . Finally,  $\bar{K}_t^\epsilon \geq 1$ .

In the second step of the proof, we give the dynamics of the process  $\bar{K}^\epsilon$  using Ito's calculus:

$$d\bar{K}_t^\epsilon = \bar{K}_{t-}^\epsilon \left( -d\Delta_\epsilon \widetilde{M}_t^c + \sum_{i=1}^d (e^{-\frac{(v_t^i - v_t^1)}{\epsilon}} - 1) d\widetilde{N}_t^i + dA_t \right)$$

where  $A$  is an increasing process. Since  $\bar{K}^\epsilon$  is a positive  $\mathbb{Q}^{*,1}$ -submartingale, we obtain from (43) and  $\bar{K}_t^\epsilon \geq 1$  :

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \langle \Delta_\epsilon \widetilde{M}^c \rangle_T \right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \int_0^T (\bar{K}_t^\epsilon)^2 d\langle \Delta_\epsilon \widetilde{M}^c \rangle_t \right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ (\bar{K}_T^\epsilon)^2 \right] \leq C_K$$

then we conclude:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \langle \Delta_\epsilon \widetilde{M}^c \rangle_T \right] \leq C_K.$$

Finally, by using concavity property we have shown that:  $|\Delta_\epsilon V_t| \leq |V_t^2 - V_t^1|$ , for all  $t \in [0, T]$ , then:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{t \in [0, T]} |\Delta_\epsilon V_t|^2 \right] \leq \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{t \in [0, T]} |V_t^2 - V_t^1|^2 \right] \leq 2 \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{t \in [0, T]} |V_t^1|^2 + \sup_{t \in [0, T]} |V_t^2|^2 \right]$$

Therefore, since the process  $V^1, V^2 \in D_0^{\text{exp}}$  and  $Z^{\mathbb{Q}^{*,1}}$  belongs to  $\mathbf{L}^{\mathbf{P}}$ , we get by using Cauchy Schwarz inequality that there exists a constant  $C$  such that:

$$\mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \sup_{t \in [0, T]} |\Delta_\epsilon V_t|^2 \right] \leq C.$$

□

**Proof of Theorem 4:** Let recall first the equality :

$$\frac{1}{2}(\langle M^{\epsilon,c} \rangle - \langle M^{1,c} \rangle) = \frac{1}{2}\langle M^{\epsilon,c} - M^{1,c} \rangle + \langle M^{\epsilon,c}, M^{1,c} \rangle - \langle M^{1,c} \rangle,$$

then the equation (39) may be written as:

$$\begin{aligned} \Delta_\epsilon V_t - \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds &= \frac{1}{\epsilon} \left( \frac{1}{2} \langle M^{\epsilon,c} - M^{1,c} \rangle_t + \langle M^{\epsilon,c}, M^{1,c} \rangle_t - \langle M^{1,c} \rangle_t \right) \\ &\quad + \sum_{i=1}^d \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{\epsilon,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta_\epsilon v_s^i \right] \lambda_s^i ds + \Delta_\epsilon M_t^c \\ &\quad + \sum_{i=1}^d \int_0^t \Delta_\epsilon v_s^i (dN_s^i - (e^{v_s^{1,i}} - 1) \lambda_s^i ds) \\ &= \frac{1}{2\epsilon} \langle M^{\epsilon,c} - M^{1,c} \rangle_t + \sum_{i=1}^d \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{\epsilon,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta_\epsilon v_s^i \right] \lambda_s^i ds \\ &\quad + (\Delta_\epsilon M_t^c + \langle \Delta_\epsilon M^c, M^{1,c} \rangle_t) + \sum_{i=1}^d \int_0^t \Delta_\epsilon v_s^i (dN_s^i - (e^{v_s^{1,i}} - 1) \lambda_s^i ds). \end{aligned}$$

By Girsanov theorem, the processes  $\Delta_\epsilon \widetilde{M}^c := \Delta_\epsilon M^c + \langle \Delta_\epsilon M^c, M^{1,c} \rangle$  and  $\widetilde{N}^i := N^i - \int_0^\cdot (e^{-v_s^{1,i}} - 1) \lambda_s^i ds$  are  $\mathbb{Q}^{1,*}$ -martingales. It follows that the process  $(\Delta_\epsilon V_t - \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds)_{t \geq 0}$  is a  $\mathbb{Q}^{1,*}$ -submartingale.

$$\begin{aligned} \Delta_\epsilon V_t - \int_0^t (\delta_s \Delta_\epsilon V_s - \Delta_\epsilon U_s) ds &= \frac{\epsilon}{2} \langle \Delta_\epsilon \widetilde{M}^c \rangle_t + \sum_{i=1}^d \int_0^t \left[ \frac{1}{\epsilon} (e^{-v_s^{\epsilon,i}} - e^{-v_s^{1,i}}) + e^{-v_s^{1,i}} \Delta_\epsilon v_s^i \right] \lambda_s^i ds + \Delta_\epsilon \widetilde{M}_t^c \\ &\quad + \sum_{i=1}^d \int_0^t \Delta_\epsilon v_s^i d\widetilde{N}_s^i, \quad \mathbb{Q}^{1,*}\text{-a.s.} \end{aligned} \tag{45}$$

Moreover, by using the uniform estimate (40), we get:

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \frac{\epsilon}{2} \langle \Delta_\epsilon \widetilde{M}^c \rangle_T \right) \leq C_p \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{2} = 0, \tag{46}$$

and using the expansion of the functional  $x \rightarrow e^x$ , we get:

$$\begin{aligned} 0 &\leq \lim_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \int_0^T \left[ \frac{e^{-v_s^{\epsilon,i}} - e^{-v_s^{1,i}}}{\epsilon} + e^{-v_s^{1,i}} \Delta_\epsilon v_s^i \right] \lambda_s^i ds \right) = \lim_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \int_0^T \sum_{p=2}^{+\infty} \frac{\epsilon^{p-1}}{p!} (\Delta_\epsilon v_s^i)^p \widetilde{\lambda}_s^i ds \right) \\ &\leq \sum_{p=2}^{\infty} \epsilon^{p-1} \mathbb{E}^{\mathbb{Q}^{1,*}} \left( \int_0^T \frac{|\Delta_\epsilon v_s^i|^p}{p!} \widetilde{\lambda}_s^i ds \right) \leq \sum_{p=2}^{\infty} C \epsilon^{p-1} = \frac{C\epsilon}{1-\epsilon}, \end{aligned}$$

thus, passing to the limit as  $\epsilon \rightarrow 0$ , we conclude that:

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{Q}^{*,1}} \left( \int_0^T \frac{1}{\epsilon} (e^{-v_s^{\epsilon,i}} - e^{-v_s^{1,i}}) + e^{-v^{1,i}} \Delta_\epsilon v_s^i \lambda_s^i ds \right) = 0, \quad 1 \leq i \leq d. \quad (47)$$

Moreover, the estimate (40) ensures that the sequence  $(\Delta_\epsilon V, \Delta_\epsilon \widetilde{M}^c, \Delta_\epsilon v)_{\epsilon > 0}$  is bounded in  $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$ . As a consequence, we can extract a subsequence  $(\Delta_{\epsilon_k} V, \Delta_{\epsilon_k} \widetilde{M}^c, \Delta_{\epsilon_k} v)_{k \in \mathbb{N}}$  which converges weakly in  $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$  and by Banach-Mazur Lemma, one may construct a sequence  $(\widehat{V}^\epsilon, \widehat{M}^{\epsilon,c}, \widehat{v}^\epsilon)_{\epsilon > 0}$  of convex combinations of elements in  $(\Delta_{\epsilon_k} V, \Delta_{\epsilon_k} \widetilde{M}^c, \Delta_{\epsilon_k} v)_{k \in \mathbb{N}}$  of the form

$$\widehat{V}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha^{\epsilon_j} \Delta_{\epsilon_j} V, \quad \widehat{M}^{\epsilon,c} := \sum_{j=1}^{N_\epsilon} \alpha^{\epsilon_j} \Delta_{\epsilon_j} \widetilde{M}^c, \quad \widehat{v}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha^{\epsilon_j} \Delta_{\epsilon_j} v$$

such that  $(\widehat{V}^\epsilon, \widehat{M}^{\epsilon,c}, \widehat{v}^\epsilon)_{\epsilon > 0}$  converges strongly in  $\mathcal{H}^2(\mathbb{P}) \times \mathcal{M}_0^2(\mathbb{P}) \times \mathcal{L}^2(\lambda, \mathbb{P})$  to  $(\partial_\epsilon V, \partial_\epsilon \widetilde{M}^c, \partial_\epsilon v)$ . Moreover, the triple  $(\widehat{V}^\epsilon, \widehat{M}^{\epsilon,c}, \widehat{v}^\epsilon)$  satisfies the BSDE's (45) associated with  $(\widehat{U}^\epsilon, \widehat{\bar{U}}^\epsilon)$  where

$$\widehat{U}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha^{\epsilon_j} \Delta_{\epsilon_j} U, \quad \widehat{\bar{U}}^\epsilon := \sum_{j=1}^{N_\epsilon} \alpha^{\epsilon_j} \Delta_{\epsilon_j} \bar{U}.$$

Therefore, passing to the limit in this equation, thanks to (46), (47) and the dominated convergence theorem, we get that  $(\partial_\epsilon V, \partial_\epsilon \widetilde{M}^c, \partial_\epsilon v)$  solves the BSDE's

$$\begin{cases} d\partial_\epsilon V_t &= (\delta_t \partial_\epsilon V_t - U'(c_t^1)(c_t^2 - c_t^1))dt + d\partial_\epsilon \widetilde{M}_t^c + \sum_{i=1}^d \partial_\epsilon v_t^i d\widetilde{N}_t^i, \quad \mathbb{Q}^{*,1}\text{-a.s.} \\ \partial_\epsilon V_T &= \bar{U}'(\psi^1)(\psi^2 - \psi^1). \end{cases}$$

Therefore  $(S_t^\delta \partial_\epsilon V_t + \int_0^t S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds)_{t \geq 0}$  is a  $\mathbb{Q}^{*,1}$  martingale which can be written as:

$$S_t^\delta \partial_\epsilon V_t + \int_0^t S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds = \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ S_T^\delta \partial_\epsilon V_T + \int_0^T S_s^\delta U'(c_s^1)(c_s^2 - c_s^1)ds \middle| \mathcal{G}_t \right].$$

Hence we get:

$$\partial_\epsilon V_t = \mathbb{E}^{\mathbb{Q}^{*,1}} \left[ \frac{S_T^\delta}{S_t^\delta} \bar{U}'(\psi^1)(\psi^2 - \psi^1) + \int_t^T \frac{S_s^\delta}{S_s^\delta} U'(c_s^1)(c_s^2 - c_s^1)ds \middle| \mathcal{G}_t \right].$$

□

## References

- [1] Anderson E., Hansen L. P., and Sargent T. : A quartet of semigroups for model specification, robustness, prices of risk, and model detection. *Journal of the European Economic Association* **1**, 68–123 (2003).

- [2] Bordigoni G. : Stochastic control and BSDEs in a robust utility maximization problem with an entropic penalty term. *PhD Thesis of Politecnico di Milano (2005)*.
- [3] Bordigoni, G., Matoussi, A., Schweizer, M. : A Stochastic control approach to a robust utility maximization problem. *F. E. Benth et al. (eds.), Stochastic Analysis and Applications. Proceedings of the Second Abel Symposium, Oslo, 2005, Springer, 125-151 (2007)* .
- [4] Barrieu, P. and El Karoui, N. : Pricing, Hedging and Optimally Designing Derivatives via Minimization of Risk Measures. in the book "*Indifference Pricing: Theory and Applications*" edited by RenÁl Carmona, Springer-Verlag (2008).
- [5] Dellacherie, C., Meyer, P.A. : Probabilités et Potentiel. Chap. I-IV. *Hermann, Paris (1975)*.
- [6] Duffie, D., Skiadas, C. : Continuous-time security pricing: a utility gradient approach. *J. Math. Econom.* **23**, 107-131 (1994).
- [7] Faidi, W., Matoussi, A., Mnif M. : Maximization of Recursive Utilities: A Dynamic Maximum Principle Approach. *Preprint of University Maine (2009)*.
- [8] El Karoui, N., Peng S., Quenez M.C. : Backward Stochastic Differential Equations in Finance, *Mathematical Finance* **7**, 1-71 (1997).
- [9] El Karoui, N. Peng, S. and Quenez, M.-C. : A dynamic maximum principle for the optimization of recursive utilities under constraints. *Annals of Applied Probability* **11**, 664-693 (2001).
- [10] El Karoui, N., Hamadène, S. : BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Processes and their Applications* **107**, 145-169 (2003).
- [11] Gundel, A. : Robust utility maximization for complete and incomplete market models. *Finance and Stochastics* **9**, 151-176 (2005).
- [12] Hansen, L.P., Sargent, T.J., Turmuhambetova, G.A., Williams, N.: *Robust control and model misspecification*, *Journal of Economic Theory* **128**, 45-90, 2006.
- [13] Kobylanski, M. : Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth, *Annals of Probability* **28**, 558-602 (2000).
- [14] Kramkov, D. and Schachermayer, W. : The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability* **9** , 904-950 (1999).
- [15] Kusuoka, S.: A remark on default risk models, *Adv. Math. Econ.*, 1,69-82,1999
- [16] Lazrak, A. and Quenez, M.-C. : A generalized stochastic differential utility. *Mathematics of Operations Research*, **28**, 154-180 (2003).
- [17] Maenhout, P. (2004) : Robust portfolio rules and asset pricing. *Review of Financial Studies* **17**, 951-983 (2004).

- [18] Mania, M. and Schweizer, M. : Dynamic exponential utility indifference valuation. *Annals of Applied Probability* **15**, 2113-2143 (2005).
- [19] Morlais, M.-A. : Utility maximization in a jump market model. *Stochastics* **81**, no. 1, 1-27 (2009).
- [20] Morlais, M.-A. : Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance Stoch.* **13**, no. 1, 121-150 (2009).
- [21] Mrad, M. : Utilités progressives. *Thèse de Doctorat de l'École Polytechnique* (2009).
- [22] Quenez, M.-C. : Optimal portfolio in a multiple-priors model. in: R. Dalang, M. Dozzi and F. Russo (eds.), *Seminar on Stochastic Analysis, Random Fields and Applications IV, Progress in Probability* **58**, Birkhäuser, p. 291-321 (2004).
- [23] Schied, A.: Optimal investments for risk- and ambiguity-averse preferences: a duality approach. *Finance and Stochastics* **11**, 107-129 (2007).
- [24] Schied, A. and Wu, C.-T. : Duality theory for optimal investments under model uncertainty. *Sat. Decisions* **23**, 199-217 (2005).
- [25] Schroder, M. and Skiadas, C. : Optimal consumption and portfolio selection with stochastic differential utility. *Journal of Economic Theory* **89**, 68-126 (1999).
- [26] Schroder, M, Skiadas, C. : Optimal lifetime consumption-portfolio strategies under trading constraints and generalized recursive preferences. *Stochastic Processes and their Applications* **108**, 155-202 (2003).
- [27] Schroder, M., Skiadas, C. : Lifetime consumption-portfolio choice under trading constraints, recursive preferences, and nontradeable income. *Stochastic Processes and their Applications* **115**, 1-30 (2005).
- [28] Skiadas C. : Robust control and recursive utility. *Finance and Stochastics* **7**, 475-489 (2003).

Monique JEANBLANC  
Laboratoire d'Analyse et Probabilités  
Université d'Evry Val d'Essonne  
Rue du Père Jarlan  
F-91025 Evry Cedex, FRANCE  
and  
Europlace Institute of Finance  
e-mail: monique.jeanblanc@univ-evry.fr

Anis MATOUSSI  
Laboratoire Manceau de Mathématiques  
Université du Maine  
Avenue Olivier Messiaen  
72085 Le Mans Cedex 9 -FRANCE  
email : anis.matoussi@univ-lemans.fr  
and  
CMAP, Ecole Polytechnique, Palaiseau

Armand NGOUPEYOU  
Laboratoire d'Analyse et Probabilités  
Université d'Evry Val d'Essonne  
Département de Mathématiques  
rue du Père Jarlan, F-91025 Evry Cedex,  
FRANCE  
e-mail: armand.ngoupeyou@univ-evry.fr