

# Random times with given survival probability and their $\mathbb{F}$ -martingale decomposition formula\*

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**Abstract** The problem concerns credit risk modelling. We are given a filtered probability space  $(\Omega, \mathbb{F}, \mathbb{P})$ , where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  is a filtration, an  $\mathbb{F}$ -adapted continuous increasing process  $\Lambda$  and a positive  $\mathbb{P}$ - $\mathbb{F}$  local martingale  $N$  such that  $Z_t := N_t e^{-\Lambda_t} \leq 1, t \geq 0$ , and we try to construct model of default time, i.e., a probability measure  $\mathbb{Q}$  and a random time  $\tau$  on an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$ , such that the survival probability satisfies  $\mathbb{Q}[\tau > t | \mathcal{F}_t] = Z_t, t \geq 0$ . In this paper, we show that there can exist various different models with a same survival probability and that the increasing family of martingales, combined with the stochastic differential equation, constitutes a natural way to construct these models. Our models will be equipped with an enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  with  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ . We show that all  $\mathbb{P}$ - $\mathbb{F}$  martingales remains  $\mathbb{G}$ -semimartingale and give an explicite semimartingale decomposition formula. At the end we show how this decomposition formula is intimately linked with stochastic differential equation.

## 1 Introduction

### 1.1 Bref description

Our research was motivated by credit risk modelling : for so-called intensity based models, the starting point is a given filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  and an  $\mathbb{F}$ -adapted increasing continuous process  $\Lambda$ . Then, one constructs a random time  $\tau$ , called the default time, such that  $(H_t - \Lambda_{t \wedge \tau}, t \geq 0)$  is a  $\mathbb{G}$ -martingale, where  $H_t = \mathbb{1}_{\tau \leq t}$ , and  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  with  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ . The "canonical" construction is the Cox model, where  $\tau := \inf\{t : \Lambda_t \geq \Theta\}$ , where  $\Theta$  is a r.v. independent of  $\mathcal{F}_\infty$ , with unit exponential law. In Cox model, the survival probability is given by  $\mathbb{Q}[\tau > t | \mathcal{F}_t] = e^{-\Lambda_t}$ .

Our aim here is to take into account a second parameter  $N$ , a positive local  $\mathbb{F}$ -martingale, and to consider models with a survival probability of the form  $N_t e^{-\Lambda_t}, t \geq 0$  (the multiplicative decomposition of the supermartingale  $\mathbb{Q}[\tau > t | \mathcal{F}_t]$ ), which is a more general form that a survival probability may have than that one appearing in the Cox model. We ask then two questions. The first one is whether there exist a probability  $\mathbb{Q}$  and a random time  $\tau$  which solve the

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equation :

$$\mathbb{Q}[\tau > t | \mathcal{F}_t] = N_t e^{-\Lambda_t}, \quad t \geq 0.$$

Note that, if the solution  $(\mathbb{Q}, \tau)$  exists, the process  $(H_t - \Lambda_{t \wedge \tau}, t \geq 0)$  is a local  $\mathbb{G}$ -martingale. This model existence problem has been studied and solved (under adequate conditions) in Gapeev et al. [2] and Jeanblanc and Song [4] where for each given survival probability  $N e^{-\Lambda}$ , one particular model has been constructed (see below, Theorem 3.1). In this paper we consider the set of all solution models. We will prove that there may exist various different models with a same conditional survival probability.

Our second question concerns the impact of the factor  $N$  on our models. We know that, for Cox model, the  $\mathbb{P}$ - $\mathbb{F}$  martingales remains  $\mathbb{Q}$ - $\mathbb{G}$  martingales, the well-known immersion property. This property, however, is not preserved in general. A study has been initiated in [4] where it is proved that, for their model, a  $\mathbb{P}$ - $\mathbb{F}$  martingale  $X$  is a  $\mathbb{Q}$ - $\mathbb{G}$  semimartingale whose drift is written out as a function of the  $\mathbb{P}$ - $\mathbb{F}$  predictable bracket  $\langle N, X \rangle$ , just as in the classical example of the progressive enlargement of filtration by a honest time (see Barlow, Jeulin and Yor [1, 5]). (Note, however, that the default time in [4] is not a  $\mathbb{F}$ -honest time.) In particular the model satisfies the (H') property of enlargement of filtration. In this paper we continue that study for more general models. We will prove again the (H') property and provide an explicite  $\mathbb{Q}$ - $\mathbb{G}$  semimartingale decomposition formula for the  $\mathbb{P}$ - $\mathbb{F}$  martingales. We notice that, in general, the semimartingale decomposition formula can depend on factors other than  $\langle N, X \rangle$ . We get a new type formulae.

The paper is organized as follows. We give below the precise definition of our problem. In Section 2 we introduce the notion of  $iM_Z$  and show the equivalent relation between the  $iM_Z$  and the solutions of our problem. In section 3 we explain how to construct  $iM_Z$ . The stochastic differential equation is the key point in these constructions. In section 4 we prove the (H') property and establish the semimartingale decomposition formula. Once again the results are obtained by an adequate use of the stochastic differential equation which defines the  $iM_Z$ . We show in addition that any solution of our problem possessing the same semimartingale decomposition formula must satisfy the same stochastic differential equation.

## 1.2 The problem and conventions

Now let us formulate precisely the problem. We begin with the notion of extension. Let  $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$  and  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q})$  be two filtered probability spaces, where  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  (resp.  $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$ ) is a filtration of sub- $\sigma$ -fields in  $\mathcal{A}$  (resp. in  $\hat{\mathcal{A}}$ ). Let  $\pi$  be a measurable map from  $\hat{\Omega}$  into  $\Omega$ . We say that  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi)$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$ , if  $\hat{\mathcal{F}}_t = \pi^{-1}(\mathcal{F}_t)$  for all  $0 \leq t \leq \infty$ , and  $\mathbb{P} = \mathbb{Q}|_{\hat{\mathcal{F}}_\infty} \circ \pi^{-1}$ . Note that if the extension holds, we shall simply identify  $\mathbb{F}, \mathbb{P}$  with  $\hat{\mathbb{F}}, \mathbb{Q}|_{\hat{\mathcal{F}}_\infty}$  and equally we shall consider a random variable  $Y$  on  $(\Omega, \mathcal{F}_\infty)$  as the random variable  $Y \circ \pi$  on  $(\hat{\Omega}, \hat{\mathcal{A}})$ . With this identification a process on  $\Omega$  is a  $\mathbb{P}$ - $\mathbb{F}$ -martingale if and only if it is a  $\mathbb{Q}$ - $\hat{\mathbb{F}}$ -martingale.

Notice that in this paper calling  $a$  a positive number means that  $a \geq 0$  and calling  $f(t), t \in \mathbb{R}$ , an increasing function means  $f(s) \leq f(t)$  for  $s \leq t$ .

Throughout this paper, we fix  $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$  a filtered probability space satisfying the usual conditions. Let  $\Lambda$  be an  $\mathbb{F}$ -adapted continuous increasing real process with  $\Lambda_0 = 0$  and  $N$  a càdlàg positive  $\mathbb{P}$ - $\mathbb{F}$  local martingale. We postulate that  $0 \leq N_t e^{-\Lambda_t} \leq 1$  for all  $0 \leq t < \infty$ .

The problem we consider is :

**Problem-A.** Construct an extension  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi)$  of  $(\Omega, \mathbb{F}, \mathbb{P})$  and a random time  $\tau$  on  $(\hat{\Omega}, \hat{\mathcal{A}})$

such that  $\mathbb{Q}[\tau > t | \mathcal{F}_t] = N_t e^{-\Lambda t}$  for all  $0 \leq t < \infty$ .

Note that in the above expression we identify  $\hat{\mathbb{F}}$  with  $\mathbb{F}$  and we consider the random variables on  $\Omega$  as random variable on  $\hat{\Omega}$ . Normally a solution of the problem should be indicated by an expression  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi, \tau)$ . But we shall also write a solution simply by  $(\hat{\Omega}, \mathbb{Q}, \tau)$  or even by  $\mathbb{Q}$  alone, if no confusion is possible.

Our solution to the problem- $\mathcal{A}$  will be constructed on the product space  $[0, \infty] \times \Omega$ . This space will exclusively be equipped with the product  $\sigma$ -field  $\mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ , with the map  $\pi : \pi(s, \omega) = \omega$  and the map  $\tau : \tau(s, \omega) = s$ , with the filtration  $\hat{\mathbb{F}} = \pi^{-1}(\mathbb{F})$  which will immediately be identified with  $\mathbb{F}$ . In such a setting, for  $([0, \infty] \times \Omega, \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty, \hat{\mathbb{F}}, \mathbb{Q}, \pi)$  to be an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$ , or for  $([0, \infty] \times \Omega, \mathbb{Q}, \tau)$  to be a solution of the problem- $\mathcal{A}$ , the only element we have to determine is the probability  $\mathbb{Q}$ . Therefore, in these cases, we shall simply say that  $\mathbb{Q}$  is an extension, or that  $\mathbb{Q}$  is a solution.

When  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi, \tau)$  is a solution of the problem- $\mathcal{A}$ , we equip systematically the space  $\hat{\Omega}$  with the progressively enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  made from  $\mathbb{F}$  with the random time  $\tau$ , i.e.  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ ,  $t \geq 0$ . This fix the framework for our discussion on the enlargement of filtration problem.

We shall notice that, for our solution of the problem- $\mathcal{A}$ , the filtration  $\hat{\mathbb{F}}$  on the product space  $[0, \infty] \times \Omega$  is not necessarily complete. In fact, we shall not really need this completeness, because the completeness will be only used to deal with limits calculus or stopping times. But, if the only limits are the limits of the  $\mathcal{F}_\infty$  measurable random variables or if the stopping times are  $\mathbb{F}$  stopping times, they can be worked on the initial space  $(\Omega, \mathbb{F}, \mathbb{P})$  which satisfies the usual condition.

In this paper, if a relation between two random variables is written without mention, it is understood to be an almost sure relation with respect to the underground probability. However, sometimes, it is important to distinguish the almost sure relation from the true one. It will be then specially mentioned.

## 2 Increasing family of positive and bounded martingales

We shall construct the solutions of the problem- $\mathcal{A}$  on the product space  $[0, \infty] \times \Omega$ . Any probability  $\mathbb{Q}$  on this product space can be disintegrated into the probability  $\mathbb{P}$  and the conditional law  $\mathbb{Q}[\tau \in du | \mathcal{F}_\infty]$ , provided  $\mathbb{Q}$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$ . A natural idea is to regard  $\mathbb{Q}[\tau \in du | \mathcal{F}_\infty]$  as the terminal term of the probability measure valued martingale  $\mathfrak{M}_t = \mathbb{Q}[\tau \in du | \mathcal{F}_t]$ ,  $0 \leq t \leq \infty$  and to hope that, when  $\mathbb{Q}$  is a solution of the problem- $\mathcal{A}$ ,  $\mathfrak{M}$  can be computed from the survival probability  $Z = N e^{-\Lambda}$ . On the one hand there is the relation, for  $u \geq t$ ,

$$\mathbb{Q}[\tau > u | \mathcal{F}_t] = \mathbb{Q}[Z_u | \mathcal{F}_t] = \mathbb{Q}[Z_\infty | \mathcal{F}_t] + \mathbb{Q}\left[\int_u^\infty Z_s d\Lambda_s | \mathcal{F}_t\right]$$

This yields effectively

$$\mathbb{Q}[\tau \in du, t \leq \tau \leq \infty | \mathcal{F}_t] = \mathbb{Q}[Z_u d\Lambda_u \mathbb{1}_{\{t \leq u < \infty\}} + Z_\infty \delta_\infty(du) | \mathcal{F}_t]$$

On the other hand, one realizes rapidly that no one-to-one map exists between  $Z$  and  $\mathbb{Q}[\tau \in du, 0 \leq \tau \leq t | \mathcal{F}_t]$ . (For a given  $Z$ , there may exist uncountable many solutions of the problem.)

Our approach consists in two steps. First of all we find (in this section) a necessary and sufficient condition on the space  $(\Omega, \mathbb{F}, \mathbb{P})$ , which guarantees the existence of a solution of the problem- $\mathcal{A}$ . Then, this condition being handleable, methods are discovered (in next section) to produce variable solutions of the problem.

## 2.1 Family $iM$ associated with a pair $(\mathbb{Q}, \tau)$

Here, we consider pairs  $(\mathbb{Q}, \tau)$  where the probability  $\mathbb{Q}$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$  and  $\tau$  is a random variable defined on the same space as  $\mathbb{Q}$ . We ask how to read such a pair from the space  $(\Omega, \mathbb{F}, \mathbb{P})$ . The answer is synthesized in the following notion.

**An increasing family of positive martingales bounded by 1** (in short  $iM$ ) is a family of processes  $(M^u : 0 < u < \infty)$  satisfying the following conditions :

1. Each  $M^u = (M_t^u)_{u \leq t \leq \infty}$  is a càdlàg  $\mathbb{P}$ - $\mathbb{F}$  martingale on  $[u, \infty]$ . (In particular  $M_\infty^u = \lim_{t \rightarrow \infty} M_t^u$ .)
2. For any  $u$ , the martingale  $M^u$  is everywhere positive and bounded by 1
3. For each fixed  $0 < t \leq \infty$ ,  $u \in (0, t) \rightarrow M_t^u$  is everywhere a right continuous increasing map (in particular, for  $0 < u < t < \infty$ ,  $M_t^u = \lim_{\epsilon \rightarrow 0} M_{t+\epsilon}^u \leq \lim_{\epsilon \rightarrow 0} M_{t+\epsilon}^t = M_t^t$   $\mathbb{P}$ -almost surely).

The theorem below gives the link between  $iM$  and  $(\mathbb{Q}, \tau)$ . To read it we recall that we identify the elements on  $(\Omega, \mathbb{F}, \mathbb{P})$  with elements on its extension.

**Theorem 2.1** *1. If  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi)$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$ , if  $\tau$  is a random time on  $(\hat{\Omega}, \hat{\mathcal{A}})$ , then there exists a unique family  $iM = (M^u : 0 < u < \infty)$  such that, for  $0 < u < \infty, u \leq t \leq \infty$ ,*

$$M_t^u = \mathbb{Q}[\tau \leq u | \mathcal{F}_t].$$

*We shall say that  $iM$  is associated with  $(\mathbb{Q}, \tau)$ .*

2. *Let  $(M^u : 0 < u < \infty)$  be an  $iM$ . Then, there is a unique probability measure  $\mathbb{Q}$  on  $([0, \infty] \times \Omega, \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty)$  which extends  $(\Omega, \mathbb{F}, \mathbb{P})$  and satisfies  $\mathbb{Q}[\tau \leq u | \mathcal{F}_t] = M_t^u$  for  $0 < u < \infty, u \leq t \leq \infty$ . We shall say that  $\mathbb{Q}$  is associated with  $iM$ .*

**Proof :** Consider the first assertion. For each  $0 < u < \infty$ , let  $(G_t^u, t \geq 0)$  be a càdlàg version of the  $\mathbb{P}$ - $\mathbb{F}$  martingale  $\mathbb{Q}[\tau \leq u | \mathcal{F}_t]$ . We insist that the random variable  $G_t^u$  is chosen really  $\mathcal{F}_t$ -measurable (not merely "almost  $\mathcal{F}_t$ -measurable"). It is clear that, for  $u < v$ ,  $0 \leq t \leq \infty$ , one has  $0 \leq G_t^u \leq G_t^v \leq 1$   $\mathbb{P}$ -almost surely (recalling that  $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}$ ). For  $0 < u < \infty, 0 \leq t \leq \infty$  set

$$M_t^u = \inf\{(G_t^w \wedge 1)^+ : w \in \mathbb{Q}_+, u < w\}.$$

We have immediately the following properties :

- For  $0 < u < \infty$ , the process  $M^u = (M_t^u)_{0 \leq t \leq \infty}$  is  $\mathbb{F}$ -optional.
- For  $0 \leq t \leq \infty$ ,  $u \in (0, \infty) \rightarrow M_t^u$  is everywhere increasing and right continuous.
- For  $0 \leq t \leq \infty, 0 < u < v < \infty$ ,  $0 \leq M_t^u \leq M_t^v \leq 1$  everywhere.

Let  $0 < u < \infty$ , and  $T$  be an  $\mathbb{F}$ -stopping time. We can write

$$\mathbb{Q}[\tau \leq u] = \mathbb{Q}[G_T^u] \leq \mathbb{Q}[M_T^u] = \inf_{v \in \mathbb{Q}_+, u < v} \mathbb{Q}[G_T^v] = \inf_{v \in \mathbb{Q}_+, u < v} \mathbb{Q}[\tau \leq v] = \mathbb{Q}[\tau \leq u]$$

This shows that  $M_T^u = G_T^u$ ,  $\mathbb{P}$ -almost surely. Consequently,  $G^u$  and  $M^u$  are  $\mathbb{P}$ -indistinguishable and, therefore,  $M^u$  itself is a càdlàg  $\mathbb{P}$ - $\mathbb{F}$  uniformly integrable martingale on  $[0, \infty)$ . We have

$$M_\infty^u = G_\infty^u = \lim_{t \rightarrow \infty} G_t^u = \lim_{t \rightarrow \infty} M_t^u$$

$\mathbb{P}$ -almost surely. The family of processes  $(M^u : 0 < u < \infty)$  defines therefore a  $iM$  satisfying the first assertion.

Consider the second assertion. Let

$$M_\infty^0 = \lim_{u \rightarrow 0} M_\infty^u, \quad M_\infty^\infty = \lim_{u \rightarrow \infty} M_\infty^u$$

The map  $u \in (0, \infty) \rightarrow M_\infty^u - M_\infty^0$  being increasing and right continuous, we denote by  $d_u M_\infty^u$  the associated random measure on  $(0, \infty)$ . Define a probability measure on  $([0, \infty] \times \Omega, \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty)$  by

$$\mathbb{Q}[F] := \mathbb{P}\left[\int_{[0, \infty]} F(u, \cdot) (M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty) \delta_\infty(du))\right]$$

where  $F(t, \omega) \in \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ ,  $F(t, \omega) \geq 0$ . We compute. Firstly for  $A \in \mathcal{F}_\infty$  :

$$\mathbb{Q}[A] = \mathbb{Q}[A \cap \{0 \leq \tau \leq \infty\}] = \mathbb{P}[\mathbb{I}_A \int_{[0, \infty]} (M_\infty^0 \delta_0(du) + d_u M_\infty^u + (1 - M_\infty^\infty) \delta_\infty(du))] = \mathbb{P}[A]$$

Secondly let  $0 < u < \infty, u \leq t \leq \infty, A \in \mathcal{F}_t$ . We have

$$\mathbb{Q}[A \cap \{\tau \leq u\}] = \mathbb{P}[\mathbb{I}_A \int_{[0, u]} (M_\infty^0 \delta_0(ds) + d_s M_\infty^s)] = \mathbb{P}[\mathbb{I}_A M_\infty^u] = \mathbb{P}[\mathbb{I}_A M_t^u] = \mathbb{Q}[\mathbb{I}_A M_t^u]$$

These results prove that  $\mathbb{Q}$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$  and  $\mathbb{Q}[\tau \leq u | \mathcal{F}_t] = M_t^u$  for  $0 < u < \infty, u \leq t \leq \infty$ . The uniqueness of  $\mathbb{Q}$  is also immediate. ■

**Remark :** We can ask whether, for a given  $iM$ , its associated probability  $\mathbb{Q}$  can be constructed on the measurable space  $(\Omega, \mathcal{F}_\infty)$  instead of the space  $([0, \infty] \times \Omega, \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty)$ . This is equivalent to ask if there exists a random variable  $L$  on  $(\Omega, \mathcal{F}_\infty)$  such that  $M_t^u = \mathbb{Q}[L \leq u | \mathcal{F}_t]$ . But if it is the case, we must have  $M_\infty^u = \mathbb{1}_{\{L \leq u\}}$ . This is a fairly restrictive condition on the family  $iM$ . We understand therefore, to have a perfect correspondance between the families  $iM$  and the probability-time pair  $(\mathbb{Q}, \tau)$ , we have to accept to work on enlarged spaces like the space  $[0, \infty] \times \Omega$ .

## 2.2 Families $iM_Z$ and solutions to the problem- $\mathcal{A}$

Let  $Z_t := N_t e^{-\Lambda t}, 0 \leq t < \infty$ . We introduce the following definition :

**An increasing family of positive martingales bounded precisely by  $1 - Z$**  (abbreviation  $iM_Z$ ) is an increasing family of positive martingales  $(M^u : 0 < u < \infty)$  satisfying the initial value condition :  $M_u^u = 1 - Z_u$   $\mathbb{P}$ -almost surely for  $0 < u < \infty$ .

Notice that, for  $u < t < \infty$ ,  $M_t^u \leq M_t^t = 1 - Z_t$   $\mathbb{P}$ -almost surely. The theorem below establishes that a solution of the problem- $\mathcal{A}$  exists if and only if an  $iM_Z$  exists. It is an immediate consequence of the Theorem 2.1.

**Theorem 2.2** 1. If  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi, \tau)$  is a solution of the problem- $\mathcal{A}$ , the family  $iM$  associated with  $(\mathbb{Q}, \tau)$  is an  $iM_Z$ .

2. Let  $(M^u : 0 < u < \infty)$  be an  $iM_Z$  and  $\mathbb{Q}$  be the associated probability measure on  $([0, \infty] \times \Omega, \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty)$ . Then,  $\mathbb{Q}$  gives a solution of the problem- $\mathcal{A}$ . ■

### 3 Constructions of $iM_Z$

Now, to solve the problem- $\mathcal{A}$ , it is enough to find  $iM_Z$ . That is what we solve in this section. Let us begin with some basic properties of the survival probability  $Z_t = N_t e^{-\Lambda_t}$ ,  $0 \leq t < \infty$ . First of all  $Z_\infty = \lim_{t \rightarrow \infty} Z_t$  exists always. The process  $Z$  is a  $\mathbb{P}$ - $\mathbb{F}$  supermartingale whose canonical decomposition is :

$$dZ_s = e^{-\Lambda_s} dN_s - Z_s d\Lambda_s$$

where the random variable  $\int_0^\infty Z_s d\Lambda_s$  is integrable and the martingale  $e^{-\Lambda} \cdot N$  is in BMO. When  $Z$  is a potential, i.e., when  $Z_\infty = 0$ , we have the relation

$$Z_t = \mathbb{P}\left[\int_t^\infty Z_s d\Lambda_s \mid \mathcal{F}_t\right]$$

Since  $N_\infty$  exists and finite,  $Z$  will be a potential whenever  $\Lambda_\infty = \infty$ .

#### 3.1 A basic $iM_Z$

We introduce the hypothesis :

**Hypothesis Hy**( $N, \Lambda$ )

1. For all  $0 < t < \infty$ ,  $0 \leq Z_t < 1$ ,  $0 \leq Z_{t-} < 1$  (strictly smaller than 1).
2.  $\mathbb{P}$ -almost surely,  $\int_a^b \frac{d\Lambda_t}{1-Z_t} < \infty$  for any  $0 < a < b < \infty$ .

The following theorem is borrowed from [4].

**Theorem 3.1** *Assume **Hy**( $N, \Lambda$ ). The family*

$$B_t^u = (1 - Z_t) \exp\left(-\int_u^t \frac{Z_s d\Lambda_s}{1 - Z_s}\right), \quad 0 < u < \infty, u \leq t \leq \infty,$$

*defines an  $iM_Z$ .*

**Proof :** It is enough to show that  $B^u$  is a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. But this is an immediate consequence of Itô's formula. ■

#### 3.2 More $iM_Z$ when $1 - Z > 0$

We assume in this subsection the hypothesis **Hy**( $N, \Lambda$ ) and  $Z_0 = 1$ , as well as the following one :

**Hypothesis Hy(C)** : All  $\mathbb{P}$ - $\mathbb{F}$  martingales are continuous.

Note that under this hypothesis, for  $0 < u < \infty$ , the stochastic integrals  $\int_u^t \frac{e^{-\Lambda_s}}{1-Z_s} dN_s$ ,  $u \leq t < \infty$ , exist as a continuous  $\mathbb{P}$ - $\mathbb{F}$  local martingale.

##### 3.2.1 The generating equation

We need the following lemma.

**Lemma 3.1** *Let  $0 < u < \infty$ . Let  $M$  be a  $\mathbb{P}$ - $\mathbb{F}$  local martingale on  $[u, \infty)$  such that  $M_u \leq 1 - Z_u$ . Then,  $M_t \leq (1 - Z_t)$  on  $t \in [u, \infty)$  if and only if the local time at zero  $L^0(M - (1 - Z))$  of  $M - (1 - Z)$  on  $[u, \infty)$  is identically null. Here the local time is taken right continuous in  $a \rightarrow L_t^a(M - (1 - Z))$ .*

**Proof :** a) Sufficient condition. If  $L^0(M - (1 - Z)) \equiv 0$ , for any  $\mathcal{F}$  stopping time  $T \geq u$  such that everything in the following calculus is integrable, we compute with Tanaka's formula :

$$\begin{aligned} \mathbb{P}[(M_T - (1 - Z_T))^+] &= \mathbb{P}[(M_u - (1 - Z_u))^+] \\ &+ \mathbb{P}\left[\int_u^T \mathbb{I}_{\{M_s - (1 - Z_s) > 0\}} d(M_s - (1 - Z_s))\right] + \frac{1}{2} \mathbb{P}[L_T^0(M - (1 - Z))] \leq 0 \end{aligned}$$

because  $M - (1 - Z)$  is a  $\mathbb{P}$ - $\mathbb{F}$  supermartingale. This proves the suffisance of the condition.

Necessary condition. If  $(M_t - (1 - Z_t)) \leq 0$  on  $t \in [u, \infty)$ , Tanaka's formula implies immediately  $L^{0+}(M - (1 - Z)) \equiv 0$ . ■

Recall that an  $iM_Z$  must satisfy the conditions :  $0 \leq M^u \leq 1 - Z$ ,  $M^u \leq M^v$  if  $u < v$ , and  $M_u^u = 1 - Z_u$ . We notice that the stochastic differential equation is a natural tool to deal with these conditions.

**Equation** (‡) Let  $Y$  be a  $\mathbb{P}$ - $\mathbb{F}$  local martingale and  $f$  be a bounded Lipschitz function with  $f(0) = 0$ . For any  $0 < u < \infty$  we consider the equation

$$(‡_u) \begin{cases} dM_t = M_t \left( -\frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(M_t - (1 - Z_t)) dY_t \right), & u \leq t < \infty \\ M_u = x \end{cases}$$

where  $x$  can be any  $\mathcal{F}_u$ -measurable random variable.

**Remark 3.1** Our method remains valid if  $f(M_t - (1 - Z_t))$  is replaced by some more general function  $f(M_t - (1 - Z_t), M_t, t, \omega)$  such that

$$|f(M_t - (1 - Z_t), M_t, t, \omega)| \leq K |M_t - (1 - Z_t)|$$

or if some extra term  $g(M_t - (1 - Z_t), M_t, t, \omega)$  is added. But we shall not involve this generality in this paper. Instead we prefer to well explain the relation between the dynamic defined by the equation (‡) and the decomposition formula in enlargement of filtration problem. A simpler function  $f(M_t^u - (1 - Z_t))$  will exhibit this relation better. ■

**Theorem 3.2** *Let  $0 < u < \infty$ . The equation (‡<sub>u</sub>) has a unique solution  $M$  for each given initial value  $x$ . If  $M_u \leq 1 - Z_u$ ,  $M$  is bounded by  $(1 - Z)$  on  $[u, \infty)$ .*

**Proof** For the existence and the uniqueness of the solution of the equation (‡<sub>u</sub>), we refer to Protter [6]. To see that the solution is bounded by  $(1 - Z)$  on  $[u, \infty)$ , we introduce the process  $\Delta = M - (1 - Z)$ . According to Lemma 3.1, we prove that the local time  $L^0(\Delta)$  is identically null. To do this, we calculate  $\langle \Delta \rangle$  using the fact that, from Itô's calculus

$$d\Delta_t = -\Delta_t \frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + M_t f(\Delta_t) dY_t - Z_t d\Lambda_t$$

Therefore,

$$\begin{aligned} d\langle\Delta\rangle_t &= \Delta_t^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + M_t^2 f(\Delta_t)^2 d\langle Y\rangle_t - 2\Delta_t \frac{e^{-\Lambda t}}{1-Z_t} M_t f(\Delta_t) d\langle N, Y\rangle_t \\ &\leq 2\Delta_t^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + 2M_t^2 f(\Delta_t)^2 d\langle Y\rangle_t \leq 2\Delta_t^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + 2M_t^2 K^2 \Delta_t^2 d\langle Y\rangle_t \end{aligned} \quad (1)$$

From this we can write

$$\int_0^t \mathbb{I}_{\{0 < \Delta_s < \epsilon\}} \frac{1}{\Delta_s^2} d\langle\Delta\rangle_s < \infty, \quad 0 < \epsilon, 0 < t < \infty.$$

According to Revuz-Yor [7],  $L^0(\Delta) \equiv 0$ . The theorem is proved. ■

**Theorem 3.3** *Let  $0 < u < \infty$ . Let  $M, L$  be two solutions of the equation  $(\mathfrak{h}_u)$  with initial conditions  $M_u = x < y = L_u$ . Then,  $M_t \leq L_t$  for all  $u \leq t \leq \infty$ .*

**Proof** Let  $L^0(M-L)$  denote the local time at zero of  $M-L$ . We have, denoting  $\Delta = M-(1-Z)$ , and  $\Delta^L = L-(1-Z)$

$$d(M_t - L_t) = (M_t - L_t) \left( -\frac{e^{-\Lambda t}}{1-Z_t} dN_t \right) + (M_t f(\Delta_t) - L_t f(\Delta_t^L)) dY_t$$

So, using the same computation as in (1), we obtain

$$d\langle M - L\rangle_t \leq 2(M_t - L_t)^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + 2(M_t f(\Delta_t) - L_t f(\Delta_t^L))^2 d\langle Y\rangle_t$$

Then, using the fact that

$$M_t f(\Delta_t) - L_t f(\Delta_t^L) = (M_t - L_t) f(\Delta_t) + L_t (f(\Delta_t) - f(\Delta_t^L))$$

we obtain

$$\begin{aligned} d\langle M - L\rangle_t &\leq 2(M_t - L_t)^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + 4(M_t - L_t)^2 f^2(\Delta_t) d\langle Y\rangle_t + 4L_t^2 (f(\Delta_t) - f(\Delta_t^L))^2 d\langle Y\rangle_t \\ &\leq 2(M_t - L_t)^2 \left( \frac{e^{-\Lambda t}}{1-Z_t} \right)^2 d\langle N\rangle_t + 4(M_t - L_t)^2 f^2(\Delta_t) d\langle Y\rangle_t + 4L_t^2 K^2 (M_t - L_t)^2 d\langle Y\rangle_t \end{aligned}$$

It yields that

$$\int_0^t \mathbb{I}_{\{0 < M_s - L_s < \epsilon\}} \frac{1}{(M_s - L_s)^2} d\langle M - L\rangle_s < \infty, \quad 0 < \epsilon < \infty, 0 < t < \infty$$

$L^0(M-L)$  therefore is identically null. The asserted property is proved. ■

**Corollary 3.1** *Let  $0 < u < \infty$ . Let  $M$  be a solution of the equation  $(\mathfrak{h}_u)$  with initial conditions  $M_u^u = x \geq 0$ . Then,  $M_t^u \geq 0$  for all  $u \leq t \leq \infty$ .*

**Proof** We notice that 0 is the solution of equation  $(\mathfrak{h}_u)$  with initial condition 0. ■

As a consequence of Theorem 3.2 and Corollary 3.1, the local martingale  $M$  is bounded between 0 and 1. It is therefore a true  $\mathbb{P}$ - $\mathbb{F}$  uniformly integrable martingale.

**Corollary 3.2** *For  $0 < u < \infty$ , let  $L^u$  denote the solution of the equation  $(\mathfrak{h}_u)$  with the initial condition  $L_u^u = 1 - Z_u$ . Then, for  $u < v \leq b < \infty$ ,  $L_b^u \leq L_b^v$ .*

**Proof** This comparison relation is because that  $L^u, L^v$  satisfy the same equation  $\mathfrak{h}_v$  on  $[v, \infty)$  and  $L_v^u \leq (1 - Z_v) = L_v^v$ . ■

### 3.2.2 The $iM_Z$ associated with the Equation (4)

For  $0 < u < \infty$ , let  $L^u$  be as in the preceding corollary. We know already that  $L^u$  is a bounded  $\mathbb{P}$ - $\mathbb{F}$  martingales,  $L_b^u \leq L_b^v$  if  $u < v$ , and  $0 \leq L_b^u \leq (1 - Z_b)$  for  $b \geq u$ . Set  $L_\infty^u = \lim_{b \rightarrow \infty} L_b^u$  and

$$\begin{aligned} M_u^u &= (1 - Z_u) \\ M_b^u &= \inf\{(L_b^v)^+ \wedge (1 - Z_b) : v \in \mathbb{Q}_+, u < v \leq b\}, \quad u < b \leq \infty \end{aligned}$$

**Theorem 3.4** *Each  $M^u$  is  $\mathbb{P}$ -indistinguishable to  $L^u$  and  $(M^u : 0 < u < \infty)$  is an  $iM_Z$ . We shall say that this  $iM_Z$  is associated with the equation(4).*

**Proof** We need only to prove the martingale property of  $M^u$ . Let  $0 < u < \infty$ . Let  $T$  be a  $\mathbb{F}$ -stopping time such that  $T \geq u$  and everything concerned in the following computation is integrable. We have

$$\begin{aligned} \mathbb{P}[M_T^u - L_T^u] &= \mathbb{P}[\mathbb{I}_{\{u < T\}} \inf_{v \in \mathbb{Q}, u < v \leq T} L_T^v - \mathbb{I}_{\{u < T\}} L_T^u] \\ &= \inf_{v \in \mathbb{Q}, u < v} \mathbb{P}[\mathbb{I}_{\{u < T\}} L_T^{v \wedge T} - \mathbb{I}_{\{u < T\}} L_T^u] \\ &= \inf_{v \in \mathbb{Q}, u < v} \mathbb{P}[\mathbb{I}_{\{u < T\}} (1 - Z_{v \wedge T})] - \mathbb{P}[\mathbb{I}_{\{u < T\}} (1 - Z_u)] \\ &= 0 \end{aligned}$$

This shows that  $M^u, L^u$  are  $\mathbb{P}$ -indistinguishable on  $[u, \infty]$  and in particular  $M^u$  is a continuous  $\mathbb{P}$ - $\mathbb{F}$  uniformly integrable martingale. ■

### 3.3 An $iM_Z$ in case of possible zero of $1 - Z$

In this section we study the case where  $1 - Z$  may take the value zero. Consequently the analysis made in the preceding section does not work anymore, because, for example,  $\frac{e^{-\Lambda}}{1-Z} \cdot N$  can be undefined. New method is needed to construct  $iM_Z$ .

Let  $\mathbf{H} = \{s : 1 - Z_s = 0\}$  and define, for  $0 < t < \infty$ , the random times

$$\begin{aligned} g_t &:= \sup\{0 \leq s < t : s \in \mathbf{H}\}, \\ d_t &:= \inf\{s > t : s \in \mathbf{H}\}. \end{aligned}$$

We introduce the hypothesis :

**Hypothesis Hy(H)** The set  $\mathbf{H}$  is not empty and is closed. The measure  $d\Lambda$  has a decomposition  $d\Lambda_s = dV_s + dA_s$  where  $V, A$  are continuous increasing processes such that  $dV$  charges only  $\mathbf{H}$  while  $dA$  charges its complementary  $\mathbf{H}^c$ . Moreover, we suppose

$$\mathbb{I}_{\{g_t \leq u < t\}} \int_u^t \frac{Z_s}{1 - Z_s} dA_s = \mathbb{I}_{\{u < t \leq d_u\}} \int_u^t \frac{Z_s}{1 - Z_s} dA_s < \infty$$

for any  $0 < u < t < \infty$ .

We suppose in this section **Hy(H)** and  $Z_0 = 1$ .

**Lemma 3.2** *Let  $E_t^u = \exp\left(-\int_u^t \frac{Z_s}{1 - Z_s} dA_s\right)$ . We have the identity, for  $0 < u \leq t \leq \infty$*

$$\mathbb{I}_{\{g_t \leq u\}} E_t^u (1 - Z_t) = (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u e^{-\Lambda_s} dN_s$$

**Proof** It is rather straightforward to prove that, for  $0 < u \leq t \leq d_u$

$$d_t(E_t^u(1 - Z_t)) = E_t^u(Z_t dV_t - e^{-\Lambda t} dN_t) = -E_t^u e^{-\Lambda t} dN_t$$

We use the balayage formula (see [7]) to calculate, for  $u < t < \infty$

$$\begin{aligned} \mathbb{I}_{\{g_t \leq u\}} E_t^u(1 - Z_t) &= \mathbb{I}_{\{g_t \leq u\}} \left( (1 - Z_u) - \int_u^t E_s^u e^{-\Lambda s} dN_s \right) \\ &= (1 - Z_u) - \int_u^t \mathbb{I}_{\{g_s \leq u\}} E_s^u e^{-\Lambda s} dN_s \end{aligned}$$

The lemma is proved. ■

Set

$$M_t^u = \mathbb{I}_{\{g_t \leq u\}} E_t^u(1 - Z_t), \quad 0 < u < \infty, u \leq t \leq \infty.$$

We see immediately that the process  $M^u$  is càdlàg, positive and bounded by  $(1 - Z_t)$  on  $[u, \infty]$ . According to Lemma 3.2,  $M^u$  is a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. So it is an  $\mathbb{P}$ - $\mathbb{F}$  uniformly integrable martingale. The initial value of  $M^u$  is  $M_u^u = (1 - Z_u)$  whilst the terminal value is

$$\mathbb{I}_{\{g \leq u\}} \exp\left\{-\int_u^\infty \frac{Z_s}{1 - Z_s} dA_s\right\} (1 - Z_\infty).$$

where  $g := \lim_{t \rightarrow \infty} g_t$ . It yields that, for  $0 < t \leq \infty$ , the map  $u \in (0, t) \rightarrow M_t^u$  is increasing and right continuous.

We define for  $0 < t \leq \infty$ ,

$$M_t^0 = \lim_{u \downarrow 0} M_t^u$$

We have

$$\mathbb{P}[M_t^0] = \lim_{u \downarrow 0} \mathbb{Q}[M_t^u] = \lim_{u \downarrow 0} \mathbb{Q}[M_u^u] = \lim_{u \downarrow 0} \mathbb{Q}[1 - Z_u] = 0,$$

therefore,  $M^0 \equiv 0$ . We have on the other hand

$$M_\infty^\infty = \mathbb{I}_{\{g < \infty\}} \mathbb{I}_B(1 - Z_\infty) = \mathbb{I}_B(1 - Z_\infty)$$

where

$$B = \left\{ \inf_{0 \leq u < \infty} \int_u^\infty \frac{Z_s}{1 - Z_s} dA_s < \infty \right\}$$

We conclude

**Theorem 3.5** *The family  $(M^u : 0 < u < \infty)$  is an  $iM_Z$ . ■*

Notice that, if  $1 - Z_u$  is not identically null,  $\int_u^\infty \frac{Z_s}{1 - Z_s} dA_s$  is not identically infinite.

Now we have an  $iM_Z$ . This implies automatically a solution of the problem- $\mathcal{A}$ . However for this  $iM_Z$  the solution has an interesting form we believe useful to write down explicitly. We define  $\mathbb{Q}$  a probability measure on  $[0, \infty] \times \Omega$  by the following relations : for  $A \in \mathcal{F}_\infty, 0 < u < \infty$ ,

$$\begin{aligned} \mathbb{Q}[A\{\tau \leq u\}] &= \mathbb{P}[\mathbb{I}_A \exp\left\{-\int_u^\infty \frac{Z_s}{1 - Z_s} dA_s\right\} (1 - Z_\infty)], \\ \mathbb{Q}[A\{\tau = \infty\}] &= \mathbb{P}[\mathbb{I}_A (\mathbb{I}_B Z_\infty + \mathbb{I}_{B^c})]. \end{aligned}$$

This definition implies immediately the equality  $\mathbb{Q}[A] = \mathbb{P}[A]$ ,  $A \in \mathcal{F}_\infty$ , i.e.  $\mathbb{Q}$  is an extension of  $\mathbb{P}$ . Now, for  $0 < u < \infty$  and  $A \in \mathcal{F}_\infty$ ,

$$\mathbb{Q}[A\{g \vee \tau \leq u\}] = \mathbb{P}[\mathbb{I}_A \mathbb{I}_{\{g \leq u\}} \exp\{-\int_u^\infty \frac{Z_s}{1-Z_s} dA_s\}(1-Z_\infty)] = \mathbb{P}[\mathbb{I}_A M_\infty^u]$$

In particular, if  $A \in \mathcal{F}_u$ ,

$$\mathbb{Q}[A\{g \vee \tau \leq u\}] = \mathbb{P}[\mathbb{I}_A M_u^u] = \mathbb{P}[\mathbb{I}_A(1-Z_u)]$$

i.e.  $\mathbb{Q}[g \vee \tau > u | \mathcal{F}_u] = Z_u$ . We conclude

**Theorem 3.6** ( $\mathbb{Q}, g \vee \tau$ ) is a solution of the problem- $\mathcal{A}$ . ■

## 4 Enlargement of filtration problem

In the preceding sections we have constructed families  $iM_Z$  which generates solutions of the problem- $\mathcal{A}$ . In this section we shall study the enlargement of filtration problem for these solutions. Let us recall the setting. For a solution  $(\hat{\Omega}, \hat{\mathcal{A}}, \hat{\mathbb{F}}, \mathbb{Q}, \pi, \tau)$  of the problem- $\mathcal{A}$ , we equip it with the progressively enlarged filtration  $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$  where  $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau \wedge t)$ ,  $t \geq 0$ . We want to know if a  $\mathbb{P}$ - $\mathbb{F}$  local martingale  $X$  remains a  $\mathbb{Q}$ - $\mathbb{G}$  semimartingale and if it is the case, what is its  $\mathbb{Q}$ - $\mathbb{G}$  semimartingale decomposition. The problem can be completely solved for the period before the time  $\tau$  by [5]. We have the lemma.

**Lemma 4.1** Suppose **Hy**( $B$ ). Suppose  $\mathbb{Q}[0 < \tau < \infty] = 1$ . Let  $X$  be a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. We denote by  $B$  the  $\mathbb{Q}$ - $\mathbb{F}$  predictable dual projection of the jump process  $\Delta X_\tau \mathbb{I}_{\{\tau \leq t\}}$ ,  $0 \leq t < \infty$ . Let  $\langle N, X \rangle$  be the  $\mathbb{P}$ - $\mathbb{F}$ -predictable dual projection of  $[N, X]$  ( $\langle N, X \rangle$  exists always). Then,

$$X_{\cdot \wedge \tau} - \int_0^{\cdot \wedge \tau} \frac{1}{Z_{s-}} (e^{-\Lambda_s} d\langle N, X \rangle_s + dB_s)$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale. ■

**Remark 4.1** that, if  $\mathbb{Q}[\tau = T] = 0$  for all  $\mathbb{F}$ -stopping time  $T$ ,  $B \equiv 0$ . More generally the process  $B$  is linked with  $\Lambda$  by the formula :  $dB_s = H_s^X Z_s d\Lambda_s$ , where  $H^X$  is a  $\mathbb{F}$ -predictable process such that  $H_\tau^X = \mathbb{Q}[\Delta_\tau X | \mathcal{F}_{\tau-}]$ . ■

Therefore the real problem for us is about  $X_{t \vee \tau} - X_\tau$  for the period after the time  $\tau$ . Let us mention some examples where the problem for the period after  $\tau$  have been studied with success. They are the case of honest time (see Barlow, Jenlin and Yor [1, 5]) and the case of initial time (see Jeanblanc and Le Cam [3]). The situation studied here is different. We work in the framework fixed by a solution of the problem- $\mathcal{A}$ . Fortunately, the families  $iM_Z$  of our solutions satisfy good stochastic differential equations. This makes applicable a classical idea from Yor [8]. The problem will be thus solved. To complete our program, we remark that the reasoning process can be nicely reversed. We shall prove that, if for a given family  $iM_Z$  a good semimartingale decomposition formula holds for the model constructed with it, then the  $iM_Z$  must be the solutions of a certain stochastic differential equation.

#### 4.1 The case of the basic $iM_Z$

The following theorem has been proved in [4] :

**Theorem 4.1** *Suppose  $\mathbf{Hy}(N, \Lambda)$ . Consider the  $iM_Z$  defined by the family (Theorem 3.1)*

$$B_t^u = (1 - Z_t) \exp \left( - \int_u^t \frac{Z_s d\Lambda_s}{1 - Z_s} \right) \quad 0 < u < \infty, u \leq t \leq \infty,$$

Let  $\mathbb{Q}$  be the probability measure on  $[0, \infty] \times \Omega$  associated with the family  $(B^u : 0 < u < \infty)$ . Then, for any  $\mathbb{P}$ - $\mathbb{F}$  local martingale  $X$ ,

$$X. - \int_0^\cdot \mathbb{I}_{\{s \leq \tau\}} \frac{e^{-\Lambda_s} d\langle N, X \rangle_s}{Z_{s-}} + \int_0^\cdot \mathbb{I}_{\{\tau < s\}} \frac{e^{-\Lambda_s} d\langle N, X \rangle_s}{1 - Z_{s-}}$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale.

#### 4.2 From the equation (†) to the decomposition formula in enlargement of filtration

In this subsection we suppose  $\mathbf{Hy}(N, \Lambda)$  and  $\mathbf{Hy}(C)$  and  $Z_0 = 1, Z_\infty = 0$ . We consider a generating equation(†) (see subsection 3.2.1) :  $0 < u < \infty$ ,

$$(\dagger_u) \begin{cases} dM_t = M_t \left( -\frac{e^{-\Lambda_t}}{1 - Z_t} dN_t + f(M_t - (1 - Z_t)) dY_t \right), & u \leq t < \infty \\ M_u = x \end{cases}$$

with  $f$  being a continuously differentiable Lipschitz function and  $Y$  being a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. Let  $(M^u, 0 < u < \infty)$  be the  $iM_Z$  family associated to the above equation (†). We suppose in addition the following hypothesis:

**Hy(Mc)** : For each  $0 < t < \infty$ , the map  $u \rightarrow M_t^u$  is continuous on  $(0, t]$ .

We can write by monotone convergence theorem :

$$\mathbb{P}[\lim_{v \rightarrow \infty} M_\infty^v] = \lim_{v \rightarrow \infty} \mathbb{P}[M_\infty^v] = \lim_{v \rightarrow \infty} \mathbb{P}[M_v^v] = \lim_{v \rightarrow \infty} \mathbb{P}[(1 - Z_v)] = 1.$$

But  $M_t^v \leq 1 - Z_t \leq 1$  for all  $v \leq t \leq \infty$ . So the above equality yields  $M_\infty^\infty = 1$ . It is also true that  $M_\infty^0 = 0$ . We denote by  $d_u M_t^u$  the random measure on  $(0, t)$  induced by the map  $u \in (0, t) \rightarrow M_t^u$ , for  $0 < t \leq \infty$ . The following lemma is straightforward.

**Lemma 4.2** *For  $0 < u < \infty$ , for  $h$  a bounded Borel function on  $[0, \infty]$ , the process :*

$$t \in [u, \infty] \rightarrow \int_0^u h(v) d_v M_t^v$$

is a bounded  $\mathbb{P}$ - $\mathbb{F}$  martingale on  $[u, \infty]$ . ■

Let  $\mathbb{Q}$  be the probability on the product space  $[0, \infty] \times \Omega$  associated with the  $iM_Z$ . Note that since  $M_\infty^\infty = 1$  and  $M_\infty^0 = 0$ ,  $\mathbb{Q}[0 < \tau < \infty] = 1$ . And also the **Hy(Mc)** implies  $\mathbb{Q}[\tau = T] = 0$  for any  $\mathbb{F}$  stopping time  $T$ . We need another lemma.

**Lemma 4.3** Let  $V_s(\omega), 0 \leq s < \infty, \omega \in \Omega$ , be a function such that, for fixed  $s$ ,  $V_s$  is  $\mathcal{F}_\infty$  measurable and for fixed  $\omega$ ,  $s \rightarrow V_s(\omega)$  is càdlàg and increasing. We denote by  $dV_s$  the induced measure on  $[0, \infty)$ . Let  $F_s(t, \omega), 0 \leq s < \infty, 0 \leq t \leq \infty, \omega \in \Omega$ , be a positive function measurable with respect to  $\mathcal{B}[0, \infty] \otimes \mathcal{B}[0, \infty] \otimes \mathcal{F}_\infty$ . Suppose  $\mathbb{Q}[\int_0^\infty dV_s] < \infty$ . Then,

$$\mathbb{Q}[\int_0^\infty F_s dV_s] = \mathbb{Q}[\int_0^\infty \left( \int_0^\infty F_s(v, \cdot) d_v M_\infty^v \right) dV_s]$$

**Proof.** By monotone class theorem, we need only to check the relation for a function of form  $F_s(t, \omega) = h(s)H(t, \omega)$ . Recall that  $\mathbb{Q}$  is an extension of  $(\Omega, \mathbb{F}, \mathbb{P})$  so that it is identified as  $\mathbb{P}$  on  $\mathcal{F}_\infty$ . We compute

$$\begin{aligned} \mathbb{Q}[\int_0^\infty F_s dV_s] &= \mathbb{Q}[H \int_0^\infty h(s) dV_s] \\ &= \mathbb{Q}[\int_0^\infty H(v, \cdot) d_v M_\infty^v \int_0^\infty h(s) dV_s] \\ &= \mathbb{Q}[\int_0^\infty \left( \int_0^\infty F_s(v, \cdot) d_v M_\infty^v \right) dV_s] \end{aligned}$$

The lemma is proved. ■

**Theorem 4.2** Let  $X$  be a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. Then, the process

$$\begin{aligned} X_t - X_0 - \int_0^t \mathbb{1}_{\{s \leq \tau\}} \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X \rangle_s \\ - \int_0^t \mathbb{1}_{\{\tau < s\}} \left( -\frac{e^{-\Lambda_s}}{1-Z_s} \right) d\langle N, X \rangle_s \\ - \int_0^t \mathbb{1}_{\{\tau < s\}} \left( f(M_s^\tau - (1-Z_s)) + M_s^\tau f'(M_s^\tau - (1-Z_s)) \right) d\langle Y, X \rangle_s \end{aligned}$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale. (For a later use, we denote by  $A^{[Y;f]}(X)_t; 0 \leq t < \infty$  the sum of the three stochastic integrals so that the above formula can be written as  $X_t - X_0 - A^{[Y;f]}(X)_t$ )

**Proof :** We could adopt the reasoning in Yor [8]. But we prefer to give another proof which reveals clearly the implication of the stochastic differential equation (4) in the semimartingale decomposition formula. We write

$$X_t - X_0 = X_{\tau \vee t} - X_\tau + X_{\tau \wedge t} - X_0$$

The decomposition for  $X_{\tau \wedge t} - X_0$  is given by Lemma 4.1 with  $B \equiv 0$ . We need only to prove that

$$\begin{aligned} X_{\tau \vee t} - X_\tau - \int_0^t \mathbb{1}_{\{\tau < s\}} \left( -\frac{e^{-\Lambda_s}}{1-Z_s} \right) d\langle N, X \rangle_s \\ - \int_0^t \mathbb{1}_{\{\tau < s\}} \left( f(M_s^\tau - (1-Z_s)) + M_s^\tau f'(M_s^\tau - (1-Z_s)) \right) d\langle Y, X \rangle_s \end{aligned}$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale. Without loss of generality we suppose that  $X$  is stopped so that everything in the calculus below is integrable. In particular we assume that

$$\mathbb{Q}[\int_\tau^\infty \frac{e^{-\Lambda_s}}{1-Z_s} |d\langle N, X \rangle|_s] = \mathbb{Q}[\int_0^\infty e^{-\Lambda_s} |d\langle N, X \rangle|_s]$$

is finite. Let  $0 \leq s < t < \infty, 0 < u < \infty$  and  $A \in \mathcal{F}_s$ .

$$\begin{aligned} \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\tau \leq u\}}(X_{\tau \vee t} - X_{\tau \vee s})] &= \lim_{n \rightarrow \infty} \mathbb{Q}[\mathbb{1}_A \sum_{k=1}^n \mathbb{1}_{\{\frac{(k-1)u}{n} < \tau \leq \frac{ku}{n}\}}(X_{\tau \vee t} - X_{\tau \vee s})] \\ &= \lim_{n \rightarrow \infty} \mathbb{Q}[\mathbb{1}_A \sum_{k=1}^n \mathbb{1}_{\{\frac{(k-1)u}{n} < \tau \leq \frac{ku}{n}\}}(X_{\frac{ku}{n} \vee t} - X_{\frac{ku}{n} \vee s})] \end{aligned}$$

Now, by integration by parts formula, we compute the terms under the summation and the limit signs, setting  $t_n = \frac{ku}{n} \vee t$  and  $s_n = \frac{ku}{n} \vee s$

$$\begin{aligned} A_n(k) &:= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\frac{(k-1)u}{n} < \tau \leq \frac{ku}{n}\}}(X_{t_n} - X_{s_n})] = \mathbb{Q}[\mathbb{1}_A (M_{\infty}^{\frac{ku}{n}} - M_{\infty}^{\frac{(k-1)u}{n}})(X_{t_n} - X_{s_n})] \\ &= \mathbb{Q}[\mathbb{1}_A \int_{s_n}^{t_n} d\langle M_{\infty}^{\frac{ku}{n}}, X \rangle_r] - \mathbb{Q}[\mathbb{1}_A \int_{s_n}^{t_n} d\langle M_{\infty}^{\frac{(k-1)u}{n}}, X \rangle_r] \end{aligned}$$

We now compute the predictable brackets using the equation(†) satisfied by  $M^v$  :

$$\begin{aligned} A_n(k) &= \mathbb{Q}[\mathbb{1}_A \int_{s_n}^{t_n} M_r^{\frac{ku}{n}} (-\frac{e^{-\Lambda_r}}{1 - Z_r} d\langle N, X \rangle_r + f(M_r^{\frac{ku}{n}} - (1 - Z_r))d\langle Y, X \rangle_r)] \\ &\quad - \mathbb{Q}[\mathbb{1}_A \int_{s_n}^{t_n} M_r^{\frac{(k-1)u}{n}} (-\frac{e^{-\Lambda_r}}{1 - Z_r} d\langle N, X \rangle_r + f(M_r^{\frac{(k-1)u}{n}} - (1 - Z_r))d\langle Y, X \rangle_r)] \\ &= \mathbb{Q}[\mathbb{1}_A \int_s^t \mathbb{1}_{\{\frac{ku}{n} < r\}} \frac{e^{-\Lambda_r}}{1 - Z_r} (-M_{\infty}^{\frac{ku}{n}} + M_{\infty}^{\frac{(k-1)u}{n}}) d\langle N, X \rangle_r] \\ &\quad + \mathbb{Q}[\mathbb{1}_A \int_s^t \mathbb{1}_{\{\frac{ku}{n} < r\}} \left( M_r^{\frac{ku}{n}} f(M_r^{\frac{ku}{n}} - (1 - Z_r)) - M_r^{\frac{(k-1)u}{n}} f(M_r^{\frac{(k-1)u}{n}} - (1 - Z_r)) \right) d\langle Y, X \rangle_r] \end{aligned}$$

We next compute  $\lim_{n \rightarrow \infty} \sum_{k=1}^n A_n(k)$  as sum of two limits. The first limit is given by

$$\begin{aligned} &\lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{Q}[\mathbb{1}_A (-M_{\infty}^{\frac{ku}{n}} + M_{\infty}^{\frac{(k-1)u}{n}}) \int_s^t \mathbb{1}_{\{\frac{ku}{n} < r\}} \frac{e^{-\Lambda_r}}{1 - Z_r} d\langle N, X \rangle_r] \\ &= \lim_{n \rightarrow \infty} \mathbb{Q}[\mathbb{1}_A \sum_{k=1}^n \mathbb{1}_{\{\frac{(k-1)u}{n} < \tau \leq \frac{ku}{n}\}} \int_s^t \mathbb{1}_{\{\frac{ku}{n} < r\}} (-\frac{e^{-\Lambda_r}}{1 - Z_r}) d\langle N, X \rangle_r] \\ &= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\tau \leq u\}} \int_s^t \mathbb{1}_{\{\tau < r\}} (-\frac{e^{-\Lambda_r}}{1 - Z_r}) d\langle N, X \rangle_r] \end{aligned}$$

The second limit is computed by

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{k=1}^n \mathbb{Q}[\mathbb{1}_A \int_s^t \mathbb{1}_{\{\frac{ku}{n} < r\}} \\ & \quad (M_r^{\frac{ku}{n}} f(M_r^{\frac{ku}{n}} - (1 - Z_r)) - M_r^{\frac{(k-1)u}{n}} f(M_r^{\frac{(k-1)u}{n}} - (1 - Z_r))) d\langle Y, X \rangle_r] \\ = & \lim_{n \rightarrow \infty} \mathbb{Q}[\mathbb{1}_A \int_s^t \sum_{k=1}^{n'} \\ & \quad (M_r^{\frac{ku}{n}} f(M_r^{\frac{ku}{n}} - (1 - Z_r)) - M_r^{\frac{(k-1)u}{n}} f(M_r^{\frac{(k-1)u}{n}} - (1 - Z_r))) d\langle Y, X \rangle_r] \end{aligned}$$

where  $n' = [\frac{nr}{u}] \wedge n$

$$= \lim_{n \rightarrow \infty} \mathbb{Q}[\mathbb{1}_A \int_s^t M_r^{\frac{n'u}{n}} f(M_r^{\frac{n'u}{n}} - (1 - Z_r)) d\langle Y, X \rangle_r]$$

since  $M_r^0 = 0$

$$\begin{aligned} & = \mathbb{Q}[\mathbb{1}_A \int_s^t M_r^{r \wedge u} f(M_r^{r \wedge u} - (1 - Z_r)) d\langle Y, X \rangle_r] \\ & = \mathbb{Q}[\mathbb{1}_A \int_s^t \int_0^{r \wedge u} (f(M_r^v - (1 - Z_r)) + M_r^v f'(M_r^v - (1 - Z_r))) d_v M_r^v d\langle Y, X \rangle_r] \\ & = \mathbb{Q}[\mathbb{1}_A \int_s^t \int_0^\infty \mathbb{1}_{\{v < u\}} \mathbb{1}_{\{v < r\}} (f(M_r^v - (1 - Z_r)) + M_r^v f'(M_r^v - (1 - Z_r))) d_v M_\infty^v d\langle Y, X \rangle_r] \\ & = \mathbb{Q}[\mathbb{1}_A \int_s^t \mathbb{1}_{\{\tau < u\}} \mathbb{1}_{\{\tau < r\}} (f(M_r^\tau - (1 - Z_r)) + M_r^\tau f'(M_r^\tau - (1 - Z_r))) d\langle Y, X \rangle_r] \end{aligned}$$

We resume the situation : (note that  $\mathbb{Q}[\tau = u] = 0$ )

$$\begin{aligned} & \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\tau \leq u\}} (X_{\tau \vee t} - X_{\tau \vee s})] \\ = & \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\tau \leq u\}} \int_s^t \mathbb{1}_{\{\tau < r\}} (-\frac{e^{-\Lambda r}}{1 - Z_r}) d\langle N, X \rangle_r] \\ & + \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\tau \leq u\}} \int_s^t \mathbb{1}_{\{\tau < r\}} (f(M_r^\tau - (1 - Z_r)) + M_r^\tau f'(M_r^\tau - (1 - Z_r))) d\langle Y, X \rangle_r] \end{aligned}$$

for all  $0 < u < \infty$  and  $A \in \mathcal{F}_s$ . As  $\mathcal{G}_s \subset \mathcal{F}_s \otimes \sigma(\tau)$ , the theorem is proved. ■

### 4.3 Reciprocal relation between the equation (h) and the decomposition formula in enlargement of filtration

In the preceding section, we started with the dynamics of an  $iM_Z$  and we arrive at a semimartingale decomposition formula. Here we shall consider an inversed situation. We begin with a probability  $\mathbb{Q}$  on  $[0, \infty] \times \Omega$  which is a solution of the problem- $\mathcal{A}$ . Let  $(M^u : 0 < u < \infty)$  be the  $iM_Z$  family associated with  $(\mathbb{Q}, \tau)$ . We assume that the semimartingale decomposition formula in the Theorem 4.2 holds under  $\mathbb{Q}$ . We want to know the behaviour of the associated  $iM$  family. We have the following theorem :

**Theorem 4.3** *Assume the hypothesis  $\mathbf{Hy}(N, \Lambda)$ ,  $\mathbf{Hy}(C)$ ,  $Z_0 = 1, Z_\infty = 0$  and  $\mathbf{Hy}(Mc)$ . Suppose that there exist a  $\mathbb{P}$ - $\mathbb{F}$  local martingale  $Y$  and a continuous differentiable Lipschitz function  $f$  such that  $X_t - X_0 - A^{[Y;f]}(X)_t; 0 \leq t < \infty$  is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale. Then, the martingales  $M^u \in iM_Z, 0 < u < \infty$ , are solution of the equation (h<sub>u</sub>).*

**Proof.** Let  $0 < u < t < \infty$ . Let  $X$  be any  $\mathbb{P}$ - $\mathbb{F}$  uniformly integrable martingale. Let  $T$  be any  $\mathbb{F}$ -stopping time such that  $u \leq T$ , and, stopped at  $T$ , everything in the following computations are integrable. Recall  $M_t^u = \mathbb{Q}[\mathbb{1}_{\{\tau \leq u\}} | \mathcal{F}_t]$ . We compute

$$\begin{aligned} & \mathbb{Q}[M_t^u X_t^T] \\ = & \mathbb{Q}[\mathbb{1}_{\{\tau \leq u\}} X_u^T + \mathbb{1}_{\{\tau \leq u\}} (X_t^T - X_u^T)] \\ = & \mathbb{Q}[\mathbb{1}_{\{\tau \leq u\}} X_u^T - \mathbb{1}_{\{\tau \leq u\}} \int_u^t \frac{e^{-\Lambda s}}{1 - Z_s} d\langle N, X^T \rangle_s] \\ & + \mathbb{Q}[\mathbb{1}_{\{\tau \leq u\}} \int_u^t (f(M_s^\tau - (1 - Z_s)) + M_s^\tau f'(M_s^\tau - (1 - Z_s))) d\langle Y, X^T \rangle_s] \end{aligned}$$

Due to the martingale property of  $X$ , the first term becomes

$$\mathbb{Q}[\mathbb{1}_{\{\tau \leq u\}} X_u^T - \mathbb{1}_{\{\tau \leq u\}} \int_u^t \frac{e^{-\Lambda_s}}{1 - Z_{s-}} d\langle N, X^T \rangle_s] = \mathbb{Q}[(1 - Z_u) X_t^T - X_t^T \int_u^t M_s^u \frac{e^{-\Lambda_s}}{1 - Z_s} dN_s]$$

The second term becomes

$$\begin{aligned} & \mathbb{Q}[\int_u^t \int_0^u (f(M_s^v - (1 - Z_s)) + M_s^v f'(M_s^v - (1 - Z_s))) d_v M_\infty^v d\langle Y, X^T \rangle_s] \\ &= \mathbb{Q}[\int_u^t \int_0^u (f(M_s^v - (1 - Z_s)) + M_s^v f'(M_s^v - (1 - Z_s))) d_v M_s^v d\langle Y, X^T \rangle_s] \\ &= \mathbb{Q}[\int_u^t M_s^u f(M_s^u - (1 - Z_s)) d\langle Y, X^T \rangle_s] \\ &= \mathbb{Q}[X_t^T \int_u^t M_s^u f(M_s^u - (1 - Z_s)) dY_s] \end{aligned}$$

Putting these together we obtain

$$\mathbb{Q}[M_t^u X_t^T] = \mathbb{Q}\left[\left((1 - Z_u) - \int_u^t M_s^u \frac{e^{-\Lambda_s}}{1 - Z_s} dN_s + \int_u^t M_s^u f(M_s^u - (1 - Z_s)) dY_s\right) X_t^T\right]$$

This implies

$$M_t^u = (1 - Z_u) - \int_u^t M_s^u \frac{e^{-\Lambda_s}}{1 - Z_s} dN_s + \int_u^t M_s^u f(M_s^u - (1 - Z_s)) dY_s$$

i.e., the equation  $(\natural_u)$ . ■

#### 4.4 Enlargement of filtration in case of possible zero of $1 - Z$

This is our last section. We note that the eventual zeros of  $1 - Z$  make the family  $iM_Z$  different. We would like to see if the enlargement of filtration problem will be different. We suppose the  $\mathbf{Hy}(H)$  and  $Z_0 = 1$ . We consider the  $iM_Z$  constructed in Theorem 3.5 and its associated probability measure  $\mathbb{Q}$  on  $[0, \infty) \times \Omega$ .

**Lemma 4.4** *Let  $C$  be an  $\mathcal{F}$  predictable integrable increasing process. For  $0 < u < \infty$ , we have*

$$\mathbb{Q}[\mathbb{1}_{\{g \vee \tau \leq u\}} \int_u^\infty \frac{dC_s}{1 - Z_{s-}}] = \mathbb{Q}[\int_u^\infty \mathbb{1}_{\{g_s \leq u\}} \exp\{-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\} dC_s] \leq \mathbb{Q}[C_\infty] < \infty.$$

**Proof** Notice that for  $u \leq s < \infty$

$$\mathbb{Q}[M_s^u | \mathcal{F}_{s-}] = \mathbb{1}_{\{g_s \leq u\}} \exp\{-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\} (1 - Z_{s-})$$

Let  $0 < \epsilon$ . We compute

$$\begin{aligned} & \mathbb{Q}[\mathbb{1}_{\{g \vee \tau \leq u\}} \int_u^\infty \frac{dC_s}{1 - Z_{s-} + \epsilon}] \\ &= \mathbb{Q}[\mathbb{1}_{\{g \leq u\}} \exp\{-\int_u^\infty \frac{Z_v}{1 - Z_v} dA_v\} (1 - Z_\infty) \int_u^\infty \frac{dC_s}{1 - Z_{s-} + \epsilon}] \\ &= \mathbb{Q}[\int_u^\infty \mathbb{1}_{\{g_s \leq u\}} \exp\{-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\} (1 - Z_{s-}) \frac{dC_s}{1 - Z_{s-} + \epsilon}] \\ &\leq \mathbb{Q}[\int_u^\infty \mathbb{1}_{\{g_s \leq u\}} \exp\{-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\} dC_s] \\ &= \mathbb{Q}[C_\infty] < \infty \end{aligned}$$

Now let  $\epsilon \downarrow 0$  to achieve the proof. ■

**Theorem 4.4** Let  $X$  be a  $\mathbb{P}$ - $\mathbb{F}$  local martingale. Then, denoting by  $H^X$  a  $\mathbb{F}$ -predictable process such that  $H_{g \vee \tau}^X = \mathbb{Q}[\Delta_{g \vee \tau} X | \mathcal{F}_{g \vee \tau -}]$ ,

$$X - X_0 - \int_0^\cdot \mathbb{1}_{\{s \leq g \vee \tau\}} \left( \frac{e^{-\Lambda_s}}{Z_s} d\langle N, X \rangle_s + H_s^X d\Lambda_s \right) + \int_0^\cdot \mathbb{1}_{\{g \vee \tau < s\}} \frac{e^{-\Lambda_s}}{1 - Z_{s-}} d\langle N, X \rangle_s$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale.

**Proof.** We make the same reasoning as in Theorem 4.2, but in a simpler form. We need only to prove

$$X_{g \vee \tau \vee \cdot} - X_{g \vee \tau} + \int_0^\cdot \mathbb{1}_{\{g \vee \tau < s\}} \frac{e^{-\Lambda_s}}{1 - Z_{s-}} d\langle N, X \rangle_s$$

is a  $\mathbb{Q}$ - $\mathbb{G}$  local martingale. Let  $0 < u < \infty, 0 < a < b < \infty, A \in \mathcal{F}_a$ . Let  $T$  be a  $\mathbb{F}$  stopping time which makes the things integrable. We compute

$$\begin{aligned} & \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{g \vee \tau \leq u\}} (X_{g \vee \tau \vee b}^T - X_{g \vee \tau \vee a}^T)] \\ &= \lim_{n \uparrow \infty} \mathbb{Q}[\mathbb{1}_A \sum_{k=1}^n \mathbb{1}_{\{\frac{(k-1)u}{n} < g \vee \tau \leq \frac{ku}{n}\}} (X_{\frac{ku}{n} \vee b}^T - X_{\frac{ku}{n} \vee a}^T)] \\ &= \lim_{n \uparrow \infty} \sum_{k=1}^n \mathbb{Q}[\mathbb{1}_A (M_{\infty}^{\frac{ku}{n}} - M_{\infty}^{\frac{(k-1)u}{n}}) (X_{\frac{ku}{n} \vee b}^T - X_{\frac{ku}{n} \vee a}^T)] \end{aligned}$$

By Lemma 3.2,

$$M_t^u = (1 - Z_u) - \int_u^t \mathbb{1}_{\{g_s \leq u\}} \exp\left\{-\int_u^s \frac{Z_v}{1 - Z_v} dA_v\right\} e^{-\Lambda_s} dN_s$$

So

$$\begin{aligned} & \mathbb{Q}[\mathbb{1}_A (M_{\infty}^{\frac{ku}{n}} - M_{\infty}^{\frac{(k-1)u}{n}}) (X_{\frac{ku}{n} \vee b}^T - X_{\frac{ku}{n} \vee a}^T)] \\ &= \mathbb{Q}[\mathbb{1}_A \int_{\frac{ku}{n} \vee a}^{\frac{ku}{n} \vee b} \mathbb{1}_{\{g_s \leq \frac{ku}{n}\}} \exp\left\{-\int_{\frac{ku}{n}}^s \frac{Z_v}{1 - Z_v} dA_v\right\} (-e^{-\Lambda_s}) d\langle N, X^T \rangle_s] \\ & \quad - \mathbb{Q}[\mathbb{1}_A \int_{\frac{ku}{n} \vee a}^{\frac{ku}{n} \vee b} \mathbb{1}_{\{g_s \leq \frac{(k-1)u}{n}\}} \exp\left\{-\int_{\frac{(k-1)u}{n}}^s \frac{Z_v}{1 - Z_v} dA_v\right\} (-e^{-\Lambda_s}) d\langle N, X^T \rangle_s] \\ &= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{g \vee \tau \leq \frac{ku}{n}\}} \int_{\frac{ku}{n} \vee a}^{\frac{ku}{n} \vee b} \frac{(-e^{-\Lambda_s})}{1 - Z_{s-}} d\langle N, X^T \rangle_s] - \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{g \vee \tau \leq \frac{(k-1)u}{n}\}} \int_{\frac{ku}{n} \vee a}^{\frac{ku}{n} \vee b} \frac{(-e^{-\Lambda_s})}{1 - Z_{s-}} d\langle N, X^T \rangle_s] \\ & \quad \text{according to Lemma 4.4} \\ &= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\frac{(k-1)u}{n} < g \vee \tau \leq \frac{ku}{n}\}} \int_a^b \mathbb{1}_{\{\frac{ku}{n} < s\}} \frac{(-e^{-\Lambda_s})}{1 - Z_{s-}} d\langle N, X^T \rangle_s] \end{aligned}$$

Now we can calculate the limit

$$\begin{aligned} & \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{g \vee \tau \leq u\}} (X_{g \vee \tau \vee b}^T - X_{g \vee \tau \vee a}^T)] \\ &= \lim_{n \uparrow \infty} \sum_{k=1}^n \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{\frac{(k-1)u}{n} < g \vee \tau \leq \frac{ku}{n}\}} \int_a^b \mathbb{1}_{\{\frac{ku}{n} < s\}} \frac{(-e^{-\Lambda_s})}{1 - Z_{s-}} d\langle N, X^T \rangle_s] \\ &= \mathbb{Q}[\mathbb{1}_A \mathbb{1}_{\{g \vee \tau \leq u\}} \int_a^b \mathbb{1}_{\{g \vee \tau < s\}} \frac{(-e^{-\Lambda_s})}{1 - Z_{s-}} d\langle N, X^T \rangle_s] \end{aligned}$$

This proves the theorem.  $\blacksquare$

We remark that the semimartingale decomposition formula here takes the same form as in the preceding situations.

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