On last passage times of linear diffusions to curved boundaries

Christophe PROFETA[1]

Abstract: The aim of this paper is to study the law of the last passage time of a linear diffusion to a curved boundary. We start by giving a general expression for the density of such a random variable under some regularity assumptions. Following Robbins & Siegmund, we then show that this expression may be computed for some implicit boundaries via a martingale method. Finally, we discuss some links between first hitting times and last passage times via time inversion, and present an integral equation (which we solve in some particular cases) satisfied by the density of the last passage time. Many examples are given in the Brownian and Bessel frameworks.

Keywords: Last passage times; linear diffusions; hitting times; Brownian motion; Bessel processes.

1 Introduction

1.1 Motivation

Let $\ell \in [-\infty, +\infty[$ and consider a linear regular conservative diffusion $(X_t, t \geq 0)$ taking values in $I = (\ell, +\infty[$ with $+\infty$ a natural boundary.

Let $f : [0, +\infty[\longrightarrow [\ell, +\infty[$ be a continuous function and define $\zeta(f) := \inf\{t \geq 0\,;\, f(t) = \ell\} \in]0, +\infty]$. The aim of this paper is to compute the law of the last passage time of X to the boundary f before time $\zeta(f)$:

$$\begin{aligned} G_f &:= \sup\{0 \leq t \leq \zeta(f)\,;\, X_t = f(t)\}, \\ &(= 0 \quad \text{if } \{0 \leq t \leq \zeta(f)\,;\, X_t = f(t)\} = \varnothing). \end{aligned}$$

This problem was essentially addressed (in a far more general framework) in the literature through the study of additive functionals. We refer in particular to Getoor & Sharpe [GS73, Proposition 3.3] where the law of the last exit time from a Borel set D of a general Markov process is computed, with the help of the potential kernel of an additive functional associated to D. In [Por67], the case of $\mathbb{R}^n$-valued symmetric stable Lévy processes is tackled, and the law of the last exit time from a Borel set $D \subset \mathbb{R}^n$ is obtained in term of the equilibrium measure of D (see also Takeuchi [Tak67] for a study of moments).

In the set-up of diffusions (which is our concern), one of the main result is due to Pitman & Yor [PY81], who compute the law of G_f when $f(t) = a$ is a constant boundary and $(X_t, t \geq 0)$ is a transient diffusion going to $+\infty$ a.s. (Another proof is given in [PRY10, Chapter 2, p.38] using a formula for the last passage time of a continuous local martingale to a constant boundary.) We shall recover their result in Example 2 below.

On the other hand, much emphasis has been put, quite naturally, on the study of the first passage time of X to the boundary f:

$$\begin{aligned} T_f &:= \inf\{t \geq 0\,;\, X_t = f(t)\}, \\ &(= +\infty \quad \text{if } \{t \geq 0\,;\, X_t = f(t)\} = \varnothing). \end{aligned}$$

[1]Laboratoire d'Analyse et Probabilités, Université d'Évry - Val d'Essonne, Bâtiment I.B.G.B.I., 3ème étage, 23 Bd. de France, 91037 EVRY CEDEX.
E-mail: christophe.profeta@univ-evry.fr

Many ideas have emerged to solve this problem: changes of measure and partial differential equations [Sal88, Gro89], integral equations [RSS84, BNR87, GNRS89], martingale methods [RS70, Nov81], inversion of time [PY81, AP10]... We shall see that some of these methods may be applied to get information on last passage times.

1.2 Notations

Let $(X_t, t \geq 0)$ be a linear regular diffusion taking values in $I = (\ell, +\infty[$ with $+\infty$ a natural boundary. We assume that $(X_t, t \geq 0)$ is conservative, in the sense that it has an infinite life-time, but we make no assumption on the nature of the boundary point ℓ. Let $\mathbb{P}_x$ and $\mathbb{E}_x$ denote, respectively, the probability measure and the expectation associated with X when started from $x \geq \ell$. We assume that X is defined on the canonical space $\Omega := \mathcal{C}(\mathbb{R}^+ \to I)$ and we denote by $(\mathcal{F}_t, t \geq 0)$ its natural filtration.

We denote by $m(dx) = \rho(x)dx$ its speed measure, which is assumed to be absolutely continuous with respect to the Lebesgue measure and by s its scale function, which we assume to be of $\mathcal{C}^2$ class. With these notations, the infinitesimal generator of $(X_t, t \geq 0)$ reads:

$$\mathcal{G} = \frac{\partial^2}{\rho(x)\partial x\, s'(x)\partial x} \qquad \text{for } (t,x) \in \mathbb{R}^+ \times I.$$

It is known from Itô & McKean [IM74, p.149] that $(X_t, t \geq 0)$ admits a transition density $q(t,x,y)$ with respect to m, which is jointly continuous and symmetric in x and y, that is: $q(t,x,y) = q(t,y,x)$. In the remainder of the paper, we shall always assume that q is a $\mathcal{C}^{1,2,2}$ class function on $]0,+\infty[\times I \times I$. In particular, for any $y \in]\ell, +\infty[$, the function $(t,x) \longmapsto q(t,x,y)$ is a solution of the partial differential equation:

$$\mathcal{G}q = \frac{\partial q}{\partial t} \qquad \text{on }]0,+\infty[\times]\ell,+\infty[. \tag{1}$$

1.3 Organization of the paper

- We start in Section 2 by giving, under some regularity assumptions, a general expression for the density of the r.v. G_f:
$$\mathbb{P}_x(G_f \in dt) = \Phi(t) q(t,x,f(t))dt, \qquad (0 < t < \zeta(f)) \tag{2}$$
where
$$\Phi(t) = \frac{1}{s'(f(t))} \frac{\partial}{\partial y} \mathbb{P}_y(T_{f(t+\bullet)} = +\infty)|_{y=f(t)}.$$
Observe that the dependence on the initial state only appears through the transition density. Several examples of application of this formula are given, involving Brownian motion and Bessel processes with linear, squared root and square boundaries.

- In Section 3, we follow Robbins & Siegmund [RS70] and present a martingale method for computing the function Φ which appears in the previous formula. This gives us the law of G_f for a large class of implicit boundaries f.

- We then discuss, in Section 4, some relations between first hitting times and last passage times through time inversion.

- Finally, for all initial states x such that $\mathbb{P}_x(G_f > 0) = 1$ (i.e. the law of G_f under $\mathbb{P}_x$ has no atoms), Formula (2) implies that:
$$\int_0^{\zeta(f)} \Phi(t) q(t,x,f(t))dt = 1.$$

We shall see in Section 5 that in some cases, this relation characterizes uniquely the function Φ, hence the law of G_f.

2 The density of G_f

We start by establishing a general formula for the density of the last passage time G_f. Let $(\theta_t, t \geq 0)$ denote the translation operator defined by:

$$\forall u \geq 0, \qquad (f \circ \theta_t)(u) = f(t+u).$$

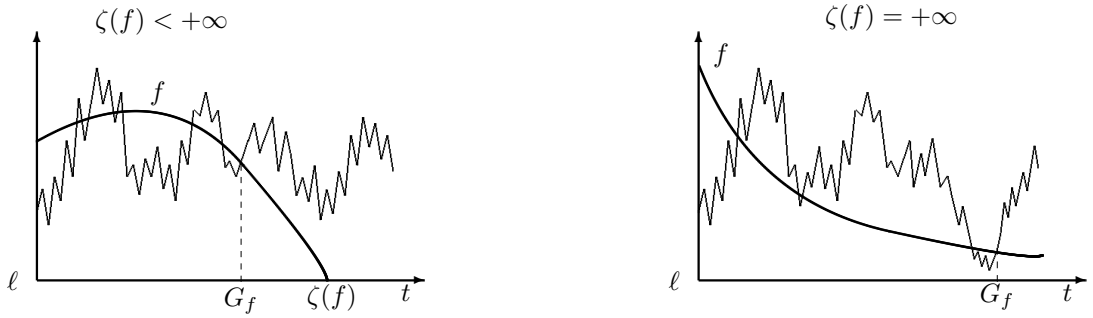
In the following, we shall distinguish two cases, depending on whether the process $(X_t, t \geq 0)$ remains above the boundary f after the last passage time G_f (we shall say that f is a lower boundary) or under the boundary f (resp. f is an upper boundary).

2.1 Lower boundaries

Let $f : [0, +\infty[\longrightarrow [\ell, +\infty[$ be a continuous function, which is of $\mathcal{C}^1$ class on $]0, +\infty[$ and define $\zeta(f) := \inf\{t \geq 0\,;\, f(t) = \ell\} \in]0, +\infty]$. If $\zeta(f) = +\infty$, we suppose (to ensure that $G_f \neq +\infty$ a.s.) that:

$$\forall x \in I, \qquad \mathbb{P}_x\left(\lim_{t \to \zeta(f)} X_t - f(t) > 0\right) = 1. \tag{3}$$

This implies in particular that, for any value of $\zeta(f) \in]0, +\infty]$, the diffusion $(X_t, t \geq 0)$ remains above the boundary f on the time interval $]G_f, \zeta(f)[$.



Let H be the function defined by:

$$H : (t, y) \longmapsto \mathbb{P}_y(T_{f \circ \theta_t} = +\infty).$$

We start with a general formula which gives the density of the r.v. G_f.

Theorem 1.
Assume that H is of $\mathcal{C}^{1,2}$ class on $]0, \zeta(f)[\times]\ell, +\infty[$ and is such that, for every $x \in]\ell, +\infty[$ and $t > 0$:

$$\lim_{y \to +\infty} \frac{\partial H(t,y)}{s'(y)\partial y} q(t, x, y) = 0. \tag{4}$$

Then, the density of the r.v. G_f under $\mathbb{P}_x$ is given by:

$$\mathbb{P}_x(G_f \in dt) = \frac{q(t, x, f(t))}{s'(f(t))} \, \frac{\partial H(t,y)}{\partial y}|_{y=f(t)} \, dt \qquad (0 < t < \zeta(f)). \tag{5}$$

Note that for $x > f(0)$, this density is defective.

Proof.
From Assumption (3), it is clear that $G_f < \zeta(f)$ a.s. Let $0 < t < \zeta(f)$:

$$\begin{aligned}
\mathbb{P}_x\left(G_f \le t\right) &= \mathbb{P}_x\left(G_f \le t \cap X_t \ge f(t)\right) \\
&= \int_{f(t)}^{+\infty} \mathbb{P}_x\left(G_f \le t | X_t = y\right) q(t,x,y) m(dy) \\
&= \int_{f(t)}^{+\infty} \mathbb{P}_y\left(T_{f\circ\theta_t} = +\infty\right) q(t,x,y) m(dy) \qquad \text{(from the Markov property)} \\
&= \int_{f(t)}^{+\infty} H(t,y) q(t,x,y) m(dy). \qquad (6)
\end{aligned}$$

Observe that, still from the Markov property:

$$\mathbb{P}_y\left(T_f = +\infty | \mathcal{F}_t\right) = \mathbb{P}_{X_t}\left(T_{f\circ\theta_t} = +\infty\right) 1_{\{T_f > t\}} = H(t \wedge T_f, X_{t\wedge T_f}). \qquad (7)$$

Set $(Z_t = s(X_{t\wedge T_\ell}), t < \zeta(f))$. By construction of the scale function s, the process Z is a continuous local martingale. But, for $X_0 > f(0)$, $T_f \wedge \zeta(f) \le T_\ell \wedge \zeta(f)$, hence the process $(X_{t\wedge T_f} = s^{-1}(Z_{t\wedge T_f}), t < \zeta(f))$ is a semimartingale. Therefore, applying Itô's formula to (7), we deduce that the term with finite variation vanishes, i.e. H is a solution to the partial differential equation:

$$\mathcal{G}H + \frac{\partial H}{\partial t} = 0 \qquad \text{on the domain } \{(t,y);\ y > f(t)\}. \qquad (8)$$

Now, let us differentiate (6) with respect to t:

$$\mathbb{P}_x(G_f \in dt) = \int_{f(t)}^{+\infty} \frac{\partial H(t,y)}{\partial t} q(t,x,y) m(dy) + \int_{f(t)}^{+\infty} H(t,y) \frac{\partial q(t,x,y)}{\partial t} m(dy) \qquad (9)$$

since $H(t, f(t)) = 0$. Integrating by part the second integral, we obtain:

$$\begin{aligned}
\int_{f(t)}^{+\infty} H(t,y) \frac{\partial q(t,x,y)}{\partial t} m(dy) &= \int_{f(t)}^{+\infty} H(t,y) \mathcal{G} q(t,x,y) m(dy) \qquad \text{(from (1))} \\
&= \int_{f(t)}^{+\infty} H(t,y) \frac{\partial^2 q(t,x,y)}{\partial y s'(y) \partial y} dy \\
&= \left[H(t,y) \frac{\partial q(t,x,y)}{s'(y)\partial y} \right]_{f(t)}^{+\infty} - \int_{f(t)}^{+\infty} \frac{\partial H(t,y)}{\partial y} \frac{\partial q(t,x,y)}{s'(y)\partial y} dy \\
&= -\int_{f(t)}^{+\infty} \frac{\partial H(t,y)}{s'(y)\partial y} \frac{\partial q(t,x,y)}{\partial y} dy \\
&\quad \text{(since } H \text{ is bounded and } +\infty \text{ is a natural boundary, see [BS02, p.20])} \\
&= -\left[\frac{\partial H(t,y)}{s'(y)\partial y} q(t,x,y) \right]_{f(t)}^{+\infty} + \int_{f(t)}^{+\infty} \frac{\partial^2 H(t,y)}{\partial y s'(y) \partial y} q(t,x,y) dy \\
&= \frac{q(t,x,f(t))}{s'(f(t))} \frac{\partial H(t,y)}{\partial y}|_{y=f(t)} + \int_{f(t)}^{+\infty} \mathcal{G}H(t,y) q(t,x,y) m(dy).
\end{aligned}$$

It only remains to plug this relation in (9) and to use (8) to get the desired result.

□

Example 2 (Transient diffusions and constant boundaries)**.**
Let $(X_t, t \ge 0)$ be a transient diffusion whose scale function is of $\mathcal{C}^2$ class, with the normalization $s(+\infty) = 0$. Let $a > \ell$ and choose $f(t) = a$. Then, for $a < y < b$, we have, by definition of the scale function s:

$$\mathbb{P}_y\left(T_b < T_a\right) = \frac{s(y) - s(a)}{s(b) - s(a)}$$

and, letting $b \to +\infty$, since $+\infty$ is a natural boundary:

$$\mathbb{P}_y\left(T_a = +\infty\right) = 1 - \frac{s(y)}{s(a)}$$

so we recover the well-known formula:

$$\mathbb{P}_x(G_a \in dt) = -\frac{q(t,x,a)}{s(a)}dt, \qquad (0 < t < +\infty)$$

see Pitman & Yor [PY81]. Note that in this case, Equation (4) reduces to $\lim_{y\to+\infty} q(t,x,y) = 0$, which is of course always satisfied in our set-up.

Remark 3. More generally, it can be proven that, for monotone functions f, Equation (4) is automatically satisfied, provided that H is smooth enough.

- Assume first that f is increasing. Let t be fixed and set $a = f(t)$. Then, for $y \geq a$, by the continuity of paths and the strong Markov property:

$$\begin{aligned}
\mathbb{P}_y(T_a = +\infty) &= \mathbb{P}_y(T_{f\circ\theta_t} = +\infty) + \mathbb{P}_y(T_a = +\infty \cap T_{f\circ\theta_t} < +\infty) \\
&= H(t,y) + \int_0^{+\infty} \mathbb{P}_y(T_a = +\infty | T_{f\circ\theta_t} = s)\mathbb{P}_y(T_{f\circ\theta_t} \in ds) \\
&= H(t,y) + \int_0^{+\infty} \mathbb{P}_{f(t+s)}(T_a = +\infty)\mathbb{P}_y(T_{f\circ\theta_t} \in ds) \\
&\quad \text{(see Peskir [Pes02])} \\
&= H(t,y) + \mathbb{E}_y[\Psi(T_{f\circ\theta_t})] \quad \text{where } \Psi(s) = \mathbb{P}_{f(t+s)}(T_a = +\infty).
\end{aligned}$$

  Observe now that, since Ψ is increasing, all three functions are increasing functions of y. Therefore, we deduce, with the same normalization as in the previous example (f is increasing hence X must be transient), that:

$$0 \leq \frac{\partial H(t,y)}{s'(y)\partial y} \leq \frac{\partial \mathbb{P}_y(T_a = +\infty)}{s'(y)\partial y} = -\frac{1}{s(a)},$$

  which implies, since $\lim_{y\to+\infty} q(t,x,y) = 0$, that Equation (4) is satisfied.

- Assume now that f is decreasing. In this case, the function $t \longmapsto H(t,y)$ is increasing, hence, from Equation (8):

$$\frac{\partial^2 H(t,y)}{\partial y s'(y)\partial y} = -\rho(y)\frac{\partial H(t,y)}{\partial t},$$

  we deduce that the function $y \longmapsto \dfrac{\partial H(t,y)}{s'(y)\partial y}$ is a positive and decreasing function. Therefore it is bounded at $+\infty$ and Equation (4) is satisfied.

We now give a few examples involving Brownian motion and Bessel processes. From Theorem 7, we only need to compute $\mathbb{P}_y(T_{f\circ\theta_t} = +\infty)$ to obtain the density of the last passage time to the boundary f.

Example 4 (Bessel processes and straight lines: $f(t) = a - bt$)**.**
Let $(R_t, t \geq 0)$ be a Bessel process of index $\nu > -1$. $(R_t, t \geq 0)$ is a diffusion whose generator reads:

$$\mathcal{G}^{(\nu)} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{2\nu+1}{2x}\frac{\partial}{\partial x}.$$

Its speed measure $m^{(\nu)}$ and scale function $s^{(\nu)}$ are given by:

$$\begin{cases} m^{(\nu)}(dx) = 2x^{2\nu+1}dx, \\ (s^{(\nu)})'(x) = x^{-2\nu-1}. \end{cases}$$

We denote by $\mathbb{P}_x^{(\nu)}$ the law of $(R_t, t \geq 0)$ when started at x and by $q^{(\nu)}$ its transition density function with respect to $m^{(\nu)}$:

$$q^{(\nu)}(t,x,y) = \frac{1}{2t}(xy)^{-\nu} \exp\left(-\frac{x^2+y^2}{2t}\right) I_\nu\left(\frac{xy}{t}\right)$$

with I_ν the modified Bessel function of the third kind.
Choose $f(t) = a - bt$ with $a, b > 0$, and thus $\zeta(f) = \dfrac{b}{a}$. Letting λ tend toward 0 in Theorem 5.1 of Alili & Patie [AP10], we deduce that, for $y \geq a$:

$$\mathbb{P}_y^{(\nu)}(T_f < +\infty) = \exp\left(\frac{b}{2a}(a^2-y^2)\right)\frac{y^{-\nu}}{a^{-\nu}}\int_0^{+\infty}\frac{K_\nu(\sqrt{2}yz)}{K_\nu(\sqrt{2}az)}q^{(\nu)}\left(\frac{b}{2a},\frac{b}{\sqrt{2}},z\right)m^{(\nu)}(dz)$$

where K_ν denotes the McDonald function of index ν. Then, since:

$$\mathbb{P}_y^{(\nu)}(T_{f\circ\theta_t} < +\infty) = \mathbb{P}_y^{(\nu)}(T_{a-bt-b_\bullet} < +\infty)$$

we obtain :

$$\frac{\partial H(t,y)}{\partial y}|_{y=a-bt} = b + \int_0^{+\infty}\sqrt{2}z\frac{K_{\nu+1}}{K_\nu}\left(\sqrt{2}(a-bt)z\right)q^{(\nu)}\left(\frac{b}{2(a-bt)},\frac{b}{\sqrt{2}},z\right)m^{(\nu)}(dz)$$

and finally, for $0 < t < \dfrac{a}{b}$:

$$\mathbb{P}_x^{(\nu)}(G_{a-b_\bullet} \in dt) = \frac{1}{2t}(a-bt)^{\nu+1}x^{-\nu}\exp\left(-\frac{x^2+(a-bt)^2}{2t}\right)I_\nu\left(\frac{x(a-bt)}{t}\right)\frac{\partial H(t,y)}{\partial y}|_{y=a-bt}\,dt.$$

Example 5 (Reflected Brownian motion and straight lines: $f(t) = a - bt$)**.**
In particular, when $\nu = -\frac{1}{2}$ (i.e. when $(X_t, t \geq 0)$ is a Brownian motion reflected at 0), then $K_{-\frac{1}{2}} = K_{\frac{1}{2}}$ and the above formula simplifies to:

$$\begin{aligned}&\mathbb{P}_x^{(-\frac{1}{2})}(G_{a-b_\bullet} \in dt)\\ &\quad= \left(b + \sqrt{\frac{b}{2\pi(a-bt)}}e^{-b(a-bt)} + b\mathrm{Erf}\left(\sqrt{\frac{b(a-bt)}{2}}\right)\right)\left(e^{-\frac{(x+a-bt)^2}{2t}} + e^{-\frac{(x-a+bt)^2}{2t}}\right)\frac{dt}{2\sqrt{2\pi t}}\end{aligned}$$

where Erf denotes the error function: $\mathrm{Erf}(z) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^z e^{-x^2}dx$.

Example 6 (Bessel processes and squared root boundaries: $f(t) = a\sqrt{1+2\gamma t}$)**.**
Let $(R_t, t \geq 0)$ be a Bessel process of index $\nu > -1$ and choose $f(t) = a\sqrt{1+2\gamma t}$ with $a > 0$, $\gamma < 0$, and thus $\zeta(f) = -\dfrac{1}{2\gamma}$. Define

$$\tau(t) = \frac{e^{2\gamma t}-1}{2\gamma}.$$

Then, it is known that the process $\left(e^{-\gamma t}R_{\tau(t)}, t \geq 0\right)$ has the same law as a radial Ornstein-Uhlenbeck process with parameters ν and γ, i.e.

$$(X_t = e^{-\gamma t}R_{\tau(t)}, t \geq 0)$$

is a linear diffusion with characteristics:

$$\begin{cases} m^{(\nu,\gamma)}(dx) = 2x^{2\nu+1}e^{-\gamma x^2}dx, \\ s^{(\nu,\gamma)}(x) = -\displaystyle\int_x^{+\infty} y^{-2\nu-1}e^{\gamma y^2}dy. \end{cases}$$

Now, we obtain by time change, since τ is increasing:

$$\begin{aligned}\inf\left\{u \geq 0,\ R_u = a\sqrt{1+2\gamma u}\right\} &= \tau\left(\inf\left\{t \geq 0,\ R_{\tau(t)} = ae^{\gamma t}\right\}\right)\\ &= \tau\left(\inf\left\{t \geq 0,\ X_t = a\right\}\right)\end{aligned}$$

and we deduce from Example 2 that:

$$\mathbb{P}_y^{(\nu)}(T_f = +\infty) = 1 - \frac{s^{(\nu,\gamma)}(y)}{s^{(\nu,\gamma)}(a)}.$$

Then, since

$$\left(f \circ \theta_t\right)(u) = a\sqrt{1+2\gamma(t+u)} = a\sqrt{1+2\gamma t}\,\sqrt{1+\frac{2\gamma}{1+2\gamma t}u},$$

we obtain:

$$\mathbb{P}_y^{(\nu)}(T_{f\circ\theta_t} = +\infty) = 1 - \frac{s^{(\nu,\frac{2\gamma}{1+2\gamma t})}(y)}{s^{(\nu,\frac{2\gamma}{1+2\gamma t})}\left(a\sqrt{1+2\gamma t}\right)},$$

and finally, for $0 < t < -\dfrac{1}{2\gamma}$:

$$\mathbb{P}_x^{(\nu)}\left(G_f \in dt\right) = -\frac{e^{2\gamma a^2}}{s^{(\nu,\frac{2\gamma}{1+2\gamma t})}\left(a\sqrt{1+2\gamma t}\right)} q^{(\nu)}\left(t, x, a\sqrt{1+2\gamma t}\right) dt.$$

2.2 Upper boundaries

We now study the case when the process $(X_t, t \geq 0)$ remains under the boundary f after the last passage time G_f. Let $\ell \in [-\infty, +\infty[$. We make the following assumption on the nature of the boundary point ℓ (see [BS02, p.19-20]):

- If $\ell = -\infty$, we assume that ℓ is natural.
- If $\ell > -\infty$, we assume that ℓ is entrance-not-exit.

These hypotheses ensure that for every $t > 0$ and $x > \ell$:

$$\lim_{y\to\ell} \frac{\partial q(t,x,y)}{s'(y)\partial y} = 0.$$

Let $f : [0, +\infty[\longrightarrow]\ell, +\infty[$ be a continuous function, which is of $\mathcal{C}^1$ class on $]0, +\infty[$, and such that $\zeta(f) := \inf\{t \geq 0\,;\, f(t) = \ell\} = +\infty$. We assume that :

$$\forall x \in I, \qquad \mathbb{P}_x\left(\lim_{t\to+\infty} X_t - f(t) < 0\right) = 1.$$

This implies that after G_f, the diffusion $(X_t, t \geq 0)$ remains under the boundary f.
Let H be the function defined by:

$$H : (t, y) \longmapsto \mathbb{P}_y(T_{f\circ\theta_t} = +\infty).$$

Theorem 7.
Assume that the function H is of $\mathcal{C}^{1,2}$ class on $]0, +\infty[\times]\ell, +\infty[$ and is such that:

$$\lim_{y\to\ell} \frac{\partial H(t,y)}{s'(y)\partial y} q(t,x,y) = 0. \tag{10}$$

Then, the density of the r.v. G_f under $\mathbb{P}_x$ is given by:

$$\mathbb{P}_x(G_f \in dt) = -\frac{q(t,x,f(t))}{s'(f(t))}\ \frac{\partial H(t,y)}{\partial y}\Big|_{y=f(t)}\ dt \qquad (t > 0). \tag{11}$$

Proof.
The proof is of course very similar to the previous one. For $0 < t < +\infty$, we have:

$$\mathbb{P}_x\left(G_f \leq t\right) = \int_\ell^{f(t)} H(t,y) q(t,x,y) m(dy)$$

with the function H solution of

$$\mathcal{G}H + \frac{\partial H}{\partial t} = 0 \qquad \text{on the domain } \{(t,y);\ y < f(t)\}. \tag{12}$$

(Note that, by hypothesis, $T_\ell = +\infty$ a.s. hence, since s is a function of $\mathcal{C}^2$ class, X is a semimartingale). Then, we make the same integrations by parts, noticing that the terms between brackets cancel since:

i) H is bounded and $\dfrac{\partial q(t,x,y)}{s'(y)\partial y} \xrightarrow[y\to\ell]{} 0$,

ii) $\lim\limits_{y\to\ell} \dfrac{\partial H(t,y)}{s'(y)\partial y} q(t,x,y) = 0$ by hypothesis.

□

Example 8 (Bessel processes and straight lines: $f(t) = a + bt$)**.**
Let $(R_t, t \geq 0)$ be a Bessel process of index $\nu > 0$ and choose $f(t) = a + bt$ with $a, b > 0$. Then, from Theorem 5 of Alili & Patie [AP10], for $y < a$:

$$\mathbb{P}_y^{(\nu)}(T_{a+b\bullet} \in du) = \exp\left(\frac{b}{2a}(y^2 - a^2) - \frac{b^2}{2}u\right)\left(1 + \frac{b}{a}u\right)^{\nu-1} \sum_{k=1}^{+\infty} \frac{y^{-\nu} j_{\nu,k} J_\nu(j_{\nu,k}\frac{y}{a})}{a^{2-\nu} J_{\nu+1}(j_{\nu,k})} \exp\left(-\frac{u\, j_{\nu,k}^2}{2a(a+bu)}\right) du,$$

where J_ν denotes the Bessel function of the first kind and $(j_{\nu,k})_{k\geq 0}$ the ordered sequence of its positive zeroes. Then,

$$H(t,y) = 1 - \mathbb{P}_y^{(\nu)}(T_{a+bt+b\bullet} < +\infty)$$

and, for $x \geq 0$:

$$\begin{aligned}
\mathbb{P}_x^{(\nu)}(G_{a+b\bullet} \in dt) &= -\frac{1}{2t}(a+bt)^{\nu+1} x^{-\nu} \exp\left(-\frac{x^2 + (a+bt)^2}{2t}\right) I_\nu\left(\frac{x(a+bt)}{t}\right) \frac{\partial H(t,y)}{\partial y}|_{y=a+bt}\, dt, \\
&= -\frac{1}{2t}(a+bt)^{\nu+1} x^{-\nu} \exp\left(-\frac{x^2 + (a+bt)^2}{2t}\right) I_\nu\left(\frac{x(a+bt)}{t}\right) \left(-b + \psi'(a+bt)\right) dt
\end{aligned}$$

with

$$\psi(y) = \int_0^{+\infty} \exp\left(-\frac{b^2}{2}u\right)\left(1 + \frac{b}{a+bt}u\right)^{\nu-1} \sum_{k=1}^{+\infty} \frac{y^{-\nu} j_{\nu,k} J_\nu(j_{\nu,k}\frac{y}{a+bt})}{(a+bt)^{2-\nu} J_{\nu+1}(j_{\nu,k})} \exp\left(-\frac{u\, j_{\nu,k}^2}{2(a+bt)(a+b(t+u))}\right) du.$$

Example 9 (Brownian motion and square boundary: $f(t) = a + bt^2$)**.**
Let $(B_t, t \geq 0)$ be a Brownian motion and define $f(t) = a + bt^2$ with $a, b > 0$. From Salminen [Sal88], we know that:

$$\forall y < a, \qquad \mathbb{P}_y(T_f \in du) = 2(bc)^2 \exp\left(-\frac{2}{3}b^2u^3\right) \sum_{k=0}^{+\infty} \exp\left(\frac{\lambda_k}{c}u\right) \frac{\mathrm{Ai}(\lambda_k + 2bc(a-y))}{\mathrm{Ai}'(\lambda_k)} du$$

where Ai denotes the Airy function, $(\lambda_k)_{k\geq 0}$ its zeroes on the negative half-line and $c = (2b^2)^{-\frac{1}{3}}$. Now fix $t \geq 0$. Applying the Cameron-Martin formula, we deduce that:

$$\mathbb{P}_{y-bt^2}^{(-2bt)}(T_f \in du) = \exp\left(-2bt(a - y + bt^2) - (2bt)bu^2 - \frac{(2bt)^2}{2}u\right) \mathbb{P}_{y-bt^2}(T_f \in du),$$

where $\mathbb{P}_x^{(\mu)}$ denotes the law of a Brownian motion with drift μ started at x. But, since

$$\inf\{u \geq 0;\ (y - bt^2) + B_u - 2btu = a + bu^2\} = \inf\{u \geq 0;\ y + B_u = a + b(u + t)^2\},$$

we obtain that $\mathbb{P}_y(T_{f\circ\theta_t} < +\infty) = \mathbb{P}_{y-bt^2}^{(-2bt)}(T_f < +\infty)$ and

$$H(t,y) = 1 - 2(bc)^2 \exp\left(-\frac{4}{3}b^2t^3 - 2bt(a-y)\right) \times \int_0^{+\infty} \exp\left(-\frac{2}{3}b^2(u+t)^3\right) \sum_{k=0}^{+\infty} \exp\left(\frac{\lambda_k}{c}u\right) \frac{\mathrm{Ai}(\lambda_k + 2bc(a - y + bt^2))}{\mathrm{Ai}'(\lambda_k)} du.$$

Finally, for $x \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}_x(G_f \in dt) &= -\frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{(x-a-bt^2)^2}{2t}\right) \frac{\partial H(t,y)}{\partial y}|_{y=a+bt^2}\, dt, \\ &= -\frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{(x-a-bt^2)^2}{2t}\right) \left(-2bt + \psi'(a+bt^2)\right)\end{aligned}$$

with

$$\psi(y) = -2(bc)^2 \exp\left(\frac{2}{3}b^2t^3\right) \int_0^{+\infty} \exp\left(-\frac{2}{3}b^2(u+t)^3\right) \sum_{k=0}^{+\infty} \exp\left(\frac{\lambda_k}{c}u\right) \frac{\mathrm{Ai}(\lambda_k + 2bc(a - y + bt^2))}{\mathrm{Ai}'(\lambda_k)} du.$$

3 Martingales methods

We now present a method to obtain explicit expressions for the function H (associated with an a priori implicit boundary). Let $\ell \in [-\infty, +\infty[$ and $f : [0, +\infty[\longmapsto]\ell, +\infty[$ be a continuous function. In this section, we shall restrict our attention to lower boundaries, and make the following Assumption:

Assumption 10. *If $\zeta(f) = +\infty$, we assume that:*

$$\lim_{t\to+\infty} X_t = +\infty \ \ a.s \qquad \textit{and} \qquad \mathbb{P}_x\left(\lim_{t\to+\infty} X_t - f(t) > 0\right) = 1.$$

Consider the domains:

$$\mathcal{D} = \{(t,y) \in]0, \zeta(f)[\times]\ell, +\infty[;\ y \geq f(t)\} \quad \text{and} \quad \partial\mathcal{D} := \{(t,y) \in]0, \zeta(f)[\times]\ell, +\infty[;\ y = f(t)\}.$$

Lemma 11. *Assume that Assumption 10 holds and that there exists a function $\overline{H} : \mathcal{D} \longmapsto [0,1]$ of $\mathcal{C}^{1,2}$ class on $\mathcal{D}$ which is solution of the following problem:*

$$\begin{cases} \mathcal{G}\overline{H} + \dfrac{\partial \overline{H}}{\partial t} = 0, \qquad \textit{with the boundary condition } \overline{H}(t,y) = 1 \textit{ on } \partial\mathcal{D}, \\ \overline{H} \textit{ is decreasing in } y \\ \begin{array}{c|c} \textit{If } \zeta(f) = +\infty, & \textit{If } \zeta(f) < +\infty, \\ \lim\limits_{t,y\to+\infty} \overline{H}(t,y) = 0 & \forall y \in]\ell, +\infty[,\ \lim\limits_{t\to\zeta(f)} \overline{H}(t,y) = 0. \end{array} \end{cases} \tag{13}$$

Then,

$$\overline{H}(t,y) = \mathbb{P}_y(T_{f\circ\theta_t} < +\infty).$$

Proof.
Let $t < \zeta(f)$ be fixed. Applying Itô's formula, we deduce that the process $\left(\overline{H}(t+s, X_s), s < (\zeta(f) - t) \wedge T_{f\circ\theta_t}\right)$ is a positive and continuous local martingale. Let $s < \zeta(f) - t$ be fixed. From Doob' stopping theorem, with $f(t) \leq y \leq a$:

$$\begin{aligned}\overline{H}(t,y) &= \mathbb{E}_y\left[\overline{H}(t+s\wedge T_a \wedge T_{f\circ\theta_t}, X_{s\wedge T_a\wedge T_{f\circ\theta_t}})\right] \\ &= \mathbb{E}_y\left[\overline{H}(t+s, X_s)1_{\{s<T_a\wedge T_{f\circ\theta_t}\}}\right] + \mathbb{E}_y\left[\overline{H}(t+T_a, a)1_{\{T_a<s\wedge T_{f\circ\theta_t}\}}\right] + \mathbb{P}_y\left(T_{f\circ\theta_t} < s\wedge T_a\right).\end{aligned}$$

We first let $a \to +\infty$ to obtain, applying the dominated convergence theorem:

$$\overline{H}(t,y) = \mathbb{E}_y\left[\overline{H}(t+s, X_s)1_{\{s<T_{f\circ\theta_t}\}}\right] + \mathbb{P}_y\left(T_{f\circ\theta_t} < s\right).$$

Then, we must distinguish between two cases.

1. If $\zeta(f) = +\infty$, then, we assumed that $\lim_{s\to+\infty} X_s = +\infty$ a.s., so:

   $$\overline{H}(t+s, X_s) \xrightarrow[s\to+\infty]{} 0 \quad \text{a.s.}$$

   and the result follows from the dominated convergence theorem.

2. If $\zeta(f) < +\infty$, then, since $\mathbb{P}_y(X_{\zeta(f)-t} = \ell) = 0$:

   $$\lim_{s\to\zeta(f)-t} \overline{H}(t+s, X_s) = 0 \quad \text{a.s.}$$

   and the result follows once again from the dominated convergence theorem.

□

We shall now give some examples of functions $\overline{H}$ which are solutions of this problem.

3.1 Martingales constructed on the resolvant

We follow the idea of Robbins & Siegmund [RS70]. Let us define, for $\lambda > 0$, the resolvent kernel of $(X_t, t \geq 0)$ by (see [BS02, p.19]):

$$u_\lambda(x,y) = \int_0^\infty e^{-\lambda t} q(t,x,y)dt.$$

Let F be a finite measure on $[0, +\infty[$, and define:

$$\overline{H}(t,y) = \int_0^{+\infty} e^{-\lambda t} u_\lambda(0,y) F(d\lambda).$$

We assume furthermore that:

1. if $\ell = -\infty$, then ℓ is a natural boundary,
2. if $\ell > -\infty$, then ℓ is an entrance-not-exit boundary.

This implies that $\lim_{y\to\ell} u_\lambda(0,y) = +\infty$. Now, since the function $y \longmapsto u_\lambda(0,y)$ is strictly decreasing, we may define a function $f : [0, +\infty[\longrightarrow]\ell, +\infty[$ by

$$\forall t \geq 0, \qquad \overline{H}(t, f(t)) = 1,$$

which is such that $\zeta(f) = +\infty$. Then:

Proposition 12. *The density of G_f is given by:*

$$\mathbb{P}_x(G_f \in dt) = -\frac{q(t,x,f(t))}{s'(f(t))}\left(\int_0^{+\infty} e^{-\lambda t}\frac{\partial u_\lambda(0,y)}{\partial y}|_{y=f(t)}F(d\lambda)\right)dt \qquad (0<t<+\infty).$$

Proof. We need to check that the function $\overline{H}$ satisfies the hypotheses of Lemma 11.
By construction, $\overline{H}$ is a solution to the partial differential equation (13). It is also a decreasing function of y and, since $+\infty$ is a natural boundary, we have from the monotone convergence theorem, $\lim_{t,y\to+\infty}\overline{H}(t,y) = 0$.

□

Example 13 (Brownian motion with drift)**.**
Let $(X_t, t\geq 0)$ be a Brownian motion with drift μ, and choose $F(d\lambda) = 2\mu a\delta_0(d\lambda) + 2\sqrt{2b+\mu^2}\,\delta_b(d\lambda)$. Then:

$$\overline{H}(t,y) = ae^{-2\mu y} + e^{-bt-\left(\sqrt{2b+\mu^2}+\mu\right)y}.$$

Define:

$$\varphi(y) = -\frac{\left(\sqrt{2b+\mu^2}+\mu\right)y + \ln(1-ae^{-2\mu y})}{b} \qquad \text{and} \quad f(t) = \varphi^{-1}(t).$$

f is a decreasing function such that $\lim_{t\to+\infty} f(t) = \dfrac{\ln(a)}{2\mu}$. Then:

$$\mathbb{P}_x(G_f \in dt) = -\frac{q(t,x,f(t))}{s'(f(t))}\left(a\left(\sqrt{2b+\mu^2}-\mu\right)e^{-2\mu f(t)} - \sqrt{2b+\mu^2}-\mu\right)dt.$$

Example 14 (Bessel process of dimension 3)**.**
Let $(X_t, t\geq 0)$ be a Bessel process of dimension 3, and choose $F(d\lambda) = a\delta_0(d\lambda) + \delta_b(d\lambda)$. Then:

$$\overline{H}(t,y) = \frac{a}{y} + \frac{e^{-bt-\sqrt{2b}y}}{y}.$$

Define:

$$\varphi(y) = -\frac{\sqrt{2b}y + \ln(y-a)}{b} \qquad \text{and} \quad f(t) = \varphi^{-1}(t).$$

f is a decreasing function such that $\lim_{t\to+\infty} f(t) = a$. Then :

$$\mathbb{P}_x(G_f \in dt) = \frac{q(t,x,f(t))}{s'(f(t))}\left(\frac{1}{f(t)} + \sqrt{2b}\left(1-\frac{a}{f(t)}\right)\right)dt.$$

3.2 Martingales constructed on the transition density

We assume in this subsection that $(X_t, t\geq 0)$ is defined on $(0,+\infty[$.

3.2.1 Preliminaries

We start by recalling a few properties of the transition density $(t,x) \longmapsto q(t,x,0)$. From Kotani & Watanabe [KW82], it is known that :

$$\lim_{t\to 0}(-2t)\log\left(q(t,x,y)\right) = \left(\int_x^y \sqrt{\frac{\rho(z)s'(z)}{2}}dz\right)^2. \tag{14}$$

In [KW82], this formula was obtained in the special case $s(x) = x$, but, assuming that s is a strictly increasing function of $\mathcal{C}^1$ class, Formula (14) follows easily from the fact that $s(X)$ is a diffusion on natural scale. In particular, for $x > 0$, we deduce that:

$$\lim_{t\to 0} q(t, x, 0) = 0. \tag{15}$$

Next, we define a new diffusion $(\overline{X}_t, t \geq 0)$ whose speed measure $\overline{m}(dx) = \overline{\rho}(x)dx$ and scale function $\overline{s}$ are given by Biane's transform:

$$\begin{cases} \overline{\rho}(x) = (m([0,x]))^2 s'(x) \\ \overline{s}(x) = \dfrac{1}{m([0,+\infty[)} - \dfrac{1}{m([0,x])}. \end{cases}$$

It is known from [PRY10, Chapter 8] that the transition densities of X and $\overline{X}$ satisfy the following relation:

$$q(t, y, 0) = \int_y^\infty \overline{q}(t, 0, z) m([0,z]) s'(z) dz, \tag{16}$$

which implies in particular that for every $t > 0$, the function $y \longmapsto q(t, y, 0)$ is decreasing and tends toward 0 as $y \longrightarrow +\infty$.

3.2.2 First example

Let $\zeta, c > 0$ and consider the function:

$$\overline{H}(t, y) = \frac{1}{c} q(\zeta - t, y, 0).$$

When $c < \inf_{t<\zeta} q(\zeta - t, 0, 0)$ we may define a boundary f by:

$$q(\zeta - t, f(t), 0) = c.$$

In this set-up, $\zeta = \inf\{t \geq 0;\ f(t) = 0\} = \zeta(f)$ from (15). Then:

Proposition 15. *The density of G_f is given by:*

$$\mathbb{P}_x(G_f \in dt) = -\frac{q(t, x, f(t))}{c s'(f(t))} \frac{\partial q(\zeta - t, y, 0)}{\partial y}|_{y=f(t)} dt \qquad (0 < t < \zeta).$$

Proof. From (1), the function $\overline{H}$ is a solution of the PDE (13), which is decreasing from (16) and, for every $y > 0$, $\lim_{t\to\zeta} \overline{H}(t, y) = 0$ from (15).

□

Note that the martingale $(\overline{H}(t, X_t), t < \zeta)$ appears as a density when constructing diffusion bridges via Doob's h-transform, see Fitzsimmons, Pitman & Yor [FPY93].

Example 16 (Radial Ornstein-Uhlenbeck processes)**.**
Let $(X_t, t \geq 0)$ be a squared radial Ornstein-Uhlenbeck process with parameters $\nu > -1$ and $\gamma < 0$, see Example 6. Its transition density function reads (see [BS02, p.142]):

$$q^{(\nu,\gamma)}(t, y, 0) = \frac{\gamma^{\nu+1} e^{\gamma(\nu+1)t}}{2^{\nu+1}\Gamma(\nu+1)(\sinh(\gamma t))^{\nu+1}} \exp\left(-\frac{\gamma e^{-\gamma t} y^2}{2\sinh(\gamma t)}\right).$$

Let $\alpha \in]0,1[$, $\zeta \in]0, +\infty[$ and choose:

$$c = \alpha q^{(\nu,\gamma)}(\zeta, 0, 0)$$

With these values, the boundary $f^{(\nu,\gamma)}$ is then defined, for $t < \zeta$ by:

$$f^{(\nu,\gamma)}(t) = \sqrt{-\frac{2}{\gamma}\sinh(\gamma(\zeta-t))e^{\gamma(\zeta-t)}\ln\left(\alpha\left(\frac{\sinh(\gamma(\zeta-t))}{\sinh(\gamma\zeta)}\right)^{\nu+1}e^{\gamma(\nu+1)t}\right)}$$

and the density of the r.v. G_f is given by:

$$\mathbb{P}_x(G_{f^{(\nu,\gamma)}} \in dt) = \frac{\gamma e^{-\gamma(\zeta-t)}}{c\sinh(\gamma(\zeta-t))}\left(f^{(\nu,\gamma)}(t)\right)^{2\nu+2}e^{-\gamma\left(f^{(\nu,\gamma)}(t)\right)^2}q^{(\nu,\gamma)}(t,x,f^{(\nu,\gamma)}(t))dt \qquad (0<t<\zeta).$$

Example 17 (Bessel process)**.**
Letting $\gamma \to 0$ in the previous example, we obtain the following boundary for the Bessel process of index ν:

$$f^{(\nu)}(t) = \sqrt{-2(\zeta-t)\ln\left(\alpha\left(1-\frac{t}{\zeta}\right)^{\nu+1}\right)},$$

and the density for the last passage time to $f^{(\nu)}$ reads:

$$\mathbb{P}_x^{(\nu)}(G_{f^{(\nu)}} \in dt) = \frac{1}{c(\zeta-t)}\left(f^{(\nu)}(t)\right)^{2\nu+2}q^{(\nu)}(t,x,f^{(\nu)}(t))dt, \qquad (0<t<\zeta),$$

with $c = \dfrac{\alpha}{2^{\nu+1}\Gamma(\nu+1)t^{\nu+1}}$.

3.2.3 Second example

We assume that 0 is an entrance-not-exit boundary point and that $(X_t, t \geq 0)$ is transient and goes towards $+\infty$ as $t \to +\infty$. Let $h : [0,+\infty[\longrightarrow]0,+\infty[$ be a bounded function of $\mathcal{C}^1$ class with bounded derivative. We consider the function:

$$\overline{H}(t,y) = \int_0^{+\infty} h(t+u)q(u,y,0)du.$$

Since we have assumed that 0 is entrance-not-exit, $u_\lambda(0,y) \xrightarrow[y\to 0]{} +\infty$, i.e. $q(t,0,0)$ is not integrable at 0.
Therefore, for every $t \geq 0$, the function $y \longmapsto \overline{H}(t,y)$ is decreasing from $+\infty$ to 0, and we may define a function f by:

$$\int_0^{+\infty} h(t+u)q(u,f(t),0)du = 1.$$

Note that by construction, $\zeta(f) = +\infty$.

Proposition 18. *The density of G_f is given by:*

$$\mathbb{P}_x(G_f \in dt) = -\frac{q(t,x,f(t))}{s'(f(t))}\left(\int_0^{+\infty} h(u+t)\frac{\partial q(u,y,0)}{\partial y}|_{y=f(t)}du\right)dt \qquad (t>0).$$

Remark 19. We may recover (partly) the result of Subsection 3.1 by taking $h(u) = \int_0^{+\infty} e^{-\lambda u}F(d\lambda)$. Indeed, in this case, from Fubini-Tonelli:

$$\int_0^{+\infty} h(u+t)q(u,y,0)du = \int_0^{+\infty}\left(\int_0^{+\infty} e^{-\lambda(t+u)}F(d\lambda)\right)q(u,y,0)du = \int_0^{+\infty} e^{-\lambda t}u_\lambda(y,0)F(d\lambda).$$

Proof. We need to prove that $\overline{H}$ satisfies the hypotheses of Lemma 11.
Since h is bounded, and for $y > 0$, $u \longmapsto q(u,y,0)$ is integrable, the dominated convergence theorem implies that :

$$\lim_{t,y\to+\infty} \overline{H}(t,y) = 0.$$

Then, integrating by parts, for $y > 0$:

$$\begin{aligned}\frac{\partial \overline{H}(t,y)}{\partial t} &= \int_0^{+\infty} h'(t+u)q(u,y,0)du \qquad \text{(which is finite since } h' \text{ is bounded)}\\ &= \Big[h(t+u)q(u,y,0)\Big]_0^{+\infty} - \int_0^{+\infty} h(t+u)\frac{\partial q(u,y,0)}{\partial u}du\\ &= -\left(\int_0^{+\infty} h(t+u)\mathcal{G}q(u,y,0)du\right)\\ &= -\mathcal{G}\overline{H}(t,y),\end{aligned}$$

which ends the proof.

□

Remark 20. We may remove the hypothesis $(X_t, t \geq 0)$ is transient if we replace the assumption on h by : h is a decreasing and integrable function of $\mathcal{C}^1$ class. Indeed, since for $y > 0$ the function $u \longmapsto q(u,y,0)$ is bounded, $\overline{H}$ is well-defined and so is $\int_0^{+\infty} h'(t+u)q(u,y,0)du$. Besides, we still have $\lim_{t,y\to+\infty} \overline{H}(t,y) = 0$ from monotone convergence.

4 Inverting time

Consider a diffusion $(X_t, t \geq 0)$ enjoying the inversion property in the sense of Watanabe [Wat75], i.e. such that the process $(\overline{X}_t = tX_{\frac{1}{t}}, t \geq 0)$ is also a linear regular conservative diffusion. Let f be a continuous function and define:

$$\overline{f} : t \longmapsto tf\left(\frac{1}{t}\right).$$

Then, if $\overline{X}_0 \neq \lim_{t\to 0} \overline{f}(t)$:

$$\begin{aligned}\inf\left\{t \geq 0;\, \overline{X}_t = \overline{f}(t)\right\} &= \inf\left\{t \geq 0;\, tX_{\frac{1}{t}} = tf\left(\frac{1}{t}\right)\right\}\\ &= \inf\left\{t \geq 0;\, X_{\frac{1}{t}} = f\left(\frac{1}{t}\right)\right\}\\ &= \frac{1}{\sup\{u \geq 0;\, X_u = f(u)\}}.\end{aligned}$$

In particular, under the hypotheses of Theorem 1 or 7, the density of the first hitting time of $\overline{f}$ by $(\overline{X}_t, t \geq 0)$ admits the expression:

$$\overline{\mathbb{P}}_{\overline{x}}(T_{\overline{f}} \in dt) = \frac{1}{t^2}\Phi\left(\frac{1}{t}\right) q\left(\frac{1}{t}, x, f\left(\frac{1}{t}\right)\right) dt.$$

4.1 Brownian motion

Consider a Brownian motion $(B_t, t \geq 0)$ started from 0. Then, it is well-known that $(\overline{B}_t, t \geq 0)$ is also a Brownian motion started at 0.

Example 21. Take $f(u) = a + bu^2$ with $a, b > 0$. Then, $\overline{f}(t) = at + \dfrac{b}{t}$ and :

$$\inf\left\{t \geq 0;\, \overline{B}_t = at + \frac{b}{t}\right\} = \frac{1}{\sup\{u \geq 0;\, B_u = a + bu^2\}}$$

From Example 9, we deduce that:

$$\overline{\mathbb{P}}_0(T_{\overline{f}} \in dt) = \Phi\left(\frac{1}{t}\right) \frac{1}{2t\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(at + \frac{b}{t}\right)^2\right) dt$$

where Φ is defined by :

$$\Phi\left(\frac{1}{t}\right) = \frac{2b}{t} - \psi'\left(a + \frac{b}{t^2}\right)$$

with

$$\psi(y) = -2(bc)^2 \exp\left(\frac{2}{3}\frac{b^2}{t^3}\right) \int_0^{+\infty} \exp\left(-\frac{2}{3}b^2\left(u + \frac{1}{t}\right)^3\right) \sum_{k=0}^{+\infty} \exp\left(\frac{\lambda_k}{c}u\right) \frac{\mathrm{Ai}\left(\lambda_k + 2bc\left(a - y + \frac{b}{t^2}\right)\right)}{\mathrm{Ai}'(\lambda_k)} du.$$

with $(\lambda_k)_{k\geq 0}$ the negative zeroes of the Airy function Ai and $c = (2b^2)^{-1/3}$.

4.2 Bessel processes with drift

Consider a Bessel process $(R_t, t \geq 0)$ with index $\nu > 0$ and drift $c \geq 0$ started from x. $(R_t, t \geq 0)$ is a diffusion whose generator is given by:

$$\mathcal{G}^{(\nu,c)} = \frac{1}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{2\nu+1}{2x} + c\frac{I_{\nu+1}(cx)}{I_\nu(cx)}\right)\frac{\partial}{\partial x}.$$

This process was first introduced by Watanabe in [Wat75] as a generalization of Bessel processes. For integer dimension of $n = 2(\nu+1) \in \mathbb{N}$, $(R_t, t \geq 0)$ has the same law as the norm $\|\vec{B}_t + \vec{\mu}\,.\,\vec{t}\|$ where $(\vec{B}_t, t \geq 0)$ is an n-dimensional Brownian motion and $\|\vec{\mu}\| = c$.
The scale function of R is given by:

$$(s^{(\nu,c)})'(x) = \frac{1}{(\Gamma(\nu+1))^2}\left(\frac{c}{2}\right)^{2\nu}\frac{1}{xI_\nu^2(cx)}$$

and its speed measure by:

$$m^{(\nu,c)}(dx) = 2(\Gamma(\nu+1))^2\left(\frac{2}{c}\right)^{2\nu} xI_\nu^2(cx)dx.$$

From Watanabe [Wat75, Theorem 2.1], the process $\left(\overline{R}_t = tR_{\frac{1}{t}}, t \geq 0\right)$ is a Bessel process with index ν and drift x started from c.

Example 22. Take $f(u) = a + bu$ with $a, b > 0$. Then $\overline{f}(t) = b + at$ and

$$\inf\{t \geq 0;\, \overline{R}_t = b + at\} = \frac{1}{\sup\{u \geq 0;\, R_u = a + bu\}}$$

which leads to the density of the r.v. $T_{b+a.}$ under the form:

$$\mathbb{P}_0^{(\nu,x)}(T_{b+a.} \in dt) = \frac{1}{2t}\Phi\left(\frac{1}{t}\right)\left(x\left(a + \frac{b}{t}\right)\right)^{-\nu} \exp\left(-\frac{x^2t^2 + (at+b)^2}{2t}\right) I_\nu\left(x(at+b)\right) dt$$

where the function Φ is given by (see Example 8):

$$\Phi\left(\frac{1}{t}\right) = \left(a + \frac{b}{t}\right)^{2\nu+1}\left(b - \psi'\left(a + \frac{b}{t}\right)\right)$$

with

$$\psi(y) = \int_0^{+\infty}\left(1 + \frac{bt}{at+b}u\right)^{\nu-1}\sum_{k=1}^{+\infty} t^2\frac{(yt)^{-\nu}j_{\nu,k}J_\nu(j_{\nu,k}\frac{yt}{at+b})}{(at+b)^{2-\nu}J_{\nu+1}(j_{\nu,k})}\exp\left(-\frac{u\,t^2\,j_{\nu,k}^2}{2(at+b)(at+b(1+tu))} - \frac{b^2}{2}u\right)du,$$

where J_ν denotes the Bessel function of the first kind and $(j_{\nu,k})_{k\geq 0}$ the ordered sequence of its positive zeroes.

5 On an integral equation

We set:

$$\Phi(t) = \frac{1}{s'(f(t))} \frac{\partial}{\partial y} \mathbb{P}_y \left(T_{f \circ \theta_t} = +\infty \right) |_{y=f(t)}.$$

Integrating (5) and (11) with respect to t, we deduce the following corollary:

Corollary 23. *Assume that the hypotheses of Theorem 1, resp. 7, are satisfied. Then, the function Φ is a solution of the following Fredholm equation of the first kind:*

- *For lower boundaries:*

$$\int_0^{\zeta(f)} \Phi(t) q(t, x, f(t)) dt = 1, \qquad x < f(0).$$

- *Resp, for upper boundaries:*

$$\int_0^{+\infty} \Phi(t) q(t, x, f(t)) dt = 1, \qquad x > f(0).$$

In some particular cases, this equation may charaterize uniquely Φ, hence the law of G_f. We give below a few examples of this situation, where time inversion is involved.

5.1 A link with time inversion

Theorem 24.

i) Let $(B_t^{(\mu)}, t \geq 0)$ be a Brownian motion with drift $\mu > 0$ started from x and f be a continuous function on $[0, +\infty[$ such that

$$f(0) = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} \frac{f(t)}{t} < \mu.$$

Assume that the equation :

$$\forall x < 0, \qquad \int_0^{+\infty} \Phi(t) q(t, x, f(t)) dt = 1,$$

admits a unique solution Φ. Then :

$$\mathbb{P}_x^{(\mu)} \left(G_f \in dt \right) = \Phi(t) q(t, x, f(t)) dt.$$

ii) Let $(R_t, t \geq 0)$ be a Bessel process with index $\nu > -1$ and drift $c \geq 0$ started from $x > 0$, and let f be a continuous function on $[0, +\infty[$ such that:

$$f(0) = 0 \qquad \text{and} \qquad \lim_{t \to +\infty} \frac{f(t)}{t} > c.$$

Assume that the equation

$$\forall x > 0, \qquad \int_0^{+\infty} \Phi(t) q^{(\nu,c)}(t, x, f(t)) dt = 1,$$

admits a unique solution Φ. Then for $x > 0$:

$$\mathbb{P}_x^{(\nu,c)} \left(G_f \in dt \right) = \Phi(t) q^{(\nu,c)}(t, x, f(t)) dt.$$

Proof.
i) The transition density of $(B_t^{(\mu)}, t \geq 0)$ reads:

$$q(t,z,y) = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\mu(z+y) - \frac{\mu^2}{2}t - \frac{(z-y)^2}{2t}\right).$$

Observe first that:

$$q(t,z,y) = q(t,0,y) \exp\left(-\frac{z^2}{2t} + \frac{zy}{t} - \mu z\right),$$

hence, by hypothesis, Φ is the unique solution of the equation:

$$\forall z < 0, \qquad \int_0^{+\infty} \Phi(t) q(t,0,f(t)) \exp\left(-\frac{z^2}{2t} + \frac{zf(t)}{t}\right) dt = e^{\mu z},$$

or, with the change of variable $s = \frac{1}{t}$:

$$\forall z < 0, \qquad \int_0^{+\infty} \Phi\left(\frac{1}{s}\right) q\left(\frac{1}{s}, 0, f\left(\frac{1}{s}\right)\right) \exp\left(-\frac{z^2}{2}s + zsf\left(\frac{1}{s}\right)\right) \frac{ds}{s^2} = e^{\mu z}.$$

Now, let $\overline{f}(s) = sf\left(\frac{1}{s}\right)$ and consider, for $z < 0$, the exponential Brownian martingale $\left(\exp\left(zB_s - \frac{z^2}{2}s\right), s \geq 0\right)$. Doob' stopping theorem implies, since $\mu > \lim_{s\to 0} \overline{f}(s)$ by hypothesis:

$$e^{z\mu} = \mathbb{E}_\mu\left[\exp\left(zB_{T_{\overline{f}}} - \frac{z^2}{2}T_{\overline{f}}\right)\right] = \int_0^{+\infty} e^{z\overline{f}(s) - \frac{z^2}{2}s} \, \mathbb{P}_\mu\left(T_{\overline{f}} \in ds\right).$$

Since $\overline{f}$ is a positive function, $T_{\overline{f}}$ admits a density and by unicity of the solution Φ, we have:

$$\mathbb{P}_\mu\left(T_{\overline{f}} \in ds\right) = \frac{1}{s^2}\Phi\left(\frac{1}{s}\right) q\left(\frac{1}{s}, 0, f\left(\frac{1}{s}\right)\right) ds.$$

Next, the Cameron-Martin formula gives:

$$\mathbb{P}_\mu^{(x)}\left(T_{\overline{f}} < s\right) = \mathbb{E}_\mu\left[\exp\left(xB_{T_{\overline{f}}} - \frac{x^2}{2}T_{\overline{f}} - \mu x\right) 1_{\{T_{\overline{f}} < s\}}\right],$$

i.e. the density of the r.v. $T_{\overline{f}}$ equals:

$$\mathbb{P}_\mu^{(x)}\left(T_{\overline{f}} \in ds\right) = \exp\left(x\overline{f}(s) - \frac{x^2}{2}s - \mu x\right) \frac{1}{s^2}\Phi\left(\frac{1}{s}\right) q\left(\frac{1}{s}, 0, f\left(\frac{1}{s}\right)\right) ds = \frac{1}{s^2}\Phi\left(\frac{1}{s}\right) q\left(\frac{1}{s}, x, f\left(\frac{1}{s}\right)\right) ds.$$

The result then follows from the time inversion property.

ii) We shall proceed similarly for Bessel processes with drift. The transition density of $(R_t, t \geq 0)$ reads:

$$q^{(\nu,c)}(t,z,y) = \frac{1}{2t(\Gamma(\nu+1))^2}\left(\frac{c}{2}\right)^{2\nu} \frac{\exp\left(-\frac{c^2 t}{2}\right)}{I_\nu(cz)I_\nu(cy)} \exp\left(-\frac{z^2+y^2}{2t}\right) I_\nu\left(\frac{zy}{t}\right),$$

hence, let Φ be the unique solution of the equation:

$$\forall z > 0, \qquad \int_0^{+\infty} \frac{\Phi(t)}{2t(\Gamma(\nu+1))^2}\left(\frac{c}{2}\right)^{2\nu} \frac{\exp\left(-\frac{c^2 t}{2} - \frac{(f(t))^2}{2t}\right)}{I_\nu(cf(t))} \exp\left(-\frac{z^2}{2t}\right) \frac{I_\nu\left(\frac{zf(t)}{t}\right)}{I_\nu(cz)} dt = 1.$$

Consider the local martingale under $\mathbb{P}_c^{(\nu,x)}$:

$$\left(M_t = e^{-\lambda t} \frac{I_\nu\left(X_t\sqrt{2\lambda + x^2}\right)}{I_\nu(xX_t)}, t \geq 0 \right).$$

Applying Doob' stopping theorem to $(M_{T_{\overline{f}} \wedge T_\varepsilon \wedge t}, t \geq 0)$ and letting $t \to +\infty$ and $\varepsilon \to 0$, we obtain (since $c < \lim_{t\to 0} \overline{f}(t)$ and $\overline{f}(t) \underset{+\infty}{=} o(t)$ by hypothesis):

$$\frac{I_\nu\left(c\sqrt{2\lambda + x^2}\right)}{I_\nu(xc)} = \mathbb{E}_c^{(\nu,x)}\left[M_{T_{\overline{f}}}\right] = \int_0^{+\infty} e^{-\lambda t} \frac{I_\nu\left(\overline{f}(t)\sqrt{2\lambda + x^2}\right)}{I_\nu(x\overline{f}(t))} \mathbb{P}_c^{(\nu,x)}(T_{\overline{f}} \in dt).$$

Now, from Watanabe [Wat75]:

$$\mathbb{P}_c^{(\nu,x)}\left(\lim_{t\to+\infty} \frac{X_t}{t} = x\right) = 1,$$

hence, since $\overline{f}(t) \underset{+\infty}{=} o(t)$, we deduce that, for $x > 0$, the r.v. $T_{\overline{f}}$ admits a density (see also [Leh02]) and, setting $2\lambda + x^2 = z^2$, we obtain from the unicity of the solution Φ:

$$\frac{I_\nu(cx)}{I_\nu(x\overline{f}(t))} e^{\frac{x^2}{2}t} \mathbb{P}_c^{(\nu,x)}\left(T_{\overline{f}} \in dt\right) = \frac{1}{t^2} \frac{t\Phi\left(\frac{1}{t}\right)}{2(\Gamma(\nu+1))^2} \left(\frac{c}{2}\right)^{2\nu} \frac{\exp\left(-\frac{c^2}{2t} - \frac{t}{2}\left(f\left(\frac{1}{t}\right)\right)^2\right)}{I_\nu\left(cf\left(\frac{1}{t}\right)\right)} dt.$$

The result then follows once again from a time inversion argument.
Observe that, if $\mathbb{P}_c^{(\nu,0)}(T_{\overline{f}} < +\infty) = 1$, (for instance if f is of $\mathcal{C}^1$ class in the neighborhood of 0), then the result also holds for $x = 0$.

□

It might be noticed that for these two processes, the proof above shows anew the phenomenon of separation of variables which appears in the law of G_f. We shall now give an example of each situation.

5.2 Brownian motion with drift and $f(t) = a + b\sqrt{t}$

Let $(B_t^{(\mu)}, t \geq 0)$ be a Brownian motion with drift $\mu > 0$ and choose $f(t) = a + b\sqrt{t}$ with $a, b \in \mathbb{R}$.

Proposition 25. *The density of the last passage time $G_{a+b\sqrt{\cdot}}$ is given, for every $x \in \mathbb{R}$, by:*

$$\mathbb{P}_x^{(\mu)}\left(G_{a+b\sqrt{\cdot}} \in dt\right) = \varphi(t) \exp\left(-\frac{(x-a)^2}{2t} + \frac{(x-a)b}{2\sqrt{t}} - \mu(x-a) - \frac{b^2}{4}\right) dt$$

where the function φ has Mellin's transform:

$$\int_0^{+\infty} t^{\lambda-1}\varphi(t)dt = \frac{1}{\mu^{2\lambda-1} D_{-2\lambda+1}(b)}.$$

In particular, if $x = a$, we deduce that:

$$\mathbb{E}_a\left[G_{a+b\sqrt{\cdot}}^{\lambda-1}\right] = \frac{\exp\left(-\frac{b^2}{4}\right)}{\mu^{2\lambda-1} D_{-2\lambda+1}(b)}.$$

Proof. We need to prove that the equation:

$$\forall z > 0, \qquad \int_0^{+\infty} \Phi(t) q(t, 0, b\sqrt{t}) \exp\left(-\frac{z^2}{2t} - \frac{zb}{\sqrt{t}}\right) dt = \exp\left(-\mu z + 2\mu a\right), \tag{17}$$

admits a unique solution Φ, in order to apply Theorem 24. Let us recall the following formula ([GR07, 3.462-1]):

$$\int_0^{+\infty} z^{\nu-1} \exp\left(-\frac{\beta}{2}z^2 - \gamma z\right) dz = \frac{\Gamma(\nu)}{(\beta)^{\nu/2}} \exp\left(\frac{\gamma^2}{4\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{\beta}}\right) \qquad \text{with } \beta, \nu > 0, \text{ and } \gamma \in \mathbb{R},$$

where $D_{-\nu}$ denotes the parabolic cylinder function of index $-\nu$. Integrating Equation (17) with respect to $z^{\nu-1}dz$ and applying Fubini-Tonelli, we obtain:

$$\int_0^{+\infty} \Phi(t) q(t, 0, b\sqrt{t}) \Gamma(\nu) t^{\nu/2} \exp\left(\frac{b^2}{4}\right) D_{-\nu}(b)\, dt = \frac{\Gamma(\nu)}{\mu^\nu} \exp(2\mu a)$$

which, by setting $\lambda - 1 = \frac{\nu}{2}$, gives the Mellin's transform:

$$\int_0^{+\infty} \Phi(t) q(t, 0, b\sqrt{t}) t^{\lambda-1} dt = \frac{\exp\left(2\mu a - \frac{b^2}{4}\right)}{\mu^{2\lambda-1} D_{-2\lambda+1}(b)}.$$

Since Mellin's transform is injective, the result follows from Theorem 24.

□

5.3 Bessel process with drift and $f(t) = \sqrt{at^2 + bt}$

Let $(R_t, t \geq 0)$ be a Bessel process with index $\nu > -1$ and drift $c \geq 0$ started from x, and choose $f(t) = \sqrt{at^2 + bt}$ with $b > 0$ and $\sqrt{a} > c$.

Proposition 26. *The density of the last passage time $G_{\sqrt{a.^2+b.}}$ is given, for every $x \geq 0$, by:*

$$\mathbb{P}_x^{(\nu,c)}\left(G_{\sqrt{a.^2+b.}} \in dt\right) = \varphi\left(\ln\left(1 + \frac{b}{at}\right)\right) \frac{b}{ct} \frac{1}{\sqrt{at^2+bt}} \frac{\exp\left(-\frac{x^2}{2t}\right)}{I_\nu(cx)} I_\nu\left(\frac{x\sqrt{at^2+bt}}{t}\right) dt$$

where the function φ has Laplace transform:

$$\int_0^{+\infty} e^{-\lambda t} \varphi(t) dt = \frac{\exp\left(\frac{c^2 b}{4a}\right) M_{-\lambda,\nu/2}\left(\frac{c^2 b}{2a}\right)}{\exp\left(\frac{b}{4}\right) M_{-\lambda,\nu/2}\left(\frac{b}{2}\right)}. \tag{18}$$

Proof. We need to prove that the equation :

$$\forall x > 0, \qquad \int_0^{+\infty} \frac{\Phi(t)}{2t(\Gamma(\nu+1))^2} \left(\frac{c}{2}\right)^{2\nu} \frac{e^{-\frac{c^2 t}{2}}}{I_\nu(cf(t))} \exp\left(-\frac{x^2 + (f(t))^2}{2t}\right) I_\nu\left(\frac{x\sqrt{at^2+bt}}{t}\right) dt = I_\nu(cx), \tag{19}$$

admits a unique solution Φ in order to apply Theorem 24. To simplify the notation, set

$$\Psi(t) = \frac{\Phi(t)}{2t(\Gamma(\nu+1))^2} \left(\frac{c}{2}\right)^{2\nu} \frac{e^{-\frac{c^2 t}{2}}}{I_\nu(cf(t))} \exp\left(-\frac{(f(t))^2}{2t}\right).$$

Recall the formula [GR07, 6.643-2], for $\alpha > 0, \beta \in \mathbb{R}$ and $\lambda + \nu + \frac{1}{2} > 0$:

$$\int_0^{+\infty} x^{2\lambda} e^{-\alpha x^2} I_{2\nu}(2\beta x) dx = \frac{\Gamma(\lambda + \nu + \frac{1}{2})}{2\Gamma(2\nu+1)} \frac{1}{\beta \alpha^\lambda} \exp\left(\frac{\beta^2}{2\alpha}\right) M_{-\lambda,\nu}\left(\frac{\beta^2}{\alpha}\right)$$

where $M_{-\lambda,\nu}$ denotes the Whittaker function. We integrate (19) with respect to $x^{2\lambda} e^{-\frac{a}{2b}x^2} dx$, to obtain:

$$\int_0^{+\infty} \Psi(t) \frac{2t}{\sqrt{at^2+bt}} \frac{e^{\frac{b}{4}}}{(\frac{a}{2b} + \frac{1}{2t})^\lambda} M_{-\lambda,\nu/2}\left(\frac{b}{2}\right) dt = \frac{2}{c} \left(\frac{2b}{a}\right)^\lambda \exp\left(\frac{c^2 b}{4a}\right) M_{-\lambda,\nu/2}\left(\frac{c^2 b}{2a}\right).$$

This expression simplifies to:

$$\int_0^{+\infty} \Psi(t)\sqrt{\frac{t}{at+b}}\frac{1}{(1+\frac{b}{at})^{\lambda}}dt = \frac{1}{c}\exp\left(\frac{c^2 b}{4a}-\frac{b}{4}\right)\frac{M_{-\lambda,\nu/2}\left(\frac{c^2 b}{2a}\right)}{M_{-\lambda,\nu/2}\left(\frac{b}{2}\right)}.$$

Finally, we make the change of variable $e^u = 1+\dfrac{b}{at}$:

$$\int_0^{+\infty} e^{-\lambda u}\Psi\left(\frac{b}{a(e^u-1)}\right)\frac{e^{\frac{u}{2}}}{(e^u-1)^2}du = \frac{a\sqrt{a}}{bc}\exp\left(\frac{c^2 b}{4a}-\frac{b}{4}\right)\frac{M_{-\lambda,\nu/2}\left(\frac{c^2 b}{2a}\right)}{M_{-\lambda,\nu/2}\left(\frac{b}{2}\right)}$$

which gives the Laplace transform of Ψ up to a few transformations.

□

Remark 27. Taking $\lambda = \alpha - \frac{\nu+1}{2}$, we obtain, since $\sqrt{a} > c$:

$$\int_0^{+\infty} e^{-\alpha t}e^{\frac{\nu+1}{2}t}\varphi(t)dt = \frac{\exp\left(\frac{c^2 b}{4a}\right)M_{-\alpha+\frac{\nu+1}{2},\nu/2}\left(\frac{c^2 b}{2a}\right)}{\exp\left(\frac{b}{4}\right)M_{-\alpha+\frac{\nu+1}{2},\nu/2}\left(\frac{b}{2}\right)} = \left(\frac{a}{c^2}\right)^{\frac{\nu+1}{2}}\mathbb{Q}^{(\nu,\frac{1}{2})}_{c\sqrt{\frac{b}{a}}}\left[e^{-\alpha T_{\sqrt{b}}}\right]$$

where $\mathbb{Q}^{(\nu,\gamma)}_x$ denotes the law of a radial Ornstein-Uhlenbeck process with parameters ν and γ started at x. Therefore:

$$\varphi(t)dt = \exp\left(-\frac{\nu+1}{2}t\right)\left(\frac{a}{c^2}\right)^{\frac{\nu+1}{2}}\mathbb{Q}^{(\nu,\frac{1}{2})}_{c\sqrt{\frac{b}{a}}}\left(T_{\sqrt{b}}\in dt\right).$$

Corollary 28. *Take $f(u) = \sqrt{au^2+bu}$ with $b>0$ and $\sqrt{a}>c$. Then $\overline{f}(t) = \sqrt{a+bt}$ and:*

$$\inf\{t\geq 0;\, \overline{R}_t = \sqrt{a+bt}\} = \frac{1}{\sup\{u\geq 0;\, R_u = \sqrt{au^2+bu}\}}$$

which leads to the density of the r.v. $T_{\sqrt{a+b\bullet}}$ under the form:

$$\mathbb{P}^{(\nu,x)}_c(T_{\sqrt{a+b\bullet}}\in dt) = \varphi\left(\ln\left(1+\frac{bt}{a}\right)\right)\frac{b}{c}\frac{1}{\sqrt{a+bt}}\frac{\exp\left(-\frac{x^2}{2}t\right)}{I_\nu(cx)}I_\nu\left(x\sqrt{a+bt}\right)dt$$

where the function φ is given by (18).

References

[AP10] L. Alili and P. Patie. Boundary crossing identities for diffusions having the time-inversion property. *J. Theoret. Probab.*, 23(1):65–84, 2010.

[BNR87] A. Buonocore, A. G. Nobile, and L. M. Ricciardi. A new integral equation for the evaluation of first-passage-time probability densities. *Adv. in Appl. Probab.*, 19(4):784–800, 1987.

[BS02] A. N. Borodin and P. Salminen. *Handbook of Brownian motion—facts and formulae*. Probability and its Applications. Birkhäuser Verlag, Basel, second edition, 2002.

[FPY93] P. Fitzsimmons, J. Pitman, and M. Yor. Markovian bridges: construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)*, volume 33 of *Progr. Probab.*, pages 101–134. Birkhäuser Boston, Boston, MA, 1993.

[GNRS89] V. Giorno, A. G. Nobile, L. M. Ricciardi, and S. Sato. On the evaluation of first-passage-time probability densities via non-singular integral equations. *Adv. in Appl. Probab.*, 21(1):20–36, 1989.

[GR07] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, seventh edition, 2007.

[Gro89] P. Groeneboom. Brownian motion with a parabolic drift and Airy functions. *Probab. Theory Related Fields*, 81(1):79–109, 1989.

[GS73] R. K. Getoor and M. J. Sharpe. Last exit times and additive functionals. *Ann. Probability*, 1:550–569, 1973.

[IM74] K. Itô and H. P. McKean. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[KW82] S. Kotani and S. Watanabe. Kreĭn's spectral theory of strings and generalized diffusion processes. In *Functional analysis in Markov processes (Katata/Kyoto, 1981)*, volume 923 of *Lecture Notes in Math.*, pages 235–259. Springer, Berlin, 1982.

[Leh02] A. Lehmann. Smoothness of first passage time distributions and a new integral equation for the first passage time density of continuous Markov processes. *Adv. in Appl. Probab.*, 34(4):869–887, 2002.

[Nov81] A. A. Novikov. A martingale approach to first passage problems and a new condition for Wald's identity. In *Stochastic differential systems (Visegrád, 1980)*, volume 36 of *Lecture Notes in Control and Information Sci.*, pages 146–156. Springer, Berlin, 1981.

[Pes02] G. Peskir. On integral equations arising in the first-passage problem for Brownian motion. *J. Integral Equations Appl.*, 14(4):397–423, 2002.

[Por67] S. C. Port. Hitting times for transient stable processes. *Pacific J. Math.*, 21:161–165, 1967.

[PRY10] C. Profeta, B. Roynette, and M. Yor. *Option prices as probabilities*. Springer Finance. Springer-Verlag, Berlin, 2010. A new look at generalized Black-Scholes formulae.

[PY81] J. Pitman and M. Yor. Bessel processes and infinitely divisible laws. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 285–370. Springer, Berlin, 1981.

[RS70] H. Robbins and D. Siegmund. Boundary crossing probabilities for the Wiener process and sample sums. *Ann. Math. Statist.*, 41:1410–1429, 1970.

[RSS84] L. M. Ricciardi, L. Sacerdote, and S. Sato. On an integral equation for first-passage-time probability densities. *J. Appl. Probab.*, 21(2):302–314, 1984.

[Sal88] P. Salminen. On the first hitting time and the last exit time for a Brownian motion to/from a moving boundary. *Adv. in Appl. Probab.*, 20(2):411–426, 1988.

[Tak67] J. Takeuchi. Moments of the last exit times. *Proc. Japan Acad.*, 43:355–360, 1967.

[Wat75] S. Watanabe. On time inversion of one-dimensional diffusion processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:115–124, 1974/75.