Global existence for a damped wave equation and convergence towards a solution of the Navier-Stokes problem

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Abstract

In two and three space dimensions, and under suitable assumptions on the initial data, we show global existence for a damped wave equation which approaches, in some sense, the Navier-Stokes problem. The proofs are based on a refinement of the energy method in [2].

In this paper, we improve the results of [2] and [8]. We relax the regularity of the initial data of the former, even though we still use energy methods as a principal tool. Regarding [8], the improvement consists in requiring less regularity for the convergence to the Navier-Stokes problem and in the simplicity of the proofs. In particular, we do not use any Strichartz estimate. Nevertheless, we lose the natural Strichartz regularity for the global existence results.

1 Introduction

First, let us recall the Navier-Stokes equations that describe the motion of a viscous, homegeneous and incompressible fluid

$$\left\{\begin{array}{l} \partial_t v - \Delta v = -\nabla : v \otimes v - \nabla p \\ \nabla . v = 0 \\ v|_{t=0} = v_0, \end{array}\right. \tag{1}$$

where v is the velocity of the fluid and p the pressure, assumed to vanish at infinity, in the sense that

$$\frac{1}{|B(0,R)|}\int_{B(0,R)} p \ \longrightarrow \ 0 \ , \quad R \to +\infty. \tag{2}$$

The hyperbolic version of the Navier-Stokes equations studied here has been obtained after relaxation of the Euler equations and rescaling variables (see [2] for the details):

$$\left\{\begin{array}{l} \varepsilon \partial_{tt} u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon = -\nabla : u^\varepsilon \otimes u^\varepsilon - \nabla p^\varepsilon \\ \nabla . u^\varepsilon = 0 \\ (u^\varepsilon, \partial_t u^\varepsilon)|_{t=0} = (u_0^\varepsilon, u_1^\varepsilon). \end{array}\right. \tag{3}$$

For details on the justification of this approximation, see [2, 7] and references therein. This type of approximation is the same as the hyperbolic perturbation of the heat equation introduced and investigated by Cattaneo in the fifties (see [3] and further works).

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In [2], Brenier, Natalini and Puel proved global existence and uniqueness for the perturbed Navier-Stokes equation (3) with initial data in $H^2(\mathbb{T}^2)^2 \times H^1(\mathbb{T}^2)^2$, where $\mathbb{T}^2$ is the unit periodic square $\mathbb{R}^2/\mathbb{Z}^2$, if $\varepsilon < \varepsilon_0$ and ε_0 depends only on the initial data $(u_0^\varepsilon, u_1^\varepsilon)$. Moreover, they proved the convergence of the solution to (3) towards a smooth solution of (1) in the $L^\infty(]0,T[;L^2(\mathbb{R}^2))$ norm, provided that v_0 is smooth.

In [8], Paicu and Raugel considered the same relaxed model (3). In their paper, they proved, again if ε is small enough, global existence and uniqueness results for (3) with significantly improved regularity for the initial data. In fact, they need only $H^1(\mathbb{R}^2)^2 \times L^2(\mathbb{R}^2)^2$ regularity, thanks to a Strichartz estimate. In the three-dimensional case, they state a global existence result under a natural smallness condition on the initial data in $H^{1+\delta}(\mathbb{R}^3)^3 \times H^\delta(\mathbb{R}^3)^3$, for $\delta > 0$ and ε small enough. Furthermore, they announced improved error estimates in the $H^1(\mathbb{R}^2)^2$ norm for $u^\varepsilon - v$ and in the $L^2(\mathbb{R}^2)^2$ norm for $\partial_t\left(t(u^\varepsilon - v)\right)$ in the two-dimensional case, where u^ε is as above and v is the solution to (1) with $v_0 = u_0^\varepsilon \in H^1(\mathbb{R}^2)$ or $v_0 = u_0^\varepsilon \in H^{1+\delta}(\mathbb{R}^3)$.

In two and three space dimensions and under suitable smallness assumptions on the initial data, we prove global existence for the Cauchy problem (3) in $\dot{H}^{\frac{n}{2}+\delta} \cap \dot{H}^{\frac{n}{2}-1+\delta}(\mathbb{R}^n)^n$ in dimensions $n = 2$ and $n = 3$.

Moreover, for all positive T, we prove the convergence in the $L^\infty(]0,T[;L^2(\mathbb{R}^n)^n)$ norm, with $n = 2, 3$, of solutions to (3) towards solutions to the Navier-Stokes problem (1) with $v_0 \in H^{\frac{n}{2}-1+s}(\mathbb{R}^n)^n$ and $s > 0$. More precisely, we prove the following two theorems.

Theorem 1. *Let $T \geq 0$ and $0 < s, \delta < 1$. Let $v_0 \in H^s(\mathbb{R}^2)^2$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta}(\mathbb{R}^2)^2 \times H^\delta(\mathbb{R}^2)^2$ be a sequence of initial data for problem (3). Assume*

$$\begin{cases} \|u_0^\varepsilon - v_0\|_{L^2} + \varepsilon\|u_1^\varepsilon\|_{L^2} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^1} + \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{1+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^\delta} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^\delta} = o\left(1\right). \end{cases} \tag{4}$$

Then, for ε small enough, there exists a global solution u^ε to system (3) that converges, when ε goes to 0, in the $L^\infty(]0,T[;L^2(\mathbb{R}^2)^2)$ norm, towards the unique solution v to the incompressible Navier-Stokes equations (1), with v_0 as initial data. Moreover, there exists a constant C_T, depending only on T and v, such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} |u^\varepsilon - v|^2\, dx \leq C_T \varepsilon^{\left(\frac{s}{2}\right)^-}. \tag{5}$$

Theorem 2. *Let $T \geq 0$ and $0 < s, \delta < 1$. Let $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ be a divergence-free vector field and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$ be a sequence of initial data for problem (3) such that $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$. Assume*

$$\begin{cases} \|u_0^\varepsilon - v_0\|_{L^2} + \varepsilon\|u_1^\varepsilon\|_{L^2} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^1} + \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \\ \varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} + \varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} = o\left(1\right). \end{cases} \tag{6}$$

Then there exists a global solution u^ε to system (3) that converges, when ε goes to 0, in the $L^\infty(]0,T[;L^2(\mathbb{R}^3)^3)$ norm, towards the solution v to the incompressible Navier-Stokes equations (1), with v_0 as initial data. Moreover, there exists a constant C_T, depending only on T and v, such that

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} |u^\varepsilon - v|^2\, dx \leq C_T \varepsilon^{\min(s,\frac{1}{2})}. \tag{7}$$

Remark 1. *As a consequence of the assumptions $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \frac{1}{16}$ and $\|u_0^\varepsilon - v_0\|_{L^2} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we obtain the smallness of $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$, which is a necessary condition to the existence of global solutions to the Navier-Stokes equation.*

Remark 2. *We prove the convergence for initial data $v_0 \in H^{s+\frac{n}{2}-1}(\mathbb{R}^n)^n$, where $s > 0$ and $H^{\frac{n}{2}-1}(\mathbb{R}^n)^n$ is the critical space for the Navier-Stokes equation.*

Before going further, we should check up that, given v_0, there exists a couple of functions $(u_0^\varepsilon, u_1^\varepsilon)$ satisfying all assumptions (4) in Theorem 1, for example. First, since u_1^ε is not involved in the condition $\|u_0^\varepsilon - v_0\|_{L^2} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right)$, we can take $u_1^\varepsilon \equiv 0$. Then let u_0^ε be defined by

$$\widehat{u_0^\varepsilon}(\xi) = \widehat{v_0}(\xi)\ \mathbb{1}_{|\xi|<\frac{1}{\sqrt{\varepsilon}}}.$$

Now a Bernstein inequality (see [6] page 24) gives

$$\|u_0^\varepsilon\|_{\dot{H}^\sigma} \leq \left(\sqrt{\varepsilon}\right)^{s-\sigma} \|v_0\|_{\dot{H}^s}$$

and easy calculations lead to the Jackson inequality

$$\|u_0^\varepsilon - v_0\|_{L^2} \leq \left(\sqrt{\varepsilon}\right)^{s} \|v_0\|_{\dot{H}^s}.$$

Therefore, the conditions (4) are fulfilled by at least this particular choice of $(u_0^\varepsilon, u_1^\varepsilon)$. The same arguments lead to the existence of a couple of functions satisfying (6).

The energy method we use in this paper is inspired by [2] and refined in order to take into account the loss of regularity of the initial data. Therefore, we frequently have to use tame estimates and interpolations; this makes the proofs more technical than in [2]. Moreover, we introduce a new energy, inspired from the classical one in [2,8], which is more convenient for the spaces we work in.

This paper is organized as follows. In the next section, we treat the two-dimensional case and prove Theorem 1. In the last section, we adapt the results to the three-dimensional case and prove Theorem 2.

2 The two-dimensional case : proof of Theorem 1

This section includes three subsections. In the first one, we prove the existence of a local solution to (3) using the Duhamel formula. Then we show that this solution is global and finally we prove the convergence to a solution to the Navier-Stokes equation, which is the last part of the statement of Theorem 1.

Notation: In the following, let $\mathbb{P}$ denote the Leray projection that maps a vector field to its zero-divergence part and, to alleviate the notations, let $L^p_T\mathcal{E}$ denote the space $L^p\left((0,T);\mathcal{E}\right)$.

Since $\mathbb{P}$ is a convolution operator on $\mathbb{R}^2$, it commutes with the differential operators. So, applying $\mathbb{P}$ to problem (3), we obtain the damped (nonlinear) wave equation

$$(NLW_\varepsilon)\left\{\begin{array}{rcl} \varepsilon\partial_{tt}u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon &=& -\mathbb{P}\nabla : u^\varepsilon \otimes u^\varepsilon \\ (u^\varepsilon, \partial_t u^\varepsilon)|_{t=0} &=& (u_0^\varepsilon, u_1^\varepsilon). \end{array}\right. \tag{8}$$

which we consider with $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta}(\mathbb{R}^2) \times H^\delta(\mathbb{R}^2)$. Local existence for this equation is admitted in [8]. For the sake of completeness, we give the proof in the next subsection. First, we shall perform a scale change, so that we will have to study the same equation with parameter $\varepsilon = 1$:

$$(NLW)\left\{\begin{array}{rcl} \partial_{tt}u + \partial_t u - \Delta u &=& -\mathbb{P}\nabla : u \otimes u \\ (u, \partial_t u)|_{t=0} &=& (u_0, u_1) \in H^{1+\delta}(\mathbb{R}^2) \times H^\delta(\mathbb{R}^2). \end{array}\right.$$

In this purpose, let us set

$$u^\varepsilon(\tau, y) = \frac{1}{\sqrt{\varepsilon}} u(\frac{\tau}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}). \tag{9}$$

This scaling transforms system (NLW_ε) into system (NLW) with initial data

$$\begin{aligned} u_0(x) &= \sqrt{\varepsilon} u_0^\varepsilon \left(\sqrt{\varepsilon} x\right) \\ u_1(x) &= \varepsilon^{\frac{3}{2}}\ u_1^\varepsilon \left(\sqrt{\varepsilon} x\right) \end{aligned}$$

As usual, let us denote by $\Box$ the D'Alembert operator : $\Box = \partial_{tt} - \Delta$ and rewrite (NLW) as follows :

$$(NLW) \left\{ \begin{array}{rcl} \Box u &=& -\partial_t u - \mathbb{P}\nabla : u \otimes u =: F(u) \\ (u, \partial_t u)|_{t=0} &=& (u_0, u_1) \in H^{1+\delta}(\mathbb{R}^2) \times H^\delta(\mathbb{R}^2). \end{array} \right. \tag{10}$$

Duhamel formula for u solution to (NLW) is then

$$u(t) = \cos(t\Lambda) u_0 + \frac{\sin(t\Lambda)}{\Lambda} u_1 + \int_0^t \frac{\sin\left((t-s)\Lambda\right)}{\Lambda} F\left(u(s)\right)\, ds =: \phi\left(u\right)(t)\ , \tag{11}$$

where Λ is the differentiation operator $\Lambda = \sqrt{-\Delta}$, defined in Fourier variables by

$$\widehat{\Lambda f}(\xi) = |\xi| \hat{f}(\xi).$$

2.1 Local existence in $\dot{H}^{1+\delta} \cap \dot{H}^\delta(\mathbb{R}^2, \mathbb{R}^2)$

We show local existence for (NLW) in the complete metric space

$$\begin{aligned} X_T(a) &= \Big\{ (f, \partial_t f) \in L^\infty_T(\dot{H}^{1+\delta} \cap \dot{H}^\delta)(\mathbb{R}^2) \times L^\infty_T \dot{H}^\delta(\mathbb{R}^2)\ :\ \nabla . f = 0\ , \\ &\qquad \|f\|_{X_T}\ :=\ \|f\|_{L^\infty_T \dot{H}^{1+\delta}} + \|f\|_{L^\infty_T \dot{H}^\delta} + \|\partial_t f\|_{L^\infty_T \dot{H}^\delta} \leq a \Big\}, \end{aligned} \tag{12}$$

with $a > 0$ and $0 < T < 1$ to be chosen later. Let $0 \leq t \leq T$.

In the following, we use two facts: first, $\dot{H}^{1+\delta} \cap \dot{H}^\delta(\mathbb{R}^2)$ is an algebra (see the book by Alinhac and Gérard [1]) and the inequality

$$\|fg\|_{\dot{H}^{1+\delta} \cap \dot{H}^\delta} \leq C \left(\|f\|_{\dot{H}^{1+\delta}} \|g\|_{\dot{H}^\delta} + \|g\|_{\dot{H}^{1+\delta}} \|f\|_{\dot{H}^\delta} \right) \tag{13}$$

holds for all $f, g \in \dot{H}^{1+\delta} \cap \dot{H}^\delta$. Secondly, since u is divergence-free, one can easily check

$$\nabla : u \otimes u = u . \nabla u.$$

We want to apply the Picard iteration theorem. So we shall start by looking for suitable reals a and T such that $X_T(a)$ is stable under ϕ.
Let $(u, \partial_t u) \in X_T(a)$. Using Duhamel formula (11), we have

$$\begin{aligned} \|\phi(u)(t)\|_{\dot{H}^{1+\delta}} &\leq \|u_0\|_{\dot{H}^{1+\delta}} + \left\| \frac{\sin\left(t\Lambda\right)}{\Lambda} u_1 \right\|_{\dot{H}^{1+\delta}} + \int_0^t \left\| \frac{\sin\left((t-s)\Lambda\right)}{\Lambda} F\left(u(s)\right) \right\|_{\dot{H}^{1+\delta}} ds \\ &\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta} + \int_0^t \|F\left(u(s)\right)\|_{\dot{H}^\delta}\, ds \\ &\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta} + \int_0^t \|\partial_t u(s) + \mathbb{P}(u.\nabla u)\|_{\dot{H}^\delta}\, ds. \end{aligned}$$

Now let us consider the integral term.

$$\begin{aligned} \int_0^t \|\partial_t u(s) + \mathbb{P}(u.\nabla u)\|_{\dot{H}^\delta}\, ds &\leq \int_0^t \left(\|\partial_t u(s)\|_{\dot{H}^\delta} + \|\mathbb{P}(u.\nabla u)\|_{\dot{H}^\delta} \right) ds \\ &\leq T \left(\|\partial_t u\|_{L^\infty_T \dot{H}^\delta} + \|\mathbb{P}(u.\nabla u)\|_{L^\infty_T \dot{H}^\delta} \right) \end{aligned}$$

Using the identity

$$\|\vec{f}.\nabla\vec{f}\|_{\dot{H}^\delta} = \left\|\sum_{i=1}^n f_i\partial_i\vec{f}\right\|_{\dot{H}^\delta} = \left\|\sum_{i=1}^n \partial_i(f_i\vec{f})\right\|_{\dot{H}^\delta}, \tag{14}$$

that holds for any divergence-free vector $\vec{f}$ valued in $\mathbb{R}^n$, we have

$$\|\mathbb{P}(u.\nabla u)\|_{\dot{H}^\delta} \leq \sum_{i=1}^2 \|\partial_i(u_i u)\|_{\dot{H}^\delta} \leq \sum_{i=1}^2 \|u_i u\|_{\dot{H}^{1+\delta}} \leq 4C\|u\|^2_{\dot{H}^{1+\delta}\cap\dot{H}^\delta}$$

thanks to inequality (13). Thus, we have the following estimate

$$\|\phi(u)(t)\|_{\dot{H}^{1+\delta}} \leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta} + Ta\,(1+4Ca)\,. \tag{15}$$

In order to estimate the $\dot{H}^\delta(\mathbb{R}^2)$ norm of $\phi(u)(t)$, we use that the operator $\dfrac{\sin(t\Lambda)}{t\Lambda}$ is bounded for all positive t and its norm is less than 1.

$$\begin{aligned}
\|\phi(u)(t)\|_{\dot{H}^\delta} &\leq \|u_0\|_{\dot{H}^\delta} + \|\frac{\sin(t\Lambda)}{\Lambda}u_1\|_{\dot{H}^\delta} + \int_0^t \left\|\frac{\sin((t-s)\Lambda)}{\Lambda}F(u(s))\right\|_{\dot{H}^\delta} ds \\
&\leq \|u_0\|_{\dot{H}^\delta} + T\left\|\frac{\sin(t\Lambda)}{t\Lambda}u_1\right\|_{\dot{H}^\delta} + \int_0^t (t-s)\left\|\frac{\sin((t-s)\Lambda)}{(t-s)\Lambda}F(u(s))\right\|_{\dot{H}^\delta} ds \\
&\leq \|u_0\|_{\dot{H}^\delta} + T\|u_1\|_{\dot{H}^\delta} + \int_0^t (t-s)\|F(u(s))\|_{\dot{H}^\delta}\, ds.
\end{aligned}$$

Now,

$$\begin{aligned}
\int_0^t (t-s)\|F(u(s))\|_{\dot{H}^\delta}\, ds &\leq T\int_0^t \left(\|\partial_t u\|_{\dot{H}^\delta} + \|u.\nabla u\|_{\dot{H}^\delta}\right)(s)\, ds \\
&\leq T\int_0^t \left(\|u\|_{X_T} + 4C\|u\|^2_{\dot{H}^{1+\delta}\cap\dot{H}^\delta}\right)(s)\, ds \\
&\leq T^2\left(\|u\|_{X_T} + 4C\|u\|^2_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^\delta)}\right).
\end{aligned}$$

Finally,

$$\|\phi(u)(t)\|_{\dot{H}^\delta} \leq \|u_0\|_{\dot{H}^\delta} + T\|u_1\|_{\dot{H}^\delta} + T^2\left(a + 4Ca^2\right). \tag{16}$$

Similarly, we have

$$\begin{aligned}
\|\partial_t\left[\phi(u)\right](t)\|_{\dot{H}^\delta} &\leq \|-\Lambda\sin(t\Lambda)u_0\|_{\dot{H}^\delta} + \|\cos(t\Lambda)u_1\|_{\dot{H}^\delta} + \int_0^t \|\cos\left((t-s)\Lambda\right)F(u)(s)\|_{\dot{H}^\delta}\, ds \\
&\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta} + T\left(\|\partial_t u\|_{L^\infty_T\dot{H}^\delta} + 4C\|u\|^2_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^\delta)}\right) \\
&\leq \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta} + Ta\,(1+4Ca)\,.
\end{aligned}$$

Finally, we obtain the estimate

$$\|\phi(u)\|_{X_T} \leq 2\left(\|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta}\right) + \|u_0\|_{\dot{H}^\delta} + T\|u_1\|_{\dot{H}^\delta} + aT(2+T)(1+4Ca). \tag{17}$$

Let us set $a = 4\left(\|u_0\|_{\dot{H}^\delta} + \|u_0\|_{\dot{H}^{1+\delta}} + \|u_1\|_{\dot{H}^\delta}\right)$ and $T < 1$ such that :

$$T\frac{a}{4} + aT(2+T)(1+4Ca) \leq \frac{a}{2}. \tag{18}$$

Since $T < 1$, it is sufficient for T to satisfy

$$T \leq \frac{1}{12(1+4Ca)}. \tag{19}$$

With a and T chosen that way, the space $X_T(a)$ is stable under ϕ.
It remains to prove that $\phi : X_T(a) \longrightarrow X_T(a)$ is contractive. So let $(u, \partial_t u)$ and $(v, \partial_t v)$ be two elements of $X_T(a)$. Again using Duhamel formula (11), we have

$$\begin{aligned}
[\phi(u) - \phi(v)](t) &= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} \left(\partial_t v + \mathbb{P}v.\nabla v - \partial_t u - \mathbb{P}u.\nabla u\right)(s)\, ds \\
&= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} \left[\partial_t (v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\right](s)\, ds.
\end{aligned}$$

Let us calculate the X_T norm of $\phi(u) - \phi(v)$.

$$\begin{aligned}
\|[\phi(u) - \phi(v)](t)\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} &\leq \int_0^t \|\partial_t(v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\|_{\dot{H}^{\delta}(\mathbb{R}^2)}\, ds \\
&\leq T\left(\|\partial_t(v-u)\|_{L^\infty_T \dot{H}^{\delta}(\mathbb{R}^2)} + \|\nabla : (v\otimes v - u\otimes u)\|_{L^\infty_T \dot{H}^{\delta}(\mathbb{R}^2)}\right) \\
&\leq T\left(\|v-u\|_{X_T} + \|\nabla : (v\otimes v - u\otimes u)\|_{L^\infty_T \dot{H}^{\delta}(\mathbb{R}^2)}\right).
\end{aligned}$$

Now, since u and v are divergence-free, one can check easily that the identity

$$\nabla : (v\otimes v - u\otimes u) = (v-u).\nabla v + u.\nabla(v-u) \tag{20}$$

holds and we can write

$$\begin{aligned}
\|[\phi(u) - \phi(v)](t)\|_{\dot{H}^{1+\delta}} &\leq T\left(\|v-u\|_{X_T} + \|(v-u).\nabla v\|_{L^\infty_T \dot{H}^{\delta}} + \|u.\nabla(v-u)\|_{L^\infty_T \dot{H}^{\delta}}\right) \\
&\leq T\Big(\|v-u\|_{X_T} + 4C\|v-u\|_{X_T}\|v\|_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})} + \\
&\qquad + 4C\|u\|_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})}\|v-u\|_{X_T}\Big) \\
&\leq T\left(1 + 4C\|v\|_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})} + 4C\|u\|_{L^\infty_T(\dot{H}^{1+\delta}\cap\dot{H}^{\delta})}\right)\|v-u\|_{X_T} \\
&\leq T\,(1+8Ca)\,\|v-u\|_{X_T}.
\end{aligned}$$

Besides, we estimate the $\dot{H}^{\delta}$ norm as follows :

$$\begin{aligned}
\|[\phi(u) - \phi(v)](t)\|_{\dot{H}^{\delta}} &\leq \int_0^t (t-s)\left\|\frac{\sin((t-s)\Lambda)}{(t-s)\Lambda}\left(\partial_t(v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\right)(s)\right\|_{\dot{H}^{\delta}} ds \\
&\leq T\int_0^t \left(\|\partial_t(v-u)\|_{\dot{H}^{\delta}} + \|\nabla(v\otimes v - u\otimes u)\|_{\dot{H}^{\delta}}\right)(s)\, ds \\
&\leq T^2\left(\|v-u\|_{X_T} + \|(v-u).\nabla v\|_{L^\infty_T \dot{H}^{\delta}} + \|u.\nabla(v-u)\|_{L^\infty_T \dot{H}^{\delta}}\right) \\
&\leq T^2\,(1+8Ca)\,\|v-u\|_{X_T}.
\end{aligned}$$

Now, computing $\partial_t[\phi(u) - \phi(v)](t)$ and estimating its $\dot{H}^{\delta}(\mathbb{R}^2)$ norm, we obtain

$$\begin{aligned}
\|\partial_t[\phi(u) - \phi(v)](t)\|_{\dot{H}^{\delta}} &\leq \int_0^t \|\cos((t-s)\Lambda)\left[\partial_t(v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\right]\|_{\dot{H}^{\delta}}(s)\, ds \\
&\leq T\left(\|\partial_t(v-u)\|_{L^\infty_T \dot{H}^{\delta}} + \|(v-u).\nabla v\|_{L^\infty_T \dot{H}^{\delta}} + \|u.\nabla(v-u)\|_{L^\infty_T \dot{H}^{\delta}}\right) \\
&\leq T\,(1+8Ca)\,\|v-u\|_{X_T}.
\end{aligned}$$

By similar computations and using (20), we have the estimate

$$\|\phi(u) - \phi(v)\|_{X_T} \leq T(2+T)\,(1+8Ca)\,\|v-u\|_{X_T} \tag{21}$$

and ϕ is contractive if

$$T(2+T) < \frac{1}{(1+8Ca)}, \tag{22}$$

or, since $T < 1$, if

$$T < \frac{1}{3(1+8Ca)}. \tag{23}$$

Combining conditions (19) and (23) above, we deduce the local existence of a solution to (NLW), defined on $[0,T[$, for all positive T such that

$$T \leq \frac{1}{12 + 48C\left(\|u_0\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} + \|u_0\|_{\dot{H}^{\delta}(\mathbb{R}^2)} + \|u_1\|_{\dot{H}^{\delta}(\mathbb{R}^2)}\right)}. \tag{24}$$

In particular, as long as $\|u(t)\|_X := \|u(t)\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} + \|u(t)\|_{\dot{H}^{\delta}(\mathbb{R}^2)} + \|\partial_t u(t)\|_{\dot{H}^{\delta}(\mathbb{R}^2)}$ remains bounded, we can iterate the fixed point argument and extend the solution.

2.2 Globalization

Let us resume the initial equation (NLW_ε). On ecan easily check that

$$\|u(t)\|_X = \varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} + \|u^\varepsilon\|_{\dot{H}^{\delta}(\mathbb{R}^2)} + \varepsilon\|\partial_t u^\varepsilon\|_{\dot{H}^{\delta}(\mathbb{R}^2)}\right). \tag{25}$$

Then, for all non negative real δ and all non negative t, we define the energy

$$E_\varepsilon^\delta(t) = \int_{\mathbb{R}^2}\left(\frac{1}{2}|\Lambda^\delta(u^\varepsilon + \varepsilon\partial_t u^\varepsilon)|^2 + \frac{\varepsilon^2}{2}|\Lambda^\delta\partial_t u^\varepsilon|^2 + \varepsilon|\Lambda^{1+\delta}u^\varepsilon|^2\right)dx \tag{26}$$

and we have

$$\varepsilon^{\frac{\delta}{2}}\left(\sqrt{\varepsilon}\|u^\varepsilon\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} + \|u^\varepsilon\|_{\dot{H}^{\delta}(\mathbb{R}^2)} + \varepsilon\|\partial_t u^\varepsilon\|_{\dot{H}^{\delta}(\mathbb{R}^2)}\right) \leq C\varepsilon^{\frac{\delta}{2}}\sqrt{E_\varepsilon^\delta}. \tag{27}$$

In this subsection, we shall prove the following inequality that yields the globality of the solution.

$$E_\varepsilon^\delta(t) \leq C_0\varepsilon^{-\delta}, \tag{28}$$

where C_0 is a universal constant.

First, let us point out that $\dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$ is an algebra and that tame estimate

$$\|fg\|_{\dot{H}^{1+\delta}(\mathbb{R}^2)} \leq C_1\left(\|f\|_{\dot{H}^{1+\delta}}\|g\|_\infty + \|g\|_{\dot{H}^{1+\delta}}\|f\|_\infty\right). \tag{29}$$

holds (see [1]) for all functions $f, g \in \dot{H}^{1+\delta} \cap L^\infty(\mathbb{R}^2)$. Moreover, we know that the homogeneous Besov[1] space $\dot{B}^1_{2,1}(\mathbb{R}^2)$ embeds into $L^\infty(\mathbb{R}^2)$ and, interpolating, we obtain

$$\|f\|_\infty \;\leq\; \tilde{C}\|f\|_{\dot{B}^1_{2,1}} \;\leq\; C_2\,\|f\|^{\delta}_{\dot{H}^\delta}\cdot\|f\|^{1-\delta}_{\dot{H}^{1+\delta}}. \tag{30}$$

Lemma 1. *Assume the following, when ε goes to zero :*

$$(H)\begin{cases} i) & \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^{1+\delta}} + \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot{H}^\delta} = o(1) \\ ii) & \varepsilon^{\frac{1}{2}}\|u_0^\varepsilon\|_{\dot{H}^1} + \varepsilon\|u_1^\varepsilon\|_{L^2} = o(1). \end{cases}$$

Let us define $0 \leq T \leq T_\varepsilon^{\max}$ by

$$T = \sup\left\{0 \leq \tau \leq T_\varepsilon^{\max} \;:\; \forall\, t \in [0,\tau[,\ \|u^\varepsilon(t)\|_\infty < \frac{1}{8\ C_1\sqrt{\varepsilon}}\right\}. \tag{31}$$

Then, for ε small enough, there exists a large number N, only depending on δ and $\|u_0^\varepsilon\|_{L^2}$ (which is arbitrary), such that, for all $0 \leq t \leq T$,

$$E_\varepsilon^\delta(t) \leq E_\varepsilon^\delta(0)\left(2\|u_0^\varepsilon\|_{L^2}^2 + 1\right)^N. \tag{32}$$

[1]For definitions and properties, see the book by P.-G. Lemarié-Rieusset [6]

Proof. Let us compute the (time) derivative of E_ε^δ.

$$\begin{aligned}\frac{dE_\varepsilon^\delta}{dt}(t) &= \int \Lambda^\delta(\varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon).\Lambda^\delta(u^\varepsilon+\varepsilon\partial_t u^\varepsilon)+\varepsilon^2\Lambda^\delta(\partial_t u^\varepsilon).\Lambda^\delta(\partial_{tt}u^\varepsilon)+2\varepsilon\Lambda^{\delta+1}(\partial_t u^\varepsilon).\Lambda^{\delta+1}(u^\varepsilon)\\ &= \int \Lambda^\delta(\varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon-\Delta u^\varepsilon).\Lambda^\delta(u^\varepsilon+2\varepsilon\partial_t u^\varepsilon)-\varepsilon|\Lambda^\delta\partial_t u^\varepsilon|^2-|\Lambda^{\delta+1}u^\varepsilon|^2\\ &= \int -\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon-2\varepsilon\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta\partial_t u^\varepsilon-\varepsilon|\Lambda^\delta\partial_t u^\varepsilon|^2-|\Lambda^{\delta+1}u^\varepsilon|^2\end{aligned}$$

Performing a classical Young inequality $2ab\le a^2+b^2$ and rearranging the terms, we obtain

$$\frac{dE_\varepsilon^\delta}{dt}(t) \le -\int\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon+\left(\varepsilon\int|\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon)|^2-\int|\Lambda^{\delta+1}u^\varepsilon|^2\right)$$

Now, let us recall that

$$u^\varepsilon.\nabla u^\varepsilon=\sum_{i=1}^2 u_i^\varepsilon\partial_i u^\varepsilon=\sum_{i=1}^2\partial_i(u_i^\varepsilon u^\varepsilon) \tag{33}$$

since u^ε is divergence-free. Now, (29) yields

$$\|u^\varepsilon.\nabla u^\varepsilon\|_{\dot H^\delta}\le\sum_{i=1}^2\|\partial_i(u_i^\varepsilon u^\varepsilon)\|_{\dot H^\delta}\le 4C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{1+\delta}} \tag{34}$$

Thus

$$\begin{aligned}\left|\int\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon\,dx\right| &\le \|u^\varepsilon.\nabla u^\varepsilon\|_{\dot H^\delta}\|u^\varepsilon\|_{\dot H^\delta}\\ &\le 4C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{1+\delta}}\|u^\varepsilon\|_{\dot H^\delta}\\ &\le 4C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{1+\delta}}\|u^\varepsilon\|_{\dot H^\delta}\\ &\le 4C_1C_2\|u^\varepsilon\|_{\dot H^\delta}^\delta\|u^\varepsilon\|_{\dot H^\delta}\|u^\varepsilon\|_{\dot H^{1+\delta}}^{2-\delta}.\end{aligned}$$

The interpolation inequality

$$\|u^\varepsilon\|_{\dot H^\delta}\le C_3\|u^\varepsilon\|_2^{1-\delta}\|u^\varepsilon\|_{\dot H^1}^\delta \tag{35}$$

yields finally

$$\left|\int\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon).\Lambda^\delta u^\varepsilon\,dx\right|\le 4C_1C_2C_3\|u^\varepsilon\|_2^{1-\delta}\left(\|u^\varepsilon\|_{\dot H^1}\|u^\varepsilon\|_{\dot H^\delta}\right)^\delta\|u^\varepsilon\|_{\dot H^{1+\delta}}^{2-\delta}. \tag{36}$$

Besides, since we assume i) and using inequality (30), we can write, for ε small enough,

$$\|u_0^\varepsilon\|_\infty\le\frac{1}{8C_1\sqrt{\varepsilon}}. \tag{37}$$

Now, by continuity of the (local) solution u^ε with respect to t, we deduce $T>0$ and the inequality

$$\begin{aligned}\varepsilon\int|\Lambda^\delta(u^\varepsilon.\nabla u^\varepsilon)|^2-\int|\Lambda^{\delta+1}u^\varepsilon|^2 &\le \left(16C_1^2\varepsilon\|u^\varepsilon\|_\infty^2-1\right)\int|\Lambda^{\delta+1}u^\varepsilon|^2\\ &\le -\frac14\|u^\varepsilon\|_{\dot H^{1+\delta}}^2\end{aligned} \tag{38}$$

holds on $[0,T[$.
Hence, for all $0\le t<T$,

$$\frac{d}{dt}E_\varepsilon^\delta(t)\le-\frac14\|u^\varepsilon\|_{\dot H^{1+\delta}}^2+4C_1C_2C_3\|u^\varepsilon\|_2^{1-\delta}\left(\|u^\varepsilon\|_{\dot H^1}\|u^\varepsilon\|_{\dot H^\delta}\right)^\delta\|u^\varepsilon\|_{\dot H^{1+\delta}}^{2-\delta}. \tag{39}$$

Now, consider the second term on the right hand side and use Young inequality

$$ab \leq \frac{\delta}{2}\, a^{\frac{2}{\delta}} + \frac{2-\delta}{2}\, b^{\frac{2}{2-\delta}} \tag{40}$$

with $b = 2^{\delta-2}\|u^\varepsilon\|^{2-\delta}_{\dot{H}^{1+\delta}}$. We obtain

$$\begin{aligned}\frac{d}{dt}E^\delta_\varepsilon(t) &\leq \frac{1}{4}\left(\frac{2-\delta}{2}-1\right)\|u^\varepsilon\|^2_{\dot{H}^{1+\delta}} + \frac{\delta}{2}\left(2^{2-\delta}4C_1C_2C_3\|u^\varepsilon\|_2^{1-\delta}\|u^\varepsilon\|^\delta_{\dot{H}^1}\|u^\varepsilon\|^\delta_{\dot{H}^\delta}\right)^{\frac{2}{\delta}}\\ &\leq \frac{\delta}{2}\, 2^{2\frac{2-\delta}{\delta}}(4C_1C_2C_3)^{\frac{2}{\delta}}\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\|u^\varepsilon\|^2_{\dot{H}^1}\|u^\varepsilon\|^2_{\dot{H}^\delta}.\end{aligned}$$

Then the inequality

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2 \geq 2\|b\|^2 \tag{41}$$

with $a = \dfrac{u^\varepsilon}{2} + \varepsilon\partial_t u^\varepsilon$ and $b = \dfrac{u^\varepsilon}{2}$ yields, for all non negative δ, the estimate

$$E^\delta_\varepsilon(t) \;\geq\; \|u^\varepsilon + \varepsilon\partial_t u^\varepsilon\|^2_{\dot{H}^\delta} + \|\varepsilon\partial_t u^\varepsilon\|^2_{\dot{H}^\delta} \;\geq\; \frac{1}{2}\|u^\varepsilon\|^2_{\dot{H}^\delta} \tag{42}$$

from which we deduce

$$\frac{d}{dt}E^\delta_\varepsilon(t) \leq \delta\; 2^{2\frac{2-\delta}{\delta}}(4C_1C_2C_3)^{\frac{2}{\delta}}\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\|u^\varepsilon\|^2_{\dot{H}^1}E^\delta_\varepsilon(t). \tag{43}$$

Instead of showing that $t \mapsto E^\delta_\varepsilon(t)$ is decreasing, we will prove that $E^\delta(E^0_\varepsilon+1)^N$ decreases if N is large enough. Even though this new energy does not seem natural, it is more convenient than $E^\delta + NE^0_\varepsilon$ since the latter would have required a smallness assumption on the L^2 norm of u^ε_0. But this would yield a weaker result than one might hope as the Navier-Stokes equation with initial data in $L^2(\mathbb{R}^2)$ unconditionally has a global solution.

First of all, let us notice that $\int_{\mathbb{R}^n} f.(f.\nabla f) = 0$ for all divergence-free function f and that (see inequality (38))

$$\frac{dE^0_\varepsilon}{dt}(t) \;\leq\; -\frac{1}{4}\|u^\varepsilon(t)\|^2_{\dot{H}^1}. \tag{44}$$

We use it in (45) below.

$$\begin{aligned}\frac{d}{dt}\left[(E^0_\varepsilon+1)^N E^\delta_\varepsilon\right] &= N\frac{dE^0_\varepsilon}{dt}(E^0_\varepsilon+1)^{N-1}E^\delta_\varepsilon + (E^0_\varepsilon+1)^N\frac{dE^\delta_\varepsilon}{dt}\\ &\leq \frac{1}{4}\,(E^0_\varepsilon+1)^{N-1}E^\delta_\varepsilon\|u^\varepsilon\|^2_{\dot{H}^1}\times\\ &\quad\times\left[-N+\delta\,(16C_1C_2C_3)^{\frac{2}{\delta}}(E^0_\varepsilon+1)\|u^\varepsilon\|_2^{2\frac{1-\delta}{\delta}}\right].\end{aligned} \tag{45}$$

Under the assumption *iii*), we have

$$\begin{aligned}E^0_\varepsilon(0) &= \int_{\mathbb{R}^2}\left(\frac{|u^\varepsilon_0+\varepsilon u^\varepsilon_1|^2+\varepsilon^2|u^\varepsilon_1|^2}{2}+\varepsilon|\nabla u^\varepsilon_0|^2\right)dx\\ &\leq \|u^\varepsilon_0\|^2_2 + \frac{3\varepsilon^2}{2}\|u^\varepsilon_1\|^2_2 + \varepsilon\|u^\varepsilon_0\|^2_{\dot{H}^1}\\ &< 2\,\|u^\varepsilon_0\|^2_2\end{aligned}$$

if ε is small enough. Now, (42) with $\delta = 0$ yields

$$\|u^\varepsilon(t)\|^2_2 \;\leq\; 2E^0_\varepsilon(t) \;<\; 4\|u^\varepsilon_0\|^2_2 \tag{46}$$

for all $t \in (0,T)$. We have obtained

$$\frac{d}{dt}\left[(E_\varepsilon^0+1)^N E_\varepsilon^\delta\right] \;\le\; \frac{1}{4}(E_\varepsilon^0+1)^{N-1}E_\varepsilon^\delta \|u^\varepsilon\|_{\dot H^1}^2\left[-N+\delta\,(16C_1C_2C_3)^{\frac{2}{\delta}}(4\|u_0^\varepsilon\|_2^2)^{\frac{1-\delta}{\delta}}(2\|u_0^\varepsilon\|_2^2+1)\right]. \tag{47}$$

We deduce that, for N large enough (depending only on δ and $\|u_0^\varepsilon\|_2$), the right hand side is negative and

$$E_\varepsilon^\delta(t) \le E_\varepsilon^\delta(0)(E_\varepsilon^0(0)+1)^N < E_\varepsilon^\delta(0)(2\|u_0^\varepsilon\|_2^2+1)^N \tag{48}$$

for all $0 \le t < T$. □

We want to reiterate the reasoning, *i.e.* to keep the control of $\|u^\varepsilon(t)\|_\infty$. The aim of Lemma 2 is to ensure this control throughout the time.

Lemma 2. *Assume the limit*

$$\varepsilon^{1+\frac{\delta}{2}}\|u_1^\varepsilon\|_{\dot H^\delta} \longrightarrow 0\ ,\quad \varepsilon \to 0 \tag{49}$$

in addition to the assumptions (H) *in Lemma 1. Then the inequality*

$$E_\varepsilon^\delta(0) \;<\; \frac{2^{-\delta}}{(16C_1C_2)^2}\,\varepsilon^{-\delta}(2\|u_0^\varepsilon\|_2^2+1)^{-N}. \tag{50}$$

holds for ε *small enough. Moreover, for all* $t \in [0,T[$,

$$\|u^\varepsilon\|_\infty \;\le\; \frac{1}{16C_1\sqrt{\varepsilon}}. \tag{51}$$

Proof. Easy calculations show that (50) is true as a result of (H) and (49). Now, let $t \in [0,T[$ and recall interpolation inequality (30)

$$\|u^\varepsilon\|_\infty \le C_2\|u^\varepsilon\|_{\dot H^\delta}^{\delta}\|u^\varepsilon\|_{\dot H^{1+\delta}}^{1-\delta}. \tag{52}$$

Thanks to inequality (42), we have

$$\|u^\varepsilon(t)\|_\infty \le C_2\sqrt{2E_\varepsilon^\delta(t)}^{\delta}\sqrt{\varepsilon^{-1}E_\varepsilon^\delta(t)}^{1-\delta} = C_2 2^{\frac{\delta}{2}}\varepsilon^{\frac{\delta-1}{2}}\sqrt{E_\varepsilon^\delta(t)}. \tag{53}$$

Using (32) (conclusion of Lemma 1) and (50), we obtain

$$\|u^\varepsilon(t)\|_\infty \le C_2 2^{\frac{\delta}{2}}\varepsilon^{\frac{\delta-1}{2}}\frac{2^{-\frac{\delta}{2}}}{16C_1C_2}\varepsilon^{-\frac{\delta}{2}} = \frac{1}{16C_1}\varepsilon^{-\frac{1}{2}} \tag{54}$$

for all $0 < t < T$. □

Remark 3. *Under the assumptions* (4) *in Theorem 1, the conditions* (H) *in Lemma 1 and* (49) *in Lemma 2 are fulfilled.*

Now, we shall prove that these estimations (on $\|u^\varepsilon(t)\|_\infty$ and, consequently, on $E_\varepsilon^\delta(t)$) remain true on the whole existence interval $[0,T_\varepsilon^{\max})$, where $T_\varepsilon^{\max}$ is the existence time given by the Picard iteration in subsection 2.1. We have already proved that $T > 0$. Assume $T < T_\varepsilon^{\max}$. Then

$$\|u^\varepsilon(T)\|_\infty = \frac{1}{8C_1\sqrt{\varepsilon}}. \tag{55}$$

On the other hand, (51) in Lemma 2 yields

$$\|u^\varepsilon(T)\|_\infty \le \frac{1}{16C_1\sqrt{\varepsilon}} < \frac{1}{8C_1\sqrt{\varepsilon}}, \tag{56}$$

which contradicts (55) so $T \ge T_\varepsilon^{\max}$. We deduce then from Lemma 1 and Lemma 2 that E_ε^δ satisfies inequality (28) on the existence interval $[0,T_\varepsilon^{\max})$.Therefore the (NLW_ε) equation has a global solution.

2.3 Convergence towards a solution to Navier-Stokes problem

Let us recall that Brenier, Natalini and Puel showed in [2] that, under suitable assumptions, the solutions to (NLW_ε), with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^2 \times H^1(\mathbb{T}^2)$, converge to the solutions to (NS) with smooth initial data v_0 when ε goes to 0. The authors used the modulated energy method (see [2] and references therein) to show an error estimate in the $L^\infty_T L^2(\mathbb{T}^2)$ norm, for all positive T.

In this section, we show a similar result with less regularity on the initial data $(u_0^\varepsilon, u_1^\varepsilon)$ and v_0, in $\mathbb{R}^2$ instead of $\mathbb{T}^2$. Indeed, we prove that, under suitable assumptions (less restrictive than those in [2]) and for any positive δ, the solutions to (NLW_ε), with initial data $(u_0^\varepsilon, u_1^\varepsilon) \in H^{1+\delta} \times H^\delta(\mathbb{R}^2)$, converge in the $L^2(\mathbb{R}^2)^2$ norm, when ε goes to 0, to the solutions to (NS) with initial data $v_0 \in \dot{H}^s(\mathbb{R}^2)^2$, $s > 0$. We recall here that the critical space for the Navier-Stokes equations in $\mathbb{R}^2$ is L^2.
Because of the loss of regularity, the proof is not straightforward anymore. Interpolation and tame estimates will help to get around the difficulties.

We shall start the proof like in [2], introducing the so-called Dafermos modulated energy, which is the total energy of the wave equation (NLW_ε), modulated by a divergence-free vector $(t,x) \mapsto v(t,x)$

$$E_{\varepsilon,v}(t) = \int_{\mathbb{R}^2} \left(\frac{1}{2}|u^\varepsilon - v(t,x) + \varepsilon \partial_t u^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon|^2 + \varepsilon |\nabla u^\varepsilon|^2 \right) dx. \tag{57}$$

This energy satisfies the inequality

$$\int_{\mathbb{R}^2} |u^\varepsilon - v|^2 \, dx \leq 4 E_{\varepsilon,v}(t). \tag{58}$$

Via a Gronwall inequality, we shall show that, for all positive t such that $t < T$, the modulated energy $E_{\varepsilon,v}(t)$ converges to 0 when ε goes to 0. In this purpose, let us compute the time derivative of $E_{\varepsilon,v}$.

Lemma 3 (see [2])**.** *If u^ε and v are divergence-free functions, the Dafermos modulated energy satisfies the identity*

$$\begin{aligned} \frac{d}{dt} E_{\varepsilon,v}(t) &= \int v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v) - \varepsilon \int |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \\ &\quad - \varepsilon \int \partial_t v . \partial_t u^\varepsilon + \int \left(\varepsilon |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 - |\nabla (u^\varepsilon - v)|^2 \right) \\ &\quad + \int (\partial_t v + v.\nabla v - \Delta v)(v - u^\varepsilon). \end{aligned} \tag{59}$$

Remark 4. *This lemma is proved in [2] but the computations are somewhat different here.*

Proof.

$$\begin{aligned} \frac{dE_{\varepsilon,v}}{dt}(t) &= \int_{\mathbb{R}^2} \left((u^\varepsilon - v + \varepsilon \partial_t u^\varepsilon).(\partial_t u^\varepsilon - \partial_t v + \varepsilon \partial_{tt} u^\varepsilon) + \varepsilon^2 \partial_t u^\varepsilon . \partial_{tt} u^\varepsilon + 2\varepsilon \nabla u^\varepsilon . \partial_t \nabla u^\varepsilon \right) dx \\ &= \int_{\mathbb{R}^2} (\partial_t u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon).(u^\varepsilon + \varepsilon \partial_t u^\varepsilon) - \int_{\mathbb{R}^2} v.(\partial_t u^\varepsilon - \partial_t v + \varepsilon \partial_{tt} u^\varepsilon) \\ &\quad - \int_{\mathbb{R}^2} u^\varepsilon . \partial_t v - \varepsilon \int_{\mathbb{R}^2} \partial_t u^\varepsilon . \partial_t v + \varepsilon^2 \int_{\mathbb{R}^2} \partial_t u^\varepsilon . \partial_{tt} u^\varepsilon - 2\varepsilon \int_{\mathbb{R}^2} \Delta u^\varepsilon . \partial_t u^\varepsilon \\ &= \int_{\mathbb{R}^2} (\varepsilon \partial_{tt} u^\varepsilon + \partial_t u^\varepsilon - \Delta u^\varepsilon).(u^\varepsilon + 2\varepsilon \partial_t u^\varepsilon) + \int_{\mathbb{R}^2} \Delta u^\varepsilon . u^\varepsilon - \varepsilon \int_{\mathbb{R}^2} |\partial_t u^\varepsilon|^2 \\ &\quad + \int_{\mathbb{R}^2} (v - u^\varepsilon).\partial_t v - \int_{\mathbb{R}^2} v.(\partial_t u^\varepsilon + \varepsilon \partial_{tt} u^\varepsilon) - \varepsilon \int_{\mathbb{R}^2} \partial_t v . \partial_t u^\varepsilon . \end{aligned}$$

Now, recall that u^ε is a solution to (NLW_ε). So

$$\begin{aligned}\frac{dE_{\varepsilon,v}}{dt}(t) &= -\int_{\mathbb{R}^2} \nabla : (u^\varepsilon \otimes u^\varepsilon).(u^\varepsilon + 2\varepsilon\partial_t u^\varepsilon) - \int_{\mathbb{R}^2} |\nabla u^\varepsilon|^2 - \varepsilon \int_{\mathbb{R}^2} |\partial_t u^\varepsilon|^2 \\ &\quad + \int_{\mathbb{R}^2} (v-u^\varepsilon).(\partial_t v + v.\nabla v - \Delta v) - \int_{\mathbb{R}^2} (v-u^\varepsilon).(v.\nabla v - \Delta v) \\ &\quad - \int_{\mathbb{R}^2} v.(\partial_t u^\varepsilon + \varepsilon\partial_{tt} u^\varepsilon) - \varepsilon \int_{\mathbb{R}^2} \partial_t v.\partial_t u^\varepsilon.\end{aligned}$$

Then gather the perfect squares.

$$\begin{aligned}\frac{dE_{\varepsilon,v}}{dt}(t) &= -\varepsilon \int_{\mathbb{R}^2} |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 + \varepsilon \int_{\mathbb{R}^2} |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \\ &\quad + \int_{\mathbb{R}^2} (v-u^\varepsilon).(\partial_t v + v.\nabla v - \Delta v) + \int_{\mathbb{R}^2} u^\varepsilon.(v.\nabla v) - \int_{\mathbb{R}^2} |\nabla v|^2 \\ &\quad + \int_{\mathbb{R}^2} \nabla u^\varepsilon.\nabla v - \int_{\mathbb{R}^2} v.\,(\Delta u^\varepsilon - \nabla : (u^\varepsilon \otimes u^\varepsilon)) - \varepsilon \int_{\mathbb{R}^2} \partial_t v.\partial_t u^\varepsilon \\ &= -\varepsilon \int_{\mathbb{R}^2} |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 + \varepsilon \int_{\mathbb{R}^2} |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 \\ &\quad + \int_{\mathbb{R}^2} (v-u^\varepsilon).(\partial_t v + v.\nabla v - \Delta v) - \varepsilon \int_{\mathbb{R}^2} \partial_t v.\partial_t u^\varepsilon - \int_{\mathbb{R}^2} |\nabla(u^\varepsilon - v)|^2 \\ &\quad + \int_{\mathbb{R}^2} u^\varepsilon.(v.\nabla v) + \int_{\mathbb{R}^2} v.(u^\varepsilon.\nabla u^\varepsilon).\end{aligned}$$

It remains to show that

$$\int_{\mathbb{R}^2} u^\varepsilon.(v.\nabla v) + \int_{\mathbb{R}^2} v.(u^\varepsilon.\nabla u^\varepsilon) = \int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v). \tag{60}$$

In this purpose, let us write

$$u^\varepsilon.(v.\nabla v) = \sum_{i,j} u_j^\varepsilon v_i (\partial_i v_j) \tag{61}$$

and, using that u^ε is divergence-free,

$$\begin{aligned}\int v.(u^\varepsilon.\nabla u^\varepsilon) &= \sum_{i,j} \int u_i^\varepsilon (\partial_i u_j^\varepsilon) v_j \\ &= -\sum_{i,j} \int u_i^\varepsilon u_j^\varepsilon (\partial_i v_j) \; - \sum_j \int \sum_i (\partial_i u_i^\varepsilon) u_j^\varepsilon v_j \\ &= -\sum_{i,j} \int u_i^\varepsilon u_j^\varepsilon (\partial_i v_j).\end{aligned}$$

Adding the two expressions, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} (u^\varepsilon.(v.\nabla v) + v.(u^\varepsilon.\nabla u^\varepsilon)) &= \sum_{i,j} \int_{\mathbb{R}^2} u_j^\varepsilon (\partial_i v_j)(v_i - u_i^\varepsilon) \\ &= -\sum_{i,j} \int_{\mathbb{R}^2} (\partial_i u_j^\varepsilon) v_j (v_i - u_i^\varepsilon) \;\; \text{car } \nabla.u^\varepsilon = \nabla.v = 0 \\ &= \int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v) - \sum_{i,j} \int_{\mathbb{R}^2} (\partial_i v_j) v_j (v_i - u_i^\varepsilon).\end{aligned}$$

Since $u^\varepsilon - v$ is divergence-free, integrating by parts gives

$$\sum_{i,j} \int_{\mathbb{R}^2} (\partial_i v_j) v_j (v_i - u_i^\varepsilon) = -\sum_{i,j} \int_{\mathbb{R}^2} v_j (\partial_i v_j)(v_i - u_i^\varepsilon), \tag{62}$$

from which we deduce (60). □

From now on, let v be a solution to the Cauchy problem

$$\partial_t v = \Delta v - \mathbb{P}\nabla : v \otimes v \ , \quad \nabla . v = 0 \ , \quad v|_{t=0} = v_0 \in H^s(\mathbb{R}^2), \tag{63}$$

where s is a positive real, so that the last term of (59) vanishes. Now, since the second term in the identity (59) is negative, (59) writes

$$\begin{aligned} \frac{d}{dt} E_{\varepsilon,v}(t) \quad \leq \quad & -\varepsilon \int_{\mathbb{R}^2} \partial_t v . \partial_t u^\varepsilon \, dx + \int_{\mathbb{R}^2} v . \nabla : (u^\varepsilon - v) \otimes (u^\varepsilon - v) \, dx \\ & + \int_{\mathbb{R}^2} \left(\varepsilon |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2 - |\nabla (u^\varepsilon - v)|^2 \right) dx. \end{aligned} \tag{64}$$

We shall treat the three terms on the right hand side in the following three subsections.

First, let us recall the Duhamel formula for Navier-Stokes equation (63):

$$v(t,.) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\nabla : (v \otimes v)(s,.) \, ds. \tag{65}$$

Computing the time derivative of (65), we obtain

$$\partial_t v(t,.) = \Delta e^{t\Delta} v_0 - \mathbb{P}\nabla : (v \otimes v)(t,.) - \int_0^t \Delta e^{(t-s)\Delta} \mathbb{P}\nabla : (v \otimes v)(s,.) \, ds. \tag{66}$$

Remark 5. *If $v_0 \in H^s(\mathbb{R}^2)$, then, for all positive T, the solution v to (63) satisfies*

$$v \in L^2_T H^{s+1}(\mathbb{R}^2) \cap L^\infty_T H^s(\mathbb{R}^2). \tag{67}$$

2.3.1 Estimating $\varepsilon \int_0^T \int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v \, dt \, dx$

In order to bound the integral by a positive power of ε, we shall find spaces $L^p_T \dot{H}^\sigma(\mathbb{R}^2)$ and $L^q_T \dot{H}^{\sigma'}(\mathbb{R}^2)$ containing respectively $\partial_t v$ and $\varepsilon \partial_t u^\varepsilon$.

Step 1: Regularity of $\partial_t v$.

Lemma 4. *Let $\theta \in [0,1[$ and $v(t,x)$ be a solution to equation (63). Then, for all $0 < \tilde{\varepsilon} < s$,*

$$\partial_t v \in L^p_T \dot{H}^\sigma(\mathbb{R}^2), \tag{68}$$

where $\dfrac{1}{p} = \dfrac{1-\theta}{2} + \theta = \dfrac{1+\theta}{2}$ and $\sigma = (1-\theta)(s-1) + \theta(s-\tilde{\varepsilon}) = s - 1 + \theta(1-\tilde{\varepsilon})$.

Proof. Recall that

$$\partial_t v(t,.) = \Delta e^{t\Delta} v_0 - \nabla : (v \otimes v)(t,.) - \int_0^t \Delta e^{(t-s)\Delta} \nabla : (v \otimes v)(s,.) \, ds. \tag{69}$$

We shall start considering $e^{t\Delta} v_0$.

$$\begin{aligned} \|e^{t\Delta} v_0\|^2_{L^2_T \dot{H}^{s+1}} \quad &= \quad \int_0^T \int_{\xi \in \mathbb{R}^2} |\xi|^{2s+2} e^{-2t|\xi|^2} |\widehat{v_0}(\xi)|^2 \, d\xi \, dt \\ &= \quad \int_{\xi \in \mathbb{R}^2} |\xi|^{2s} |\widehat{v_0}(\xi)|^2 \left(\int_0^T |\xi|^2 e^{-2t|\xi|^2} \, dt \right) d\xi \\ &= \quad \int_{\xi \in \mathbb{R}^2} |\xi|^{2s} |\widehat{v_0}(\xi)|^2 \left(-\frac{1}{2} e^{-2T|\xi|^2} + \frac{1}{2} \right) d\xi \\ &\leq \quad \int_{\xi \in \mathbb{R}^2} |\xi|^{2s} |\widehat{v_0}(\xi)|^2 \, d\xi \ = \ \|v_0\|^2_{\dot{H}^s}. \end{aligned}$$

Therefore $\Delta e^{t\Delta}v_0 \in L^2_T\dot{H}^{s-1}$. Furthermore, if $0 < \tilde{\varepsilon} < s$, we have

$$\begin{aligned}
\|\Delta e^{t\Delta}v_0\|^2_{\dot{H}^{s-\tilde{\varepsilon}}} &= \int_{\xi\in\mathbb{R}^2} |\xi|^{2s-2\tilde{\varepsilon}+4} e^{-2t|\xi|^2} |\widehat{v_0}(\xi)|^2 \, d\xi \\
&= \int_{\xi\in\mathbb{R}^2} \left(|\xi|^{2s}|\widehat{v_0}(\xi)|^2\right)\left(|\xi|^{4-2\tilde{\varepsilon}} e^{-2t|\xi|^2}\right) d\xi \\
&\leq \sup_{\xi\in\mathbb{R}^2}\left(|\xi|^{4-2\tilde{\varepsilon}} e^{-2t|\xi|^2}\right) \times \int_{\xi\in\mathbb{R}^2} |\xi|^{2s}|\widehat{v_0}(\xi)|^2 \, d\xi \\
&= \frac{1}{2t^{2-\tilde{\varepsilon}}} \, \|v_0\|^2_{\dot{H}^s}.
\end{aligned}$$

Thus

$$\|\Delta e^{t\Delta}v_0\|_{L^1_T\dot{H}^{s-\tilde{\varepsilon}}} \leq \frac{1}{\sqrt{2}} \, \|v_0\|_{\dot{H}^s} \int_0^T \frac{1}{t^{1-\frac{\tilde{\varepsilon}}{2}}} \, dt < \infty \tag{70}$$

and

$$\Delta e^{t\Delta}v_0 \in L^2_T\dot{H}^{s+1} \cap L^1_T\dot{H}^{s-\tilde{\varepsilon}}. \tag{71}$$

Let us consider now the term $\nabla : (v \otimes v)$. Using remark 5 and a classical Sobolev embedding, we have $v \in L^2_T L^\infty \cap L^\infty_T H^s$. Since $L^\infty \cap \dot{H}^s$ is an algebra (see [1] page 98), we deduce that

$$(v \otimes v)_{i,j} = v_i v_j \in L^2_T\dot{H}^s \tag{72}$$

Consequently,

$$\nabla : (v \otimes v) \in L^2_T\dot{H}^{s-1}. \tag{73}$$

Moreover, using that $v \in L^2_T H^{s+1}(\mathbb{R}^2)$ and $H^{s+1}(\mathbb{R}^2)$ being an algebra (see [1]), we obtain

$$(v \otimes v)_{i,j} = v_i v_j \in L^1_T H^{s+1}(\mathbb{R}^2), \tag{74}$$

and $\nabla : (v \otimes v) \in L^1_T H^s(\mathbb{R}^2)$.
Interpolating between $L^2_T\dot{H}^{s-1}(\mathbb{R}^2)$ and $L^1_T\dot{H}^{s-\tilde{\varepsilon}}$ and taking $\theta \in [0,1]$, we have

$$\Delta e^{t\Delta}v_0 \ , \quad \mathbb{P}\nabla : (v \otimes v) \in L^{\frac{2}{1+\theta}}\dot{H}^{s-1+\theta(1-\tilde{\varepsilon})}. \tag{75}$$

The following theorem (proved in [6] page 64), applied to $\mathbb{P}\nabla : (v \otimes v)$, allows us to conclude immediately that the integral term is also in $L^{\frac{2}{1+\theta}}\dot{H}^{s-1+\theta(1-\tilde{\varepsilon})}$ for all $\theta \in [0,1)$.

Theorem 3 (Maximal L^pL^q regularity for the heat kernel)**.** *Let A be an operator defined by*

$$f(t,x) \mapsto Af(t,x) = \int_0^t e^{(t-s)\Delta}\Delta f(s,x) \, ds. \tag{76}$$

Then A is bounded from $L^p_T L^q(\mathbb{R}^d)$ to itself, for all reals $T > 0$, $1 < p < \infty$ and $1 < q < \infty$.

□

Step 2: Regularity of $\varepsilon\partial_t u^\varepsilon$.

Lemma 5. *Let $0 \leq \theta \leq 1$ and u^ε be a solution to (NLW_ε). Then*

$$\varepsilon\partial_t u^\varepsilon \in L^q_T\dot{H}^{\sigma'}(\mathbb{R}^2) \tag{77}$$

where $\dfrac{1}{q} = \dfrac{1-\theta}{2}$ and $\sigma' = \theta\delta$ and

$$\|\varepsilon\partial_t u^\varepsilon\|_{L^q_T\dot{H}^{\sigma'}} = \mathcal{O}\left(\varepsilon^{\frac{1-\theta}{2}-\frac{\theta\delta}{2}}\right). \tag{78}$$

Proof. First, let us show that

$$\varepsilon\partial_t u^\varepsilon \in L^2_T L^2(\mathbb{R}^2, \varepsilon^{-1/2}dx) \cap L^\infty_T \dot{H}^\delta(\mathbb{R}^2, \varepsilon^{\delta/2}dx). \tag{79}$$

In this purpose, let us consider the energy

$$E^0_\varepsilon(t) =: \int_{\mathbb{R}^2} \left(\frac{1}{2}|u^\varepsilon + \varepsilon\partial_t u^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon|^2 + \varepsilon|\nabla u^\varepsilon|^2 \right) dx \tag{80}$$

and differentiate it. A simple calculation gives us

$$\frac{d}{dt}E^0_\varepsilon(t) + \varepsilon\int_{\mathbb{R}^2} |\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\, dx + \int_{\mathbb{R}^2} \left(|\nabla u^\varepsilon|^2 - \varepsilon|\nabla(u^\varepsilon \otimes u^\varepsilon)|^2 \right) dx = 0. \tag{81}$$

Thanks to the control of the norm $\|u^\varepsilon\|_\infty$ by $\dfrac{1}{2\sqrt{\varepsilon}}$, the last term in the right hand side is positive and we obtain

$$E^0_\varepsilon(T) + \int_0^T\!\!\int_{\mathbb{R}^2} \varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\, dx\, dt + \int_0^T\!\!\int_{\mathbb{R}^2} \frac{1}{2}|\nabla u^\varepsilon|^2\, dx\, dt \le E^0_\varepsilon(0) \le C_0. \tag{82}$$

Therefore, we have that $\int_0^T\!\int_{\mathbb{R}^2} \varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\, dx\, dt \le C$ and $\int_0^T\!\int_{\mathbb{R}^2} \frac{1}{2}|\nabla u^\varepsilon|^2\, dx\, dt \le C$ which gives

$$\varepsilon\int_0^T\!\!\int_{\mathbb{R}^2} |\nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\, dx\, dt \le \varepsilon\|u^\varepsilon\|_\infty \int_0^T\!\!\int_{\mathbb{R}^2} |\nabla u^\varepsilon|^2\, dx\, dt \le C. \tag{83}$$

Consequently,

$$\begin{aligned} \|\sqrt{\varepsilon}\partial_t u^\varepsilon\|^2_{L^2_T L^2} &\le 2\int_0^T\!\!\int_{\mathbb{R}^2} \varepsilon|\partial_t u^\varepsilon + \nabla : (u^\varepsilon \otimes u^\varepsilon)|^2\, dx\, dt + 2\varepsilon\int_0^T\!\!\int_{\mathbb{R}^2} |u^\varepsilon.\nabla u^\varepsilon|^2\, dx\, dt \\ &\le 2C + 2C = 4C, \end{aligned} \tag{84}$$

i.e. $\sqrt{\varepsilon}\partial_t u^\varepsilon \in L^2_T L^2$ uniformly in parameter ε.

Moreover, directly from the expression of E^δ_ε and from (28), we deduce that

$$\|\partial_t u^\varepsilon\|_{L^\infty_T \dot{H}^\delta(\mathbb{R}^2)} = \mathcal{O}\left(\varepsilon^{-1-\frac{\delta}{2}}\right). \tag{85}$$

Now, interpolating between the two spaces in (79), we obtain (77). □

We will now choose un appropriate $0 < \theta < 1$, such that

$$\varepsilon\int_0^T\!\!\int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v\, dt\, dx \longrightarrow 0\ , \quad \varepsilon \to 0. \tag{86}$$

So we should have

$$-1 + \theta(1+\delta) < 0 \Longleftrightarrow 0 < \theta < \frac{1}{1+\delta}, \tag{87}$$

so that the power of ε controlling the integral is negative. Moreover, $L^q_T H^{\sigma'}$ is the dual space of $L^p_T H^\sigma$ if $\sigma' = -\sigma$, *i.e.*

$$\theta = \frac{1-s}{1+\delta-\tilde{\varepsilon}}. \tag{88}$$

This value of θ satisfies condition (87) and gives us

$$\varepsilon\int_0^T\!\!\int_{\mathbb{R}^2} \partial_t u^\varepsilon \partial_t v\, dt\, dx = \mathcal{O}\left(\sqrt{\varepsilon}^{(s(1+\delta)-\tilde{\varepsilon})/(1+\delta-\tilde{\varepsilon})}\right) = \mathcal{O}\left(\varepsilon^\nu\right), \tag{89}$$

with $\nu = \dfrac{s(1+\delta)-\tilde{\varepsilon}}{2(1+\delta-\tilde{\varepsilon})} > 0$. Notice that $0 < \nu < \frac{s}{2}$ and that ν can be chosen arbitrarily close to $\frac{s}{2}$.

2.3.2 Estimating $\int_0^T\int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\,dt\,dx$

Lemma 6. *Let v and u^ε be as above. We have*

$$\int_0^T\int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\,dt\,dx \leq 2\int_0^T \|v\|^2_{BMO}E_{\varepsilon,v}(t)\,dt + \frac{1}{2}\int_0^T \|\nabla(u^\varepsilon - v)\|^2_{L^2}. \quad (90)$$

Proof. Let us recall that $v(t)\in\dot{H}^1(\mathbb{R}^2)\subset BMO(\mathbb{R}^2)$ (see remark 5) and that the solution of the wave equation (NLW_ε), $u^\varepsilon(t)$, is an $L^2(\mathbb{R}^2)$ divergence-free vector. The following theorem applies.

Theorem 4 (div-curl lemma, see [4])**.** *Let f be an $L^2(\mathbb{R}^2)$ divergence-free vector and $g\in\dot{H}^1(\mathbb{R}^2)$. Then*

$$f.\nabla g \in \mathcal{H}^1(\mathbb{R}^2), \quad (91)$$

where $\mathcal{H}^1(\mathbb{R}^2)$ is the Hardy space constructed on $L^1(\mathbb{R}^2)$.

Using the duality of Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ and $BMO(\mathbb{R}^2)$ (proved in [5]) and the div-curl theorem (Theorem 4), then performing a Young inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^2} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\,dx &\leq \|v\|_{BMO}\|u^\varepsilon - v\|_{L^2}\|\nabla(u^\varepsilon - v)\|_{L^2}\\ &\leq \frac{1}{2}\|v\|^2_{BMO}\|u^\varepsilon - v\|^2_{L^2} + \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}\\ &\overset{(58)}{\leq} 2\|v\|^2_{BMO}E_{\varepsilon,v}(t) + \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}.\end{aligned}$$

Therefore, we have showed (90). □

At this level, we have the following inequality

$$E_{\varepsilon,v}(T) - E_{\varepsilon,v}(0) \leq 2\int_0^T \|v\|^2_{BMO}E_{\varepsilon,v}(t)\,dt + \mathcal{O}(\varepsilon^\nu) + \int_0^T\left(\varepsilon\|\nabla(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2} - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}\right)dt. \quad (92)$$

Let us call $\tilde{A}^\varepsilon$ the term that remains to estimate :

$$\tilde{A}^\varepsilon = \int_0^T A^\varepsilon\,dt = \int_0^T\left(\varepsilon\|\nabla(u^\varepsilon\otimes u^\varepsilon)\|^2_{L^2} - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}\right)dt. \quad (93)$$

2.3.3 Estimating $\int_0^T A^\varepsilon\,dt$

The aim of this part is to prove the following lemma

Lemma 7. *Let v and u^ε be as above. Then, for all positive T and for some positive μ, we have*

$$\int_0^T A^\varepsilon\,dt = \int_0^T\int_{\mathbb{R}^2}\left(\varepsilon|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2 - \frac{1}{2}|\nabla(u^\varepsilon - v)|^2\right)dt\,dx = \mathcal{O}(\varepsilon^\mu). \quad (94)$$

Proof. First, let us write

$$\nabla(u^\varepsilon\otimes u^\varepsilon) = u^\varepsilon.\nabla u^\varepsilon = u^\varepsilon.\nabla(u^\varepsilon - v) + u^\varepsilon.\nabla v \quad (95)$$

then perform a Young inequality

$$|\nabla(u^\varepsilon\otimes u^\varepsilon)|^2 \leq 2|u^\varepsilon.\nabla(u^\varepsilon - v)|^2 + 2|u^\varepsilon.\nabla v|^2. \quad (96)$$

So we have that

$$\begin{aligned} A^\varepsilon &= \varepsilon \int_{\mathbb{R}^2} |\nabla(u^\varepsilon \otimes u^\varepsilon)|^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \\ &\leq 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon.\nabla(u^\varepsilon - v)|^2 + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon.\nabla v|^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \\ &\leq \left(2\varepsilon\|u^\varepsilon\|_{L^\infty}^2 - \frac{1}{2}\right) \int_{\mathbb{R}^2} |\nabla(u^\varepsilon - v)|^2 + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon.\nabla v|^2 \\ &\leq \left(\frac{1}{2C_1^2} - \frac{1}{2}\right) \int_{\mathbb{R}^2} |\nabla(u^\varepsilon - v)|^2 + 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon.\nabla v|^2 \\ &\leq 2\varepsilon \int_{\mathbb{R}^2} |u^\varepsilon.\nabla v|^2 \ \leq \ 2\varepsilon\|u^\varepsilon\|_{L^\infty}^2 \int_{\mathbb{R}^2} |\nabla v|^2 \ = \ \mathcal{O}\left(\varepsilon^s\right) \end{aligned}$$

since we have (see (30) and assumptions (4))

$$\|u_0^\varepsilon\|_{L^\infty} \leq C_2 \|u_0^\varepsilon\|_{\dot{H}^\delta}^{\delta} \|u_0^\varepsilon\|_{\dot{H}^{1+\delta}}^{1-\delta} \lesssim \varepsilon^{-\frac{\delta}{2}\delta} \times \varepsilon^{\frac{s}{2}\delta} \times \varepsilon^{-\frac{1+\delta}{2}(1-\delta)} \times \varepsilon^{\frac{s}{2}(1-\delta)} = \varepsilon^{\frac{s-1}{2}}. \tag{97}$$

Then we have obtained the lemma, with $\mu = s$:

$$\int_0^T A^\varepsilon \, dt = \mathcal{O}(\varepsilon^s). \tag{98}$$

□

2.3.4 Conclusion

Gathering the results in the previous three subsections, we obtain

$$E_{\varepsilon,v}(T) \leq 2 \int_0^T \|v\|_{BMO}^2 E_{\varepsilon,v}(t) \, dt + \mathcal{O}\left(\varepsilon^\nu\right) + \mathcal{O}(\varepsilon^s) + E_{\varepsilon,v}(0). \tag{99}$$

Assuming that

$$\|u_0^\varepsilon - v_0\|_{L^2} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right) \tag{100}$$

in addition to assumptions (H) and (49), we have

$$E_{\varepsilon,v}(0) = \mathcal{O}\left(\varepsilon^s\right) \tag{101}$$

by the triangle inequality.
It follows that

$$E_{\varepsilon,v}(T) \leq 2 \int_0^T \|v_0\|_{BMO}^2 E_{\varepsilon,v}(t) \, dt + \mathcal{O}\left(\varepsilon^\nu\right) + \mathcal{O}\left(\varepsilon^s\right). \tag{102}$$

Now $\nu = \dfrac{s(1+\delta) - \tilde{\varepsilon}}{2(1+\delta-\tilde{\varepsilon})} \leq \dfrac{s}{2}$ since $s \leq 1$ so

$$E_{\varepsilon,v}(T) \leq 2 \int_0^T \|v_0\|_{BMO}^2 E_{\varepsilon,v}(t) \, dt + \mathcal{O}\left(\varepsilon^\nu\right). \tag{103}$$

Gronwall lemma (and $v \in L_T^2 BMO(\mathbb{R}^2)$) yields

$$E_{\varepsilon,v}(T) = \mathcal{O}\left(\varepsilon^\nu\right) \tag{104}$$

for all positive T and, using inequality (58)

$$\int_{\mathbb{R}^2} |u^\varepsilon - v|^2 \, dx \leq 4E_{\varepsilon,v}(t), \tag{105}$$

we deduce the convergence in the $L_T^\infty L^2(\mathbb{R}^2)$ norm of u^ε solution to (NLW_ε) towards v solution to the (NS) equations with initial data in $H^s(\mathbb{R}^2)^2$, where $0 < s \leq 1$. Theorem 1 is now proved.

3 The three-dimensional case : proof of Theorem 2

In this section, we shall follow the plan of the previous one: we shall start by showing local existence for the damped wave equation (NLW_ε) then we shall globalize the local solution u^ε and finally prove the convergence of u^ε towards the solution to Navier-Stokes problem with initial data in $H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$ and $0 < s < 1$. Let us recall again that the critical space for (NS) in $\mathbb{R}^3$ is $\dot{H}^{\frac{1}{2}}$.

3.1 Local existence in $\dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3, \mathbb{R}^3)$

We apply the same fixed point method as in the two-dimensional case. We perform the same scale change (9) and retrieve system (NLW) with initial data $(u_0, u_1) \in H^{\frac{3}{2}+\delta}(\mathbb{R}^3)^3 \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$. Let us recall Duhamel formula for the wave equation (NLW) :

$$\phi(u)(t) = \cos(t\Lambda)u_0 + \frac{\sin(t\Lambda)}{\Lambda}u_1 + \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda}F(u(s))\,ds, \tag{106}$$

where $F(u) = -\partial_t u - \mathbb{P}\nabla : u \otimes u$.
We shall prove local existence for (NLW) in the complete metric space

$$\begin{aligned} X_T(a) \quad = \quad & \Big\{(v, \partial_t v) \in L^\infty_T(\dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}) \times L^\infty_T \dot{H}^{\frac{1}{2}+\delta} \ : \ \nabla.v = 0\ , \\ & \|v\|_{X_T} := \|v\|_{L^\infty_T \dot{H}^{\frac{3}{2}+\delta}} + \|v\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}} + \|\partial_t v\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}} \leq a\Big\}, \end{aligned} \tag{107}$$

with $a > 0$ and $0 \leq T < 1$ to be chosen later. Let $0 \leq t \leq T$.

As in subsection 2.1, for the two-dimensional case, we shall use the fact that the space $\dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)$ is an algebra and that the inequality

$$\|fg\|_{\dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}} \leq C\left(\|f\|_{\dot{H}^{\frac{3}{2}+\delta}}\|g\|_{\dot{H}^{\frac{1}{2}+\delta}} + \|f\|_{\dot{H}^{\frac{1}{2}+\delta}}\|g\|_{\dot{H}^{\frac{3}{2}+\delta}}\right) \tag{108}$$

holds for all $f, g \in \dot{H}^{\frac{3}{2}+\delta} \cap \dot{H}^{\frac{1}{2}+\delta}$.

Let us start the estimates on the X_T norm of $\phi(u)$.

$$\begin{aligned} \|\phi(u)(t)\|_{\dot{H}^{\frac{3}{2}+\delta}} \quad \leq \quad & \|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}} + \int_0^t \|\frac{\sin\left((t-s)\Lambda\right)}{\Lambda}F\left(u(s)\right)\|_{\dot{H}^{\frac{3}{2}+\delta}}\,ds \\ \leq \quad & \|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}} + \int_0^t \|F\left(u(s)\right)\|_{\dot{H}^{\frac{1}{2}+\delta}}\,ds. \end{aligned}$$

Now consider the integral term.

$$\begin{aligned} \int_0^t \|F\left(u(s)\right)\|_{\dot{H}^{\frac{1}{2}+\delta}}\,ds \quad \leq \quad & \int_0^t \|\partial_t u(s) + \mathbb{P}(u.\nabla u)\|_{\dot{H}^{\frac{1}{2}+\delta}}\,ds \\ \leq \quad & \int_0^t \left(\|\partial_t u(s)\|_{\dot{H}^{\frac{1}{2}+\delta}} + \|\mathbb{P}(u.\nabla u)\|_{\dot{H}^{\frac{1}{2}+\delta}}\right)\,ds \\ \leq \quad & T\left(\|\partial_t u\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}} + \|u.\nabla u\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}}\right). \end{aligned}$$

Using (14), we have

$$\|u.\nabla u\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq \sum_{i=1}^3 \|\partial_i(u_i u)\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq \sum_{i=1}^3 \|u_i u\|_{\dot{H}^{\frac{3}{2}+\delta}} \leq 3 \times 2C\|u\|_{\dot{H}^{\frac{1}{2}+\delta}}\|u\|_{\dot{H}^{\frac{3}{2}+\delta}}$$

since (108) holds. So

$$\int_0^t \|F\left(u(s)\right)\|_{\dot{H}^{\frac{1}{2}+\delta}}\,ds \leq T\left(\|\partial_t u\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}} + 6C\|u\|_{L^\infty_T \dot{H}^{\frac{1}{2}+\delta}}\|u\|_{L^\infty_T \dot{H}^{\frac{3}{2}+\delta}}\right) \tag{109}$$

which yields

$$\|\phi(u)(t)\|_{\dot{H}^{\frac{3}{2}+\delta}} \leq \|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}} + Ta\,(1+6Ca)\,. \tag{110}$$

By similar computations and using (108), we obtain

$$\|\phi(u)\|_{X_T} \leq 2\|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}} + 2\|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}} + T\|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}} + aT(2+T)(1+6Ca). \tag{111}$$

We choose

$$a = 4\left(\|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}\right) \tag{112}$$

and $T < 1$ such that

$$T\frac{a}{4} + aT(2+T)(1+6Ca) < \frac{a}{2}. \tag{113}$$

Since $T < 1$, it is sufficient to take

$$T < \frac{1}{12(1+6Ca)}. \tag{114}$$

With a and T chosen this way, the space $X_T(a)$ is stable under ϕ.

All it remains to do now is to prove that the application $\phi : X_T(a) \longrightarrow X_T(a)$ is contractive. So let $(u, \partial_t u)$ and $(v, \partial_t v)$ be in $X_T(a)$. By Duhamel formula, we have the identity

$$\begin{aligned}
[\phi(u) - \phi(v)](t) &= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} \left(\partial_t v + \mathbb{P}(v.\nabla v) - \partial_t u - \mathbb{P}(u.\nabla u)\right)(s)\, ds \\
&= \int_0^t \frac{\sin((t-s)\Lambda)}{\Lambda} \left[\partial_t(v-u) + \mathbb{P}\nabla : (v\otimes v - u\otimes u)\right](s)\, ds.
\end{aligned} \tag{115}$$

Moreover, since u and v are divergence-free vector fields, the identity

$$\nabla : (v\otimes v - u\otimes u) = (v-u).\nabla v + u.\nabla(v-u) \tag{116}$$

holds.
Using (115) and (116), we estimate the X_T norm of $\phi(u) - \phi(v)$. We obtain

$$\|\phi(u) - \phi(v)\|_{X_T} \leq T(2+T)\,(1+12Ca)\,\|v-u\|_{X_T}. \tag{117}$$

So ϕ is contractive if, and only if, $T < 1$ satisfies

$$T(2+T) < \frac{1}{1+12Ca}. \tag{118}$$

Since $T < 1$, it is sufficient to take T such that

$$T < \frac{1}{3(1+12Ca)}. \tag{119}$$

We conclude then that there exists a local solution u to (NLW), defined on the time interval $[0, T[$, for all positive real T satisfying conditions (114) and (119), for example

$$T \leq \frac{1}{12 + 72C\left(\|u_0\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u_0\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|u_1\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}\right)}. \tag{120}$$

In particular, while $\|u(t)\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)}$ is bounded, we can reiterate the fixed point argument and extend the solution. Let $T^{\max}$ be the maximal existence time. We shall prove that $T^{\max} = +\infty$. Following the same reasoning by contradiction as in the previous section, we assume that $T^{\max} < +\infty$. This would imply

$$\|u(t)\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \|\partial_t u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \longrightarrow +\infty\ ,\quad t \to T^{\max}. \tag{121}$$

Let us resume our wave equation (NLW_ε) with parameter ε. The scaling (9) gives

$$\|u(t)\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}} + \|\partial_t u(t)\|_{\dot{H}^{\frac{1}{2}+\delta}} = \varepsilon^{\frac{\delta}{2}} \left(\sqrt{\varepsilon} \|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} + \varepsilon \|\partial_t u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} \right). \tag{122}$$

Indeed, for a non negative real s, we have

$$\begin{aligned} \|u^\varepsilon\|^2_{\dot{H}^s(\mathbb{R}^3)} &= \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u^\varepsilon}(\xi)|^2 \, d\xi \\ &= \varepsilon^2 \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{u}(\sqrt{\varepsilon}\xi)|^2 \, d\xi \\ &= \varepsilon^2 \int_{\mathbb{R}^3} |\frac{\eta}{\sqrt{\varepsilon}}|^{2s} |\hat{u}(\eta)|^2 \, \varepsilon^{-\frac{3}{2}} d\eta \\ &= \varepsilon^{\frac{1}{2}-s} \|u\|^2_{\dot{H}^s(\mathbb{R}^3)}. \end{aligned}$$

Similarly, writing $\partial_\tau u^\varepsilon(\tau, y) = \varepsilon^{-\frac{3}{2}} \partial_t u(\frac{\tau}{\varepsilon}, \frac{y}{\sqrt{\varepsilon}}) = \varepsilon^{-\frac{3}{2}} \partial_t u(t, \frac{y}{\sqrt{\varepsilon}})$, we obtain

$$\begin{aligned} \|\partial_\tau u^\varepsilon\|^2_{\dot{H}^s(\mathbb{R}^3)} &= \|\varepsilon^{-\frac{3}{2}} \partial_t u(t, \frac{y}{\sqrt{\varepsilon}})\|^2_{\dot{H}^s(\mathbb{R}^3)} \\ &= \varepsilon^{-3} \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{\partial_t u \left(\frac{\cdot}{\sqrt{\varepsilon}} \right)}(\xi)|^2 \, d\xi \\ &= \varepsilon^{-3} \int_{\mathbb{R}^3} |\xi|^{2s} |\varepsilon^{\frac{3}{2}} \widehat{\partial_t u}(\sqrt{\varepsilon}\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^3} |\frac{\eta}{\sqrt{\varepsilon}}|^{2s} |\widehat{\partial_t u}(\eta)|^2 \, \varepsilon^{-\frac{3}{2}} d\eta \\ &= \varepsilon^{-\frac{3}{2}-s} \|\partial_t u\|^2_{\dot{H}^s(\mathbb{R}^3)}. \end{aligned}$$

Let us define, for positive δ, the energy

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2} |\Lambda^{\frac{1}{2}+\delta}(u^\varepsilon + \varepsilon \partial_t u^\varepsilon)|^2 + \frac{\varepsilon^2}{2} |\Lambda^{\frac{1}{2}+\delta} \partial_t u^\varepsilon|^2 + \varepsilon |\Lambda^{\frac{3}{2}+\delta} u^\varepsilon|^2 \right) dx. \tag{123}$$

Then

$$\varepsilon^{\frac{\delta}{2}} \left(\sqrt{\varepsilon} \|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}+\delta}(\mathbb{R}^3)} + \|u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} + \varepsilon \|\partial_t u^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}(\mathbb{R}^3)} \right) \leq C \varepsilon^{\frac{\delta}{2}} \sqrt{E_\varepsilon^\delta}. \tag{124}$$

If we prove that $\varepsilon^{\frac{\delta}{2}} \sqrt{E_\varepsilon^\delta}$ is bounded, we can deduce that the solution u^ε is global.

3.2 Globalization

Let $\delta > 0$ and consider the energy $E_\varepsilon^{\frac{1}{2}+\delta}$ defined as above. We shall prove that there exists a positive constant C_0 such that

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq C_0 \varepsilon^{-\delta} \tag{125}$$

for all time $t > 0$. In this purpose, we adapt the method used in the two-dimensional case to three space dimensions. To get around the difficluties, we use mainly interpolations and tame estimates.

First, let us recall that $\dot{H}^{\frac{3}{2}+\delta} \cap L^\infty(\mathbb{R}^3)$ is an algebra (see [1]) and that the inequality

$$\|fg\|_{\dot{H}^{\frac{3}{2}+\delta}} \leq C_1 \left(\|f\|_{L^\infty} \|g\|_{\dot{H}^{\frac{3}{2}+\delta}} + \|g\|_{L^\infty} \|f\|_{\dot{H}^{\frac{3}{2}+\delta}} \right) \tag{126}$$

holds for all $f, g \in \dot{H}^{\frac{3}{2}+\delta} \cap L^\infty(\mathbb{R}^3)$. Moreover, using the embedding $\dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$ and the interpolation inequality

$$\|f\|_{\dot{B}^{3/2}_{2,1}(\mathbb{R}^3)} \leq C\|f\|^{\delta}_{\dot{H}^{\frac{1}{2}+\delta}} \|f\|^{1-\delta}_{\dot{H}^{\frac{3}{2}+\delta}}, \tag{127}$$

we obtain

$$\|f\|_\infty \leq C_2\|f\|^{\delta}_{\dot{H}^{\frac{1}{2}+\delta}} \|f\|^{1-\delta}_{\dot{H}^{\frac{3}{2}+\delta}}. \tag{128}$$

Remark 6. *In $\mathbb{R}^3$, the Navier-Stokes problem with initial data v_0 in $H^{\frac{1}{2}}$ has a global solution if $\|v_0\|_{\dot{H}^{\frac{1}{2}}}$ is small enough. So we could show that $E_\varepsilon^{\frac{1}{2}+\delta} + NE_\varepsilon^{\frac{1}{2}}$ decreases and deduce the globality of the solution but this method requires a smallness assumption on the $\dot{H}^{\frac{1}{2}}$ norm of u_0^ε which depends on δ and we would have*

$$\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} \to 0$$

when δ goes to 0.

We shall prove that $E_\varepsilon^{\frac{1}{2}+\delta}\left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$ decreases if N is large enough and if the $\dot{H}^{\frac{1}{2}}$ norm of u^ε is small enough. First, we have to prove that $E_\varepsilon^{\frac{1}{2}}$ decreases. In this purpose, let us compute the time derivative of the energy.

$$\begin{aligned}
\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) &= -\int_{\mathbb{R}^3} \Lambda^{\frac{1}{2}}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac{1}{2}}u^\varepsilon + \int_{\mathbb{R}^3}\left(\varepsilon|\Lambda^{\frac{1}{2}}(u^\varepsilon.\nabla u^\varepsilon)|^2 - |\Lambda^{\frac{3}{2}}u^\varepsilon|^2\right) \\
&\leq \int_{\mathbb{R}^3} u^\varepsilon.\nabla u^\varepsilon.\Lambda u^\varepsilon + \left((6C_1)^2\varepsilon\|u^\varepsilon\|^2_\infty - 1\right)\|u^\varepsilon\|^2_{\dot{H}^{\frac{3}{2}}}.
\end{aligned} \tag{129}$$

In order to estimate the first term on the right hand side, we state the following theorem :

Theorem 5. *Let $f \in H^{\frac{3}{2}}(\mathbb{R}^3)$. On a*

$$|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)| \leq \|\Lambda^{\frac{1}{2}} f\|_{L^2(\mathbb{R}^3)} \|\Lambda^{\frac{3}{2}} f\|^2_{L^2(\mathbb{R}^3)} \tag{130}$$

Proof. Let $u \in H^{\frac{3}{2}}(\mathbb{R}^3)$. First, we use the duality between $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$.

$$|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)| \leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)}. \tag{131}$$

We shall estimate the $\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)$ norm of $f.\nabla f$. For this sake, let us notice that $f \in H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$ by a Sobolev embedding. Thus, by Hölder inequality, we have

$$f.\nabla f \in L^{\frac{3}{2}}(\mathbb{R}^3) \tag{132}$$

since $\frac{1}{6} + \frac{1}{2} = \frac{2}{3}$. Moreover, we shall use the Sobolev embedding $H^{\frac{1}{2}}(\mathbb{R}^3) \subset L^3(\mathbb{R}^3)$ on its dual form $L^{\frac{3}{2}}(\mathbb{R}^3) \subset H^{-\frac{1}{2}}(\mathbb{R}^3)$ .Finally, we obtain

$$\begin{aligned}
|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)| &\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3)} \\
&\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f.\nabla f\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} \\
&\leq \|\Lambda f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|f\|_{L^6(\mathbb{R}^3)} \|\nabla f\|_{L^2(\mathbb{R}^3)} \\
&\leq \|f\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \|f\|^2_{\dot{H}^1(\mathbb{R}^3)}.
\end{aligned}$$

Now, we use the Gagliardo-Nirenberg inequality

$$\|f\|_{\dot{H}^1} \leq \sqrt{\|f\|_{\dot{H}^{\frac{3}{2}}} \|f\|_{\dot{H}^{\frac{1}{2}}}}\,. \tag{133}$$

Thus

$$|\int_{\mathbb{R}^3} \Lambda f.(f.\nabla f)| \leq \|f\|^2_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^3)} \|f\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)}, \tag{134}$$

which finishes the proof. □

Using theorem 5, we shall prove the following lemma.

Lemma 8. *Let us define* $0 \leq T \leq T_\varepsilon^{\max}$ *by*

$$T = \sup\left\{0 \leq \tau \leq T_\varepsilon^{\max} \ : \ \forall\ t \in [0,\tau[,\ \|u^\varepsilon(t)\|_\infty < \frac{1}{7C_1\sqrt{\varepsilon}}\right\}. \tag{135}$$

Assume $\|u_0^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} < \dfrac{1}{16}$ *and*

$$\varepsilon\|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}}} + \sqrt{\varepsilon}\|u_0^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} = o(1) \tag{136}$$

when ε *goes to* 0. *Then the energy* $E_\varepsilon^{\frac{1}{2}}$ *decreases on* $[0,T]$ *and we have*

$$\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) \leq -\frac{1}{8}\|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} \tag{137}$$

for all $t \in [0,T]$.

Proof. Let $t \in [0,T]$. Since $\|u^\varepsilon(t)\|_\infty < \dfrac{1}{7C_1\sqrt{\varepsilon}}$, inequality (129) becomes

$$\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) \leq \int_{\mathbb{R}^3} u^\varepsilon.\nabla u^\varepsilon.\Lambda u^\varepsilon - \frac{1}{4}\|u^\varepsilon\|^2_{\dot{H}^{\frac{3}{2}}}. \tag{138}$$

Now, theorem 5 (applied to first term on the right hand side) yields

$$\frac{dE_\varepsilon^{\frac{1}{2}}}{dt}(t) \leq \|u^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}}\|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} - \frac{1}{4}\|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} \leq \left(\|u^\varepsilon(t)\|_{\dot{H}^{\frac{1}{2}}} - \frac{1}{4}\right)\|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}}. \tag{139}$$

In $t = 0$, this inequality gives that $E_\varepsilon^{\frac{1}{2}}$ decreases in a neighborhood of 0. Let

$$\tau = \sup\left\{0 \leq \tilde{\tau} \leq T \ : \ E_\varepsilon^{\frac{1}{2}} \text{ decreases on } [0,\tau]\right\}. \tag{140}$$

So $0 < \tau \leq T$. Assume that $\tau < T$. On the one hand, we have

$$\|u^\varepsilon(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} < 2E_\varepsilon^{\frac{1}{2}}(\tau). \tag{141}$$

On the other hand, for ε small enough, we have

$$E_\varepsilon^{\frac{1}{2}}(0) \leq \|u_0^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} + \frac{3\varepsilon^2}{2}\|u_1^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} + \varepsilon\|u_0^\varepsilon\|^2_{\dot{H}^{\frac{3}{2}}} < 2\|u_0^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} \tag{142}$$

thanks to assumptions (136). Adding to that the energy decay on $[0,\tau]$, we obtain

$$\|u^\varepsilon(\tau)\|^2_{\dot{H}^{\frac{1}{2}}} < 2E_\varepsilon^{\frac{1}{2}}(\tau) \leq 2E_\varepsilon^{\frac{1}{2}}(0) \leq 4\|u_0^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} \tag{143}$$

so

$$\|u^\varepsilon(\tau)\|_{\dot{H}^{\frac{1}{2}}} < 2 \times \frac{1}{16} = \frac{1}{8}. \tag{144}$$

thus $E_\varepsilon^{\frac{1}{2}}$ decreases in a neighborhood of τ and this is a contradiction with its definition. Therefore $E_\varepsilon^{\frac{1}{2}}$ decreases on $[0,T]$ and the inequality (137) is true for all $t \in [0,T]$. □

Lemma 9. *Assume* $\|u_0^\varepsilon\|_{\dot H^{\frac12}} < \dfrac{1}{16}$ *and*

$$(H_1) \qquad i)\ \varepsilon^{\frac{1+\delta}{2}}\|u_0^\varepsilon\|_{\dot H^{\frac32+\delta}} = o(1)\ , \qquad ii)\ \varepsilon^{\frac{\delta}{2}}\|u_0^\varepsilon\|_{\dot H^{\frac12+\delta}} = o(1). \tag{145}$$

when ε *goes to zero.*
Recall that $0 \le T \le T_\varepsilon^{\max}$ *is defined by*

$$T = \sup\left\{0 \le \tau \le T_\varepsilon^{\max}\ :\ \forall\ t \in [0,\tau[,\ \|u^\varepsilon(t)\|_\infty < \frac{1}{7C_1\sqrt{\varepsilon}}\right\}. \tag{146}$$

Then $T > 0$ *and there exists a large number* N*, depending only on* δ*, and a constant* $C > 1$ *such that*

$$E_\varepsilon^{\frac12+\delta}(t) \le C^N\, E_\varepsilon^{\frac12+\delta}(0) \tag{147}$$

for all $t \in [0,T)$ *and* ε *small enough.*

Proof. Let us compute the derivative of the energy.

$$\begin{aligned}
\frac{d}{dt}E_\varepsilon^{\frac12+\delta}(t) &= \int_{\mathbb{R}^3}\Big(\Lambda^{\frac12+\delta}(\varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon+\varepsilon\partial_t u^\varepsilon)+\varepsilon^2\Lambda^{\frac12+\delta}\partial_t u^\varepsilon.\Lambda^{\frac12+\delta}\partial_{tt}u^\varepsilon+\\
&\qquad +2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon\Big)\\
&= \int_{\mathbb{R}^3}\Big(\Lambda^{\frac12+\delta}(\varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon-\Delta u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon+2\varepsilon\partial_t u^\varepsilon)+\\
&\qquad +\Lambda^{\frac52+\delta}u^\varepsilon.\Lambda^{\frac12+\delta}\,(u^\varepsilon+2\varepsilon\partial_t u^\varepsilon)-\varepsilon\Lambda^{\frac12+\delta}(\varepsilon\partial_{tt}u^\varepsilon+\partial_t u^\varepsilon).\Lambda^{\frac12+\delta}\partial_t u^\varepsilon+\\
&\qquad +\varepsilon^2\Lambda^{\frac12+\delta}\partial_t u^\varepsilon.\Lambda^{\frac12+\delta}\partial_{tt}u^\varepsilon+2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon\Big).
\end{aligned}$$

Now, recall that u^ε is a solution to (NLW_ε). Then

$$\begin{aligned}
\frac{d}{dt}E_\varepsilon^{\frac12+\delta}(t) &= \int_{\mathbb{R}^3}\Big(-\Lambda^{\frac12+\delta}(\nabla : u^\varepsilon\otimes u^\varepsilon).\Lambda^{\frac12+\delta}(u^\varepsilon+2\varepsilon\partial_t u^\varepsilon)-\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\,(u^\varepsilon+2\varepsilon\partial_t u^\varepsilon)\\
&\qquad -\varepsilon|\Lambda^{\frac12+\delta}\partial_t u^\varepsilon|^2+2\varepsilon\Lambda^{\frac32+\delta}u^\varepsilon.\Lambda^{\frac32+\delta}\partial_t u^\varepsilon\Big)\\
&= \int_{\mathbb{R}^3}\Big(-\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon-2\varepsilon\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}\partial_t u^\varepsilon-\\
&\qquad -\varepsilon|\Lambda^{\frac12+\delta}\partial_t u^\varepsilon|^2-|\Lambda^{\frac32+\delta}u^\varepsilon|^2\Big)\\
&\overset{\text{Young}}{\le} -\int_{\mathbb{R}^3}\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon+\int_{\mathbb{R}^3}\Big(\varepsilon|\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon)|^2-|\Lambda^{\frac32+\delta}u^\varepsilon|^2\Big).
\end{aligned}$$

Cauchy-Schwarz inequality followed by (14) and (126) yields

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3}\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon\Big| &\le \|u^\varepsilon.\nabla u^\varepsilon\|_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\
&\le 3\times 2C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\
&\le 6C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}.
\end{aligned}\tag{148}$$

Thus, back to (148), we have

$$\begin{aligned}
\Big|\int_{\mathbb{R}^3}\Lambda^{\frac12+\delta}(u^\varepsilon.\nabla u^\varepsilon).\Lambda^{\frac12+\delta}u^\varepsilon\Big| &\le 6C_1\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot H^{\frac32+\delta}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\\
&\le 6C_1C_2\|u^\varepsilon\|^{1+\delta}_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}}\\
&\le 6C_1C_2\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{\delta}_{\dot H^{\frac12+\delta}}\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}}\\
&\le 6C_1C_2C_3\|u^\varepsilon\|^{1-\delta}_{\dot H^{\frac12}}\Big(\|u^\varepsilon\|_{\dot H^{\frac32}}\|u^\varepsilon\|_{\dot H^{\frac12+\delta}}\Big)^{\delta}\|u^\varepsilon\|^{2-\delta}_{\dot H^{\frac32+\delta}},
\end{aligned}$$

where we have used another interpolation inequality between $\dot{H}^{\frac{1}{2}}$ and $\dot{H}^{\frac{3}{2}}$:

$$\|f\|_{\dot{H}^{\frac{1}{2}+\delta}} \leq C_3 \|f\|^{1-\delta}_{\dot{H}^{\frac{1}{2}}} \|f\|^{\delta}_{\dot{H}^{\frac{3}{2}}}. \tag{149}$$

Furthermore, the inequality

$$\begin{aligned} \int_{\mathbb{R}^3} \left(\varepsilon |\Lambda^{\frac{1}{2}+\delta}(u^\varepsilon.\nabla u^\varepsilon)|^2 - |\Lambda^{\frac{3}{2}+\delta} u^\varepsilon|^2 \right) &\leq \left((6C_1)^2 \varepsilon \|u^\varepsilon\|^2_\infty - 1 \right) \int_{\mathbb{R}^3} |\Lambda^{\frac{3}{2}+\delta} u^\varepsilon|^2 \\ &\leq -\frac{1}{4} \|u^\varepsilon\|^2_{\dot{H}^{\frac{3}{2}+\delta}} \end{aligned} \tag{150}$$

holds if

$$\|u^\varepsilon(t)\|_\infty \leq \frac{1}{7C_1\sqrt{\varepsilon}}. \tag{151}$$

For ε small enough, this condition is ensured in $t = 0$ thanks to assumptions (H_1). By continuity of the local solution u^ε with respect to the time variable t, we deduce that $T > 0$.
Let $t < T$.
A Young inequality yields

$$\begin{aligned} \frac{d}{dt} E_\varepsilon^{\frac{1}{2}+\delta} &\leq \delta 2^{2\frac{2-\delta}{\delta}} (6C_1C_2C_3)^{\frac{2}{\delta}} \|u^\varepsilon(t)\|^{2\frac{1-\delta}{\delta}}_{\dot{H}^{\frac{1}{2}}} \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &\leq \delta 2^{2\frac{2-\delta}{\delta}} (6C_1C_2C_3)^{\frac{2}{\delta}} \left(4\|u_0^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}} \right)^{\frac{1-\delta}{\delta}} \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &< 16\delta \left(3C_1C_2C_3\right)^{\frac{2}{\delta}} \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} E_\varepsilon^{\frac{1}{2}+\delta}(t). \end{aligned}$$

To alleviate the notations, let us set $K = K(\delta, C_1, C_2, C_3) = 16\delta \left(3C_1C_2C_3\right)^{\frac{2}{\delta}}$.

As above, in subsection 2.2, we prove that $E_\varepsilon^{\frac{1}{2}+\delta} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$ decreases on $[0, T)$ for ε small enough and N large enough.
Let $0 \leq t \leq T$ and $N \geq 0$.
We have

$$\begin{aligned} \frac{d}{dt} \left(E_\varepsilon^{\frac{1}{2}+\delta} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N \right)(t) &= N \frac{dE_\varepsilon^{\frac{1}{2}}}{dt} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta} + \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N \frac{dE_\varepsilon^{\frac{1}{2}+\delta}}{dt}(t) \\ &\leq \left[-\frac{N}{8} + K \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right) \right] \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t) \\ &\leq \left[-\frac{N}{8} + K \left(1 + 2\|u_0^\varepsilon\|^2_{\dot{H}^{\frac{1}{2}}}\right) \right] \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t). \end{aligned}$$

Inequality (143) yields

$$\frac{d}{dt} \left(E_\varepsilon^{\frac{1}{2}+\delta} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N \right)(t) \leq \left(-\frac{N}{8} + \frac{129}{128} K \right) \|u^\varepsilon(t)\|^2_{\dot{H}^{\frac{3}{2}}} \left(1 + E_\varepsilon^{\frac{1}{2}}(t)\right)^{N-1} E_\varepsilon^{\frac{1}{2}+\delta}(t). \tag{152}$$

Now, taking $N \geq 8K \times \frac{129}{128} = 129\delta \left(3C_1C_2C_3\right)^{\frac{2}{\delta}}$, we prove that $E_\varepsilon^{\frac{1}{2}+\delta} \left(1 + E_\varepsilon^{\frac{1}{2}}\right)^N$ decays on $[0, T]$. So

$$E_\varepsilon^{\frac{1}{2}+\delta}(t) \leq E_\varepsilon^{\frac{1}{2}+\delta}(0) \left(1 + E_\varepsilon^{\frac{1}{2}}(0)\right)^N \leq E_\varepsilon^{\frac{1}{2}+\delta}(0) \left(\frac{129}{128}\right)^N \tag{153}$$

for $t \in [0, T]$. □

Lemma 10. *Assume the limit*

$$\varepsilon^{1+\frac{\delta}{2}} \|u_1^\varepsilon\|_{\dot{H}^{\frac{1}{2}+\delta}} \longrightarrow 0 \ , \quad \varepsilon \to 0 \tag{154}$$

in addition to the assumptions (H_1) *and* (136)*. Then we have*

$$E_\varepsilon^{\frac{1}{2}+\delta}(0) = o\left(\varepsilon^{-\delta}\right). \tag{155}$$

Moreover, for all $t \in [0,T[$ *and* ε *small enough, the inequality*

$$\|u^\varepsilon\|_\infty \ \leq \ \frac{1}{14C_1\sqrt{\varepsilon}} \tag{156}$$

holds.

We skip the proof here as it is analogous to the one of Lemma 2. As above, this lemma implies the inequality (125) and the globality of the solution u^ε.

3.3 Convergence towards a solution to (NS) problem

The aim of this subsection is to prove an error estimate in the $L^\infty_T L^2(\mathbb{R}^3)^3$ norm like in the two-dimensional case. More precisely, let $v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3)^3$, $s>0$, be the initial data for the Navier-Stokes equation (1) and $(u_0^\varepsilon, u_1^\varepsilon) \in H^{\frac{3}{2}+\delta} \times H^{\frac{1}{2}+\delta}(\mathbb{R}^3)^3$, $\delta>0$, be the initial data for the damped wave equation (3). We will prove that, if $\|u_0^\varepsilon - v_0\|^2_{L^2} = \mathcal{O}(\varepsilon^s)$, then for all positive T, we have

$$\sup_{t\in(0,T)} \|(u^\varepsilon - v)(t)\|^2_{L^2} = \mathcal{O}\left(\varepsilon^{\min(s,\frac{1}{2})}\right). \tag{157}$$

Remark 7. *The natural space for the convergence is* $L^\infty_T \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ *but we were unable to show the error estimate in this space as we needed a Kato-Ponce type inequality we think it cannot true.*

In the following, we use again the method in section 2.3. So let us consider again the Dafermos modulated energy defined by

$$\mathcal{E}_{\varepsilon,v}(t) = \int_{\mathbb{R}^3} \left(\frac{1}{2}|u^\varepsilon - v(t,x) + \varepsilon\partial_t u^\varepsilon|^2 + \frac{\varepsilon^2}{2}|\partial_t u^\varepsilon|^2 + \varepsilon|\nabla u^\varepsilon|^2\right) dx \tag{158}$$

for all divergence-free vector v. As above, this energy satisfies the inequality

$$\int_{\mathbb{R}^3} |u^\varepsilon - v|^2\, dx \leq 4\mathcal{E}_{\varepsilon,v}(t). \tag{159}$$

We shall show that the assumptions (6) in Theorem 2 and a Gronwall inequality imply that

$$\forall\ t>0\ ,\ \ \mathcal{E}_{\varepsilon,v}(t) = \mathcal{O}\left(\varepsilon^{\min(s,\frac{1}{2})}\right). \tag{160}$$

Let us recall the derivative of $\mathcal{E}_{\varepsilon,v}$:

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{\varepsilon,v}(t) \ &= \ \int_{\mathbb{R}^3} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\, dx - \varepsilon\int_{\mathbb{R}^3} |\partial_t u^\varepsilon + \nabla : u^\varepsilon \otimes u^\varepsilon|^2\, dx \\ &\quad -\varepsilon\int_{\mathbb{R}^3} \partial_t v.\partial_t u^\varepsilon\, dx + \int_{\mathbb{R}^3} \left(\varepsilon|\nabla : (u^\varepsilon\otimes u^\varepsilon)|^2 - |\nabla(u^\varepsilon - v)|^2\right) dx \\ &\quad + \int_{\mathbb{R}^3} (\partial_t v + v.\nabla v - \Delta v).(v - u^\varepsilon)\, dx. \end{aligned} \tag{161}$$

As above in section 2.3, we take v solution to $\partial_t v + v.\nabla v - \Delta v = 0$ in the following, so that the last term in identity (161) vanishes. Moreover, the second term in (161) being negative, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_{\varepsilon,v}(t) \ &\leq \ -\varepsilon\int_{\mathbb{R}^3} \partial_t v.\partial_t u^\varepsilon\, dx + \int_{\mathbb{R}^3} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\, dx \\ &\quad + \int \left(\varepsilon|\nabla : (u^\varepsilon\otimes u^\varepsilon)|^2 - |\nabla(u^\varepsilon - v)|^2\right). \end{aligned} \tag{162}$$

We will estimate the three terms on the right hand side in the following three subsections.

3.3.1 Estimating $\varepsilon \int_0^T\int_{\mathbb{R}^3} \partial_t u^\varepsilon.\partial_t v \, dt \, dx$

First, similarly to the two-dimensional case, we know that

$$\|\varepsilon\partial_t u^\varepsilon\|_{L^2_T\dot{H}^{\frac{1}{2}}(\mathbb{R}^3} = \mathcal{O}\left(\varepsilon^{1/2}\right). \tag{163}$$

For the proof, use the decay of the energy

$$E^{\frac{1}{2}}_\varepsilon(t) = \int_{\mathbb{R}^3}\left(\frac{1}{2}|\Lambda^{\frac{1}{2}}(u^\varepsilon + \varepsilon\partial_t u^\varepsilon)|^2 + \frac{\varepsilon^2}{2}|\Lambda^{\frac{1}{2}}\partial_t u^\varepsilon|^2 + \varepsilon|\Lambda^{\frac{3}{2}}u^\varepsilon|^2\right) dx. \tag{164}$$

Now, let us consider v solution to the Cauchy problem

$$\partial_t v = \Delta v - \mathbb{P}\nabla : (v\otimes v)\ , \quad \nabla.v = 0\ , \quad v|_{t=0} = v_0 \in H^{s+\frac{1}{2}}(\mathbb{R}^3),\ s>0. \tag{165}$$

We know that

$$v \in L^2_T H^{s+\frac{3}{2}} \cap L^\infty_T H^{s+\frac{1}{2}} \tag{166}$$

so we have obviously $\Delta v \in L^2_T\dot{H}^{-\frac{1}{2}}$ and, using the sobolev embedding $H^{s+\frac{3}{2}} \subset L^\infty(\mathbb{R}^3)$ and that $L^\infty \cap \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is an algebra, we prove that

$$v\otimes v \in L^2_T\dot{H}^{\frac{1}{2}} \tag{167}$$

which implies that

$$\partial_t v \in L^2_T\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3). \tag{168}$$

(163) and (168) yield the estimation

$$\varepsilon\int_0^T\int_{\mathbb{R}^3}\partial_t u^\varepsilon.\partial_t v\, dt\, dx = \mathcal{O}\left(\sqrt{\varepsilon}\right). \tag{169}$$

3.3.2 Estimating $\int_0^T\int_{\mathbb{R}^3} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\, dt\, dx$

In order to apply the argument in the second subsection of part 2.3, let us recall that $v(t) \in H^{\frac{3}{2}}(\mathbb{R}^3) \subset BMO(\mathbb{R}^3)$ and that $u^\varepsilon, v \in H^1(\mathbb{R}^3)$ are divergence-free. The div-curl lemma and inequality (159) yield

$$\int_0^T\int_{\mathbb{R}^3} v.\nabla : (u^\varepsilon - v)\otimes(u^\varepsilon - v)\, dt\, dx \leq 2\int_0^T \|v\|^2_{BMO}\mathcal{E}_{\varepsilon,v}(t)\, dt - \frac{1}{2}\int_0^T \|\nabla(u^\varepsilon - v)\|^2_{L^2}\, dt. \tag{170}$$

At this level, we have the following estimate:

$$\begin{aligned}\mathcal{E}_{\varepsilon,v}(T) - \mathcal{E}_{\varepsilon,v}(0) \quad &\leq \quad 2\int_0^T \|v\|^2_{BMO}\mathcal{E}_{\varepsilon,v}(t)\, dt + \mathcal{O}\left(\sqrt{\varepsilon}\right) \\ &\quad + \int_0^T\left(\varepsilon\|\nabla : u^\varepsilon\otimes u^\varepsilon\|^2_{L^2} - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}\right)(t)\, dt.\end{aligned} \tag{171}$$

Let us set

$$\int_0^T A^\varepsilon(t)\, dt := \int_0^T\left(\varepsilon\|\nabla : u^\varepsilon\otimes u^\varepsilon\|^2_{L^2} - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|^2_{L^2}\right)(t)\, dt. \tag{172}$$

It remains to estimate the last term $\int_0^T A^\varepsilon(t)\, dt$.

3.3.3 Estimating $\int_0^T A^\varepsilon(t)\,dt$

We treat this term exactly the same way as in section 2.3. First, let us exhibit a term in $u^\varepsilon - v$ that is bound to vanish:

$$\nabla : (u^\varepsilon \otimes u^\varepsilon) = u^\varepsilon.\nabla u^\varepsilon = u^\varepsilon.\nabla(u^\varepsilon - v) + u^\varepsilon.\nabla v \tag{173}$$

then a Young inequality yields

$$\|\nabla : (u^\varepsilon \otimes u^\varepsilon)\|_{L^2}^2 \leq 2\|u^\varepsilon.\nabla(u^\varepsilon - v)\|_{L^2}^2 + 2\|u^\varepsilon.\nabla v\|_{L^2}^2. \tag{174}$$

So we have

$$\begin{aligned} A^\varepsilon(t) &= \varepsilon\|\nabla : u^\varepsilon \otimes u^\varepsilon\|_{L^2}^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \\ &\leq 2\varepsilon\|u^\varepsilon.\nabla(u^\varepsilon - v)\|_{L^2}^2 + 2\varepsilon\|u^\varepsilon.\nabla v\|_{L^2}^2 - \frac{1}{2}\|\nabla(u^\varepsilon - v)\|_{L^2}^2 \\ &\leq \left(2\varepsilon\|u^\varepsilon\|_\infty^2 - \frac{1}{2}\right)\|\nabla(u^\varepsilon - v)\|_{L^2}^2 + 2\varepsilon\|u^\varepsilon.\nabla v\|_{L^2}^2. \end{aligned}$$

Therefore, using that $\varepsilon\|u^\varepsilon\|_\infty^2 = \mathcal{O}(\varepsilon^s)$ and that $v \in L^2_T\dot{H}^1$, we obtain

$$\int_0^T A^\varepsilon(t)\,dt \leq 2\varepsilon\|u^\varepsilon\|_\infty^2\|\nabla v\|_{L^2_T L^2}^2 = \mathcal{O}(\varepsilon^s) \tag{175}$$

if ε is small enough.

Remark 8. *If we study the convergence in the $L^\infty_T\dot{H}^{\frac{1}{2}}$ norm, we need to estimate $\|u^\varepsilon.\nabla u^\varepsilon\|_{\dot{H}^{\frac{1}{2}}}$ by $C\|u^\varepsilon\|_\infty\|u^\varepsilon\|_{\dot{H}^{\frac{3}{2}}} + K(u^\varepsilon)$, with $K(u^\varepsilon)$ going to zero as ε goes to zero. We think this is not possible.*

3.3.4 Conclusion

Finally, we have

$$\mathcal{E}_{\varepsilon,v}(T) \leq 2\int_0^T \|v\|_{BMO}^2\mathcal{E}_{\varepsilon,v}(t)\,dt + \mathcal{O}\left(\sqrt{\varepsilon}\right) + \mathcal{O}(\varepsilon^s) + \mathcal{E}_{\varepsilon,v}(0). \tag{176}$$

Assuming that

$$\|u_0^\varepsilon - v\|_{L^2} + \varepsilon\|u_1^\varepsilon\|_{L^2} + \sqrt{\varepsilon}\|u_0^\varepsilon\|_{\dot{H}^1} = \mathcal{O}\left(\varepsilon^{\frac{s}{2}}\right), \tag{177}$$

we have immediately the estimate

$$\mathcal{E}_{\varepsilon,v}(0) = \mathcal{O}(\varepsilon^s). \tag{178}$$

Let us recall that $v \in L^2_T BMO(\mathbb{R}^3)$. Gronwall lemma implies that

$$\mathcal{E}_{\varepsilon,v}(T) = \mathcal{O}\left(\varepsilon^{\min(s,\frac{1}{2})}\right). \tag{179}$$

Now, (159) yields the announced error estimate and the proof of Theorem 1 is complete.

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