Mean-Variance Hedging on uncertain time horizon in a market with a jump

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Abstract

In this work, we study the problem of mean-variance hedging with a random horizon $T \wedge \tau$, where T is a deterministic constant and τ is a jump time of the underlying asset price process. We first formulate this problem as a stochastic control problem and relate it to a system of BSDEs with jumps. We then provide a verification theorem which gives the optimal strategy for the mean-variance hedging using the solution of the previous system of BSDEs. Finally, we prove that this system of BSDEs admits a solution via a decomposition approach coming from filtration enlargement theory.

Keywords: Mean-variance hedging, Backward SDE, random horizon, jump processes, progressive enlargement of filtration, decomposition in the reference filtration.

AMS subject classifications: 91B30, 60G57, 60H10, 93E20.

1 Introduction

In most of the financial markets, the simplifying assumption of completeness fails to be true. In particular, investors cannot always hedge the financial products that they are interested in. A possible approach is the mean-variance hedging one. It consists, for a financial product of terminal income H at a fixed horizon time T and an initial capital

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x, in finding a strategy π such that the portfolio $V^{x,\pi}$ of initial amount x and strategy π realizes the minimum of the mean square error

$$\mathbb{E}\Big[\big|V_T^{x,\pi} - H\big|^2\Big]$$

over all the possible investment strategies.

In this paper, we are concerned with the mean-variance hedging problem over a random horizon. More precisely, we consider a random time τ and a contingent claim with a gain of the form

$$H \quad = \quad H^b \mathbb{1}_{T<\tau} + H^a_\tau \mathbb{1}_{T\geq\tau}, \tag{1.1}$$

where $T < \infty$ is a fixed deterministic terminal time and study the mean-variance hedging problem over the horizon $[0, T \wedge \tau]$ defined by

$$\inf_\pi \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big]\,. \tag{1.2}$$

Financial products with gains of the form (1.1) naturally appear on financial markets, see e.g. Examples 2.1 and 2.2 presented in Subsection 2.3. Their valuations are therefore of an important interest.

The mean-variance hedging problem with deterministic horizon T is one of the classical problems from mathematical finance and has been considered by several authors via two main approaches. One of them is based on martingale theory and projection arguments and the other considers the problem as a quadratic stochastic control problem and describes the solution using BSDE theory.

The bulk of the literature primarily focuses on the continuous case where both approaches are used (see e.g. Delbaen and Schachermayer [5], Gouriéroux *et al.* [9], Laurent and Pham [24] and Schweizer [25] for the first approach, and Lim and Zhou [22] and Lim [21] for the second one).

In the discontinuous case, the mean-variance hedging problem is considered by Arai [1], Lim [23] and Jeanblanc *et al* [14]. In [1], the author uses the projection approach for general semimartingale price process model whereas in [23] the problem is considered via the stochastic control view for the case of diffusion price processes driven by Brownian motion and Poisson process. The author provides under a so-called "martingale condition" the existence of solution to the associated BSDEs. In the recent paper [14], the authors combine tools from both approaches, which allows them to work in a general semimartingale model and to give a description of the optimal solution to the mean-variance hedging via the BSDE theory. More precisely the authors prove that the value process of the mean-variance hedging problem has a quadratic structure and that the coefficients appearing in this quadratic expression are related to some BSDEs. Then, they provide an equivalence between the existence of an optimal strategy and the existence of a solution to a BSDE associated to the control problem. They also show in some specific examples, via the control problem, existence of solutions for BSDEs of interest, but let the problem open in the general case.

In this paper, we also use a stochastic control approach and describe the optimal solution by a solution to a system of BSDEs.

We consider a model of diffusion price process driven by a Brownian motion and a random jump time τ for which we study the mean-variance hedging with horizon $T \wedge \tau$ given by (1.2). We follow the progressive enlargement approach initiated by Jeulin [15], Jeulin and Yor [16] and Jacod [12], which lead to consider an enlargement of the initial information given by the Brownian motion to make τ a stopping time. We note that this approach allows to work under wide assumptions, in particular, no a priori law is fixed for the random time τ contrary to the Poisson case.

Following the quadratic form obtained in [14], we use a martingale optimality principle to get an associated system of nonstandard BSDE. We then provide a verification theorem (Theorem 3.2) which provides an explicit optimal investment strategy via the solution to the associated system of BSDEs. Our contribution is twofold.

• We link the mean-variance hedging problem on a random horizon with a system of BSDE, in a general progressive enlargement setup which avoids to suppose any a priori law for the jump part. We show that, under wide assumptions, the mean-variance hedging problem admits an optimal strategy described by the solution of the associated BSDE.

• We prove that the associated system of BSDEs, which is nonstandard, admits a solution. The main difficulty here is that the obtained system of BSDEs is nonstandard since it is driven by a Brownian motion and a jump martingale and has generators with quadratic growth in the variable z and are undefined for some values of the variable y. To solve these BSDE we follow a decomposition approach inspired by the result of Jeulin (see Proposition 2.1) which allows to consider BSDEs in the smallest filtration (see Theorem 4.3). Then using BMO properties, we provide solutions to the decomposed BSDEs which lead to the existence of a solution to the BSDE in the enlarged filtration.

We notice that, for the studied problem *i.e.* mean-variance hedging with horizon $T \wedge \tau$, the interest of our approach is that it provides a solution to the associated BSDE, without supposing any additional assumption specific to the studied BSDE as done in [23] where the author introduces the "martingale condition" to prove existence of a solution to the BSDE or in [14] where the existence of a solution to the BSDE is given in specific cases.

The rest of the paper is organized as follow. In Section 2, we present the details of the probabilistic model for the financial market, and set the mean-variance hedging on random horizon. In Section 3, we show how the construct the associated BSDEs via the martingale optimality principle. We then state the two main theorems. The first one concerns the existence of a solution to the associated system of BSDEs and the second one is a verification Theorem which gives an optimal strategy via the solution of the BSDEs. Then, Section 4 is dedicated to the proof of the existence of solution to the associated system of BSDEs. Finally, some technical results are relegated to the appendix.

2 Preliminaries and market model

2.1 The probability space

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion W and we denote by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ the right continuous complete filtration generated by W. We also consider on this space a random time τ, which modelizes for example a default time in credit risk or a death time in actuarial issues. The random time τ is not assumed to be an $\mathbb{F}$-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering $\mathbb{G}$ the smallest right continuous extension of $\mathbb{F}$ that turns τ into a $\mathbb{G}$-stopping time (see e.g. [15, 16, 12]). More precisely $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ is defined by

$$\mathcal{G}_t \;:=\; \bigcap_{\varepsilon>0} \tilde{\mathcal{G}}_{t+\varepsilon}\ ,$$

for all $t \geq 0$, where $\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(\mathbb{1}_{\tau\leq u}\ , u \in [0,s])$, for all $s \geq 0$.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{G})$) the σ-algebra of $\mathbb{F}$ (resp. $\mathbb{G}$)-predictable subsets of $\Omega \times \mathbb{R}_+$, i.e. the σ-algebra generated by the left-continuous $\mathbb{F}$ (resp. $\mathbb{G}$)-adapted processes.

We now introduce a decomposition result for $\mathcal{P}(\mathbb{G})$-measurable processes proved in [15].

Proposition 2.1. *Any $\mathcal{P}(\mathbb{G})$-measurable process $X = (X_t)_{t\geq 0}$ is represented as*

$$X_t \;=\; X_t^b \mathbb{1}_{t\leq\tau} + X_t^a(\tau)\mathbb{1}_{t>\tau}\ ,$$

for all $t \geq 0$, where X^b is $\mathcal{P}(\mathbb{F})$-measurable and X^a is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurable.

Remark 2.1. In the case where the studied process X depends on another parameter x evolving in a Borelian subset $\mathcal{X}$ of $\mathbb{R}^p$, and if X is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$, then, decomposition given by Proposition 2.1 is still true but where X^b is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathcal{X})$-mesurable and X^a is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathcal{X})$-measurable. Indeed, it is obvious for the processes generating $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$ of the form $X_t(\omega, x) = L_t(\omega)R(x)$, $(t, \omega, x) \in \mathbb{R}_+ \times \Omega \times \mathcal{X}$, where L is $\mathcal{P}(\mathbb{G})$-measurable and R is $\mathcal{B}(\mathcal{X})$-measurable. Then, the result is extended to any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathcal{X})$-measurable process by the monotone class theorem.

We then impose the following assumption, which is classical in the filtration enlargement theory.

(H) The process W remains a $\mathbb{G}$-Brownian motion.

We notice that under **(H)**, the stochastic integral $\int_0^t X_s dW_s$ is well defined for all $\mathcal{P}(\mathbb{G})$-measurable process X such that $\int_0^t |X_s|^2 ds < \infty$.

In the sequel we denote by N the process $\mathbb{1}_{\tau\leq\cdot}$ and we suppose

(Hτ) The process N admits an $\mathbb{F}$-compensator of the form $\int_0^{\cdot\wedge\tau} \lambda_t dt$, i.e. $N - \int_0^{\cdot\wedge\tau} \lambda_t dt$ is a $\mathbb{G}$-martingale, where λ is a bounded $\mathcal{P}(\mathbb{F})$-measurable process.

We then denote by M the $\mathbb{G}$-martingale defined by

$$M_t \;:=\; N_t - \int_0^{t\wedge\tau} \lambda_s ds \;,$$

for all $t \geq 0$. We also introduce the process $\lambda^{\mathbb{G}}$ which is defined by $\lambda_t^{\mathbb{G}} := (1 - N_t)\lambda_t$.

2.2 Financial model

We consider a financial market model on the time interval $[0, T]$ where $0 < T < \infty$ is a finite time horizon. We suppose that the financial market is composed by a riskless bond with interest rate zero and a risky asset S. The price process $(S_t)_{t\geq 0}$ of the risky asset is modeled by the linear stochastic differential equation

$$S_t \;=\; S_0 + \int_0^t S_{s^-}(\mu_s ds + \sigma_s dW_s + \beta_s dM_s) \;, \quad \forall t \in [0, T] \;, \tag{2.1}$$

where μ, σ and β are $\mathcal{P}(\mathbb{G})$-measurable processes. We impose the following assumptions on the coefficients μ, σ and β.

(HS)

(i) The processes μ and σ are bounded: there exists a constant $C > 0$ such that

$$|\mu_t| + |\sigma_t| \;\leq\; C \;, \quad \forall t \in [0, T] \;, \quad \mathbb{P} - a.s.$$

(ii) The process σ is uniformly invertible: there exists a constant $C > 0$ such that

$$|\sigma_t| \;\geq\; C \;, \quad \forall t \in [0, T] \;, \quad \mathbb{P} - a.s.$$

(iii) There exists a constant C such that

$$-1 \;\leq\; \beta_t \;\leq\; C \;, \quad \forall t \in [0, T] \;, \quad \mathbb{P} - a.s.$$

Under **(HS)**, we know from e.g. Theorem 1 in [8] that the process S defined by (2.1) is well defined.

2.3 Mean-variance hedging

We consider investment strategies which are $\mathcal{P}(\mathbb{G})$-measurable processes π such that

$$\int_0^{T\wedge\tau} |\pi_t|^2 dt \;<\; +\infty \;, \qquad \mathbb{P} - a.s.$$

This condition and **(HS)** ensure that the stochastic integral $\int_0^t \frac{\pi_r}{S_{r^-}} dS_r$ is well defined for such a strategy π and $t \in [0, T \wedge \tau]$. The wealth process $V^{x,\pi}$ corresponding to a pair (x, π), where $x \in \mathbb{R}$ is the initial amount, is defined by the stochastic differential equation

$$V_t^{x,\pi} \;:=\; x + \int_0^t \frac{\pi_r}{S_{r^-}} dS_r \;, \quad \forall\, t \in [0, T \wedge \tau] \;.$$

We denote by $\mathcal{A}$ the set of admissible strategies π such that

$$\mathbb{E}\Big[\int_0^{T\wedge\tau} |\pi_t|^2 dt\Big] \;<\; \infty\,.$$

For $x \in \mathbb{R}$, the problem of mean-variance hedging consists in computing the quantity

$$\inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|V^{x,\pi}_{T\wedge\tau} - H\big|^2\Big]\,, \tag{2.2}$$

where H is a bounded $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$H \;=\; H^b \mathbb{1}_{T<\tau} + H^a_\tau \mathbb{1}_{T\geq\tau}\,, \tag{2.3}$$

where H^b is an $\mathcal{F}_T$-measurable random variable valued in $\mathbb{R}$ and H^a is a continuous $\mathbb{F}$-adapted process also valued in $\mathbb{R}$ and such that

$$\big\|H^b\big\|_\infty \;<\; \infty, \quad \text{and} \quad \Big\| \sup_{t\in[0,T]} \big|H^a_t\big| \Big\|_\infty \;<\; \infty\,,$$

where we recall that $\|.\|_\infty$ is defined by

$$\|X\|_\infty \;:=\; \inf\Big\{C \geq 0 \;:\; \mathbb{P}\big(|X| \leq C\big) = 1\Big\}\,,$$

for any random variable X.

Since the problem we are interested in uses the values of the coefficients μ, σ and β only on the interval $[0, T\wedge\tau]$, we can assume by Proposition 2.1 that μ, σ and β are $\mathcal{P}(\mathbb{F})$-measurable and we shall do that in the sequel.

We end this section by two examples of financial product taking the form (2.3).

Example 2.1 (Insurance contract)**.** Consider a seller of an insurance policy which protects the buyer over the time horizon $[0,T]$ from some fixed loss L. Then if we denote by τ the time at which the loss appears, the gain of the seller is of the form

$$H \;=\; p\mathbb{1}_{T<\tau} + (p-L)\mathbb{1}_{T\geq\tau}\,,$$

where p denotes the premium that the insurance policy holder pays at time 0.

Example 2.2 (Credit contract)**.** Consider a bank which lends an amount A to a company over the period $[0,T]$. Suppose that the time horizon $[0,T]$ is divided on n subintervals $[k\frac{T}{n}, (k+1)\frac{T}{n}]$, $k = 0,\ldots,n-1$, and that the interest rate of the loan over a time subinterval is r. The company has then to pay $\frac{(1+r)^n}{n}A$ to the bank at each time $k\frac{T}{n}$, $k = 1,\ldots,n$. If we denote by τ the company default time, then the gain of the bank is given by

$$H \;=\; ((1+r)^n - 1)A\mathbb{1}_{T<\tau} + H^a(\tau)\mathbb{1}_{T\geq\tau}\,,$$

where the function H^a is given by

$$H^a_t \;=\; \sum_{k=1}^{n-1} \big(k\frac{(1+r)^n}{n} - 1\big)A\mathbb{1}_{k\frac{T}{n}<t\leq(k+1)\frac{T}{n}}\,, \quad t\in[0,T]\,.$$

3 Solution of the mean-variance problem by BSDEs

3.1 Martingale optimality principle

To find the optimal value of the problem (2.2), we follow the approach initiated by Hu *et al.* [11] to solve the exponential utility maximization problem in the pure Brownian case. More precisely, we look for a family of processes

$$\Big\{ \big(J_t^\pi\big)_{t\in[0,T]} \ : \ \pi \in \mathcal{A} \Big\}$$

satisfying the following conditions

(i) $J_{T\wedge\tau}^\pi = \big|V_{T\wedge\tau}^{x,\pi} - H\big|^2$, for all $\pi \in \mathcal{A}$.

(ii) $J_0^{\pi_1} = J_0^{\pi_2}$, for all $\pi_1, \pi_2 \in \mathcal{A}$.

(iii) $\big(J_t^\pi\big)_{t\in[0,T]}$ is a $\mathbb{G}$-submartingale for all $\pi \in \mathcal{A}$.

(iv) There exists some $\pi^* \in \mathcal{A}$ such that $\big(J_t^{\pi^*}\big)_{t\in[0,T]}$ is a $\mathbb{G}$-martingale.

Under these conditions, we have

$$J_0^{\pi^*} \ = \ \inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \, .$$

Indeed, using (i), (iii) and Doob's optional stopping theorem, we have

$$J_0^\pi \ \leq \ \mathbb{E}\big[J_{T\wedge\tau}^\pi\big] \ = \ \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \, , \tag{3.4}$$

for all $\pi \in \mathcal{A}$. Then, using (i), (iv) and Doob's optional stopping theorem, we have

$$J_0^{\pi^*} \ = \ \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi^*} - H\big|^2\Big] \, . \tag{3.5}$$

Therefore, from (ii), (3.4) and (3.5), we get for any $\pi \in \mathcal{A}$

$$\mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi^*} - H\big|^2\Big] \ = \ J_0^{\pi^*} \ = \ J_0^\pi \ \leq \ \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \, .$$

We can see that

$$J_0^{\pi^*} \ = \ \inf_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \, .$$

3.2 Related BSDEs

We now construct a family $\{(J_t^\pi)_{t\in[0,T]}, \ \pi \in \mathcal{A}\}$ satisfying the previous conditions by using BSDEs as in [11]. To this end, we define the following spaces.

– $\mathcal{S}_{\mathbb{G}}^\infty$ is the subset of $\mathbb{R}$-valued càd-làg $\mathbb{G}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^\infty} \ := \ \Big\| \sup_{t\in[0,T]} |Y_t| \Big\|_\infty \ < \ \infty \, .$$

– $\mathcal{S}^{\infty,+}_{\mathbb{G}}$ is the subset of $\mathcal{S}^{\infty}_{\mathbb{G}}$ of processes $(Y_t)_{t\in[0,T]}$ valued in $(0,\infty)$, such that

$$\Big\|\frac{1}{Y}\Big\|_{\mathcal{S}^{\infty}} \;<\; \infty\,.$$

– $L^2_{\mathbb{G}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

– $L^2(\lambda)$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{G})$-measurable processes $(U_t)_{t\in[0,T]}$ such that

$$\|U\|_{L^2(\lambda)} \;:=\; \Big(\mathbb{E}\Big[\int_0^{T\wedge\tau} \lambda_s|U_s|^2 ds\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

To construct a family $\{(J^\pi_t)_{t\in[0,T]},\ \pi\in\mathcal{A}\}$ satisfying the previous conditions, we set

$$J^\pi_t \;=\; Y_t\big|V^{x,\pi}_{t\wedge\tau} - \mathcal{Y}_t\big|^2 + \Upsilon_t\,,\quad t\in[0,T]\,,$$

where[1] (Y,Z,U) is solution in $\mathcal{S}^{\infty,+}_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$Y_t \;=\; 1+\int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{f}(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dM_s\,,\quad t\in[0,T]\,, \tag{3.6}$$

$(\mathcal{Y},\mathcal{Z},\mathcal{U})$ is solution in $\mathcal{S}^{\infty}_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$\mathcal{Y}_t \;=\; H+\int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{g}(s,\mathcal{Y}_s,\mathcal{Z}_s,\mathcal{U}_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} \mathcal{Z}_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \mathcal{U}_s dM_s\,,\quad t\in[0,T]\,, \tag{3.7}$$

and (Υ,Ξ,Θ) is solution in $\mathcal{S}^{\infty}_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ to

$$\Upsilon_t \;=\; \int_{t\wedge\tau}^{T\wedge\tau}\mathfrak{h}(s,\Upsilon_s,\Xi_s,\Theta_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} \Xi_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \Theta_s dM_s\,,\quad t\in[0,T]\,. \tag{3.8}$$

In these terms, we are bounded to choose three functions $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ for which J^π is a submartingale for all $\pi\in\mathcal{A}$, and there exists a $\pi^*\in\mathcal{A}$ such that J^{π^*} is a martingale. In order to calculate $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$, we write J^π as the sum of a (local) martingale M^π and an (not strictly) increasing process K^π that is constant for some $\pi^*\in\mathcal{A}$.
To alleviate the notation we write $\mathfrak{f}(t)$ (resp. $\mathfrak{g}(t)$, $\mathfrak{h}(t)$) for $\mathfrak{f}(t,Y_t,Z_t,U_t)$ (resp. $\mathfrak{g}(t,\mathcal{Y}_t,\mathcal{Z}_t,\mathcal{U}_t)$, $\mathfrak{h}(t,\Upsilon_t,\Xi_t,\Theta_t)$) for $t\in[0,T]$.
Define for each $\pi\in\mathcal{A}$ the process X^π by

$$X^\pi_t \;:=\; V^{x,\pi}_{t\wedge\tau} - \mathcal{Y}_t\,,\quad t\in[0,T]\,.$$

From Itô's formula, we get

$$dJ^\pi_t \;=\; dM^\pi_t + dK^\pi_t\,, \tag{3.9}$$

[1] As commonly done for the integration w.r.t. jumps processes, the integral $\int_a^b$ stands for $\int_{(a,b]}$.

where M^π and K^π are defined by

$$\begin{aligned} dM_t^\pi &:= \Big\{2X_{t^-}^\pi(\pi_t\beta_t - \mathcal{U}_t)(Y_{t^-}+U_t) + |\pi_t\beta_t - \mathcal{U}_t|^2(Y_{t^-}+U_t) + |X_{t^-}^\pi|^2U_t + \Theta_t\Big\}dM_t \\ &\quad + \Big\{2Y_tX_t^\pi(\pi_t\sigma_t - \mathcal{Z}_t) + Z_t|X_t^\pi|^2 + \Xi_t\Big\}dW_t \,, \end{aligned}$$

$$\begin{aligned} dK_t^\pi &:= \Big\{Y_t\big[2X_t^\pi(\pi_t\mu_t + \mathfrak{g}(t)) + |\pi_t\sigma_t - \mathcal{Z}_t|^2\big] - |X_t^\pi|^2\mathfrak{f}(t) + 2X_t^\pi Z_t(\pi_t\sigma_t - \mathcal{Z}_t) \\ &\quad + 2\lambda_t^{\mathbb{G}}X_t^\pi U_t(\pi_t\beta_t - \mathcal{U}_t) + \lambda_t^{\mathbb{G}}|\pi_t\beta_t - \mathcal{U}_t|^2(U_t+Y_t) - \mathfrak{h}(t)\Big\}dt \,. \end{aligned}$$

We then write dK^π in the following form

$$dK_t^\pi = K_t(\pi_t)dt \,,$$

where K is defined by

$$K_t(\pi) := A_t|\pi|^2 + B_t\pi + C_t \,, \quad \pi\in\mathbb{R} \,, \quad t\in[0,T] \,,$$

with

$$\begin{aligned} A_t &:= |\sigma_t|^2Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t) \,, \\ B_t &:= 2X_t^\pi(\mu_tY_t + \sigma_tZ_t + \lambda_t^{\mathbb{G}}\beta_tU_t) - 2\sigma_tY_t\mathcal{Z}_t - 2\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_t+U_t) \,, \\ C_t &:= -\mathfrak{f}(t)|X_t^\pi|^2 + 2X_t^\pi(Y_t\mathfrak{g}(t) - Z_t\mathcal{Z}_t - \lambda_t^{\mathbb{G}}U_t\mathcal{U}_t) + Y_t|\mathcal{Z}_t|^2 + \lambda_t^{\mathbb{G}}|\mathcal{U}_t|^2(U_t+Y_t) - \mathfrak{h}(t) \,, \end{aligned}$$

for all $t\in[0,T]$. In order to obtain a nondecreasing process K^π for any $\pi\in\mathcal{A}$ and that is constant for some $\pi^*\in\mathcal{A}$ it is obvious that K_t has to satisfy $\min_{\pi\in\mathbb{R}}K_t(\pi)=0$. Using $Y\in\mathcal{S}_{\mathbb{G}}^{\infty,+}$ and **(H**S**)** (ii), we then notice that $A_t>0$ for all $t\in[0,T]$. Indeed, we have

$$0 = \mathbb{E}[[Y_{T\wedge\tau}]^-] = \mathbb{E}\Big[\int_0^T[Y_{s^-}+U_s]^-\lambda_s^{\mathbb{G}}ds\Big] = \mathbb{E}\Big[\int_0^T[Y_s+U_s]^-\lambda_s^{\mathbb{G}}ds\Big] \,,$$

which gives $(Y_s+U_s)\lambda_s^{\mathbb{G}}\geq 0$ for $s\in[0,T]$. Therefore, the minimum of K_t over $\pi\in\mathbb{R}$ is given by

$$\underline{K}_t := \min_{\pi\in\mathbb{R}}K_t(\pi) = C_t - \frac{|B_t|^2}{4A_t} \,.$$

We then obtain from the expressions of A, B and C that

$$\underline{K}_t = \mathfrak{A}_t|X_t^\pi|^2 + \mathfrak{B}_tX_t^\pi + \mathfrak{C}_t \,,$$

with

$$\begin{aligned} \mathfrak{A}_t &:= -\mathfrak{f}(t) - \frac{|\mu_tY_t + \sigma_tZ_t + \lambda_t^{\mathbb{G}}\beta_tU_t|^2}{|\sigma_t|^2Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} \,, \\ \mathfrak{B}_t &:= 2\Big\{\frac{(\mu_tY_t + \sigma_tZ_t + \lambda_t^{\mathbb{G}}\beta_tU_t)(\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_t+U_t) + \sigma_tY_t\mathcal{Z}_t)}{|\sigma_t|^2Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} + \mathfrak{g}(t)Y_t - Z_t\mathcal{Z}_t - \lambda_t^{\mathbb{G}}U_t\mathcal{U}_t\Big\} \,, \\ \mathfrak{C}_t &:= -\mathfrak{h}(t) + |\mathcal{Z}_t|^2Y_t + \lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2 - \frac{|\sigma_tY_t\mathcal{Z}_t + \lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)} \,. \end{aligned}$$

For that the family $(J^\pi)_{\pi\in\mathcal{A}}$ satisfies the conditions (iii) and (iv) we choose $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$ such that

$$\mathfrak{A}_t \;=\; 0\,,\; \mathfrak{B}_t \;=\; 0 \quad \text{and} \quad \mathfrak{C}_t \;=\; 0\,,$$

for all $t\in[0,T]$. This leads to the following choice for the drivers $\mathfrak{f}$, $\mathfrak{g}$ and $\mathfrak{h}$

$$\left\{\begin{array}{rcl} \mathfrak{f}(t,y,z,u) & := & -\dfrac{|\mu_t y+\sigma_t z+\lambda_t^{\mathbb{G}}\beta_t u|^2}{|\sigma_t|^2 y+\lambda_t^{\mathbb{G}}|\beta_t|^2(u+y)}\,, \\[2ex] \mathfrak{g}(t,y,z,u) & := & \frac{1}{Y_t}\Big[Z_t z+\lambda_t^{\mathbb{G}}U_t u-\dfrac{(\mu_t Y_t+\sigma_t Z_t+\lambda_t^{\mathbb{G}}\beta_t U_t)(\sigma_t Y_t z+\lambda_t^{\mathbb{G}}\beta_t(U_t+Y_t)u)}{|\sigma_t|^2 Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\Big]\,, \\[2ex] \mathfrak{h}(t,y,z,u) & := & |\mathcal{Z}_t|^2 Y_t+\lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2-\dfrac{|\sigma_t Y_t\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2 Y_t+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\,. \end{array}\right.$$

We then notice that the obtained system of BSDEs is not fully coupled, which allows to study each BSDE alone as soon as we start from BSDE $(\mathfrak{f},1)$ and end with BSDE $(\mathfrak{h},0)$[2]. However the obtained generators are nonstandard since they involve the jump component and they are not Lipschitz continuous. Moreover, these generators are not defined on the whole space $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$. Therefore, we also have to deal with this additional issue.

Using a decomposition approach based on Proposition 2.1, we obtain the following result whose proof is detailed in Section 4.

Theorem 3.1. *The BSDEs* (3.6), (3.7) *and* (3.8) *admit solutions* (Y,Z,U), $(\mathcal{Y},\mathcal{Z},\mathcal{U})$ *and* (Υ,Ξ,Θ) *in* $\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$. *Moreover* $Y\in\mathcal{S}^{\infty,+}_{\mathbb{G}}$.

3.3 A verification Theorem

We now turn to the sufficient condition of optimality. As explained in Subsection 3.1, a candidate to be an optimal strategy is a process $\pi^*\in\mathcal{A}$ such that J^{π^*} is a martingale, which implies that $dK^{\pi^*}=0$. This leads to

$$\pi_t^* \;=\; \arg\min_{\pi\in\mathbb{R}} K_t(\pi)\,,$$

which gives the implicit equation in π^*

$$\pi_t^* \;=\; \big(\mathcal{Y}_{t^-}-V_{t^-}^{x,\pi^*}\big)\frac{\mu_t Y_{t^-}+\sigma_t Z_t+\lambda_t^{\mathbb{G}}\beta_t U_t}{|\sigma_t|^2 Y_{t^-}+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_{t^-})}+\frac{\sigma_t Y_{t^-}\mathcal{Z}_t+\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2 Y_{t^-}+\lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_{t^-})}\,.$$

Integrating each side of this equality w.r.t. $\frac{dS_t}{S_{t^-}}$ leads to the following SDE

$$\begin{array}{rcl} V_t^* & = & x+\displaystyle\int_0^t\big(\mathcal{Y}_{r^-}-V_{r^-}^*\big)\frac{\mu_r Y_{r^-}+\sigma_r Z_r+\lambda_r^{\mathbb{G}}\beta_r U_r}{|\sigma_r|^2 Y_{r^-}+\lambda_r^{\mathbb{G}}|\beta_r|^2(U_r+Y_{r^-})}\frac{dS_r}{S_{r^-}} \\ & & +\displaystyle\int_0^t\frac{\sigma_r Y_{r^-}\mathcal{Z}_r+\lambda_r^{\mathbb{G}}\beta_r\mathcal{U}_r(Y_{r^-}+U_r)}{|\sigma_r|^2 Y_{r^-}+\lambda_r^{\mathbb{G}}|\beta_r|^2(U_r+Y_{r^-})}\frac{dS_r}{S_{r^-}}\,,\quad t\in[0,T\wedge\tau]\,. \end{array} \tag{3.10}$$

We first study the existence of a solution to SDE (3.10).

[2]The notation BSDE (f,H) holds for the BSDE with generator f and terminal condition H.

Proposition 3.2. *The SDE* (3.10) *admits a solution* V^* *which satisfies*

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]}|V_t^*|^2\Big] \;<\; \infty\,. \tag{3.11}$$

Proof. To alleviate the notation we rewrite (3.10) under the form

$$\left\{\begin{array}{rcl} V_0^* &=& x\,,\\ dV_t^* &=& (E_tV_{t^-}^* - F_t)(\mu_t dt + \sigma_t dW_t + \beta_t dM_t)\,, \end{array}\right.$$

where E and F are defined by

$$\begin{aligned} E_t &:= -\frac{\mu_t Y_{t^-} + \sigma_t Z_t + \lambda_t^{\mathbb{G}}\beta_t U_t}{|\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t + Y_{t^-})}\,,\\ F_t &:= -\frac{\lambda_t^{\mathbb{G}}\beta_t\mathcal{U}_t(Y_{t^-}+U_t) + \mu_t Y_{t^-}\mathcal{Y}_{t^-} + \lambda_t^{\mathbb{G}}\beta_t U_t\mathcal{Y}_{t^-} + \sigma_r Z_r\mathcal{Y}_{r^-} + \sigma_r\mathcal{Z}_r Y_{r^-}}{|\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t + Y_{t^-})}\,, \end{aligned}$$

for all $t\in[0,T]$. We first notice that from **(H**S**)** (ii), $Y\in\mathcal{S}_{\mathbb{G}}^{\infty,+}$ and $\Delta Y_\tau = U_\tau$, there exists a constant $C>0$ such that

$$|\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_{t^-}) \;\geq\; C\,,\quad \mathbb{P}-a.s.$$

for all $t\in[0,T]$. Therefore, using (Y,Z,U), $(\mathcal{Y},\mathcal{Z},\mathcal{U})$, $(\Upsilon,\Xi,\Theta)\in\mathcal{S}_{\mathbb{G}}^{\infty}\times L_{\mathbb{G}}^2\times L^2(\lambda)$, we get that E and F are square integrable

$$\mathbb{E}\Big[\int_0^T\Big(|E_t|^2+|F_t|^2\Big)dt\Big] \;<\; \infty\,.$$

Using Itô's formula, we obtain that the process V^* defined by

$$\begin{aligned} V_t^* &:= \Phi_t(x+\Psi_t)\,,\quad t\in[0,T\wedge\tau)\,, \\ \text{and}\;\; V_{T\wedge\tau}^* &= \mathbb{1}_{\tau\leq T}\big[(1+E_\tau\beta_\tau)V_{\tau^-}^* - F_\tau\beta_\tau\big] + \mathbb{1}_{\tau>T}\Phi_T(x+\Psi_T)\,, \end{aligned} \tag{3.12}$$

where

$$\Phi_t := \exp\Big(\int_0^t\big(E_s(\mu_s-\lambda_s^{\mathbb{G}}\beta_s) - \frac{1}{2}|\sigma_s E_s|^2\big)ds + \int_0^t\sigma_s E_s dW_s\Big)\,,$$

and

$$\Psi_t := -\int_0^t\frac{F_s}{\Phi_s}\Big[\mu_s - \lambda_s^{\mathbb{G}}\beta_s - E_s|\sigma_s|^2\Big]ds - \int_0^t\frac{F_s}{\Phi_s}\sigma_s dW_s\,,$$

for all $t\in[0,T]$ is solution to (3.10).
We now prove that V^* defined by (3.12) satisfies (3.11).
Step 1: We prove that

$$\mathbb{E}\Big[|V_{T\wedge\tau}^*|^2\Big] \;<\; \infty\,. \tag{3.13}$$

Indeed, from the definition of Y and $\mathcal{Y}$ and Itô's formula, we have

$$\begin{aligned} Y_{T\wedge\tau}|V^*_{T\wedge\tau}-\mathcal{Y}_{T\wedge\tau}|^2 &= Y_0|x-\mathcal{Y}_0|^2+M^*_{T\wedge\tau}+\int_0^{T\wedge\tau}\Big[|\mathcal{Z}_t|^2Y_t+\lambda^{\mathbb{G}}_t(U_t+Y_t)|\mathcal{U}_t|^2 \\ &\quad -\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_t)}\Big]dt \end{aligned}$$

where M^* is a locally square integrable martingale. Therefore, there exists an increasing sequence of $\mathbb{G}$-stopping times $(\nu_i)_{i\in\mathbb{N}}$ such that $\nu_i\to+\infty$ as $i\to\infty$ and

$$\begin{aligned} \mathbb{E}\big[Y_{T\wedge\tau\wedge\nu_i}|V^*_{T\wedge\tau\wedge\nu_i}-\mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big] &= Y_0|x-\mathcal{Y}_0|^2+\mathbb{E}\int_0^{T\wedge\tau\wedge\nu_i}\Big[|\mathcal{Z}_t|^2Y_t+\lambda^{\mathbb{G}}_t(U_t+Y_t)|\mathcal{U}_t|^2 \\ &\quad -\frac{|\sigma_tY_t\mathcal{Z}_t+\lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2Y_t+\lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_t)}\Big]dt\,. \end{aligned}$$

Since $Y\in\mathcal{S}^{\infty,+}_{\mathbb{G}}$, there exists a constant C such that

$$\mathbb{E}\big[|V^*_{T\wedge\tau\wedge\nu_i}-\mathcal{Y}_{T\wedge\tau\wedge\nu_i}|^2\big] \le C\Big(|x-\mathcal{Y}_0|^2+\mathbb{E}\int_0^T\Big[|\mathcal{Z}_t|^2Y_t+\lambda^{\mathbb{G}}_t(U_t+Y_t)|\mathcal{U}_t|^2\Big]dt\Big)\,.$$

This inequality implies that there exists a constant C such that

$$\mathbb{E}\big[\big|V^{\pi^*}_{T\wedge\tau\wedge\nu_i}-\mathcal{Y}_{T\wedge\tau\wedge\nu_i}\big|^2\big] \le C\,.$$

From Fatou's lemma, we get that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau}-\mathcal{Y}_{T\wedge\tau}\big|^2\big] \le \liminf_{i\to\infty}\mathbb{E}\big[\big|V^*_{T\wedge\tau\wedge\nu_i}-\mathcal{Y}_{T\wedge\tau\wedge\nu_i}\big|^2\big] \le C\,.$$

Finally, noting that $\mathcal{Y}$ is uniformly bounded, it follows that

$$\mathbb{E}\big[\big|V^*_{T\wedge\tau}\big|^2\big] \le 2\big(C+\mathbb{E}\big[\big|\mathcal{Y}_{T\wedge\tau}\big|^2\big]\big)\,.$$

Therefore, we get (3.13).

Step 2: We prove that

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]}|V^*_t|^2\Big] < \infty\,.$$

For that we remark that $V^*_{.\wedge\tau}$ is solution to the following linear BSDE

$$V^*_{t\wedge\tau} = V^*_{T\wedge\tau}-\int_{t\wedge\tau}^{T\wedge\tau}\frac{\mu_s}{\sigma_s}z_sds-\int_{t\wedge\tau}^{T\wedge\tau}z_sdW_s-\int_{t\wedge\tau}^{T\wedge\tau}u_sdM_s\,,\quad t\in[0,T]\,,$$

with

$$z_t := \sigma_t\frac{(\mathcal{Y}_{t^-}-V^*_{t^-})(\mu_tY_{t^-}+\sigma_tZ_t+\lambda^{\mathbb{G}}_t\beta_tU_t)+\sigma_tY_{t^-}\mathcal{Z}_t+\lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2Y_{t^-}+\lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_{t^-})}\,,$$

$$u_t := \beta_t\frac{(\mathcal{Y}_{t^-}-V^*_{t^-})(\mu_tY_{t^-}+\sigma_tZ_t+\lambda^{\mathbb{G}}_t\beta_tU_t)+\sigma_tY_{t^-}\mathcal{Z}_t+\lambda^{\mathbb{G}}_t\beta_t\mathcal{U}_t(Y_{t^-}+U_t)}{|\sigma_t|^2Y_{t^-}+\lambda^{\mathbb{G}}_t|\beta_t|^2(U_t+Y_{t^-})}\,,$$

for all $t \in [0,T]$. Therefore, using (3.13), **(HS)**, and classical arguments for BSDEs, we get (3.11). □

As explained previously, we now consider the strategy π^* defined by

$$\pi_t^* \;=\; \frac{(\mathcal{Y}_{t^-} - V_{t^-}^*)(\mu_t Y_{t^-} + \sigma_t Z_t + \lambda_t^{\mathbb{G}}\beta_t U_t) + \sigma_t Y_{t^-}\mathcal{Z}_t + \lambda_t^{\mathbb{G}}\beta_t \mathcal{U}_t(Y_{t^-} + U_t)}{|\sigma_t|^2 Y_{t^-} + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t + Y_{t^-})}\,, \quad (3.14)$$

for all $t \in [0,T]$. We first notice from the expression of π^* and V^* that

$$V_t^{x,\pi^*} \;=\; V_t^*\,, \quad (3.15)$$

for all $t \in [0,T]$. Using (3.11) and (3.15), we have

$$\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]} |V_t^{x,\pi^*}|^2\Big] \;<\; \infty\,. \quad (3.16)$$

We can now state our verification Theorem which is the main result of this section.

Theorem 3.2. *The strategy π^* given by* (3.14) *belongs to the set $\mathcal{A}$ and is optimal for the mean-variance problem* (2.2)*. Thus we have*

$$\mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi^*} - H\big|^2\Big] \;=\; \min_{\pi\in\mathcal{A}} \mathbb{E}\Big[\big|V_{T\wedge\tau}^{x,\pi} - H\big|^2\Big] \;=\; Y_0|x - \mathcal{Y}_0|^2 + \Upsilon_0\,,$$

where $Y, \mathcal{Y}$ and Υ are solutions to (3.6)-(3.7)-(3.8)*.*

To prove this verification Theorem, we first need of the following lemma.

Lemma 3.1. *For any $\pi \in \mathcal{A}$, the process $M_{\cdot\wedge\tau}^{\pi}$ defined by* (3.10) *is a $\mathbb{G}$-local martingale.*

Proof. Fix $\pi \in \mathcal{A}$. Then from the definition of $V^{x,\pi}$, **(HS)** and BDG-inequality, we have

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|V_{t\wedge\tau}^{x,\pi}\big|^2\Big] \;<\; \infty\,. \quad (3.17)$$

Define the sequence of $\mathbb{G}$-stopping times $(\nu_n)_{n\geq 1}$ by

$$\nu_n \;:=\; \inf\Big\{s \geq 0 \;:\; \sup_{r\in[0,s]} \big|V_{r\wedge\tau}^{x,\pi}\big| \geq n\Big\}\,,$$

for all $n \geq 1$. First, notice that $(\nu_n)_{n\geq 1}$ is increasing and goes to infinity as n goes to infinity from (3.17). Then, since $\pi \in \mathcal{A}$, $Y, \mathcal{Y} \in \mathcal{S}_{\mathbb{G}}^{\infty}$ and $Z, \mathcal{Z}, \Xi \in L_{\mathbb{G}}^2$, we get

$$\mathbb{E}\Big[\int_0^{\tau\wedge\nu_n\wedge T} \Big|2Y_t X_t^{\pi}(\pi_t\sigma_t - \mathcal{Z}_t) + Z_t|X_t^{\pi}|^2 + \Xi_t\Big|^2 dt\Big] \;<\; \infty\,,$$

for all $n \geq 1$. Moreover, since $U, \mathcal{U}, \Theta \in L^2(\lambda)$, we get

$$\mathbb{E}\Big[\int_0^{\tau\wedge\nu_n\wedge T} \Big|(2X_{t^-}^{\pi} + \pi_t\beta_t - \mathcal{U}_t)(\pi_t\beta_t - \mathcal{U}_t)(Y_{t^-} + U_t) + |X_{t^-}^{\pi}|^2 U_t + \Theta_t\Big|\lambda_t^{\mathbb{G}} dt\Big] \;<\; \infty\,,$$

for all $n \geq 1$. Therefore, we get that the stopped process $M_{\cdot\wedge\tau\wedge\nu_n}^{\pi}$ is a $\mathbb{G}$-martingale. □

Proof of Theorem 3.2. As explained in Subsection 3.1, we check each of the points (i), (ii), (iii) and (iv).
(i) From the definition of Y, $\mathcal{Y}$ and Υ, we have

$$J^{\pi}_{T\wedge\tau} \;=\; Y_{T\wedge\tau}\big|V^{x,\pi}_{T\wedge\tau}-H\big|^2+\Upsilon_{T\wedge\tau} \;=\; \big|V^{x,\pi}_{T\wedge\tau}-H\big|^2\,,$$

for all $\pi\in\mathcal{A}$.

(ii) From the definition of the family $(J^{\pi})_{\pi\in\mathcal{A}}$, we have

$$J^{\pi}_0 \;=\; Y_0|V^{x,\pi}_0-\mathcal{Y}_0|^2+\Upsilon_0 \;=\; Y_0|x-\mathcal{Y}_0|^2+\Upsilon_0\,,$$

for all $\pi\in\mathcal{A}$.

(iii) Fix $\pi\in\mathcal{A}$. Since $Y,\mathcal{Y},\Upsilon\in\mathcal{S}^{\infty}_{\mathbb{G}}$, we have from the definition of J^{π} and BDG inequality

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|J^{\pi}_t|\Big] \;<\; +\infty\,. \tag{3.18}$$

Now, fix $s,t\in[0,T]$ such that $s\le t$. Using the decomposition (3.9) and Lemma 3.1, there exists an increasing sequence of $\mathbb{G}$-stopping times $(\nu_i)_{i\ge1}$ such that $\nu_i\to+\infty$ as $i\to+\infty$ and

$$\mathbb{E}\Big[J^{\pi}_{t\wedge\nu_i}\big|\mathcal{G}_s\Big] \;\ge\; J^{\pi}_{s\wedge\nu_i}\,, \tag{3.19}$$

for all $i\ge1$. Then, from (3.18), we can apply the conditional dominated convergence Theorem and we get by sending i to ∞ in (3.19)

$$\mathbb{E}\Big[J^{\pi}_t\big|\mathcal{G}_s\Big] \;\ge\; J^{\pi}_s\,,$$

for all $s,t\in[0,T]$ with $s\le t$.

(iv) We now check that $\pi^*\in\mathcal{A}$ i.e. $\mathbb{E}\int_0^{T\wedge\tau}|\pi^*_s|^2ds<\infty$. Using the definition of π^* and (3.15) we have that V^{x,π^*} is solution to linear BSDE

$$V^{x,\pi*}_t \;=\; V^{x,\pi^*}_{T\wedge\tau}-\int_{t\wedge\tau}^{T\wedge\tau}\frac{\mu_s}{\sigma_s}z_sds-\int_{t\wedge\tau}^{T\wedge\tau}z_sdW_s-\int_{t\wedge\tau}^{T\wedge\tau}u_sdM_s\,,\quad t\in[0,T]\,,$$

with

$$z_t \;=\; \sigma_t\pi^*_t\quad\text{and}\quad u_t \;=\; \beta_t\pi^*_t\,,$$

for all $t\in[0,T]$. Therefore, using (3.16), **(HS)**, and classical arguments for BSDEs, we get

$$\mathbb{E}\Big[\int_0^{T\wedge\tau}|\pi^*_s|^2ds\Big] \;<\; \infty\,.$$

We now check that J^{π^*} is a $\mathbb{G}$-martingale. Since K^{π^*} is constant, we obtain from Lemma 3.1 that J^{π^*} is a $\mathbb{G}$-local martingale. Then, from the expression of J^{π^*} and since $Y,\mathcal{Y},\Upsilon\in\mathcal{S}^{\infty}_{\mathbb{G}}$, there exists a constant C such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|J^{\pi^*}_t|\Big] \;\le\; C\Big(1+\mathbb{E}\Big[\sup_{t\in[0,T\wedge\tau]}|V^{x,\pi^*}_t|^2\Big]\Big)\,.$$

Using (3.16), we get that

$$\mathbb{E}\Big[\sup_{t\in[0,T]}|J^{\pi^*}_t|\Big] \;<\; +\infty\,.$$

Therefore, J^{π^*} is a true $\mathbb{G}$-martingale and π^* is optimal. □

4 A decomposition approach for solving BSDEs in the filtration $\mathbb{G}$

We now prove Theorem 3.1 via a decomposition procedure. We first provide a general result which gives existence of a solution to a BSDE in the enlarged filtration $\mathbb{G}$ as soon as an associated BSDE in the filtration $\mathbb{F}$ admits a solution. Actually the associated BSDE is defined by the terms appearing in the decomposition of the coefficients of the BSDE in $\mathbb{G}$ given by Lemma 2.1. We therefore introduce the spaces of processes where solutions in $\mathbb{F}$ classically lie.

- $\mathcal{S}^{\infty}_{\mathbb{F}}$ is the subset of $\mathbb{R}$-valued continuous $\mathbb{F}$-adapted processes $(Y_t)_{t\in[0,T]}$ essentially bounded

$$\|Y\|_{\mathcal{S}^{\infty}} \;:=\; \Big\| \sup_{t\in[0,T]} |Y_t| \Big\|_{\infty} \;<\; \infty\,.$$

- $\mathcal{S}^{\infty,+}_{\mathbb{F}}$ is the subset of $\mathcal{S}^{\infty}_{\mathbb{F}}$ of processes $(Y_t)_{t\in[0,T]}$ valued in $(0,\infty)$, such that

$$\Big\| \frac{1}{Y} \Big\|_{\mathcal{S}^{\infty}} \;<\; \infty\,.$$

- $L^2_{\mathbb{F}}$ is the subset of $\mathbb{R}$-valued $\mathcal{P}(\mathbb{F})$-measurable processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{L^2} \;:=\; \Big(\mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big]\Big)^{\frac{1}{2}} \;<\; \infty\,.$$

Finally since the BSDEs associated to our mean-variance problem have generators with superlinear growth, we consider the additional space of BMO-martingales: $\mathrm{BMO}(\mathbb{P})$ is the subset of $(\mathbb{P},\mathbb{F})$-martingales m such that

$$\|m\|_{\mathrm{BMO}(\mathbb{P})} \;:=\; \sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}\big[\langle m\rangle_T - \langle m\rangle_\nu | \mathcal{F}_\nu\big]^{\frac{1}{2}} \;<\; \infty\,,$$

where $\mathcal{T}_{\mathbb{F}}[0,T]$ is the set of $\mathbb{F}$-stopping times on $[0,T]$. This means local martingales of the form $m_t = \int_0^t Z_s dW_s$ are $\mathrm{BMO}(\mathbb{P})$-martingale if and only if

$$\|Z\|_{\mathrm{BMO}(\mathbb{P})} \;:=\; \sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]} \Big\| \Big(\mathbb{E}\Big[\int_\nu^T |Z_t|^2 dt \Big| \mathcal{F}_\nu\Big]\Big)^{\frac{1}{2}} \Big\|_{\infty} \;<\; \infty\,.$$

In the sequel, we shall write $Z \in \mathrm{BMO}(\mathbb{P})$ for $\int_0^\cdot Z_s dW_s \in \mathrm{BMO}(\mathbb{P})$.

4.1 A general existence theorem for BSDEs with random horizon

We provide here a general result on existence of a solution to a BSDE driven by W and N with horizon $T\wedge\tau$. We consider a generator function $F : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, which is $\mathcal{P}(\mathbb{G})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable, and a terminal condition ξ which is a $\mathcal{G}_{T\wedge\tau}$-measurable random variable of the form

$$\xi \;=\; \xi^b \mathbb{1}_{T<\tau} + \xi^a_\tau \mathbb{1}_{T\geq\tau}\,, \tag{4.20}$$

where ξ^b is an $\mathcal{F}_T$-measurable bounded random variable and $\xi^a \in \mathcal{S}^\infty_{\mathbb{F}}$. From Proposition 2.1 and Remark 2.1, we can write

$$F(t,.)\mathbb{1}_{t\leq\tau} \;=\; F^b(t,.)\mathbb{1}_{t\leq\tau}\;,\quad t\geq 0\;, \tag{4.21}$$

where F^b is a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$-measurable map. We then introduce the following BSDE

$$Y^b_t \;=\; \xi^b + \int_t^T F^b(s,Y^b_s,Z^b_s,\xi^a_s - Y^b_s)ds - \int_t^T Z^b_s dW_s\;,\quad t\in[0,T]\;. \tag{4.22}$$

Theorem 4.3. *Assume that BSDE* (4.22) *admits a solution* $(Y^b,Z^b)\in\mathcal{S}^\infty_{\mathbb{F}}\times L^2_{\mathbb{F}}$. *Then BSDE*

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s,Y_s,Z_s,U_s) - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s\;,\quad t\in[0,T]\;, \tag{4.23}$$

admits a solution $(Y,Z,U)\in\mathcal{S}^\infty_{\mathbb{G}}\times L^2_{\mathbb{G}}\times L^2(\lambda)$ *given by*

$$\begin{aligned} Y_t &= Y^b_t\mathbb{1}_{t<\tau} + \xi^a_\tau\mathbb{1}_{t\geq\tau}\;,\\ Z_t &= Z^b_t\mathbb{1}_{t\leq\tau}\;,\\ U_t &= \big(\xi^a_t - Y^b_t\big)\mathbb{1}_{t\leq\tau}\;, \end{aligned} \tag{4.24}$$

for all $t\in[0,T]$.

Proof. We proceed in three steps.

Step 1: We prove that for $t\in[0,T]$, (Y,Z,U) defined by (4.24) satisfies the equation (4.23). We distinguish three cases.

Case 1: $\tau > T$.

From (4.24), we get $Y_t = Y^b_t$, $Z_t = Z^b_t$ and $U_t = \xi^a_t - Y^b_t$ for all $t\in[0,T]$. Then, using that (Y^b,Z^b) is a solution to (4.22), we have

$$Y_t \;=\; \xi^b + \int_t^T F^b(s,Y_s,Z_s,U_s)ds - \int_t^T Z^b_s dW_s\;.$$

Since the predictable processes Z and Z^b are indistinguishable on $\{\tau>T\}$, we have from Theorem 12.23 of [10], $\int_t^T Z_s dW_s = \int_t^T Z^b_s dW_s$ on $\{\tau>T\}$. Moreover since $\xi=\xi^b$ and $\int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s = 0$ on $\{\tau>T\}$ we get by using (4.21)

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s,Y_s,Z_s,U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s\;.$$

Case 2: $\tau\in(t,T]$.

From (4.24), we have $Y_t = Y^b_t$. Since (Y^b,Z^b) is solution to (4.22), we have

$$Y_t \;=\; Y^b_\tau + \int_t^\tau F^b(s,Y^b_s,Z^b_s,\xi^a_s - Y^b_s)ds - \int_t^\tau Z^b_s dW_s\;.$$

Still using (4.21) and (4.24), we get

$$Y_t \;=\; \xi^a_\tau + \int_t^\tau F(s,Y_s,Z_s,U_s)ds - \int_t^\tau Z^b_s dW_s - (\xi^a_\tau - Y^b_\tau)\;.$$

Since the predictable processes $Z\mathbb{1}_{.<\tau}$ and $Z^b\mathbb{1}_{.<\tau}$ are indistinguishable on $\{\tau > t\} \cap \{\tau \leq T\}$, we have from Theorem 12.23 of [10], $\int_t^{T\wedge\tau} Z_s dW_s = \int_t^{T\wedge\tau} Z_s^b dW_s$ on $\{\tau > t\} \cap \{\tau \leq T\}$. Therefore, we get

$$Y_t \;=\; \xi_\tau^a + \int_t^\tau F(s, Y_s, Z_s, U_s)ds - \int_t^\tau Z_s dW_s - (\xi_\tau^a - Y_\tau^b)\;.$$

Finally, we easily check from the definition of U that $\int_t^{T\wedge\tau} U_s dN_s = \xi_\tau^a - Y_\tau^b$. Therefore, we get using (4.20)

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s\;.$$

Case 3: $\tau \leq t$.
Then, from (4.24), we have $Y_t = \xi_\tau^a$. We therefore get on $\{\tau \leq t\}$ by using (4.20)

$$Y_t \;=\; \xi + \int_{t\wedge\tau}^{T\wedge\tau} F(s, Y_s, Z_s, U_s)ds - \int_{t\wedge\tau}^{T\wedge\tau} Z_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} U_s dN_s\;.$$

Step 2: We notice that Y is a càd-làg $\mathbb{G}$-adapted process and U is $\mathcal{P}(\mathbb{G})$-measurable since Y^b and ξ^a are continuous and $\mathbb{G}$-adapted. We also notice from its definition that the process Z is $\mathcal{P}(\mathbb{G})$-measurable, since Z^b is $\mathcal{P}(\mathbb{F})$-measurable.

Step 3: We now prove that the solution satisfies the integrability conditions. From the definition of Y, we have

$$|Y_t| \;\leq\; |Y_t^b| + |\xi_t^a|\;,\quad t \in [0,T]\;. \tag{4.25}$$

Since $Y^b \in \mathcal{S}_\mathbb{F}^\infty$ and $\xi^a \in \mathcal{S}_\mathbb{F}^\infty$, there exist two constants C_1, $C_2 \geq 0$ such that

$$\sup_{t\in[0,T]} |Y_t^b| \;\leq\; C_1 \quad \text{and} \quad \sup_{t\in[0,T]} |\xi_t^a| \;\leq\; C_2\;,\quad \mathbb{P}-a.s.$$

Therefore, we get from (4.25)

$$\sup_{t\in[0,T]} |Y_t| \;\leq\; C_1 + C_2\;,\quad \mathbb{P}-a.s.$$

and $\|Y\|_{\mathcal{S}^\infty} < +\infty$.
From the definition of the process Z, we have

$$\mathbb{E}\Big[\int_0^T |Z_s|^2 ds\Big] \;\leq\; \mathbb{E}\Big[\int_0^T |Z_s^b|^2 ds\Big]\;.$$

Since $Z^b \in L_\mathbb{F}^2$, we get

$$\mathbb{E}\Big[\int_0^T |Z_s|^2 ds\Big] \;<\; +\infty\;.$$

Finally from the definition of U, we have

$$|U_t| \;\leq\; |Y_t^b| + |\xi_t^a|\;,\quad t \in [0,T]\;.$$

Since $Y^b \in \mathcal{S}^\infty_{\mathbb{F}}$, $\xi^a \in \mathcal{S}^\infty_{\mathbb{F}}$ and λ is bounded, we get

$$\mathbb{E}\Big[\int_0^{T\wedge\tau} \lambda_t |U_t|^2 dt\Big] \;<\; \infty\,.$$

□

Using this abstract result we prove the existence of solutions to each of the BSDEs (3.6), (3.7) and (3.8) in the following subsections.

4.2 Solution to BSDE $(\mathfrak{f}, 1)$

According to the general existence Theorem 4.3, we consider for coefficients $(\mathfrak{f}, 1)$ the BSDE in $\mathbb{F}$: find $(Y^b, Z^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\left\{\begin{array}{rcl} dY^b_t &=& \Big\{\dfrac{|(\mu_t - \lambda_t\beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t\beta_t|^2}{|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2} - \lambda_t + \lambda_t Y^b_t\Big\}dt + Z^b_t dW_t\,, \quad t\in[0,T]\,, \\ Y^b_T &=& 1\,. \end{array}\right. \tag{4.26}$$

To solve this BSDE, we have to deal with two main issues. The first is that the generator $\mathfrak{f}$ has a superlinear growth. The second difficulty is that the generator value is not defined for all the values that the process Y can take. In particular the generator may explode if the process Y goes to zero. Taking in consideration these issues we get the following result.

Proposition 4.3. *BSDE* (4.26) *has a solution* (Y^b, Z^b) *in* $\mathcal{S}^{\infty,+}_{\mathbb{F}} \times L^2_{\mathbb{F}}$ *with* $Z^b \in \mathrm{BMO}(\mathbb{P})$.

Proof. We first notice that BSDE (4.26) can be can written under the form

$$\left\{\begin{array}{rcl} dY^b_t &=& \Big\{\dfrac{|\mu_t - \lambda_t\beta_t|^2}{|\sigma_t|^2}Y^b_t - \dfrac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t - \lambda_t\beta_t|^2 - \lambda_t + \lambda_t Y^b_t + \dfrac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_t Z^b_t + \lambda_t\beta_t) \\ && + \dfrac{\big|\sigma_t Z^b_t + \lambda_t\beta_t + (\lambda_t\beta_t - \mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2}\Big\}dt + Z^b_t dW_t\,, \quad t\in[0,T]\,, \\ Y^b_T &=& 1\,. \end{array}\right.$$

Since the variable Y^b appears in the denominator we can not directly solve this BSDE. We then proceed in four steps. We first introduce a modified BSDE with a lower bounded denominator to ensure that the generator is well defined. We then prove via a change of probability and a comparison theorem that the solution of the modified BSDE satisfies the initial BSDE.

Step 1: *Introduction of the modified BSDE.*

Let $(Y^\varepsilon, Z^\varepsilon)$ be the solution in $\mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ to the BSDE

$$\left\{\begin{array}{rcl} dY^\varepsilon_t &=& \Big\{\dfrac{|\mu_t - \lambda_t\beta_t|^2}{|\sigma_t|^2}Y^\varepsilon_t - \dfrac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t - \lambda_t\beta_t|^2 - \lambda_t + \lambda_t Y^\varepsilon_t + \dfrac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_t Z^\varepsilon_t + \lambda_t\beta_t) \\ && + \dfrac{\big|\sigma_t Z^\varepsilon_t + \lambda_t\beta_t + (\lambda_t\beta_t - \mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{|\sigma_t|^2(Y^\varepsilon_t \vee \varepsilon) + \lambda_t|\beta_t|^2}\Big\}dt + Z^\varepsilon_t dW_t\,, \quad t\in[0,T]\,, \\ Y^\varepsilon_T &=& 1\,, \end{array}\right. \tag{4.27}$$

where ε is a positive constant such that

$$\exp\Big(-\int_0^T\Big(\lambda_t+\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}\Big)dt\Big) \;\geq\; \varepsilon\,,\quad \mathbb{P}-a.s. \tag{4.28}$$

Such a constant exists from **(HS)**. Since BSDE (4.27) is a quadratic BSDE, there exists a solution $(Y^\varepsilon, Z^\varepsilon)$ in $\mathcal{S}^\infty_{\mathbb{F}}\times L^2_{\mathbb{F}}$ from [19].

Step 2: BMO *property of the solution.*

In this part we prove that $Z^\varepsilon\in$ BMO($\mathbb{P}$). Let k denote the lower bound of the uniformly bounded process Y^ε. Applying Itô's formula to $|Y^\varepsilon-k|^2$, we obtain

$$\mathbb{E}\Big[\int_\nu^T|Z^\varepsilon_s|^2ds\Big|\mathcal{F}_\nu\Big] \;=\; |1-k|^2-|Y^\varepsilon_\nu-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y^\varepsilon_s-k)f^\epsilon(s,Y^\varepsilon_s,Z^\varepsilon_s)ds\Big|\mathcal{F}_\nu\Big]\,, \tag{4.29}$$

for any stopping times $\nu\in\mathcal{T}_{\mathbb{F}}[0,T]$, with

$$\begin{aligned} f^\epsilon(t,y,z) \;=\;& \frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y-\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+\lambda_t y+\frac{2(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}(\sigma_t z+\lambda_t\beta_t)\\ &+\frac{|\sigma_t z+\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}|^2}{|\sigma_t|^2(y\vee\varepsilon)+\lambda_t|\beta_t|^2}\,, \end{aligned}$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$. We can see that

$$f^\epsilon(t,y,z) \;\geq\; I_t+G_ty+H_tz\,, \tag{4.30}$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$ where the processes I, G and H are given by

$$\begin{cases} I_t \;:=\; -\dfrac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2-\lambda_t+2\lambda_t\beta_t\dfrac{(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}\,,\\ G_t \;:=\; \dfrac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}+\lambda_t\,,\\ H_t \;:=\; 2\dfrac{(\mu_t-\lambda_t\beta_t)}{\sigma_t}\,, \end{cases}$$

for all $t\in[0,T]$. We first notice that from **(HS)**, the processes I, J and K are bounded. Using (4.29) and (4.30), we get the following inequality

$$\mathbb{E}\Big[\int_\nu^T|Z^\varepsilon_s|^2ds\Big|\mathcal{F}_\nu\Big] \;\leq\; |1-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y^\varepsilon_s-k)(I_s+G_sY^\varepsilon_s+H_sZ^\varepsilon_s)ds\Big|\mathcal{F}_\nu\Big]\,.$$

From the inequality $2ab\leq a^2+b^2$ for $a,b\geq0$, we get

$$\begin{aligned} \mathbb{E}\Big[\int_\nu^T|Z^\varepsilon_s|^2ds\Big|\mathcal{F}_\nu\Big] \;\leq\;& |1-k|^2-2\mathbb{E}\Big[\int_\nu^T(Y^\varepsilon_s-k)(I_s+G_sY^\varepsilon_s)ds\Big|\mathcal{F}_\nu\Big]\\ &+2\mathbb{E}\Big[\int_\nu^T|H_s|^2|Y^\varepsilon_s-k|^2ds\Big|\mathcal{F}_\nu\Big]+\frac{1}{2}\mathbb{E}\Big[\int_\nu^T|Z^\varepsilon_s|^2ds\Big|\mathcal{F}_\nu\Big]\,. \end{aligned}$$

Since I, G, H and Y^ε are uniformly bounded, we get

$$\mathbb{E}\Big[\int_\nu^T|Z^\varepsilon_s|^2ds\Big|\mathcal{F}_\nu\Big] \;\leq\; C\,,$$

for some constant C which does not depend on ν. Therefore, $Z^\varepsilon \in \mathrm{BMO}(\mathbb{P})$.

Step 3: *Change of probability.*

Define the process L^ε by

$$L_t^\varepsilon \;:=\; 2\frac{(\mu_t-\lambda_t\beta_t)}{\sigma_t}+2\frac{\sigma_t\big(\lambda_t\beta_t+\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}(\lambda_t\beta_t-\mu_t)\big)}{|\sigma_t|^2(Y_t^\varepsilon\vee\varepsilon)+\lambda_t|\beta_t|^2}+\frac{|\sigma_t|^2Z_t^\varepsilon}{|\sigma_t|^2(Y_t^\varepsilon\vee\varepsilon)+\lambda_t|\beta_t|^2}\,,$$

for all $t\in[0,T]$. Since $Y^\varepsilon\in\mathcal{S}_{\mathbb{F}}^\infty$, $Z^\varepsilon\in\mathrm{BMO}(\mathbb{P})$, we get from **(H**$S$**)** that $L^\varepsilon\in\mathrm{BMO}(\mathbb{P})$. Therefore, the process $\mathcal{E}(\int_0^\cdot L_s^\epsilon dW_s)$ is an $\mathbb{F}$-martingale. Applying Girsanov Theorem we get that the process $\bar W$ defined by

$$\bar W_t \;:=\; W_t+\int_0^t L_s^\varepsilon ds\,,$$

for all $t\in[0,T]$, is a Brownian motion under the probability $\mathbb{Q}$ defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} \;=\; \mathcal{E}\Big(\int_0^T L_s^\epsilon dW_s\Big)\,.$$

We also notice that under $\mathbb{Q}$, $(Y^\varepsilon,Z^\varepsilon)$ is solution to

$$\begin{aligned}Y_t^\varepsilon \;=\;& 1+\int_t^T\frac{\lambda_s|\beta_s|^2}{|\sigma_s|^4}|\mu_s-\lambda_s\beta_s|^2-\frac{|\mu_s-\lambda_s\beta_s|^2}{|\sigma_s|^2}Y_s^\varepsilon-2\lambda_s\beta_s\frac{(\mu_s-\lambda_s\beta_s)}{|\sigma_s|^2}+\lambda_s\\&-\lambda_sY_s^\varepsilon-\frac{\big|\lambda_s\beta_s+(\lambda_s\beta_s-\mu_s)\frac{\lambda_s|\beta_s|^2}{|\sigma_s|^2}\big|^2}{|\sigma_s|^2(Y_s^\varepsilon\vee\varepsilon)+\lambda_s|\beta_s|^2}ds-\int_t^TZ_s^\varepsilon d\bar W_s\,,\quad t\in[0,T]\,.\end{aligned}\tag{4.31}$$

Step 4: *Comparison under the new probability measure* $\mathbb{Q}$.

We first notice that the generator $\bar f^\epsilon$ of BSDE (4.31) admits the following lower bound

$$\begin{aligned}\bar f^\epsilon(t,y,z) \;\geq\;& \frac{\lambda_t|\beta_t|^2}{|\sigma_t|^4}|\mu_t-\lambda_t\beta_t|^2+\lambda_t-\lambda_ty-2\lambda_t\beta_t\frac{(\mu_t-\lambda_t\beta_t)}{|\sigma_t|^2}\\&-\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y-\frac{\big|\lambda_t\beta_t+(\lambda_t\beta_t-\mu_t)\frac{\lambda_t|\beta_t|^2}{|\sigma_t|^2}\big|^2}{\lambda_t|\beta_t|^2}\mathbb{1}_{\lambda_t\beta_t\neq0}\\=\;&-\lambda_ty-\frac{|\mu_t-\lambda_t\beta_t|^2}{|\sigma_t|^2}y\,,\end{aligned}$$

for all $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}$. We now study the following BSDE

$$\underline{Y}_t \;=\; 1+\int_t^T\Big[-\lambda_s-\frac{|\mu_s-\lambda_s\beta_s|^2}{|\sigma_s|^2}\Big]\underline{Y}_sds-\int_t^T\underline{Z}_sd\bar W_s\,,\quad t\in[0,T]\,.\tag{4.32}$$

Since, this BSDE is linear, it has a unique solution given by

$$\underline{Y}_t \;:=\; \mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(-\int_t^T\big(\lambda_s+\frac{|\mu_s+\lambda_s\beta_s|^2}{|\sigma_s|^2}\big)ds\Big)\Big|\mathcal{F}_t\Big]\,,\quad t\in[0,T]\,.$$

By the comparison Theorem for BSDEs (4.31) and (4.32) we have

$$Y_t^\epsilon \;\geq\; \underline{Y}_t\,,\quad t\in[0,T]\,.$$

By (4.28), we have $\varepsilon\leq\underline{Y}_t$ for any $t\in[0,T]$. Consequently, $Y_t^\varepsilon\geq\varepsilon$ for any $t\in[0,T]$, and $(Y^\varepsilon,Z^\varepsilon)$ is solution to (4.26). □

We now are able to prove that BSDE $(\mathfrak{f},1)$ admits a solution.

Proposition 4.4. *The BSDE* (3.6) *admits a solution* $(Y,Z,U) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$ *with* $Y \in \mathcal{S}^{\infty,+}_{\mathbb{G}}$.

Proof. From Theorem 4.3 and Proposition 4.3, we obtain that BSDE (3.6) admits a solution $(Y,Z,U) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$, with Y given by

$$Y_t \;=\; Y^b_t \mathbb{1}_{\tau<t} + \mathbb{1}_{\tau \geq t}\,, \quad t \in [0,T]\,.$$

with $Y^b \in \mathcal{S}^{\infty,+}_{\mathbb{F}}$ from Proposition 4.3. Therefore $Y \in \mathcal{S}^{\infty,+}_{\mathbb{G}}$. □

4.3 Solution to BSDE $(\mathfrak{g}, H)$

We first notice that BSDE $(\mathfrak{g}, H)$ can be rewritten under the form

$$\left\{\begin{array}{rcl} d\mathcal{Y}_t &=& \Big\{\dfrac{(\mu_t Y_t + \sigma_t Z_t + \lambda^{\mathbb{G}}_t \beta_t U_t)(\sigma_t Y_t \mathcal{Z}_t + \lambda^{\mathbb{G}}_t \beta_t (U_t + Y_t)\mathcal{U}_t)}{Y_t(|\sigma_t|^2 Y_t + \lambda^{\mathbb{G}}_t |\beta_t|^2 (U_t + Y_t))} - \dfrac{Z_t}{Y_t}\mathcal{Z}_t \\ && \quad - \frac{\lambda^{\mathbb{G}}_t U_t}{Y_t}\mathcal{U}_t - \lambda^{\mathbb{G}}_t \mathcal{U}_t\Big\} dt + \mathcal{Z}_t dW_t + \mathcal{U}_t dH_t\,, \quad t \in [0, T\wedge\tau]\,, \\ \mathcal{Y}_{T\wedge\tau} &=& H\,. \end{array}\right. \tag{4.33}$$

Since $Y_t \mathbb{1}_{t<\tau} = Y^b_t \mathbb{1}_{t<\tau}$ and $U_t \mathbb{1}_{t\leq\tau} = (1 - Y^b_t)\mathbb{1}_{t\leq\tau}$, we consider the associated decomposed BSDE in $\mathbb{F}$: find $(\mathcal{Y}^b, \mathcal{Z}^b) \in \mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ such that

$$\left\{\begin{array}{rcl} d\mathcal{Y}^b_t &=& \Big\{\dfrac{((\mu_t - \lambda_t\beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t\beta_t)(\sigma_t Y^b_t \mathcal{Z}^b_t + \lambda_t\beta_t H^a_t - \lambda_t\beta_t \mathcal{Y}^b_t)}{Y^b_t(|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2)} \\ && \quad - \frac{Z^b_t}{Y^b_t}\mathcal{Z}^b_t - \dfrac{\lambda_t}{Y^b_t}H^a_t + \dfrac{\lambda_t}{Y^b_t}\mathcal{Y}^b_t\Big\} dt + \mathcal{Z}^b_t dW_t\,, \quad t \in [0,T]\,, \\ \mathcal{Y}^b_T &=& H^b\,. \end{array}\right. \tag{4.34}$$

We notice that this BSDE has a Lipschitz generator w.r.t. the unknown $(\mathcal{Y}^b, \mathcal{Z}^b)$. However the Lipschitz coefficient depends on Z^b which is not necessarily bounded. Thus we cannot apply the existing results and have to deal with this issue.

Proposition 4.5. *BSDE* (4.34) *admits a solution* $(\mathcal{Y}^b, \mathcal{Z}^b)$ *in* $\mathcal{S}^\infty_{\mathbb{F}} \times L^2_{\mathbb{F}}$ *with* $\mathcal{Z}^b \in \mathrm{BMO}(\mathbb{P})$.

Proof. We first define the equivalent probability $\mathbb{Q}$ to $\mathbb{P}$ defined by its Radon-Nikodym density $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_T} = \mathcal{E}(\int_0^T \rho_t dW_t)$ where ρ is given by

$$\rho_t \;:=\; \frac{Z^b_t}{Y^b_t} - \frac{\sigma_t\big((\mu_t - \lambda_t\beta_t)Y^b_t + \sigma_t Z^b_t + \lambda_t\beta_t\big)}{|\sigma_t|^2 Y^b_t + \lambda_t|\beta_t|^2}\,, \quad t \in [0,T]\,.$$

Since $Z^b \in \mathrm{BMO}(\mathbb{P})$, $Y^b \in \mathcal{S}^{\infty,+}_{\mathbb{F}}$ and the coefficients μ, σ and β satisfy **(H**S**)**, it implies that $\rho \in \mathrm{BMO}(\mathbb{P})$. Therefore, $\bar{W}_t := W_t - \int_0^t \rho_s ds$ is a $\mathbb{Q}$-Brownian motion. Hence, BSDE (4.34) can be written

$$\left\{\begin{array}{rcl} d\mathcal{Y}^b_t &=& a_t(\mathcal{Y}^b_t - H^a_t)dt + \mathcal{Z}^b_t d\bar{W}_t\,, \quad t \in [0,T]\,, \\ \mathcal{Y}^b_{T\wedge\tau} &=& H^b\,, \end{array}\right. \tag{4.35}$$

with

$$a_t \;:=\; \frac{\lambda_t|\sigma_t|^2Y_t^b - \lambda_t\beta_t((\mu_t-\lambda_t\beta_t)Y_t^b+\sigma_tZ_t^b)}{Y_t^b(|\sigma_t|^2Y_t^b+\lambda_t|\beta_t|^2)}\;,\quad t\in[0,T]\;.$$

By definition of a we can see that $a\in \mathrm{BMO}(\mathbb{P})$ since the coefficients μ, σ, β and λ are bounded, $Y^b\in\mathcal{S}_{\mathbb{F}}^{\infty,+}$ and $Z^b\in\mathrm{BMO}(\mathbb{P})$. Using BMO-stability Theorem (see Theorem 5.4), there exists a constant $l'\geq 0$ such that $\mathbb{E}_{\mathbb{Q}}[\int_\nu^T|a_s|^2ds|\mathcal{F}_\nu]\leq l'$ for any $\nu\in\mathcal{T}_{\mathbb{F}}[0,T]$. We now prove that the process $\mathcal{Y}^b$ defined by

$$\mathcal{Y}_t^b \;:=\; \mathbb{E}_{\mathbb{Q}}\Big[\frac{\Gamma_T}{\Gamma_t}H^b+\int_t^T\frac{\Gamma_s}{\Gamma_t}a_sH_s^ads\Big|\mathcal{F}_t\Big]\;,\quad t\in[0,T]\;,$$

with $\Gamma_t:=\exp(-\int_0^t a_sds)$, is solution of this BSDE. We proceed in four steps.

Step 1. *Integrability property of the process* Γ.

We first prove that for any $p\geq 1$ there exists a constant $C>0$ such that the process Γ satisfies for any $t\in[0,T]$

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{t\leq s\leq T}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p\Big|\mathcal{F}_t\Big] \;\leq\; C\;. \tag{4.36}$$

Since $\mathbb{E}_{\mathbb{Q}}[\int_\nu^T|a_s|^2ds|\mathcal{F}_\nu]\leq l'$ for any $\theta\in\mathcal{T}_{\mathbb{F}}[0,T]$, we get from Proposition 5.9 that there exists a constant δ such that $0<\delta<\frac{1}{l'}$ and

$$\mathbb{E}_{\mathbb{Q}}\Big[\exp\Big(\delta\int_\nu^T|a_s|^2ds\Big)\Big|\mathcal{F}_\nu\Big] \;\leq\; \frac{1}{1-\delta l'}\;.$$

We get for any $0\leq t\leq s\leq T$

$$\begin{aligned}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p &\leq \exp\Big(\int_t^s(\delta|a_r|^2+\frac{p^2}{4\delta})dr\Big)\\ &\leq \exp\Big(\frac{p^2}{4\delta}T\Big)\exp\Big(\delta\int_0^T|a_r|^2dr\Big)\;.\end{aligned}$$

Consequently, we get

$$\mathbb{E}_{\mathbb{Q}}\Big[\sup_{t\leq s\leq T}\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^p\Big|\mathcal{F}_t\Big] \;\leq\; \exp\Big(\frac{p^2}{4\delta}T\Big)\frac{1}{1-\delta l'}\;.$$

Step 2. *Uniform boundedness of* $\mathcal{Y}^b$.

We now prove that $\mathcal{Y}^b\in\mathcal{S}_{\mathbb{F}}^{\infty}$. For that we remark that by definition of $\mathcal{Y}^b$ we have the following inequality

$$|\mathcal{Y}_t^b| \;\leq\; \|H^b\|_\infty\mathbb{E}_{\mathbb{Q}}\Big[\frac{\Gamma_T}{\Gamma_t}\Big|\mathcal{F}_t\Big]+\|H^a\|_\infty\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T|a_s|^2ds\Big|\mathcal{F}_t\Big]+\|H^a\|_\infty\mathbb{E}_{\mathbb{Q}}\Big[\int_t^T\Big|\frac{\Gamma_s}{\Gamma_t}\Big|^2ds\Big|\mathcal{F}_t\Big]\;.$$

Therefore, we get that $\mathcal{Y}^b\in\mathcal{S}_{\mathbb{F}}^{\infty}$.

Step 3. *Dynamics of* $\mathcal{Y}^b$.

We now prove that $\mathcal{Y}^b$ satisfies (4.35). For that we introduce the $\mathbb{Q}$-martingale m defined by

$$m_t \;:=\; \Gamma_t \mathcal{Y}^b_t + \int_0^t \Gamma_s a_s H^a_s ds\,, \quad t \in [0,T]\,.$$

We first notice that m is $\mathbb{Q}$-square integrable. Indeed, from the definition of m, there exists a constant C such that

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \;\leq\; C\Big(\mathbb{E}_{\mathbb{Q}}\Big[\big|\Gamma_t \mathcal{Y}^b_t\big|^2\Big] + \mathbb{E}_{\mathbb{Q}}\Big[\int_0^t \big|\Gamma_s a_s H^a_s\big|^2 ds\Big]\Big)\,,$$

for all $t \in [0,T]$. Since $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$, $H^a \in \mathcal{S}^\infty_{\mathbb{F}}$ and from Cauchy-Schwarz inequality there exists a constant C such that

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \;\leq\; C\Big(\mathbb{E}_{\mathbb{Q}}\Big[\big|\Gamma_t\big|^2\Big] + \sqrt{\mathbb{E}_{\mathbb{Q}}\Big[\Big(\int_0^t |a_s|^2 ds\Big)^2\Big]}\sqrt{\mathbb{E}_{\mathbb{Q}}\Big[\sup_{0\leq s\leq t}|\Gamma_s|^4\Big]}\;\Big)\,,$$

for all $t \in [0,T]$. Since $a \in \mathrm{BMO}(\mathbb{P})$ we have from Theorem 5.4 $a \in \mathrm{BMO}(\mathbb{Q})$, and we get from Proposition 5.9 and (4.36)

$$\mathbb{E}_{\mathbb{Q}}\Big[|m_t|^2\Big] \;<\; \infty\,, \quad t \in [0,T]\,.$$

Therefore, there exists a predictable process $\tilde{\mathcal{Z}}$ such that $\mathbb{E}_{\mathbb{Q}}[\int_0^T |\tilde{\mathcal{Z}}_s|^2 ds] < \infty$ and

$$\Gamma_t \mathcal{Y}^b_t + \int_0^t \Gamma_s a_s H^a_s ds \;=\; m_0 + \int_0^t \tilde{\mathcal{Z}}_s d\bar{W}_s\,, \quad t \in [0,T]\,.$$

From Itô's formula and the definition of $\mathcal{Y}^b_T$ we have

$$\mathcal{Y}^b_t \;=\; H^b - \int_t^T a_s(\mathcal{Y}^b_s - H^a_s)ds - \int_t^T \mathcal{Z}^b_t d\bar{W}_s\,, \quad t \in [0,T]\,. \tag{4.37}$$

where the process $\mathcal{Z}^b$ is defined by

$$\mathcal{Z}^b_t \;:=\; \frac{\tilde{\mathcal{Z}}_t}{\Gamma_t}\,, \quad t \in [0,T]\,.$$

We now prove that $\mathcal{Z}^b \in \mathrm{BMO}(\mathbb{Q})$. Using (4.37), there exists a constant C such that

$$\begin{aligned}\sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}_{\mathbb{Q}}\Big[\int_\nu^T |\mathcal{Z}^b_s|^2 ds\Big|\mathcal{F}_\nu\Big] \;\leq\; C\Big(&(\|\mathcal{Y}^b\|^2_{\mathcal{S}^\infty} + \|H^a\|^2_{\mathcal{S}^\infty}) \sup_{\nu\in\mathcal{T}_{\mathbb{F}}[0,T]} \mathbb{E}_{\mathbb{Q}}\Big[\int_\nu^T |a_s|^2 ds\Big|\mathcal{F}_\nu\Big] \\ &+\|H^b\|^2_{\mathcal{S}^\infty} + \|\mathcal{Y}^b\|^2_{\mathcal{S}^\infty}\Big)\,.\end{aligned}$$

Since $\mathcal{Y}^b \in \mathcal{S}^\infty_{\mathbb{F}}$, $H^a \in \mathcal{S}^\infty_{\mathbb{F}}$, $H^b \in \mathcal{S}^\infty_{\mathbb{F}}$ and $a \in \mathrm{BMO}(\mathbb{Q})$, we get that $\mathcal{Z}^b \in \mathrm{BMO}(\mathbb{Q})$. Thus, from Theorem 5.4, $\mathcal{Z}^b \in \mathrm{BMO}(\mathbb{P})$ and $\mathbb{E}[\int_0^T |\mathcal{Z}^b_t|^2 dt] < \infty$. To conclude we get from (4.37) and the definition of $\bar{W}$ that $(\mathcal{Y}^b, \mathcal{Z}^b)$ is a solution to BSDE (4.34). □

We now prove the existence of a solution to BSDE $(\mathfrak{g}, H)$.

Proposition 4.6. *The BSDE* (3.7) *admits a solution* $(\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$.

Proof. From Theorem 4.3 and Proposition 4.5, we obtain that BSDE (3.7) admits a solution $(\mathcal{Y}, \mathcal{Z}, \mathcal{U}) \in \mathcal{S}^\infty_{\mathbb{G}} \times L^2_{\mathbb{G}} \times L^2(\lambda)$. □

4.4 Solution to BSDE $(\mathfrak{h}, 0)$

We recall that BSDE $(\mathfrak{h}, 0)$ is

$$\begin{aligned}\Upsilon_t &= \int_{t\wedge\tau}^{T\wedge\tau} \Big(|\mathcal{Z}_t|^2 Y_t + \lambda_t^{\mathbb{G}}(U_t+Y_t)|\mathcal{U}_t|^2 - \frac{|\sigma_t Y_t \mathcal{Z}_t + \lambda_t^{\mathbb{G}}\beta_t \mathcal{U}_t(U_t+Y_t)|^2}{|\sigma_t|^2 Y_t + \lambda_t^{\mathbb{G}}|\beta_t|^2(U_t+Y_t)}\Big)ds \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} \Xi_s dW_s - \int_{t\wedge\tau}^{T\wedge\tau} \Theta_s dM_s\ , \quad t\in[0,T]\ . \end{aligned} \tag{4.38}$$

Using the definitions of Y, U, $\mathcal{Z}$ and $\mathcal{U}$, we therefore consider the associated decomposed BSDE in $\mathbb{F}$: find $(\Upsilon^b, \Xi^b) \in \mathcal{S}_{\mathbb{F}}^{\infty} \times L_{\mathbb{F}}^2$ such that

$$\begin{aligned}\Upsilon_t^b &= \int_t^T \Big(|\mathcal{Z}_t^b|^2 Y_t^b + \lambda_t |H_t^a - \mathcal{Y}_t^b|^2 - \frac{|\sigma_t Y_t^b \mathcal{Z}_t^b + \lambda_t\beta_t(H_t^a - \mathcal{Y}_t^b)|^2}{|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2} - \lambda_s \Upsilon_s\Big)ds \\ &\quad - \int_{t\wedge\tau}^{T\wedge\tau} \Xi_s^b dW_s\ , \qquad t\in[0,T]\ . \end{aligned}$$

Proposition 4.7. *The BSDE* (4.39) *admits a solution* $(\Upsilon^b, \Xi^b) \in \mathcal{S}_{\mathbb{F}}^{\infty} \times L_{\mathbb{F}}^2$.

Proof. Denote by R the process defined by

$$R_t := |\mathcal{Z}_t^b|^2 Y_t^b + \lambda_t|H_t^a - \mathcal{Y}_t^b|^2 - \frac{|\sigma_t Y_t^b \mathcal{Z}_t^b + \lambda_t\beta_t(H_t^a - \mathcal{Y}_t^b)|^2}{|\sigma_t|^2 Y_t^b + \lambda_t|\beta_t|^2}\ ,$$

for $t\in[0,T]$. Define the process $\tilde{\Upsilon}^b$ by

$$\tilde{\Upsilon}_t^b := \mathbb{E}\Big[\int_t^T R_s e^{\int_0^s \lambda_u du} ds \Big| \mathcal{F}_t\Big]\ , \quad t\in[0,T]\ .$$

From **(HS)**, λ is bounded, $Y^b \in \mathcal{S}_{\mathbb{F}}^{\infty}$, $H^a \in \mathcal{S}_{\mathbb{F}}^{\infty}$, $\mathcal{Y}^b \in \mathcal{S}_{\mathbb{F}}^{\infty}$ and $\mathcal{Z}^b \in \mathrm{BMO}(\mathbb{P})$, we get from Proposition 5.9 that $\tilde{\Upsilon}^b \in \mathcal{S}_{\mathbb{F}}^{\infty}$ and the process $\tilde{\Upsilon}^b + \int_0^{\cdot} R_s e^{\int_0^s \lambda_u du} ds$ is a square integrable martingale. Hence there exists a process $\Xi \in L_{\mathbb{F}}^2$ such that

$$\tilde{\Upsilon}_t^b = \int_t^T R_s e^{\int_0^s \lambda_u du} ds - \int_t^T \Xi_s dW_s\ , \quad t\in[0,T]\ .$$

From Itô's formula we get that the processes (Υ^b, Ξ^b) defined by

$$\Upsilon_t^b = \tilde{\Upsilon}_t^b e^{-\int_0^t \lambda_s ds} \quad \text{and} \quad \Xi_t^b = \tilde{\Xi}_t^b e^{-\int_0^t \lambda_s ds}$$

satisfy (4.39). Since $\tilde{\Xi} \in L_{\mathbb{F}}^2$ and λ is uniformly bounded we get that $\Xi \in L_{\mathbb{F}}^2$. Finally, since $\tilde{\Upsilon}^b \in \mathcal{S}_{\mathbb{F}}^{\infty}$ we get that $\Upsilon^b \in \mathcal{S}_{\mathbb{F}}^{\infty}$. □

Finally, we prove the existence of a solutin to BSDE $(\mathfrak{h}, 0)$.

Proposition 4.8. *The BSDE* (3.8) *admits a solution* $(\Upsilon, \Xi, \Theta) \in \mathcal{S}_{\mathbb{G}}^{\infty} \times L_{\mathbb{G}}^2 \times L^2(\lambda)$.

Proof. From Theorem 4.3 and Proposition 4.7, we obtain that BSDE (3.8) admits a solution $(Y, Z, U) \in \mathcal{S}_{\mathbb{G}}^{\infty} \times L_{\mathbb{G}}^2 \times L^2(\lambda)$. □

5 Appendix

Theorem 5.4. *(*BMO*-Stability) Let M be a local martingale and N be a* BMO$(\mathbb{P})$*-martingale. Let define the martingale measure $\mathbb{Q} \sim \mathbb{P}$ with the Radon-Nikodym density Z_T on $\mathcal{F}_T$ given by $Z_T = \mathcal{E}(N)_T$. If M is a* BMO$(\mathbb{P})$*-martinagle then $M - \langle M, N\rangle$ is a* BMO$(\mathbb{Q})$*-martingale.*

Proof. See Kazamaki [18] Theorem 3.3. □

Proposition 5.9. *Let A be a continuous increasing $\mathbb{F}$-adapted process and $t \geq 0$ such that there exists a constant $C > 0$ satisfying*

$$\mathbb{E}\big[A_t - A_s\big|\mathcal{F}_s\big] \quad \leq \quad C \ ,$$

for any $s \in [0,t]$. Then, we have for any $s \in [0,t]$ and any $p \geq 1$

$$\mathbb{E}\big[|A_t - A_s|^p|\mathcal{F}_s\big] \leq p!|C|^p$$

and

$$\mathbb{E}\Big[\exp\big(\delta(A_t - A_s)\big)\big|\mathcal{F}_s\Big] \quad \leq \quad \frac{1}{1-\delta C} \ ,$$

for any $\delta \in (0, C)$.

Proof. Let A be a continuous increasing $\mathbb{F}$-adapted process satisfying $\mathbb{E}[A_t - A_s|\mathcal{F}_s] \leq C$ for any $0 \leq s \leq t$. We first prove by iteration that $\mathbb{E}[|A_t - A_s|^p|\mathcal{F}_s] \leq p!|C|^p$ for any $p \geq 1$.

• For $p = 1$, we have by assumption $\mathbb{E}[A_t - A_s|\mathcal{F}_s] \leq C$.

• Suppose that for some $p \geq 2$, we have $\mathbb{E}[|A_t - A_s|^{p-1}|\mathcal{F}_s] \leq (p-1)!|C|^{p-1}$. Since A is a continuous increasing $\mathbb{F}$-adapted process we have

$$|A_t - A_s|^p = p\int_s^t |A_t - A_u|^{p-1} dA_u \ ,$$

for any $s \in [0,t]$. Consequently we get

$$\begin{aligned}
\mathbb{E}\big[|A_t - A_s|^p\big|\mathcal{F}_s\big] &= p\mathbb{E}\Big[\int_s^t |A_t - A_u|^{p-1} dA_u\Big|\mathcal{F}_s\Big] \\
&= p\mathbb{E}\Big[\int_s^t \mathbb{E}\Big[|A_t - A_u|^{p-1}\Big|\mathcal{F}_u\Big] dA_u\Big|\mathcal{F}_s\Big] \\
&\leq p!|C|^{p-1}\mathbb{E}[A_t - A_s|\mathcal{F}_s] \\
&\leq p!|C|^p \ .
\end{aligned}$$

• Since the result holds true for $p = 1$ and for any $p \geq 2$ as soon as it holds for $p-1$, it holds for p, we get

$$\mathbb{E}\big[|A_t - A_s|^p\big|\mathcal{F}_s\big] \leq p!|C|^p \ ,$$

for any $p \geq 1$.

From this last inequality, we get for any $\delta \in (0, \frac{1}{C})$

$$\mathbb{E}\Big[\sum_{p\geq 0}\frac{1}{p!}|\delta|^p|A_t - A_s|^p\Big|\mathcal{F}_s\Big] \quad \leq \quad \sum_{p\geq 0}|\delta C|^p \quad = \quad \frac{1}{1-\delta C} \ ,$$

which is the expected result. □

References

[1] Arai T. (2005): "An extension of mean-variance hedging to the discontinuous case", *Finance Stoch.*, **9**, 129139.

[2] Barlow M.-T. (1978): "Study of a Filtration Expanded to Include an Honest Time", *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, **44**, 307-323.

[3] Bielecki T. and M. Rutkowski (2004): "Credit risk: modelling, valuation and hedging", Springer Finance.

[4] Bielecki T., Jeanblanc M. and M. Rutkowski (2004): "Stochastic Methods in Credit Risk Modelling", Lectures notes in Mathematics, Springer, **1856**, 27-128.

[5] Delbaen F. and W. Schachermayer (1996): "The variance-optimal martingale measure for continuous processes", *Bernoulli*, **2**, 81-105.

[6] Dellacherie C. and P.-A. Meyer (1975): "Probabilités et Potentiel - Chapitres I IV", Hermann, Paris.

[7] El Karoui N., Peng S. and M.-C. Quenez (1997): "Backward Stochastic Differential Equations in Finance", *Mathematical Finance*, 1-71.

[8] Émery M. (1979): "Équations différentielles stochastiques lipschitziennes : étude de la stabilité", Séminaire de probabilité (Strasbourg), **13**, 281-293.

[9] Gouriéroux C., Laurent J.-P. and H. Pham (1998): "Mean-variance Hedging and numéraire", *Math. Finance*, **8**, 179-200.

[10] He S., Wang J. and J. Yan (1992): "Semimartingale theory and stochastic calculus", Science Press, CRC Press, New-York.

[11] Hu Y., Imkeller P. and M. Muller (2004): "Utility maximization in incomplete markets", *Annals of Applied Probability*, **15**, 1691-1712.

[12] Jacod J. (1987): "Grosissement initial, hypothèse H' et théorème de Girsanov, Séminaire de calcul stochastique", 1982-1983, Lect. Notes in Maths, vol. 1118.

[13] Jarrow R.-A. and F. Yu (2001): "Counterparty risk and the pricing of defaultable securities", *Journal of Finance*, **56**, 1765-1799.

[14] Jeanblanc M., Mania, M., Santacroce M. and M. Schweizer (2010): "Mean-variance hedging via stochastic control and bsdes for general semimartingales", forthcoming in *Annals of Applied Probability.*

[15] Jeulin T. (1980): "Semimartingales et grossissements d'une filtration", Lect. Notes in Maths, vol. 883, Springer.

[16] Jeulin T. and M. Yor (1985): "Grossissement de filtration : exemples et applications", Lect. Notes in Maths, vol. 1118, Springer.

[17] Kazamaki N. (1979): "A sufficient condition for the uniform integrability of exponential martingales", *Math. Rep. Toyama Univ.*, **2**, 111.

[18] Kazamaki N. (1994): "Continuous martingales and BMO", Lectures Notes 1579, Springer-Verlag.

[19] Kobylanski M. (2000): "Backward stochastic differential equations and partial differential equations with quadratic growth", *The Annals of Probability*, **28**, 558-602.

[20] Kohlmann M., Xiong D. and Z. Ye (2010):"Mean-variance hedging in a general jump diffusion model", *Applied Mathematical Finance*, **17**, 29-57.

[21] Lim A.E.B (2002): "Quadratic hedging and mean-variance portfolio selection with random parameters in an incomplete market", *Math. Oper. Res.*, **29**, 132-161.

[22] Lim A.-E.-B and X.-Y. Zhou (2002): "Mean-variance portfolio selection with random parameters in a complete market", *Math Oper. Res.*, **27**, 101-120.

[23] Lim A.-E.-B (2006): "Mean-variance hedging when there are jumps", *SIAM Journal on Control and Optimization*, **44**, 1893-1922.

[24] Laurent J.-P. and H. Pham (1999): "Dynamic programming and mean-variance hedging", *Finance Stoch.*, **3**, pp. 83-110.

[25] Schweizer M. (1996): "Approximation pricing and the variance-optimal martingale measure", *Ann. Probab.*, **64**, 206-236.