Kellerer's theorem revisited

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Abstract: Kellerer's theorem asserts the existence of a Markovian martingale with given marginals, assumed to increase in the convex order. It is revisited here, in the light of previous papers by Hirsch-Roynette and by G.Lowther.
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1 Introduction

1.1

First we fix the terminology.

We say that two $\mathbb{R}$-valued processes are *associated*, if they have the same one-dimensional marginals. A process which is associated with a martingale is called a 1-*martingale*.

An $\mathbb{R}$-valued process $(X_t, t \geq 0)$ is called a *peacock* (see [HPRY] for the origin of this term and many examples) if:

i) it is *integrable*, that is:

$$\forall t \geq 0, \quad \mathbb{E}[|X_t|] < \infty;$$

ii) it *increases in the convex order*, meaning that, for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \mathbb{E}[\psi(X_t)] \in (-\infty, +\infty]$$

is increasing.

Actually, it may be noted that, in the definition of a peacock, only the family $(\mu_t, t \geq 0)$ of its one-dimensional marginals is involved. In the following, we shall also call a *peacock*, a family $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}$ such that:

i) $\forall t \geq 0, \quad \int |x| \, \mu_t(\mathrm{d}x) < \infty;$

ii) for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \, \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

Likewise, a family $(\mu_t, t \geq 0)$ of probability measures on $\mathbb{R}$ and an $\mathbb{R}$-valued process $(Y_t, t \geq 0)$ will be said to be *associated* if, for every $t \geq 0$, the law of Y_t is μ_t, i.e. if $(\mu_t, t \geq 0)$ is the family of the one-dimensional marginals of $(Y_t, t \geq 0)$.

1.2

It is an easy consequence of Jensen's inequality that an $\mathbb{R}$-valued process $(X_t, t \geq 0)$ which is a 1-martingale, is a peacock. A remarkable result due to H. Kellerer ([K], 1972) states that, conversely, any $\mathbb{R}$-valued process $(X_t, t \geq 0)$ which is a peacock, is a 1-martingale. More precisely, Kellerer's result states that any peacock admits an associated martingale which is *Markovian*.

In the recent paper [HR], a new proof of Kellerer's theorem (but without the Markov property) was presented. On the other hand, G. Lowther ([L2], 2008) showed that if $(\mu_t, t \geq 0)$ is a peacock such that the map: $t \longrightarrow \mu_t$ is weakly continuous (i.e. for any $\mathbb{R}$-valued, bounded and continuous function f on $\mathbb{R}$, the map: $t \longrightarrow \int f(x)\ \mu_t(\mathrm{d}x)$ is continuous), then $(\mu_t, t \geq 0)$ is associated with a unique *strongly Markovian* martingale which moreover is *almost-continuous* (see Section 4 for definitions). Actually, this paper [L2] partially relies on [L1] (2008) and [L3] (2009), but it seems that only [L3] was published in a journal.

1.3

In this paper, our aim is two-fold:

1. to give a proof of Kellerer's theorem (including the Markov property), following essentially [HR] and [L3];

2. to present, without proof, results of [L1, L2, L3], which complete and precise Kellerer's theorem on some points.

1.4

The remainder of this paper is organised as follows:

- Section 2 is devoted to preliminary results about *call functions* (Subsection 2.1), *Lipschitz-Markov property* (Subsection 2.2), *finite-dimensional convergence* (Subsection 2.3) and *Fokker-Planck equation* (Subsection 2.4);

- in Section 3, we prove Kellerer's theorem by a two steps approximation. We first consider the *regular case* (Subsection 3.1), then the right-continuous case (Subsection 3.2) and, finally, the general case (Subsection 3.3);

- in Section 4, we gather some related results from [L1, L2, L3], concerning notably *strong Markovianity*, *almost-continuity* and *uniqueness*.

2 Preliminary results

In this section, we fix the notation and the terminology, and we gather some preliminary results which are essential in the sequel.

2.1 Call functions

In the following, we denote by $\mathcal{M}$ the set of probability measures on $\mathbb{R}$, equipped with the topology of weak convergence (with respect to the space of $\mathbb{R}$-valued, bounded, continuous functions on $\mathbb{R}$).
We denote by $\mathcal{M}_f$ the subset of $\mathcal{M}$ consisting of measures $\mu \in \mathcal{M}$ such that $\int |x| \, \mu(\mathrm{d}x) < \infty$.
We define, for $\mu \in \mathcal{M}_f$, the *call function* C_μ by:

$$\forall x \in \mathbb{R}, \quad C_\mu(x) = \int (y - x)^+ \, \mu(\mathrm{d}y).$$

The following easy (and classical) proposition holds (see e.g. [HR, Proposition 2.1]).

Proposition 2.1 *If $\mu \in \mathcal{M}_f$, then C_μ satisfies the following properties:*

a) *C_μ is a convex, nonnegative function on $\mathbb{R}$.*

b) $\lim_{x \to +\infty} C_\mu(x) = 0$.

c) *There exists $a \in \mathbb{R}$ such that $\lim_{x \to -\infty}(C_\mu(x) + x) = a$.*

Conversely, if a function C satisfies the above three properties, then there exists a unique $\mu \in \mathcal{M}_f$ such that $C = C_\mu$. This measure μ is the second derivative, in the sense of distributions, of the function C, and $a = \mathbb{E}[\mu]$.

To state the next proposition (whose proof can be found in [HR]), we now recall that a subset $\mathcal{H}$ of $\mathcal{M}$ is said to be *uniformly integrable* if

$$\lim_{c \to +\infty} \sup_{\mu \in \mathcal{H}} \int_{\{|x| \geq c\}} |x| \, \mu(\mathrm{d}x) = 0.$$

We remark that, if $\mathcal{H}$ is uniformly integrable, then

$$\mathcal{H} \subset \mathcal{M}_f \quad \text{and} \quad \sup\{\int |x| \, \mu(\mathrm{d}x); \ \mu \in \mathcal{H}\} < \infty.$$

Proposition 2.2 *Let I be a set and let $\mathcal{E}$ be a filter on I. Consider a uniformly integrable family $(\mu_i, i \in I)$ in $\mathcal{M}$, and $\mu \in \mathcal{M}$. The following properties are equivalent:*

1) $\lim_{\mathcal{E}} \mu_i = \mu$ *with respect to the topology on $\mathcal{M}$.*

2) $\mu \in \mathcal{M}_f$ *and*

$$\forall x \in \mathbb{R}, \quad \lim_{\mathcal{E}} C_{\mu_i}(x) = C_\mu(x).$$

We now fix a family $(\mu_t, t \geq 0)$ in $\mathcal{M}_f$ and we define a function $C(t,x)$ on $\mathbb{R}_+ \times \mathbb{R}$ by:

$$C(t,x) = C_{\mu_t}(x).$$

We recall (see Subsection 1.1) that the family $(\mu_t, t \geq 0)$ is called a *peacock*, if

i) $\forall t \geq 0, \quad \int |x| \, \mu_t(\mathrm{d}x) < \infty,$

ii) for every convex function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, the map:

$$t \geq 0 \longrightarrow \int \psi(x) \, \mu_t(\mathrm{d}x) \in (-\infty, +\infty]$$

is increasing.

The following characterization is easy to prove and is stated in [HPRY, Exercise 1.7].

Proposition 2.3 *The family* $(\mu_t, t \geq 0)$ *is a peacock if and only if:*

1. *the expectation* $\mathbb{E}[\mu_t]$ *does not depend on* t*,*
2. *for every* $x \in \mathbb{R}$*, the function* $t \geq 0 \longrightarrow C(t,x)$ *is increasing.*

We also have (see [HPRY, Exercise 1.1]):

Proposition 2.4 *Assume that* $(\mu_t, t \geq 0)$ *is a peacock, and let* $T > 0$*. Then, the set* $\{\mu_t;\ 0 \leq t \leq T\}$ *is uniformly integrable.*

2.2 Lipschitz-Markov property

Following [L3] (see also [K, Definition 3]), we now introduce a property, namely the Lipschitz-Markov property, which is stronger than the mere Markov property. (Note that in [L3], the Lipschitz-Markov property is simply called *Lipschitz property.*)
If $(X_t, t \geq 0)$ is an $\mathbb{R}$-valued process, we denote by $\mathcal{F}^X$ the filtration generated by X, that is:

$$\forall t \geq 0, \quad \mathcal{F}^X_t = \sigma(X_s;\ s \leq t).$$

On the other hand, for any Lipschitz continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$, we denote by $L(f)$ its Lipschitz constant.

Definition 2.1 ([L3], Definition 4.1) Let X be an $\mathbb{R}$-valued stochastic process. Then X is said to satisfy the *Lipschitz-Markov property* if, for all $0 \leq s < t$ and every bounded Lipschitz continuous $g : \mathbb{R} \longrightarrow \mathbb{R}$ with $L(g) \leq 1$, there exists a Lipschitz continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $L(f) \leq 1$ and

$$f(X_s) = \mathbb{E}[g(X_t)|\mathcal{F}_s^X].$$

The following proposition presents an important example of process satisfying the Lipschitz-Markov property (see also Proposition 4.2 below, based on [L3, Lemma 4.3]). In the sequel, we adopt the following notation: $U = (0, +\infty) \times \mathbb{R}$ and $\overline{U} = \mathbb{R}_+ \times \mathbb{R}$.

Proposition 2.5 *Let $\sigma : (t, x) \in \overline{U} \longrightarrow \sigma(t, x) \in \mathbb{R}$ be a continuous function on $\overline{U}$ such that the derivative σ'_x exists and is continuous on $\overline{U}$. Let X_0 be an integrable random variable and let $(B_t, t \geq 0)$ denote a standard Brownian motion independent of X_0. Then, the stochastic differential equation*

$$X_t = X_0 + \int_0^t \sigma(s, X_s)\, \mathrm{d}B_s \tag{1}$$

admits a unique strong solution which satisfies the Lipschitz-Markov property.

Proof It is classical that equation (1) admits a unique (non-exploding) strong solution.

Let $s \geq 0$. For any $x \in \mathbb{R}$, we denote by $(X_t^{s,x}, t \geq s)$ the strong solution (for $t \geq s$) of

$$X_t^{s,x} = x + \int_s^t \sigma(u, X_u^{s,x})\, \mathrm{d}B_u.$$

We also denote by $(U_t^{s,x}, t \geq s)$ the process defined by:

$$U_t^{s,x} = \frac{\partial}{\partial x} X_t^{s,x}.$$

We obtain easily:

$$U_t^{s,x} = \exp\left[\int_s^t \sigma'_x(u, X_u^{s,x})\, \mathrm{d}B_u - \frac{1}{2}\int_s^t \sigma'^2_x(u, X_u^{s,x})\, \mathrm{d}u\right].$$

In particular, $(U_t^{s,x}, t \geq s)$ is a positive local martingale and hence:

$$U_t^{s,x} \geq 0 \quad \text{and} \quad \mathbb{E}[U_t^{s,x}] \leq 1.$$

Let now g be a bounded Lipschitz continuous function of C^1-class with $|g'| \leq 1$ and $0 \leq s < t$. We define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$f(x) = \mathbb{E}\left[g\left(X_t^{s,x}\right)\right].$$

Then, f is a bounded C^1-function and

$$|f'(x)| = |\mathbb{E}\left[g'\left(X_t^{s,x}\right) U_t^{s,x}\right]| \leq 1.$$

It is now clear that, if $(X_t, t \geq 0)$ is solution to (1), then

$$\mathbb{E}[g(X_t)|\mathcal{F}_s^X] = \mathbb{E}\left[g\left(X_t^{s,x}\right)\right]|_{x=X_s} = f(X_s).$$

The Lipschitz-Markov property follows easily from what precedes. □

2.3 Finite-dimensional convergence

Definition 2.2 Let I be a set and let $\mathcal{E}$ be a filter on I. We consider $\mathbb{R}$-valued stochastic processes $(X^{(i)})_{i\in I}$ and X possibly defined on different probability spaces. Then we shall say that $(X^{(i)})_{i\in I}$ converges (with respect to $\mathcal{E}$) to X in the sense of finite-dimensional distributions if, for any finite subset $\{t_1, t_2, \cdots, t_n\}$ of $\mathbb{R}_+$, the distributions of $(X_{t_1}^{(i)}, X_{t_2}^{(i)}, \cdots, X_{t_n}^{(i)})$ converge weakly to the distribution of $(X_{t_1}, X_{t_2}, \cdots, X_{t_n})$, which means that, for any bounded continuous $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\lim_{\mathcal{E}} \mathbb{E}[f(X_{t_1}^{(i)}, X_{t_2}^{(i)}, \cdots, X_{t_n}^{(i)})] = \mathbb{E}[f(X_{t_1}, X_{t_2}, \cdots, X_{t_n})].$$

We also shall write:

$$\text{f.d.}\lim_{\mathcal{E}} X^{(i)} = X$$

and we shall say, in short, that $X^{(i)}$ f.d. converges (with respect to $\mathcal{E}$) to X.

The finite-dimensional convergence has important stability properties, in particular with respect to the Lipschitz-Markov property and to the martingale property. This is stated in the next propositions where we consider, like in Definition 2.2, a set I, a filter $\mathcal{E}$ on I and $\mathbb{R}$-valued stochastic processes $(X^{(i)})_{i\in I}$ and X. The following proposition extends [K, Satz 10].

Proposition 2.6 ([L3], Lemma 4.5) *Suppose that, for every $i \in I$, $X^{(i)}$ satisfies the Lipschitz-Markov property, and that $X^{(i)}$ f.d. converges to X. Then, X satisfies the Lipschitz-Markov property.*

Proof Let $0 \leq s < t$ and a bounded Lipschitz continuous $g : \mathbb{R} \longrightarrow \mathbb{R}$ with $L(g) \leq 1$. For any $i \in I$, there exists a Lipschitz continuous $f^{(i)} : \mathbb{R} \longrightarrow \mathbb{R}$ with $L(f^{(i)}) \leq 1$ and

$$f^{(i)}(X_s^{(i)}) = \mathbb{E}[g(X_t^{(i)})|\mathcal{F}_s^{X^{(i)}}].$$

Moreover, we may suppose that:

$$\forall i \in I, \quad \sup_{x\in\mathbb{R}} |f^{(i)}(x)| \leq \sup_{x\in\mathbb{R}} |g(x)|.$$

Consider an ultrafilter $\mathcal{U}$ on I which refines $\mathcal{E}$. By Ascoli's theorem, there exists a Lipschitz continuous $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $L(f) \leq 1$ and

$$\lim_{\mathcal{U}} f^{(i)} = f \quad \text{uniformly on compact sets.}$$

For every $n \in \mathbb{N}$, $0 \leq s_1 < s_2 < \cdots < s_n \leq s$, for any bounded continuous $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ and for any continuous $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support, one has, for every $i \in I$,

$$\mathbb{E}[g(X_t^{(i)})\, \theta(X_s^{(i)})\, h(X_{s_1}^{(i)}, X_{s_2}^{(i)}, \cdots, X_{s_n}^{(i)})]$$

$$= \mathbb{E}[f^{(i)}(X_s^{(i)})\, \theta(X_s^{(i)})\, h(X_{s_1}^{(i)}, X_{s_2}^{(i)}, \cdots, X_{s_n}^{(i)})].$$

Since

$$\lim_{\mathcal{U}} f^{(i)}\theta = f\,\theta \quad \text{uniformly}$$

and

$$\text{f.d.}\lim_{\mathcal{U}} X^{(i)} = X,$$

we obtain:

$$\mathbb{E}[g(X_t)\, \theta(X_s)\, h(X_{s_1}, X_{s_2}, \cdots, X_{s_n})] = \mathbb{E}[f(X_s)\, \theta(X_s)\, h(X_{s_1}, X_{s_2}, \cdots, X_{s_n})].$$

This holding for any θ with compact support, we also have:

$$\mathbb{E}[g(X_t)\, h(X_{s_1}, X_{s_2}, \cdots, X_{s_n})] = \mathbb{E}[f(X_s)\, h(X_{s_1}, X_{s_2}, \cdots, X_{s_n})],$$

which yields the desired result. □

Proposition 2.7 *Suppose that, for every $i \in I$, $X^{(i)}$ is a martingale, and that $X^{(i)}$ f.d. converges to X. Suppose moreover that, for every $t \geq 0$,*

$$\left\{X_t^{(i)};\ i \in I\right\} \quad \textit{is uniformly integrable.}$$

Then, X is a martingale.

Proof By Proposition 2.2, the process X is integrable. We now prove that it is a martingale. We set:

$$\forall p > 0, \ \ \forall x \in \mathbb{R}, \quad \varphi_p(x) = \min[\max(x, -p), p].$$

Then, φ_p is a bounded continuous function, and

$$|\varphi_p(x) - x| \leq |x| \, 1_{\{|x|>p\}}.$$

For every $n \in \mathbb{N}$, $0 \leq s_1 < s_2 < \cdots < s_n \leq s \leq t$, for any bounded continuous $h : \mathbb{R}^n \longrightarrow \mathbb{R}$, we have for every $i \in I$,

$$\mathbb{E}[h(X^{(i)}_{s_1}, \cdots, X^{(i)}_{s_n}) \, X^{(i)}_t] = \mathbb{E}[h(X^{(i)}_{s_1}, \cdots, X^{(i)}_{s_n}) \, X^{(i)}_s].$$

We set: $\| \, h \, \|_\infty = \sup\{|h(x)|; \ x \in \mathbb{R}^n\}$. Then,

$$\begin{aligned} & |\mathbb{E}[h(X_{s_1}, \cdots, X_{s_n}) \, \varphi_p(X_t)] - \mathbb{E}[h(X_{s_1}, \cdots, X_{s_n}) \, X_t]| \\ \leq \ & \| \, h \, \|_\infty \, \mathbb{E}\left[|X_t| \, 1_{\{|X_t|>p\}}\right], \ \text{ for every } p > 0, \end{aligned}$$

$$\begin{aligned} & \left|\mathbb{E}[h(X^{(i)}_{s_1}, \cdots, X^{(i)}_{s_n}) \, \varphi_p(X^{(i)}_t)] - \mathbb{E}[h(X^{(i)}_{s_1}, \cdots, X^{(i)}_{s_n}) \, X^{(i)}_t]\right| \\ \leq \ & \| \, h \, \|_\infty \, \mathbb{E}\left[|X^{(i)}_t| \, 1_{\{|X^{(i)}_t|>p\}}\right], \ \text{ for every } i \in I \text{ and every } p > 0, \end{aligned}$$

and likewise, replacing t by s. Moreover,

$$\lim_{\mathcal{E}} \mathbb{E}[h(X^{(i)}_{s_1}, \cdots, X^{(i)}_{s_n}) \, \varphi_p(X^{(i)}_t)] = \mathbb{E}[h(X_{s_1}, \cdots, X_{s_n}) \, \varphi_p(X_t)],$$

and likewise, replacing t by s. Finally, we obtain, for $p > 0$,

$$\begin{aligned} & |\mathbb{E}[h(X_{s_1}, \cdots, X_{s_n}) \, X_t] - \mathbb{E}[h(X_{s_1}, \cdots, X_{s_n}) \, X_s]| \\ \leq \ & 2 \, \| \, h \, \|_\infty \left(\sup_{i \in I} \mathbb{E}\left[|X^{(i)}_t| \, 1_{\{|X^{(i)}_t|>p\}}\right] + \sup_{i \in I} \mathbb{E}\left[|X^{(i)}_s| \, 1_{\{|X^{(i)}_s|>p\}}\right] \right), \end{aligned}$$

and the desired result follows, letting p go to ∞. □

2.4 Fokker-Planck equation

We now state M. Pierre's uniqueness theorem for a Fokker-Planck equation, which plays an important role in our proof of Kellerer's theorem in the regular case. This theorem is stated and proved in Subsection 6.1 of [HPRY].

Theorem 2.1 ([HPRY], Theorem 6.1) *Let $a : (t,x) \in \overline{U} \longrightarrow a(t,x) \in \mathbb{R}_+$ be a continuous function such that $a(t,x) > 0$ for $(t,x) \in U$, and let $\mu \in \mathcal{M}_f$. Then there exists at most one family of probability measures $(p(t,\mathrm{d}x), t \geq 0)$ such that*

$(FP1)$ *$t \geq 0 \longrightarrow p(t,\mathrm{d}x)$ is weakly continuous,*

$(FP2)$ *$p(0,\mathrm{d}x) = \mu(\mathrm{d}x)$ and*

$$\frac{\partial p}{\partial t} - \frac{\partial^2}{\partial x^2}(a\,p) = 0 \quad \textit{in } \mathcal{D}'(U)$$

(i.e. in the sense of Schwartz distributions in the open set U).

3 Kellerer's theorem

In this section, we shall give a proof of Kellerer's theorem. Following [HR], we shall proceed by a two steps approximation, starting with the regular case.

3.1 The regular case

Theorem 3.1 *Let $(X_t, t \geq 0)$ be an $\mathbb{R}$-valued integrable process such that $\mathbb{E}[X_t]$ is independent of t, and let $C : \overline{U} \longrightarrow \mathbb{R}_+$ the corresponding call function (see Subsection 2.1):*

$$C(t,x) = \mathbb{E}[(X_t - x)^+].$$

We assume:

i) C is a C^∞-function on $\overline{U}$.
We set:

$$\forall (t,x) \in \overline{U}, \quad p(t,x) = \frac{\partial^2 C}{\partial x^2}(t,x).$$

Thus, the law of X_t is $p(t,x)\mathrm{d}x$.

ii) $p > 0$ on $\overline{U}$ and $\dfrac{\partial C}{\partial t} > 0$ on U.

We set:

$$\forall (t,x) \in \overline{U}, \quad \sigma(t,x) = \left(2\,\frac{\frac{\partial C}{\partial t}(t,x)}{p(t,x)} \right)^{1/2}.$$

Then, the stochastic differential equation

$$Z_t = Z_0 + \int_0^t \sigma(s, Z_s)\, \mathrm{d}B_s$$

(where Z_0 is a random variable with law $p(0,x)\mathrm{d}x$, independent of the Brownian motion $(B_s, s \geq 0)$) admits a unique strong solution $(Y_t, t \geq 0)$, which is a martingale associated to X and satisfying the Lipschitz-Markov property.

Proof

1) We first prove that Y is associated to X. Set:

$$a = \frac{1}{2}\,\sigma^2 = \frac{\frac{\partial C}{\partial t}}{p}.$$

We have;

$$\frac{\partial^2}{\partial x^2}(a\,p) = \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial t}C = \frac{\partial}{\partial t}p$$

on $\overline{U}$. In particular, the family $(p(t,x)\mathrm{d}x, t \geq 0)$ satisfies (FP1) and (FP2) in Theorem 2.1. On the other hand, for any C^2-function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support, we have by Itô's formula:

$$\mathbb{E}[\varphi(Y_t)] = \mathbb{E}[\varphi(Y_0)] + \int_0^t \mathbb{E}[\varphi''(Y_s)\, a(s, Y_s)]\, \mathrm{d}s$$

and therefore, denoting, for every $t \geq 0$, by $q(t, \mathrm{d}x)$ the law of Y_t,

$$\frac{d}{dt}\int \varphi(x)\; q(t, \mathrm{d}x) = \int \varphi''(x)\, a(t,x)\; q(t, \mathrm{d}x).$$

Thus, the family $(q(t, \mathrm{d}x), t \geq 0)$ also satisfies properties (FP1) and (FP2). Hence, by Theorem 2.1,

$$\forall t \geq 0, \quad q(t, \mathrm{d}x) = p(t,x)\mathrm{d}x$$

and Y is associated to X.

2) We now prove that Y is a martingale. Let ϕ be a C^2-function on $\mathbb{R}$ such that $\phi(x) = 1$ for $|x| \leq 1$, $\phi(x) = 0$ for $|x| \geq 2$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}$. We set, for $k > 0$, $\phi_k(x) = x\,\phi(k^{-1}\,x)$. Fix now $0 \leq s_1 \leq \cdots \leq s_n \leq s \leq t$ and a bounded continuous $h : \mathbb{R}^n \longrightarrow \mathbb{R}$. We set:

$$\theta_k = \mathbb{E}[h(Y_{s_1}, Y_{s_2}, \cdots, Y_{s_n})\, \phi_k(Y_t)] - \mathbb{E}[h(Y_{s_1}, Y_{s_2}, \cdots, Y_{s_n})\, \phi_k(Y_s)]$$

and $m = \sup_{x \in \mathbb{R}^n} |h(x)|$. By dominated convergence,

$$\lim_{k \to \infty} \theta_k = \mathbb{E}[h(Y_{s_1}, Y_{s_2}, \cdots, Y_{s_n})\, Y_t] - \mathbb{E}[h(Y_{s_1}, Y_{s_2}, \cdots, Y_{s_n})\, Y_s)].$$

On the other hand, since Y is associated to X, Itô's formula yields:

$$|\theta_k| \leq \frac{m}{2} \int_s^t \mathbb{E}\left[|\phi_k''(Y_u)|\, \sigma^2(u, Y_u)\right]\, \mathrm{d}u = m \int \int_s^t |\phi_k''(x)|\, \frac{\partial C}{\partial u}(u, x)\, \mathrm{d}u\, \mathrm{d}x.$$

Besides,

$$\int |\phi_k''(x)|\, \mathrm{d}x = \int |x\, \phi''(x) + 2\phi'(x)|\, \mathrm{d}x$$

and $\phi_k''(x) = 0$ for $|x| \notin [k, 2k]$. Therefore, there exists a constant $\tilde{m}$ such that:

$$|\theta_k| \leq \tilde{m}\, \sup\{C(t, y) - C(s, y);\ k \leq |y| \leq 2k\}.$$

Thus, since by hypothesis $\mathbb{E}[X_s] = \mathbb{E}[X_t]$, Proposition 2.1 entails: $\lim_{k \to \infty} \theta_k = 0$, which yields the desired result.

3) Finally, by Proposition 2.5, Y satisfies the Lipschitz-Markov property.

□

Remark By now, the formula giving σ in terms of the derivatives of C, in the statement of Theorem 3.1, is common in Mathematical Finance, where it is referred to Dupire [D] or Derman-Kani [DK].

3.2 The right-continuous case

We now present our proof of Kellerer's theorem for right-continuous peacoks.

Theorem 3.2 *Let $(\mu_t, t \geq 0)$ be a peacock such that the map:*

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is right-continuous. Then there exists a càdlàg martingale associated to $(\mu_t, t \geq 0)$ and satisfying the Lipschitz-Markov property.

Proof We set, as in Subsection 2.1, $C(t, x) = C_{\mu_t}(x)$. We shall regularize, in space and time, $p(t, \mathrm{d}x) := \mu_t(\mathrm{d}x)$ considered as a distribution on U. Thus,

let α be a density of probability on $\mathbb{R}$, of C^∞-class, with compact support contained in $[0,1]$. We set, for $\varepsilon \in (0,1)$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p_\varepsilon(t,x) = \frac{1-\varepsilon}{\varepsilon} \int \alpha(u) \left[\int \alpha\left(\frac{y-x}{\varepsilon}\right) \, \mu_{t+\varepsilon u}(\mathrm{d}y) \right] \, \mathrm{d}u + \varepsilon \, g(t,x)$$

with

$$g(t,x) = \frac{1}{\sqrt{2\,\pi\,(1+t)}} \, \exp\left(-\frac{x^2}{2\,(1+t)} \right).$$

Lemma 3.1 *The function p_ε is of C^∞-class on $\mathbb{R}_+ \times \mathbb{R}$ and $p_\varepsilon(t,x) > 0$ for any (t,x). Moreover,*

$$\int p_\varepsilon(t,x) \, \mathrm{d}x = 1 \quad \textit{and} \quad \int |x| \, p_\varepsilon(t,x) \, \mathrm{d}x < \infty.$$

The proof is straightforward.
We now set:

$$\mu_t^\varepsilon(\mathrm{d}x) = p_\varepsilon(t,x) \, \mathrm{d}x.$$

By Lemma 3.1, $\mu_t^\varepsilon \in \mathcal{M}_f$ and we set:

$$C_\varepsilon(t,x) = C_{\mu_t^\varepsilon}(x).$$

Lemma 3.2 *For any $t \geq 0$, the set $\{\mu_t^\varepsilon;\ 0 < \varepsilon < 1\}$ is uniformly integrable.*

Proof Let $a = \int y \, \alpha(y) \, \mathrm{d}y$. A simple computation yields:

$$\begin{aligned} \int_{\{|x| \geq c\}} |x| \, \mu_t^\varepsilon(\mathrm{d}x) \;\; \leq \;\; & \int \alpha(u) \left[\int_{\{|y| \geq c-1\}} (|y| + a) \, \mu_{t+\varepsilon u}(\mathrm{d}y) \right] \, \mathrm{d}u \\ & + \int_{\{|x| \geq c\}} |x| \, g(t,x) \, \mathrm{d}x \end{aligned}$$

and the result follows from the uniform integrability of $\{\mu_v;\ 0 \leq v \leq t+1\}$ (Proposition 2.4). □

Lemma 3.3 *One has:*

$$\begin{aligned} C_\varepsilon(t,x) \;\; = \;\; & (1-\varepsilon) \int \int \alpha(u) \, \alpha(y) \, C(t+\varepsilon u, x+\varepsilon y) \, \mathrm{d}y \, \mathrm{d}u \\ & + \varepsilon \int_x^{+\infty} (y-x) \, g(t,y) \, \mathrm{d}y. \end{aligned}$$

The function C_ε is of C^∞-class on $\mathbb{R}_+\times\mathbb{R}$. Moreover, for any $(t,x)\in\mathbb{R}_+\times\mathbb{R}$,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x)>0 \quad \textit{and} \quad \frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x)=p_\varepsilon(t,x).$$

Proof The above expression of C_ε follows directly from the definitions. We deduce therefrom that C_ε is of C^∞-class on $\mathbb{R}_+\times\mathbb{R}$. Now, by property 2 in Proposition 2.3,

$$\frac{\partial C_\varepsilon}{\partial t}(t,x)\geq\varepsilon\,\frac{\partial}{\partial t}\left[\int_x^{+\infty}(y-x)\,g(t,y)\,\mathrm{d}y\right]=\frac{\varepsilon}{2}\,g(t,x)>0.$$

Finally, the equality:

$$\frac{\partial^2 C_\varepsilon}{\partial x^2}(t,x)=p_\varepsilon(t,x)$$

holds, since, by Proposition 2.1, it holds in the sense of distributions, and both sides are continuous. □

The following lemma is an easy consequence of the right-continuity of $(\mu_t, t\geq 0)$.

Lemma 3.4 *For $t\geq 0$,*

$$\lim_{\varepsilon\to 0}\mu_t^\varepsilon=\mu_t \quad \textit{in } \mathcal{M}.$$

By Theorem 3.1, there exists a martingale $(M_t^\varepsilon, t\geq 0)$ satisfying the Lipschitz-Markov property, which is associated to $(\mu_t^\varepsilon, t\geq 0)$. For every $n\in\mathbb{N}$ and $\tau_n=(t_1,\cdots,t_n)\in\mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(\varepsilon,n)}$ the law of $(M_{t_1}^\varepsilon,\cdots,M_{t_n}^\varepsilon)$, a probability on $\mathbb{R}^n$.

Lemma 3.5 *For every $n\in\mathbb{N}$ and $\tau_n\in\mathbb{R}_+^n$, the set of probability measures: $\{\mu_{\tau_n}^{(\varepsilon,n)};\ 0<\varepsilon<1\}$, is tight.*

Proof Let $n\in\mathbb{N}$ and $\tau_n=(t_1,\cdots,t_n)\in\mathbb{R}_+^n$. For $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$, we set $|x|:=\sup_{1\leq j\leq n}|x_j|$. Then, for $c>0$,

$$\begin{aligned}\mu_{\tau_n}^{(\varepsilon,n)}(|x|\geq c) &= \mathbb{P}\left(\sup_{1\leq j\leq n}|M_{t_j}^\varepsilon|\geq c\right)\leq\frac{1}{c}\,\mathbb{E}\left[\sup_{1\leq j\leq n}|M_{t_j}^\varepsilon|\right]\\ &\leq \frac{1}{c}\sum_{j=1}^n\mathbb{E}\left[|M_{t_j}^\varepsilon|\right]=\frac{1}{c}\sum_{j=1}^n\int|x|\;\mu_{t_j}^\varepsilon(\mathrm{d}x).\end{aligned}$$

Now, by Lemma 3.2, for $1 \leq j \leq n$,

$$\sup_{0<\varepsilon<1} \int |x| \, \mu^{\varepsilon}_{t_j}(\mathrm{d}x) < \infty.$$

Thus,

$$\lim_{c \to +\infty} \sup_{0<\varepsilon<1} \mu^{(\varepsilon,n)}_{\tau_n}(|x| \geq c) = 0,$$

which yields the tightness of $\{\mu^{(\varepsilon,n)}_{\tau_n};\ 0 < \varepsilon < 1\}$. □

Let $\mathcal{U}$ be an ultrafilter on $(0,1)$ such that $\lim_{\mathcal{U}} \varepsilon = 0$. As a consequence of the previous lemma, the family of probabilities on $\mathbb{R}^n$: $(\mu^{(\varepsilon,n)}_{\tau_n}, \varepsilon > 0)$, weakly converges (with respect to $\mathcal{U}$) to a probability which we denote by $\mu^{(n)}_{\tau_n}$. We remark that, by Lemma 3.4, for any $t \geq 0$, $\mu^{(1)}_{(t)} = \mu_t$. There exists a process $(M_t, t \geq 0)$ such that, for every $n \in \mathbb{N}$ and every $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}^n_+$, the law of $(M_{t_1}, \cdots, M_{t_n})$ is $\mu^{(n)}_{\tau_n}$. Since for any $t \geq 0$, $\mu^{(1)}_{(t)} = \mu_t$, the process $M = (M_t, t \geq 0)$ is associated to $(\mu_t, t \geq 0)$. Moreover, by Proposition 2.6, M satisfies the Lipschitz-Markov property, and by Lemma 3.2 and Proposition 2.7, M is a martingale. By the classical theory of martingales (see, for example, [DM]), almost surely, for every $t \geq 0$,

$$\widetilde{M_t} = \lim_{s \to t, s \in \mathbb{Q}, s > t} M_s$$

is well defined, and $(\widetilde{M_t}, t \geq 0)$ is a right-continuous martingale which, obviously, is associated to $(\mu_t, t \geq 0)$. Besides, it is easy to see that $\widetilde{M}$ still satisfies the Lipschitz-Markov property. Modifying $\widetilde{M}$ on a negligible set, we may assume that $\widetilde{M}$ is a càdlàg process and, moreover,

$$\forall t \in \mathbb{Q}_+, \quad \widetilde{M_t} = M_t \ \text{ a.s.}$$

□

3.3 The general case

We now obtain, by approximation, a proof of Kellerer's theorem in the general case.

Theorem 3.3 *Let $(\mu_t, t \geq 0)$ be a peacock. Then there exists a martingale associated to $(\mu_t, t \geq 0)$ and satisfying the Lipschitz-Markov property.*

Proof We consider a peacock $(\mu_t, t \geq 0)$ and we set $C(t,x) = C_{\mu_t}(x)$.

Lemma 3.6 *There exists a countable set $D \subset \mathbb{R}_+$ such that the map:*

$$t \longrightarrow \mu_t \in \mathcal{M}$$

is continuous at any $s \notin D$.

Proof By property 2 in Proposition 2.3, there exists a countable set $D \subset \mathbb{R}_+$ such that, for every $x \in \mathbb{Q}$, the map:

$$t \longrightarrow C(t,x)$$

is continuous at any $s \notin D$. Since

$$\forall x, y, t, \quad |C(t,y) - C(t,x)| \leq |y - x|,$$

this continuity property holds for every $x \in \mathbb{R}$. It suffices then to apply Proposition 2.2, taking into account Proposition 2.4. □

We may write $D = \{d_n;\ n \in \mathbb{N}\}$. For $p \in \mathbb{N}$, we denote by $(k_n^{(p)}, n \geq 0)$ the increasing rearrangement of the set:

$$\{k\, 2^{-p};\ k \in \mathbb{N}\} \cup \{d_j;\ 0 \leq j \leq p\}.$$

We define $(\mu_t^{(p)}, t \geq 0)$ by:

$$\mu_t^{(p)} = \frac{k_{n+1}^{(p)} - t}{k_{n+1}^{(p)} - k_n^{(p)}}\, \mu_{k_n^{(p)}} + \frac{t - k_n^{(p)}}{k_{n+1}^{(p)} - k_n^{(p)}}\, \mu_{k_{n+1}^{(p)}} \qquad \text{if } t \in [k_n^{(p)}, k_{n+1}^{(p)}].$$

Lemma 3.7 *The following properties hold:*

i) $(\mu_t^{(p)}, t \geq 0)$ *is a peacock and the map:* $t \longrightarrow \mu_t^{(p)} \in \mathcal{M}$ *is continuous.*

ii) For any $t \geq 0$, the set $\{\mu_t^{(p)};\ p \in \mathbb{N}\}$ is uniformly integrable.

iii) For $t \geq 0$, $\lim_{p\to\infty} \mu_t^{(p)} = \mu_t$ in $\mathcal{M}$.

Proof Properties i) and iii) are clear by construction. Property ii) follows directly from Proposition 2.4. □

By Theorem 3.2, there exists, for each p, a martingale $(M_t^{(p)}, t \geq 0)$ which is associated to $(\mu_t^{(p)}, t \geq 0)$ and satisfies the Lipschitz-Markov property. For any $n \in \mathbb{N}$ and $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, we denote by $\mu_{\tau_n}^{(p,n)}$ the law of $(M_{t_1}^{(p)}, \cdots, M_{t_n}^{(p)})$, a probability measure on $\mathbb{R}^n$. The proof of the following lemma is quite similar to that of Lemma 3.5, hence we omit this proof.

Lemma 3.8 *For every $n \in \mathbb{N}$ and $\tau_n \in \mathbb{R}_+^n$, the set of probability measures $\{\mu_{\tau_n}^{(p,n)};\ p \geq 0\}$, is tight.*

Let now $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, which refines Fréchet's filter. As a consequence of the previous lemma, for every $n \in \mathbb{N}$ and every $\tau_n \in \mathbb{R}_+^n$, $\lim_{\mathcal{U}} \mu_{\tau_n}^{(p,n)}$ exists in $\mathcal{M}$ and we denote this limit by $\mu_{\tau_n}^{(\infty,n)}$. By property iii) in Lemma 3.7, $\mu_{(t)}^{(\infty,1)} = \mu_t$. There exists a process $M = (M_t, t \geq 0)$ such that, for every $n \in \mathbb{N}$ and every $\tau_n = (t_1, \cdots, t_n) \in \mathbb{R}_+^n$, the law of $(M_{t_1}, \cdots, M_{t_n})$ is $\mu_{\tau_n}^{(\infty,n)}$. In particular, this process $(M_t, t \geq 0)$ is associated to $(\mu_t, t \geq 0)$. Moreover, by Proposition 2.6, M satisfies the Lipschitz-Markov property, and by property ii) in Lemma 3.7 and Proposition 2.7, M is a martingale. □

4 Related results

The following definition was first introduced by Lowther in [L1], and also plays a central role in [L2, L3].

Definition 4.1 Let $X = (X_t, t \geq 0)$ be an $\mathbb{R}$-valued stochastic process. Then:

1. X is *strong Markov* if for every bounded, measurable $g : \mathbb{R} \longrightarrow \mathbb{R}$ and every $t \geq 0$, there exists a measurable $f : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(\tau, X_\tau) = \mathbb{E}[g(X_{\tau+t}|\mathcal{F}_\tau]$$

   for every stopping time τ.

2. X is *almost-continuous* if it is càdlàg, continuous in probability, and given any two independent, identically distributed càdlàg processes Y, Z with the same distribution as X and for every $0 \leq s < t$ we have

$$\mathbb{P}(Y_s < Z_s\ ,\ Y_t > Z_t \text{ and } Y_u \neq Z_u \text{ for every } u \in (s,t)) = 0.$$

3. X is an *almost-continuous diffusion* (abbreviated to ACD) if it is strong Markov and almost-continuous.

The main result of [L2] is the following theorem:

Theorem 4.1 ([L2], Theorem 1.3) *Let $(\mu_t, t \geq 0)$ be a peacock such that the map:*

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is continuous. Then there exists an ACD martingale which is associated to $(\mu_t, t \geq 0)$, and it is unique in law.

On the other hand, the Lipschitz-Markov property entails the strong Markovianity:

Proposition 4.1 ([L3], Lemma 4.2) *Let X be a càdlàg, $\mathbb{R}$-valued process which satisfies the Lipschitz-Markov property. Then it is strong Markov.*

As a consequence, the martingale appearing in the statement of Theorem 3.2 is strong Markov. Note that a kind of converse of Proposition 4.1 holds (see also [L1, Theorem 1.5]):

Proposition 4.2 ([L3], Lemma 4.3) *If X is an ACD local martingale, then X satisfies the Lipschitz-Markov property.*

The following result concerns the stability with respect to the f.d. convergence.

Theorem 4.2 ([L3], Theorem 1.2) *If $(M^{(i)})_{i \in I}$ is a family of ACD martingales which f.d. converges on a dense subset of $\mathbb{R}_+$ to a process X which is càdlàg and continuous in probability, then X is an ACD.*

It follows from what precedes that if $(\mu_t, t \geq 0)$ is a peacock such that the map:

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is continuous, the martingale $\widetilde{M}$ "constructed" in the proof of Theorem 3.2 is the *unique* ACD martingale associated to $(\mu_t, t \geq 0)$ (and, in particular, it is independent of the ultrafilter $\mathcal{U}$ and of the regularization process). About the continuity of $\widetilde{M}$, we may use the following remarkable result, the proof of which relies on [L1].

Theorem 4.3 ([L3], Lemma 1.4) *Let X be an almost-continuous process. If the support of the law of X_t is connected for every t in $\mathbb{R}_+$ outside of a countable set, then X is continuous.*

Finally, we obtain from what precedes:

Theorem 4.4 *Suppose that $(\mu_t, t \geq 0)$ is a peacock such that the map:*

$$t \geq 0 \longrightarrow \mu_t \in \mathcal{M}$$

is continuous, and such that the support of μ_t is connected for every t in $\mathbb{R}_+$ outside of a countable set. Then there exists one and only one continuous, strongly Markov martingale which is associated to $(\mu_t, t \geq 0)$.

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