Counterparty Risk and Funding: The Four Wings of the TVA

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Abstract

The credit crisis and the ongoing European sovereign debt crisis have highlighted the native form of credit risk, namely the counterparty risk. The related Credit Valuation Adjustment (CVA), Debt Valuation Adjustment (DVA), Liquidity Valuation Adjustment (LVA) and Replacement Cost (RC) issues, jointly referred to in this paper as Total Valuation Adjustment (TVA), have been thoroughly investigated in the theoretical papers Crépey (2012a, 2012b). The present work provides an executive summary and numerical companion to these papers, through which the TVA pricing problem can be reduced to Markovian pre-default TVA BSDEs. The first step consists in the counterparty clean valuation of a portfolio of contracts, which is the valuation in a hypothetical situation where the two parties would be risk-free and funded at a risk-free rate. In the second step, the TVA is obtained as the value of an option on the counterparty clean value process called Contingent Credit Default Swap (CCDS). Numerical results are presented for interest rate swaps in the Vasicek, as well as in the inverse Gaussian Hull-White short rate model, also allowing one to assess the related model risk issue.

Keywords: Counterparty risk, Credit valuation adjustment (CVA), Debt valuation adjustment (DVA), Liquidity valuation adjustment (LVA), Replacement cost (RC), Backward stochastic differential equation (BSDE), Lévy process, Interest rate swap.

1 Introduction

The credit crisis and the ongoing European sovereign debt crisis have highlighted the native form of credit risk, namely the counterparty risk. This is the risk of non-payment of promised cash-flows due to the default of the counterparty in a bilateral OTC derivative transaction. The basic counterparty risk mitigation tool is a Credit Support Annex (CSA) specifying a valuation scheme of a portfolio of contracts at the default time of a party. In particular, the

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netting rules which will be applied to the portfolio are specified, as well as a collateralization scheme similar to margin accounts in futures contracts.

By extension counterparty risk is also the volatility of the price of this risk, this price being known as Credit Valuation Adjustment (CVA), see Prisco and Rosen (2005), Brigo, Morini, and Pallavicini (2013), Bielecki, Brigo, Crépey, and Herbertsson (2013). Moreover, as banks themselves have become risky, counterparty risk must be understood in a bilateral perspective, where the counterparty risk of the two parties are jointly accounted for in the modeling. Thus, in addition to the CVA, a Debt Valuation Adjustment (DVA) must be considered. In this context the classical assumption of a locally risk-free asset which is used for financing purposes of the bank is not sustainable anymore. This raises the companion issue of proper nonlinear accounting of the funding costs of a position, and a corresponding Liquidity Valuation Adjustment (LVA). The last related issue is that of Replacement Cost (RC) corresponding to the fact that at the default time of a party the contract is valued by the liquidator according to a CSA valuation scheme which can fail to reflect the actual value of the contract at that time.

The CVA/DVA/LVA/RC intricacy issues, jointly referred to henceforth as TVA (for Total Valuation Adjustment), were thoroughly investigated in the theoretical papers Crépey (2012a, 2012b), Brigo, Capponi, Pallavicini, and Papatheodorou (2011) and Burgard and Kjaer (2011). The present work is a numerical companion paper to Crépey (2012a, 2012b). Section 2 provides an executive summary of the two papers. Section 3 describes various CSA specifications. In Sections 4 and 5 we present clean valuation (clean of counterparty risk and funding costs) and TVA computations in two simple models for interest rate derivatives. We show in a series of practical examples how CVA, DVA, LVA and RC can be computed in various situations, and we also assess the related model risk issue.

2 TVA Representations

2.1 Setup

We consider a netted portfolio of OTC derivatives between two defaultable counterparties, generically referred to as the "contract between the bank and her counterparty". The counterparty is most commonly a bank as well. Counterparty risk and funding cash-flows can only be considered at a netted and global level, outside the scope of different business desks. Consequently, the price Π of the contract must be computed as a difference between the clean price P provided by the relevant business desk and a correction Θ computed by the central TVA desk. By price we mean here the cost for the bank of margining, hedging and funding ("cost of hedging" for short). By clean price we mean the price of the contract computed without taking into account counterparty risk and excess funding costs. Symmetrical considerations apply to the counterparty, but with non-symmetrical data in the sense of hedging positions and funding conditions. As a consequence and due to nonlinearities in the funding costs, the prices (costs of hedging) are not the same for the two parties. For clarity we focus on the bank's price in the sequel.

We denote by T the time horizon of the contract with promised dividends dD_t from the bank to her counterparty. Both parties are defaultable, with respective default times denoted by θ and $\overline{\theta}$. This results in an effective dividend stream $dC_t = J_t dD_t$, where $J_t = \mathbb{1}_{t<\tau}$ with $\tau = \theta \wedge \overline{\theta}$. One denotes by $\bar{\tau} = \tau \wedge T$ the effective time horizon of the contract as there are no cash-flows after $\bar{\tau}$. The case of unilateral counterparty risk (from the perspective of the bank) can be recovered by letting $\theta \equiv \infty$. After having sold

the contract to the investor at time 0, the bank sets-up a collateralization, hedging and funding portfolio ("hedging portfolio" for short). We call an external funder of the bank (or funder for short) a generic third-party, possibly composed in practice of several entities or devices, insuring funding of the position of the bank. This funder, assumed default-free for simplicity, thus plays the role of "lender/borrower of last resort" after exhaustion of the internal sources of funding provided to the bank via the dividend and funding gains on her hedge, or via the remuneration of the margin amount.

The full model filtration is given as $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^{\theta} \vee \mathbb{H}^{\overline{\theta}}$, where $\mathbb{F}$ is a reference filtration and $\mathcal{H}_t^{\theta} = \sigma(\theta \wedge t)$, $\mathcal{H}_t^{\overline{\theta}} = \sigma(\overline{\theta} \wedge t)$. A probability space $(\Omega, \mathcal{G}_T, \mathbb{P})$, where $\mathbb{P}$ is some risk-neutral pricing measure, is fixed throughout. The meaning of a risk-neutral pricing measure in this context, with different funding rates in particular (see Crépey (2012a)), will be specified by martingale conditions introduced below in the form of suitable pricing backward stochastic differential equations (BSDEs); see El Karoui, Peng, and Quenez (1997) for the seminal reference in finance.

Remark 2.1 Even though it will not appear explicitly in this paper, a pricing measure must also be such that the gain processes on the hedging assets follow martingales, see Crépey (2012a).

Moreover, we assume that $\mathbb{F}$ is immersed into $\mathbb{G}$ in the sense that an $\mathbb{F}$-martingale stopped at τ is a $\mathbb{G}$-martingale.

Remark 2.2 As discussed in Crépey (2012b), this basic assumption precludes major wrong-way risk effects such as the ones which occur with counterparty risk on credit derivatives. In particular, under these assumptions an $\mathbb{F}$-adapted càdlàg process cannot jump at τ. We refer to Crépey (2012c) for an extension of the general methodology of this paper beyond the above immersion setup to incorporate wrong-way risk when necessary.

2.2 Data

We denote by r_t an OIS rate, where OIS stands for an Overnight Indexed Swap, the best market proxy of a risk-free rate. By $\widetilde{r}_t = r_t + \gamma_t$ we denote the credit-risk adjusted rate, where γ_t is the $\mathbb{F}$-hazard intensity of τ, which is assumed to exist. Let $\beta_t = \exp(-\int_0^t r_s ds)$ and $\widetilde{\beta}_t = \exp(-\int_0^t \widetilde{r}_s ds)$ stand for the corresponding discount factors. Furthermore, $\mathbb{E}_t$ and $\widetilde{\mathbb{E}}_t$ stand for the conditional expectations given $\mathcal{G}_t$ and $\mathcal{F}_t$, respectively. The clean value process P_t of the contract with promised dividends dD_t is defined as, for $t \in [0, \bar{\tau}]$,

$$\beta_t P_t := \widetilde{\mathbb{E}}_t \left(\int_t^T \beta_s dD_s \right) = \mathbb{E}_t \big[\int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}} \big] \tag{1}$$

by immersion of $\mathbb{F}$ into $\mathbb{G}$.

Effectively, promised dividends dD_t stop being paid at τ (if $\tau < T$), at which time the last terminal cash-flow R paid by the bank closes out her position. Let an $\mathbb{F}$-adapted process Q represent the (predictable) CSA value process of the contract, and an $\mathbb{F}$-adapted process Γ stand for the value process of a CSA (cash) collateralization scheme. We denote by π a real number meant to represent the wealth of the hedging portfolio of the bank in the financial interpretation. The close-out cash-flow $R = R(\pi)$ is in fact twofold, decomposing into a close-out cash-flow R^i from the bank to the counterparty, minus, in case of default of the bank, a cash-flow $R^f = R^f(\pi)$ from the funder to the bank (depending on π). These two

cash-flows are respectively derived from the algebraic debt χ of the bank to her counterparty and $\mathfrak{X}(\pi)$ of the bank to her funder, modeled at time τ as

$$\chi = Q_\tau - \Gamma_\tau\,,\ \ \mathfrak{X}(\pi) = -(\pi - \Gamma_{\tau-}). \tag{2}$$

The close-out cash-flow, if $\tau < T$, is then modeled as $R(\pi) = R^i - \mathbb{1}_{\tau=\theta} R^f(\pi)$, where

$$\begin{aligned} R^i &= \Gamma_\tau + \mathbb{1}_{\tau=\theta}\big(\rho\chi^+ - \chi^-\big) - \mathbb{1}_{\tau=\bar\theta}\big(\overline{\rho}\chi^- - \chi^+\big) - \mathbb{1}_{\theta=\bar\theta}\chi \\ R^f(\pi) &= (1-\mathfrak{r})\mathfrak{X}^+(\pi), \end{aligned} \tag{3}$$

in which ρ and $\overline{\rho}$ stand for recovery rates between the two parties, and $\mathfrak{r}$ stands for a recovery rate of the bank to her funder. The $\mathcal{G}_\tau$-measurable exposure at default is defined in terms of R as

$$\xi(\pi) := P_\tau - R(\pi) = P_\tau - Q_\tau + \mathbb{1}_{\tau=\theta}\big((1-\rho)\chi^+ + (1-\mathfrak{r})\mathfrak{X}^+(\pi)\big) - (1-\overline{\rho})\mathbb{1}_{\tau=\bar\theta}\chi^-, \tag{4}$$

where the second equality follows by an easy algebraic manipulation.

We now consider the cash-flows required for funding the bank's position, meant in the sense of the contract and its hedging portfolio altogether. For simplicity we stick to the most common situation where the hedge is self-funded as swapped and/or traded via repo markets (see Crépey (2012a)). The OIS rate r_t is used as a reference for all other funding rates, which are thus defined in terms of corresponding bases to r_t. Given such bases b_t and $\bar b_t$ related to the collateral posted and received by the bank, and λ_t and $\bar\lambda_t$ related to external lending and borrowing, the funding coefficient $g_t(\pi)$ is defined by

$$g_t(\pi) = (b_t\Gamma_t^+ - \bar b_t\Gamma_t^-) + \lambda_t\,(\pi - \Gamma_t)^+ - \bar\lambda_t\,(\pi - \Gamma_t)^-\,. \tag{5}$$

Then $(r_t\pi + g_t(\pi))dt$ represents the bank's funding cost over $(t, t+dt)$, depending on its wealth π.

Remark 2.3 A funding basis is typically interpreted as a combination of liquidity and/or credit risk, see Filipović and Trolle (2011) and Crépey and Douady (2012). Collateral posted in foreign-currency and switching currency collateral optionalities can also be accounted for by suitable amendments to b and $\bar b$, see Fujii, Shimada, and Takahashi (2010) and Piterbarg (2012).

2.3 BSDEs

With the data ξ and g specified, the TVA process Θ can be implicitly defined on $[0, \bar\tau]$ as the solution to the following BSDE, posed in integral form and over the random time interval $[0, \bar\tau]$: For $t \in [0, \bar\tau]$,

$$\beta_t\Theta_t = \mathbb{E}_t\Big[\beta_{\bar\tau}\mathbb{1}_{\tau<T}\xi(P_\tau - \Theta_{\tau-}) + \int_t^{\bar\tau} \beta_s g_s(P_s - \Theta_s)ds\Big]. \tag{6}$$

The reader is referred to Crépey (2012b, Proposition 2.1) for the derivation of the TVA BSDE (6). In this paper for simplicity of presentation we take (6) as the definition of the TVA.

Remark 2.4 For $\mathfrak{r}=1$ the exposure at default ξ does not depend on π, and in case of a linear funding coefficient given as $g_t(P-\vartheta)=g_t^0(P)-\lambda_t^0\vartheta$, for some g^0 and λ^0, the TVA equation (6) boils down to the explicit representation

$$\beta_t^0\Theta_t=\mathbb{E}_t\Big[\beta_{\bar\tau}^0\mathbb{1}_{\tau<T}\xi+\int_t^{\bar\tau}\beta_s^0 g_s^0(P_s)ds\Big]$$

for a funding-adjusted discount factor

$$\beta_t^0=\exp(-\int_0^t(r_s+\lambda_s^0)ds)$$

The practical conclusion of Crépey (2012b) is that one can even adopt a simpler "reduced, pre-default" perspective in which defaultability of the two parties only shows up through their default intensities, see Proposition 2.5 below and equation (3.8) in Crépey (2012b). For $t\in[0,T]$ and $\pi\in\mathbb{R}$ let

$$\begin{aligned}\chi_t&=Q_t-\Gamma_t\\ \widetilde\xi_t(\pi)&=(P_t-Q_t)+p_t\big((1-\rho_t)\chi_t^+ +(1-\mathfrak{r})(\pi-\Gamma_t)^-\big)-\overline{p}_t(1-\overline{\rho}_t)\chi_t^-,\end{aligned}\tag{7}$$

in which

$$p_\tau=\mathbb{P}(\tau=\theta\,|\,\mathcal{G}_{\tau-})\,,\ \ \overline{p}_\tau=\mathbb{P}(\tau=\overline{\theta}\,|\,\mathcal{G}_{\tau-}).$$

Note that in case of unilateral counterparty risk, we have $\theta=\infty$ and consequently, $p_\tau=0$, $\overline{p}_\tau=1$.

Proposition 2.5 (TVA Reduced-Form Representation) *One has* $\Theta=\widetilde\Theta$ *on* $[0,\bar\tau)$ *and* $\Theta_{\bar\tau}=\mathbb{1}_{\tau<T}\xi$*, where*

$$\widetilde\beta_t\widetilde\Theta_t=\widetilde{\mathbb{E}}_t\Big[\int_t^T\widetilde\beta_s\big(g_s(P_s-\widetilde\Theta_s)+\gamma_s\widetilde\xi_s(P_s-\widetilde\Theta_s)\big)ds\Big],\tag{8}$$

for $t\in[0,T]$*.*

Remark 2.6 This assumes that the data of (8) are in $\mathbb{F}$, a mild condition which can always be met be passing to $\mathbb{F}$-representatives (or pre-default values) of the original data.

In differential form the pre-default TVA BSDE (8) reads as follows (cf. (Crépey 2012b, Definition 3.1))

$$\begin{cases}\widetilde\Theta_T=0,\text{and for } t\in[0,T]:\\ -d\widetilde\Theta_t=\widetilde g_t(P_t-\widetilde\Theta_t)dt-d\widetilde\mu_t,\end{cases}\tag{9}$$

where $\widetilde\mu$ is the $\mathbb{F}$-martingale component of $\widetilde\Theta$ and

$$\widetilde g_t(P_t-\vartheta)=g_t(P_t-\vartheta)+\gamma_t\widetilde\xi_t(P_t-\vartheta)-\widetilde r_t\vartheta.\tag{10}$$

Remark 2.7 In the linear pre-default case where

$$g_t(P_t-\vartheta)+\gamma_t\big(\widetilde\xi_t(P_t-\vartheta)-\vartheta\big)=\widetilde g_t^0(P)-\widetilde\lambda_t^0\vartheta$$

for some $\widetilde{g}^0$ and $\widetilde{\lambda}^0$, the pre-default TVA equation (8) boils down to the explicit representation

$$\widetilde{\beta}^0_t \widetilde{\Theta}_t = \widetilde{\mathbb{E}}_t \Big[\int_t^T \widetilde{\beta}^0_s \widetilde{g}^0_s (P^0_s) ds \Big] \tag{11}$$

for a funding-adjusted discount factor

$$\widetilde{\beta}^0_t = \exp(-\int_0^t (r_s + \widetilde{\lambda}^0_s) ds).$$

On the numerical side explicit representations such as (7) or (**??**) allow one to estimate the corresponding "linear TVAs" by standard Monte Carlo loops provided that P_t and Q_t can be computed explicitly. In general (for example as soon as $\mathfrak{r} < 1$), nonlinear TVA computations can only be done by more advanced schemes involving linearization Fujii and Takahashi (2011), nonlinear regression Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2010) and/or branching particles Henry-Labordère (2012). Deterministic schemes for the corresponding semilinear TVA PDEs can only be used in low dimension.

Remark 2.8 From (1), P satisfies the following $\mathbb{F}$-BSDE

$$\begin{cases} P_T = 0, \text{and for } t \in [0,T] : \\ -dP_t = dD_t - r_t P_t dt - dM_t, \end{cases} \tag{12}$$

for some $\mathbb{F}$-martingale M. Therefore, the following pre-default $\mathbb{F}$-BSDE in $\widetilde{\Pi} := P - \widetilde{\Theta}$ follows from (9) and (12):

$$\begin{cases} \widetilde{\Pi}_T = 0, \text{and for } t \in [0,T] : \\ -d\widetilde{\Pi}_t = dD_t - \Big(\widetilde{g}_t(\widetilde{\Pi}_t) + r_t P_t \Big) dt - d\widetilde{\nu}_t, \end{cases} \tag{13}$$

where $d\widetilde{\nu}_t = dM_t - d\widetilde{\mu}_t$. As the pre-default price BSDE (13) involves the contractual promised cash-flows dD_t, it is less user-friendly than the pre-default TVA BSDE (9). This mathematical incentive comes on top of the financial justification recalled at the beginning of Section 2 for adopting a two-stage "clean price P minus TVA correction Θ" approach to the counterparty risk and funding issues.

2.3.1 Pre-default Markov Setup

Assume

$$\widetilde{g}_t(P_t - \vartheta) = \widehat{g}(t, X_t, \theta) \tag{14}$$

for some deterministic function $\widehat{g}(t,x,\theta)$, and an $\mathbb{R}^d$-valued $\mathbb{F}$-Markov pre-default factor process X. Then $\widetilde{\Theta}_t = \widetilde{\Theta}(t, X_t)$, where the pre-default TVA pricing function $\widetilde{\Theta}(t,x)$ is the solution to the following pre-default pricing PDE:

$$\begin{cases} \widetilde{\Theta}(T,x) = 0\,, \ x \in \mathbb{R}^d \\ (\partial_t + \mathcal{X})\, \widetilde{\Theta}(t,x) + \widehat{g}(t,x,\widetilde{\Theta}(t,x)) = 0 \text{ on } [0,T) \times \mathbb{R}^d, \end{cases} \tag{15}$$

in which $\mathcal{X}$ stands for the infinitesimal generator of X. As mentioned in Remark 2.7, from the point of view of numerical solution, deterministic PDE schemes for (15) can be used provided the dimension of X is less than 3 or 4, otherwise simulation schemes for (9) are the only viable alternative.

2.4 CVA, DVA, LVA and RC

Plugging (7) into (10) and reordering terms yields

$$\begin{aligned}\widetilde{g}_t(P_t-\vartheta)+r_t\vartheta =& -\gamma_t\overline{p}_t(1-\overline{\rho})(Q_t-\Gamma_t)^- \\ &+\gamma_t p_t\big((1-\rho)(Q_t-\Gamma_t)^+ \\ &+b_t\Gamma_t^+-\bar{b}_t\Gamma_t^-+\lambda_t(P_t-\vartheta-\Gamma_t)^+-\widetilde{\lambda}_t(P_t-\vartheta-\Gamma_t)^- \\ &+\gamma_t\left(P_t-\vartheta-Q_t\right),\end{aligned} \tag{16}$$

where the coefficient $\widetilde{\lambda}_t := \bar{\lambda}_t - \gamma_t p_t(1-\mathfrak{r})$ of $(P_t-\vartheta-\Gamma_t)^-$ in the third line can be interpreted as an external borrowing basis net of credit spread. This coefficient represents the liquidity component of $\bar{\lambda}$. From the perspective of the bank, the four terms in this decomposition of the TVA Θ can respectively be interpreted as a costly (non-algebraic, strict) credit value adjustment (CVA), a beneficial debt value adjustment (DVA), a liquidity funding benefit/cost (LVA), and a replacement benefit/cost (RC). In particular, the time-0 TVA can be represented as

$$\begin{aligned}\Theta_0 =& -\mathbb{E}\Big[\int_0^T \beta_t\gamma_t(1-\overline{\rho})\overline{p}_t(Q_t-\Gamma_t)^-dt\Big] \\ &+\mathbb{E}\Big[\int_0^T \beta_t\gamma_t(1-\rho)p_t(Q_t-\Gamma_t)^+dt\Big] \\ &+\mathbb{E}\Big[\int_0^T \beta_t\big(b_t\Gamma_t^+-\bar{b}_t\Gamma_t^-+\lambda_t(P_t-\widetilde{\Theta}_t-\Gamma_t)^+-\widetilde{\lambda}_t(P_t-\widetilde{\Theta}_t-\Gamma_t)^-\big)dt\Big] \\ &+\mathbb{E}\Big[\int_0^T \beta_t\gamma_t(P_t-\widetilde{\Theta}_t-Q_t)dt\Big].\end{aligned} \tag{17}$$

The DVA and the $\gamma_t p_t(1-\mathfrak{r})(P_t-\widetilde{\Theta}_t-\Gamma_t)^-$-component of the LVA can be considered as "deal facilitating" as they increase the TVA and therefore decrease the price (cost of the hedge) the bank can consider selling the contract to her counterparty. Conversely, the CVA and the $\bar{\lambda}_t(P_t-\widetilde{\Theta}_t-\Gamma_t)^-$ components of the LVA (for $\bar{\lambda}_t$ positive) can be considered as "deal hindering" as they decrease the TVA and therefore increase the price (cost of the hedge) for the bank. The remaining terms can be interpreted likewise as "deal facilitating or hindering" depending on their sign, which is unspecified in general.

3 CSA Specifications

In the next subsections we detail various specifications of the general form (16) of $\widetilde{g}$, depending on the CSA data: the close-out valuation scheme Q, the collateralization scheme Γ and the collateral remuneration bases b and $\bar{b}$.

3.1 Clean CSA Recovery Scheme

In case of a clean CSA recovery scheme $Q=P$, (16) rewrites as follows:

$$\begin{aligned}\widetilde{g}_t(P_t-\vartheta)+\widetilde{r}_t\vartheta =& -\gamma_t\overline{p}_t(1-\overline{\rho})(P_t-\Gamma_t)^- \\ &+\gamma_t p_t\big((1-\rho)(P_t-\Gamma_t)^+ \\ &+b_t\Gamma_t^+-\bar{b}_t\Gamma_t^-+\lambda_t(P_t-\vartheta-\Gamma_t)^+-\widetilde{\lambda}_t(P_t-\vartheta-\Gamma_t)^-.\end{aligned} \tag{18}$$

Note $\widetilde{r}_t$ on the left-hand side as opposed to r_t in (16). In case of no collateralization, i.e. for $\Gamma = 0$, the right-hand-side of (18) reduces to

$$-\gamma_t \overline{p}_t(1-\overline{\rho})P_t^- \,+\, \gamma_t p_t(1-\rho)P_t^+ + \,\lambda_t\,(P_t-\vartheta)^+ \,-\, \widetilde{\lambda}_t\,(P_t-\vartheta)^-\,; \tag{19}$$

whereas in case of continuous collateralization with $\Gamma = Q = P$, it boils down to

$$-\bar{b}_t P_t^- + b_t P_t^+ + \lambda_t \vartheta^- - \widetilde{\lambda}_t \vartheta^+. \tag{20}$$

Remark 3.1 If $\lambda = \widetilde{\lambda}$ (case of equal external borrowing and lending liquidity bases), the TVA is linear for every collateralization scheme of the form $P_t - \Gamma_t = \varepsilon_t$, for some exogenous[1] residual exposure ε_t, cf. Remark 2.7. Setting

$$\widetilde{g}_t^{\lambda} = -\left(\gamma_t \overline{p}_t(1-\overline{\rho}) + \lambda_t\right)\varepsilon_t^- + \left(\gamma_t p_t(1-\rho) + \lambda_t\right)\varepsilon_t^+ + b_t\Gamma_t^+ \,-\, \bar{b}_t\Gamma_t^-\,, \tag{21}$$

one ends up, similarly to (11), with

$$\widetilde{\beta}_t^{\lambda}\widetilde{\Theta}_t = \widetilde{\mathbb{E}}_t \textstyle\int_t^T \widetilde{\beta}_s^{\lambda}\widetilde{g}_s^{\lambda} ds \tag{22}$$

for the funding-adjusted discount factor

$$\widetilde{\beta}_t^{\lambda} = \exp(-\int_0^t (\widetilde{r}_s + \lambda_s) ds). \tag{23}$$

3.2 Pre-Default CSA Recovery Scheme

In case of a pre-default CSA recovery scheme $Q = \widetilde{\Pi} = P - \widetilde{\Theta}$, (16) rewrites as follows

$$\begin{aligned} \widetilde{g}_t(\vartheta) + r_t\vartheta = &-\left(\gamma_t\overline{p}_t(1-\overline{\rho}) + \widetilde{\lambda}_t\right)(P_t - \vartheta - \Gamma_t)^- \\ &+ \left(\gamma_t p_t(1-\rho) + \lambda_t\right)(P_t - \vartheta - \Gamma_t)^+ \\ &+ b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^-\,. \end{aligned} \tag{24}$$

In case of no collateralization, i.e. for $\Gamma = 0$, the right-hand-side reduces to

$$-\left(\gamma_t\overline{p}_t(1-\overline{\rho}) + \widetilde{\lambda}_t\right)(P_t-\vartheta)^- \,+\, \left(\gamma_t p_t(1-\rho) + \lambda_t\right)(P_t-\vartheta)^+; \tag{25}$$

whereas the continuous collateralization with $\Gamma = Q = P - \widetilde{\Theta}$ yields

$$b_t(P_t-\vartheta)^+ - \bar{b}_t(P_t-\vartheta)^-. \tag{26}$$

Remark 3.2 If $b = \bar{b}$ (case of equal collateral borrowing and lending liquidity bases), the TVA is linear for every collateralization scheme of the form $P_t - \widetilde{\Theta}_t - \Gamma_t = \varepsilon_t$, for some exogenous[2] residual exposure ε_t, cf. Remark 2.7. Setting

$$\widetilde{g}_t^{b} = -\left(\gamma_t\overline{p}_t(1-\overline{\rho}) + \widetilde{\lambda}_t\right)\varepsilon_t^- + \,\, b_t\,(P_t - \varepsilon_t) + \left(\gamma_t p_t(1-\rho) + \lambda_t\right)\varepsilon_t^+, \tag{27}$$

one ends up, again similarly to (11), with

$$\widetilde{\beta}_t^{b}\widetilde{\Theta}_t = \widetilde{\mathbb{E}}_t\Big[\textstyle\int_t^T \widetilde{\beta}_s^{b}\widetilde{g}_s^{b} ds\Big] \tag{28}$$

for the funding-adjusted discount factor

$$\widetilde{\beta}_t^{b} = \exp(-\int_0^t (r_s + b_s) ds). \tag{29}$$

Remark 3.3 If $b = \bar{b} = 0$, the two continuous collateralization schemes of equations (20)) and (26) equally collapse to $\widetilde{\Theta} = 0$ and $\widetilde{\Pi} = P = \Gamma = Q$.

[1] Not depending on $\widetilde{\Theta}_t$, like with null or continuous collateralization.

[2] Not depending on $\widetilde{\Theta}_t$, like with continuous collateralization.

3.3 Full Collateralization CSA

Let us define the full collateralization CSA by $Q = \Gamma^*$ where Γ^* is given as the solution to the following $\mathbb{F}$-BSDE:

$$\begin{cases} \Gamma^*_T = P_T, \text{and for } t \in [0,T]: \\ d\Gamma^*_t - \big((r_t + b_t)(\Gamma^*_t)^+ - (r_t + \bar{b}_t)(\Gamma^*_t)^-\big)\, dt = dP_t - r_t P_t dt + d\widetilde{\mu}^*_t \end{cases} \tag{30}$$

for some $\mathbb{F}$-martingale $\widetilde{\mu}^*$. Then $\widetilde{\Theta} := P - \Gamma^*$ solves the pre-default TVA BSDE (9), or in other words $\widetilde{\Pi} = \Gamma^*$. Indeed one has by (16), that for $t \in [0,T]$

$$\widetilde{g}_t(\Gamma^*_t) + r_t(P_t - \Gamma^*_t) \quad = b_t(\Gamma^*_t)^+ - \bar{b}_t(\Gamma^*_t)^-,$$

hence

$$(r_t + b_t)(\Gamma^*_t)^+ - (r_t + \bar{b}_t)(\Gamma^*_t)^- - r_t P_t \quad = \widetilde{g}_t(\Gamma^*_t).$$

Therefore, the second line of (30) reads as

$$-(dP_t - d\Gamma^*_t) = \widetilde{g}_t(\Gamma^*_t)dt - d\widetilde{\mu}^*_t = \widetilde{g}_t(P_t - (P_t - \Gamma^*_t))dt - d\widetilde{\mu}^*_t,$$

which, together with the fact that $P - \Gamma^*$ vanishes at T, means that $P - \Gamma^*$ satisfies the pre-default TVA BSDE (9).

If $b = \bar{b}$, then $\widetilde{g}_t(\widetilde{\Pi}_t) + r_t P_t$ reduces to $(r_t + b_t)\widetilde{\Pi}_t$ in the BSDE (13) for the fully collateralized price $\widetilde{\Pi} = \Gamma^*$. This BSDE is thus equivalent to the following explicit expression for $\widetilde{\Pi}$:

$$\widetilde{\beta}^b_t \widetilde{\Pi}_t = \widetilde{\mathbb{E}}_t\Big[\int_t^T \widetilde{\beta}^b_s dD_s\Big], \tag{31}$$

where the funding-adjusted discount factor $\widetilde{\beta}^b$ is defined by (29). In the special case $b = \bar{b} = 0$, we have $\Gamma^* = P$ which is the situation already considered in Remark 3.3. This case, which yields $\widetilde{\Theta} = 0$ and $\widetilde{\Pi} = P = Q = \Gamma$, justifies the status of formula (1) as the master clean valuation formula of a fully collateralized price at an OIS collateral funding rate r_t. With such a fully collateralized CSA there is no need for pricing-and-hedging a (null) TVA. The problem boils down to the computation of a clean price P and a related hedge, see e.g. Crépey (2012b) for possible clean hedge specifications.

3.4 Pure Funding

In case $\gamma = 0$, which is a no counterparty risk, pure funding issue case, the CSA value process Q plays no actual role. In particular the dt-coefficient of the BSDE (13) for $\widetilde{\Pi}$ is given by

$$\widetilde{g}_t(\widetilde{\Pi}_t) + r_t P_t = (r_t + b_t)\Gamma^+_t - (r_t + \bar{b}_t)\Gamma^-_t + (r_t + \lambda_t)(\widetilde{\Pi} - \Gamma_t)^+ - (r_t + \widetilde{\lambda}_t)(\widetilde{\Pi} - \Gamma_t)^-,$$

where $\widetilde{\lambda} = \bar{\lambda}$ is a pure liquidity external borrowing basis. In case $\Gamma = 0$ and $\lambda = \widetilde{\lambda}$ this results in the following explicit expression of $\widetilde{\Pi}$:

$$\widetilde{\beta}^\lambda_t \widetilde{\Pi}_t = \widetilde{\mathbb{E}}_t \int_t^T \widetilde{\beta}^\lambda_s dD_s, \tag{32}$$

with the funding-adjusted discount factor $\widetilde{\beta}^\lambda$ defined in (23). In the special case $\widetilde{\lambda} = \bar{\lambda} = 0$ one recovers the classical valuation formula (1) for $\widetilde{\Pi} = P$.

3.5 Asymmetrical TVA Approach

In practice the bank can hardly hedge her jump-to-default and therefore cannot monetize and benefit from her default unless and before it actually happens). If one wants to acknowledge this, one can avoid to reckon any actual benefit of the bank from her own default by letting $\rho = \mathfrak{r} = 1$, cf. (4). Such an asymmetrical TVA approach, even though still bilateral, allows one to avoid many concerns of a general symmetrical TVA approach, such as the arbitrage issue that arises for $\mathfrak{r} < 1$, the hypothetical and paradoxical benefit of the bank at her own default time, and the puzzle for the bank of having to hedge her own jump-to-default risk in order to monetize this benefit before her default (see Crépey (2012a, 2012b)). Indeed in this case equation (16) reduces to

$$\begin{aligned}\widetilde{g}_t(P_t-\vartheta)+r_t\vartheta = &-\gamma_t\overline{p}_t(1-\overline{\rho})(Q_t-\Gamma_t)^- \\ &+ b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\left(P_t-\vartheta-\Gamma_t\right)^+ - \bar{\lambda}_t\left(P_t-\vartheta-\Gamma_t\right)^- \\ &+ \gamma_t\left(P_t-\vartheta-Q_t\right),\end{aligned} \tag{33}$$

where there is no beneficial debt valuation adjustment anymore (for $\rho = 1$ the second line of (16) vanishes), and where the borrowing funding basis $\bar{\lambda}_t$ is interpreted as a pure liquidity cost.

Note that in case of a pre-default CSA recovery scheme $Q = \widetilde{\Pi}_- = P - \widetilde{\Theta}_-$, an asymmetrical (but still bilateral) TVA approach is equivalent to a unilateral TVA approach where the bank would simply disregard her own credit risk. One has in both cases

$$\widetilde{g}_t(P_t-\vartheta)+r_t\vartheta = b_t\Gamma_t^+ - \bar{b}_t\Gamma_t^- + \lambda_t\left(P_t-\vartheta-\Gamma_t\right)^+ - \left(\bar{\lambda}_t+\gamma_t\overline{p}_t(1-\overline{\rho})\right)\left(P_t-\vartheta-\Gamma_t\right)^-,$$

where $\gamma\overline{p}$ is the intensity of default of the counterparty. Moreover, in an asymmetrical TVA approach with $Q = \widetilde{\Pi}_-$, an inspection of the related equations in Crépey (2012b) shows that a perfect TVA hedge by the bank of an isolated default of her counterparty (obtained as the solution to the last equation in Crépey (2012b)) in fact yields a perfect hedge of the TVA jump-to-default risk altogether (default of the counterparty and/or the bank). This holds at least provided that the hedging instrument which is used for that purpose (typically a clean CDS on the counterparty) does not jump in value at an isolated default time of the bank, a mild condition[3] satisfied in most models.

4 Clean Valuations

In the numerical Section 5, we shall resort to two univariate short rate models, presented in Subsections 4.2 and 4.3, for TVA computations on an interest rate swap. These are Markovian pre-default TVA models in the sense of Subsection 2.3.1, with factor process $X_t = r_t$. Our motivation for considering two different models is twofold. Firstly, we want to emphasize the fact that from an implementation point of view, the BSDE schemes that we use for TVA computations are quite model-independent (at least the backward nonlinear regression stage, after a forward simulation of the model in a first stage). Secondly, this allows one to assess the TVA model risk.

Remark 4.1 The choice of interest rate derivatives for the illustrative purpose of this paper is not innocuous. The basic credit risk reduced-form methodology of this paper is

[3] In the notation of the concluding Subsection 4.4 of Crépey (2012b), this is given by "$\mathcal{R}_t^1 = \widetilde{\mathcal{P}}_t^1$ on $\overline{\theta} < \theta \wedge T$."

suitable for situations of reasonable dependence between the reference contract and the two parties (reasonable or unknown, e.g. the dependence between interest rates and credit is not liquidly priced on the market, see Brigo and Pallavicini (2008)). For cases of strong dependence such as counterparty risk on credit derivatives, the additional tools of Crépey (2012c) are necessary.

Talking about interest rate derivatives, one should also mention the systemic counterparty risk, referring to various significant spreads which emerged since August 2007 between quantities that were very similar before, like OIS swap rates and LIBOR swap rates of different tenors. Through its discounting implications, this systemic component of counterparty risk has impacted all derivative markets. This means that in the current market conditions, one should actually use multiple-curve clean value models of interest rate derivatives in the TVA computations (see for instance Crépey, Grbac, and Nguyen (2012) and the references therein). In order not to blur the main flow of argument we postpone this to a follow-up work.

4.1 Products

In the sequel we shall deal with the following interest rate derivatives: forward rate agreements (FRAs), IR swaps and caps, whose definitions we provide below. The latter are used in Section 5 for calibration purposes. The underlying rate for all these derivatives is the LIBOR rate. We work under the usual convention that the LIBOR rate is set in advance and the payments are made in arrears. As pointed out in Remark 4.1, we do not tackle here the multiple-curve issue. Thus, we use the classical definition of the forward LIBOR rate $L_t(T, T+\delta)$, fixed at time $t \leq T$, for the future time interval $[T, T+\delta]$:

$$L_t(T, T+\delta) = \frac{1}{\delta}\left(\frac{B_t(T)}{B_t(T+\delta)} - 1\right),$$

where $B_\cdot(T)$ denotes the time-t price of a zero coupon bond with maturity T.

Definition 4.2 A forward rate agreement (FRA) is a financial contract which fixes the interest rate K which will be applied to a future time interval. Denote by $T > 0$ the future inception date, by $T+\delta$ the maturity of the contract, where $\delta \geq 0$, and by N the notional amount. The payoff of the FRA at maturity $T+\delta$ is equal to

$$P^{fra}(T+\delta; T, T+\delta, K, N) = N\delta(L_T(T, T+\delta) - K).$$

The value at time $t \in [0, T]$ of the FRA is given by

$$P^{fra}(t; T, T+\delta, K, N) = N(B_t(T) - \bar{K}B_t(T+\delta)), \tag{34}$$

where $\bar{K} = 1 + \delta K$.

Definition 4.3 An interest rate (IR) swap is a financial contract between two parties to exchange one stream of future interest payments for another, based on a specified notional amount N. A fixed-for-floating swap is a swap in which fixed payments are exchanged for floating payments linked to the LIBOR rate. Denote by $T_0 \geq 0$ the inception date, by $T_1 < \cdots < T_n$, where $T_1 > T_0$, a collection of the payment dates and by K the fixed rate. Under our sign convention recalled in Section 2 that the clean price values promised dividends dD_t from the bank to her counterparty, the time-t clean price of the swap P_t for

the bank when it pays the floating rate (case of the so-called receiver swap for the bank) is given by

$$P_t = P^{sw}(t;T_1,T_n) \quad = \quad N\left(B_t(T_0) - B_t(T_n) - K\sum_{k=1}^{n}\delta_{k-1}B_t(T_k)\right), \tag{35}$$

where $t \leq T_0$ and $\delta_{k-1} = T_k - T_{k-1}$. The swap rate K_t, i.e. the fixed rate K making the value of the swap at time t equal to zero is given by

$$K_t \quad = \quad \frac{B_t(T_0) - B_t(T_n)}{\sum_{k=1}^{n}\delta_{k-1}B_t(T_k)}. \tag{36}$$

The value of the swap from initiation onward, i.e. the time-t value, for $T_0 \leq t < T_n$, of the swap is given by

$$P^{sw}(t;T_1,T_n) \quad = \quad N\left(\left(\frac{1}{B_{T_{k_t-1}}(T_{k_t})} - K\delta_{k_t-1}\right)B_t(T_{k_t}) - B_t(T_n) - K\sum_{k=k_t+1}^{n}\delta_{k-1}B_t(T_k)\right), \tag{37}$$

where T_{k_t} is the smallest T_k (strictly) greater than t. If the bank pays the fixed rate in the swap (case of the so-called payer swap from the bank's perspective), then the corresponding clean price $\bar{P}_t$ is given by $\bar{P}_t = -P^{sw}(t;T_1,T_n)$.

Definition 4.4 An interest rate cap (respectively floor) is a financial contract in which the buyer receives payments at the end of each period in which the interest rate exceeds (respectively falls below) a mutually agreed strike. The payment that the seller has to make covers exactly the difference between the strike K and the interest rate at the end of each period. Every cap (respectively floor) is a series of caplets (respectively floorlets). The payoff of a caplet with strike K and exercise date T, which is settled in arrears, is given by

$$P^{cpl}(T;T,K) = \delta\,(L_T(T,T+\delta) - K)^+.$$

The time-t price of a caplet with strike K and maturity T is given by, with $\bar{K} = 1 + \delta K$,

$$\begin{aligned}
P^{cpl}(t;T,K) &= \delta\, B_t(T+\delta)\mathbb{E}^{\mathbb{P}^{T+\delta}}\left[(L_T(T,T+\delta) - K)^+ \,\middle|\, \mathcal{E}_t\right] \\
&= B_t(T+\delta)\mathbb{E}^{\mathbb{P}^{T+\delta}}\left[\left(\frac{1}{B_T(T+\delta)} - \bar{K}\right)^+ \,\middle|\, \mathcal{E}_t\right] \\
&= \bar{K}B_t(T)\mathbb{E}^{\mathbb{P}^T}\left[\left(\frac{1}{\bar{K}} - B_T(T+\delta)\right)^+ \,\middle|\, \mathcal{E}_t\right] \\
&= \bar{K}\mathbb{E}\left[\exp^{-\int_t^T r_s ds}\left(\frac{1}{\bar{K}} - B_T(T+\delta)\right)^+ \,\middle|\, \mathcal{E}_t\right].
\end{aligned} \tag{38}$$

The next-to-last equality is due to the fact that the payoff $\left(\frac{1}{B_T(T+\delta)} - \bar{K}\right)^+$ at time $T+\delta$ is equal to the payoff $B_T(T+\delta)\left(\frac{1}{B_T(T+\delta)} - \bar{K}\right)^+ = \bar{K}\left(\frac{1}{\bar{K}} - B_T(T+\delta)\right)^+$ at time T. The last equality is obtained by changing from the forward measure $\mathbb{P}^T$ to the spot martingale

measure $\mathbb{P}$, cf. Musiela and Rutkowski (2005, Definition 9.6.2). The above equalities say that a caplet can be seen as a put option on a zero coupon bond.

In the two models considered in the next subsections, the counterparty clean price P of an interest rate derivative satisfies, as required for (14),

$$P_t = P(t, X_t) \tag{39}$$

for all vanilla interest rate derivatives including IR swaps, caps/floors and swaptions.

4.2 Gaussian Vasicek short rate model

In the Vasicek model the evolution of the short rate r is described by the following SDE

$$dr_t = a(k - r_t)dt + dW_t^\sigma,$$

where $a, k > 0$ and W^σ is a Brownian motion with volatility $\sigma > 0$ on the filtered probability space $(\Omega, \mathcal{G}_T, \mathbb{F}, \mathbb{P})$. The unique solution to this SDE is given by

$$r_t = r_0 e^{-at} + k(1 - e^{-at}) + \int_0^t e^{-a(t-u)} dW_u^\sigma.$$

The zero coupon bond price $B_t(T)$ in this model can be written as an exponential-affine function of the current level of the short rate r. One has

$$B_t(T) = e^{m_{va}(t,T) + n_{va}(t,T) r_t}, \tag{40}$$

where

$$m_{va}(t,T) = R_\infty \left(\frac{1}{a} \left(1 - e^{-a(T-t)} \right) - T + t \right) - \frac{\sigma^2}{4a^3} \left(1 - e^{-a(T-t)} \right)^2 \tag{41}$$

with $R_\infty = k - \frac{\sigma^2}{2a^2}$, and

$$n_{va}(t,T) := -e^{at} \int_t^T e^{-au} du = \frac{1}{a} \left(e^{-a(T-t)} - 1 \right). \tag{42}$$

The clean price P for FRAs and interest rate swaps can be written as

$$P_t = P(t, r_t), \qquad t \in [0, T],$$

inserting the expression (40) for the bond price $B_t(T)$ into equations (34), (35) and (37) from Definitions 4.2 and 4.3.

In particular, the time-t price, for $T_0 \le t < T_n$, of the interest rate swap is given by

$$\begin{aligned} P_t \;=\; & N \Big(\Big(e^{-(m_{va}(T_{k_t-1}, T_{k_t}) + n_{va}(T_{k_t-1}, T_{k_t}) r_{T_{k_t-1}})} - K \delta_{k_t - 1} \Big) e^{m_{va}(t, T_{k_t}) + n_{va}(t, T_{k_t}) r_t} \\ & - e^{m_{va}(t,T_n) + n_{va}(t,T_n) r_t} - K \sum_{k=k_t+1}^{n} \delta_{k-1} e^{m_{va}(t,T_k) + n_{va}(t,T_k) r_t} \Big), \end{aligned} \tag{43}$$

which follows from (37) and (40). In the above equation $m_{va}(t, T_k)$ and $n_{va}(t, T_k)$ are given by (41) and (42).

4.2.1 Caplet

To price a caplet at time 0 in the Vasicek model, one uses (38) with

$$B_T(T+\delta) = e^{m_{va}(T,T+\delta)+n_{va}((T,T+\delta)r_T}$$

for $m_{va}(T,T+\delta)$ and $n_{va}(T,T+\delta)$ given by (41) and (42). Combining this with Proposition 11.3.1 and the formula on the bottom of page 354 in Musiela and Rutkowski (2005) yields (recall $\bar{K} = 1+\delta K$):

$$P^{cpl}(0;T,K) = B_0(T)\Phi(-d_-) - \bar{K}B_0(T+\delta)\Phi(-d_+), \tag{44}$$

where Φ is the Gaussian distribution function and

$$d_\pm = \frac{\ln\left(\frac{B_0(T+\delta)}{B_0(T)}\bar{K}\right)}{\Xi\sqrt{T}} \pm \frac{1}{2}\Xi\sqrt{T}$$

with

$$\Xi^2 T := \frac{\sigma^2}{2a^3}\left(1-e^{-2aT}\right)\left(1-e^{-a\delta}\right)^2. \tag{45}$$

4.3 Lévy Hull-White short rate model

In this section we recall a one-dimensional Lévy Hull-White model obtained within the HJM framework. Contrary to the Vasicek model, this model fits automatically the initial bond term structure $B_0(T)$.

As in Example 3.5 of Crépey, Grbac, and Nguyen (2012), we consider the Lévy Hull–White extended Vasicek model for the short rate r given by

$$dr_t = \alpha(\kappa(t) - r_t)dt + dZ^\varsigma_t, \tag{46}$$

where $\alpha > 0$ and Z^ς denotes a Lévy process described below. Furthermore,

$$\kappa(t) = f_0(t) + \frac{1}{\alpha}\partial_t f_0(t) + \psi_\varsigma\left(\frac{1}{\alpha}\left(e^{-\alpha t}-1\right)\right) - \psi'_\varsigma\left(\frac{1}{\alpha}\left(e^{-\alpha t}-1\right)\right)\frac{1}{\alpha}e^{-\alpha t}, \tag{47}$$

where $f_0(t) = -\partial_t \log B_0(t)$ and ψ_ς denotes the cumulant function of Z^ς; see Crépey et al. (2012, Example 3.5) with the volatility specification $\sigma_s(T) = e^{-\alpha(T-s)}$, $0 \le s \le T$, therein.

In this paper we shall use an inverse Gaussian (IG) process $Z^\varsigma = (Z^\varsigma_t)_{t\ge 0}$, which is a pure-jump, infinite activity, subordinator (nonnegative Lévy process), providing an explicit control on the sign of the short rates (see Crépey et al. (2012)). The IG process is obtained from a standard Brownian motion W by setting

$$Z^\varsigma_t = \inf\{s > 0 : W_s + \varsigma s > t\},$$

where $\varsigma > 0$. Its Lévy measure is given by

$$F_\varsigma(dx) = \frac{1}{\sqrt{2\pi x^3}}e^{-\frac{\varsigma^2 x}{2}}\mathbf{1}_{\{x>0\}}\,dx.$$

The distribution of Z^ς_t is $IG(\frac{t}{\varsigma}, t^2)$. The cumulant function ψ_ς exists for all $z \in [-\frac{\varsigma^2}{2}, \frac{\varsigma^2}{2}]$ (actually for all $z \in (-\infty, \frac{\varsigma^2}{2}]$ since F_ς is concentrated on $(0,\infty)$) and is given by

$$\psi_\varsigma(z) = \varsigma\left(1 - \sqrt{1 - 2\frac{z}{\varsigma^2}}\right). \tag{48}$$

Similarly to the Gaussian Vasicek model, the bond price $B_t(T)$ in the Lévy Hull-White short rate model can be written as an exponential-affine function of the current level of the short rate r:

$$B_t(T) = e^{m_{le}(t,T)+n_{le}(t,T)r_t}, \tag{49}$$

where

$$\begin{aligned} m_{le}(t,T) &:= \log\left(\frac{B_0(T)}{B_0(t)}\right) - n(t,T)\left[f_0(t) + \psi_\varsigma\left(\frac{1}{\alpha}\left(e^{-\alpha t}-1\right)\right)\right] \\ &\quad - \int_0^t \left[\psi_\varsigma\left(\frac{1}{\alpha}\left(e^{-\alpha(T-s)}-1\right)\right) - \psi_\varsigma\left(\frac{1}{\alpha}\left(e^{-\alpha(t-s)}-1\right)\right)\right] ds \end{aligned} \tag{50}$$

and

$$n_{le}(t,T) := -e^{\alpha t}\int_t^T e^{-\alpha u}du = \frac{1}{\alpha}\left(e^{-\alpha(T-t)}-1\right). \tag{51}$$

In the Lévy Hull-White model the clean price P for FRAs and interest rate swaps can be written as

$$P_t = P(t,r_t), \qquad t \in [0,T],$$

by combining the exponential-affine representation (49) of the bond price $B_t(T)$ and Definitions 4.2 and 4.3. In particular, the time-t price, for $T_0 \le t < T_n$, of the swap is given by

$$\begin{aligned} P_t &= N\Big(\Big(e^{-(m_{le}(T_{k_t-1},T_{k_t})+n_{le}(T_{k_t-1},T_{k_t})r_{T_{k_t-1}})} - K\delta_{k_t-1}\Big)e^{m_{le}(t,T_{k_t})+n_{le}(t,T_{k_t})r_t} \\ &\quad - e^{m_{le}(t,T_n)+n_{le}(t,T_n)r_t} - K\sum_{k=k_t+1}^{n}\delta_{k-1}e^{m_{le}(t,T_k)+n_{le}(t,T_k)r_t}\Big), \end{aligned} \tag{52}$$

which follows from (37) and (49). In the above equations $m_{le}(t,T_k)$ and $n_{le}(t,T_k)$ are given by (50) and (51).

4.3.1 Caplet

To calculate the price of a caplet at time 0 in the Lévy Hull-White model, one can replace $\bar{B}^*$ with B, $\bar{\Sigma}^*$ with Σ, $\bar{A}^*$ with A, and insert $\Sigma^* = 0$ and $A^* = 0$ in Subsection 4.4 of Crépey et al. (2012), thus obtaining the time-0 price of the caplet

$$\begin{aligned} P^{cpl}(0;T,K) &= B_0(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\left(\frac{1}{B_T(T+\delta)} - \bar{K}\right)^+\right] \\ &= B_0(T+\delta)\mathbb{E}^{\mathbf{P}^{T+\delta}}\left[\left(e^Y - \bar{K}\right)^+\right] \end{aligned}$$

with

$$Y := \log\frac{B_0(T)}{B_0(T+\delta)} + \int_0^T (A_s(T+\delta) - A_s(T))ds + \int_0^T (\Sigma_s(T+\delta) - \Sigma_s(T))dZ_s^\varsigma,$$

where $\Sigma_s(t) = \frac{1}{\alpha}\big(1-e^{-\alpha(t-s)}\big)$ and $A_s(t) = \psi_\varsigma(-\Sigma_s(t))$, for $0 \le s \le t$. The time-0 price of the caplet is now given by (cf. Crépey et al. (2012, Proposition 4.5))

$$P^{cpl}(0;T,K) = \frac{B_0(T+\delta)}{2\pi}\int_{\mathbb{R}} \frac{\bar{K}^{1+\mathrm{i}v-R}M_Y^{T+\delta}(R-\mathrm{i}v)}{(\mathrm{i}v-R)(1+\mathrm{i}v-R)}dv, \tag{53}$$

for $R > 1$ such that $M_Y^{T+\delta}(R) < \infty$. The moment generating function $M_Y^{T+\delta}$ of Y under the measure $\mathbb{P}^{T+\delta}$ is provided by

$$\begin{aligned} M_Y^{T+\delta}(z) &= \exp\left(-\int_0^T \psi_\varsigma(-\Sigma_s(T+\delta))ds\right) \\ &\quad \times \exp\left(z\left(\log\frac{B_0(T)}{B_0(T+\delta)} + \int_0^T \left(\psi_\varsigma(-\Sigma_s(T+\delta)) - \psi_\varsigma(-\Sigma_s(T))\right)ds\right)\right) \\ &\quad \times \exp\left(\int_0^T \psi_\varsigma\left((z-1)\Sigma_s(T+\delta) - z\Sigma_s(T)\right)ds\right), \end{aligned} \tag{54}$$

for $z \in \mathbb{C}$ such that the above expectation is finite. Alternatively, the time-0 price of the caplet can be computed as the following expectation (cf. formula (38))

$$P^{cpl}(0;T,K) = \bar{K}\mathbb{E}\left[\exp^{-\int_t^T r_s ds}\left(\frac{1}{\bar{K}} - B_T(T+\delta)\right)^+\right], \tag{55}$$

where r is given by (46) and $B_T(T+\delta)$ by (49).

4.4 Numerics

In Section 5 we shall present TVA computations on an interest rate swap with ten years maturity, where the bank exchanges the swap rate K against a floating LIBOR at the end of each year 1 to 10. In order to fairly assess the TVA model risk issue, this will be done in the Vasicek model and the Lévy Hull-White model calibrated to the same data, in the sense that they share a common initial zero-bond term structure $B_0^*(T)$ below, and produce the same price for the cap with payments at years 1 to 10 struck at K (hence there is the same level of Black implied volatility in both models at the strike level K). Specifically, we set $r_0 = 2\%$ and the following Vasicek parameters:

$$a = 0.25\,,\ \ k = 0.05\,,\ \ \sigma = 0.004$$

with related zero-coupon rates and discount factors denoted by $R_0^*(T)$ and $B_0^*(T) = \exp(-TR_0^*(T))$. It follows from (40) and after some simple calculations,

$$\begin{aligned} R_0^*(T) &= R_\infty - (R_\infty - r_0)\frac{1}{aT}\left(1-e^{-aT}\right) + \frac{\sigma^2}{4a^3T}\left(1-e^{-aT}\right)^2 \\ f_0^*(T) &= \partial_T\big(TR_0^*(T)\big) = k + e^{-aT}(r_0-k) - \frac{\sigma^2}{2a^2}\left(1-e^{-aT}\right)^2 \\ \partial_T f_0^*(T) &= -ae^{-aT}(r_0-k) - \frac{\sigma^2}{a}\left(1-e^{-aT}\right)e^{-aT}. \end{aligned}$$

An application of formula (36) at time 0 yields for the corresponding swap rate the value $K = 3.8859\%$. We choose a swap notional of $N = 310.136066\$$ so that the fixed leg of the swap is worth $100\$$ at inception.

In the Lévy Hull-White model we use $\alpha = a = 0.25$ (same speed of mean-reversion as in the Vasicek model), an initial bond term-structure $B_0(T)$ fitted to $B_0^*(T)$ by using $f_0(T) = f_0^*(T)$ above in (47), and a value of $\varsigma = 17.570728$ calibrated to the price in the above Vasicek model of the cap with payments at years 1 to 10 struck at K. The calibration is done by least square minimization based on the explicit formulas for caps in both models

reviewed in Subsections 4.2.1 and 4.3.1. After calibration the price of the cap in both models is 20.161$ (for the above notional N yielding a value of 100$ for the fixed leg of the swap).

The top panels of Figure 1 show 20 paths, expectations and 2.5/97.5-percentiles over 10000 paths, simulated in the two models by an Euler scheme $\widehat{r}$ for the short-rate r on a uniform time grid with 200 time steps over $[0, 10]$yrs. Note one does not see the jumps on the right panel because we used interpolation between the points so that one can identify better the twenty paths.

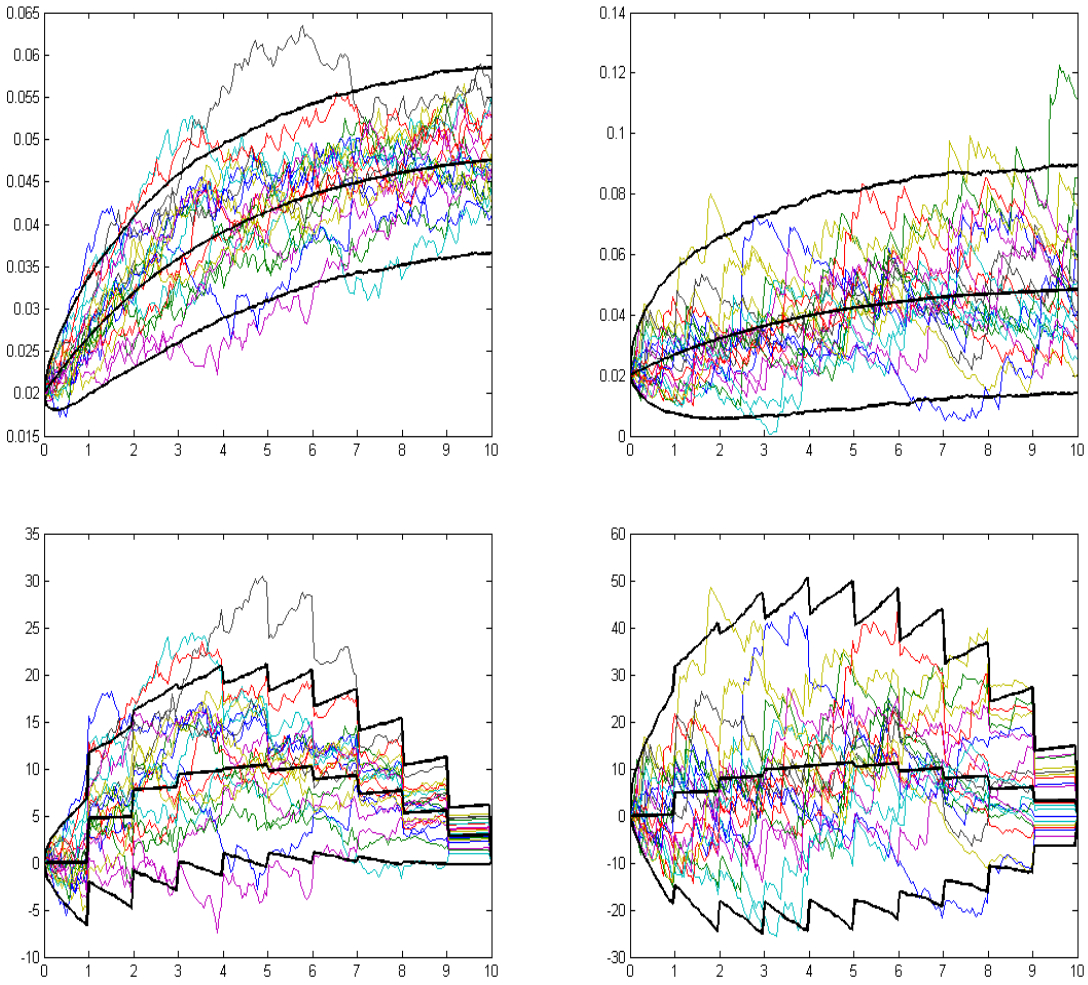


Figure 1: 20 paths with 200 time points each of the short rate r_t and of the clean price process $P_t = P(t, r_t)$ of the swap. *Left*: Vasicek model ; *Right*: LHW Model.

The top panels of Figure 2 show the initial zero-coupon rates term structure $R_0^*(T)$ and the corresponding forward curves $f_0^*(T)$, whilst the corresponding discount factors $B_0^*(T)$ can be seen on the lower left panel. This displays an increasing term structure of interest rates, meaning that the bank will on average be out-of-the-money with a positive $P_t = P_t^{sw}$ in (37), or in-the-money with a negative $\bar{P}_t = -P_t^{sw}$, depending on whether the bank pays floating (case of a receiver swap) or fixed rate (case of a payer swap) in the swap, see the

bottom panels of Figure 1. Note that the swap price processes have quite distinct profiles in the two models even though these are co-calibrated. The bottom right panel of Figure 2 shows the Lévy Hull-White mean-reversion function $\kappa(t)$ corresponding to $f_0^*(T)$ through (47).

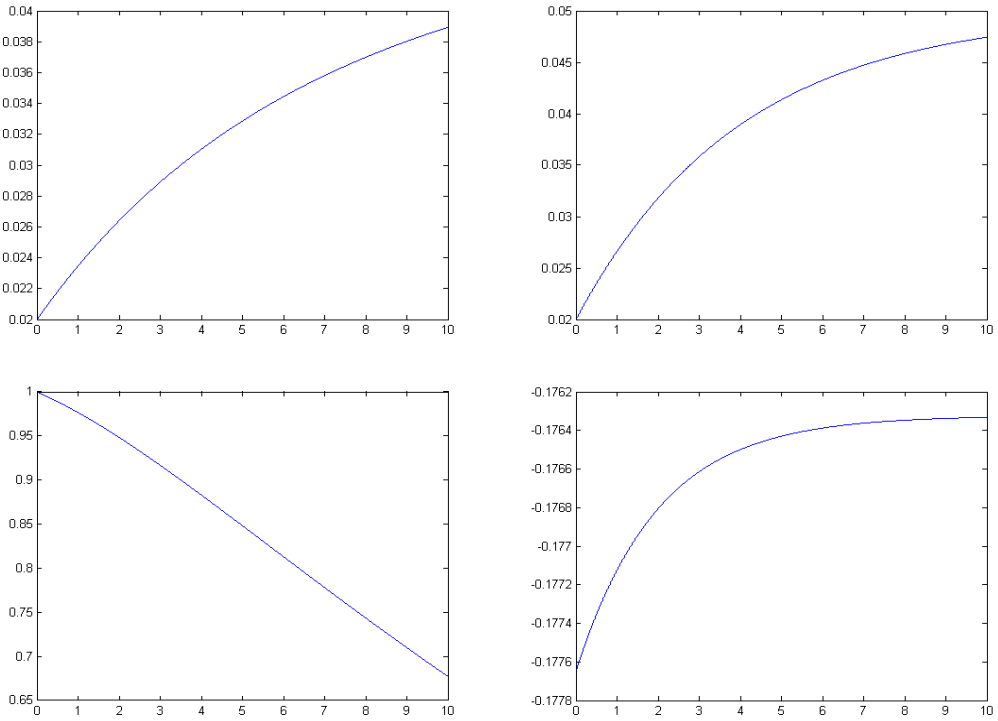


Figure 2: *Top Left*: $R_0^*(T)$; *Top Right*: $f_0^*(T)$; *Bottom Left*: $B_0^*(T)$; *Bottom Right*: $\kappa(t)$.

5 TVA Computations

In this section we show with practical examples how the CVA, DVA, LVA and RC terms defined in Section 2 can be computed for various CSA specifications of the coefficient $\widehat{g}$ in Section 3. The computation are done for an IR swap in the two models of Section 4.

5.1 TVA Equations

The generator $\mathcal{X}$ of the Gaussian Vasicek short rate r in the pre-default pricing PDE (15) is given by

$$\mathcal{X}\widetilde{\Theta}\,(t,r) = (a(k-r))\partial_r\widetilde{\Theta} + \frac{1}{2}\sigma^2\partial_{r^2}^2\widetilde{\Theta}.$$

Assuming (14), the corresponding TVA Markovian BSDE writes: $\widetilde{\Theta}(T, r_T) = 0$, and for $t \in [0, T]$:

$$-d\widetilde{\Theta}(t, r_t) = \widehat{g}(t, r_t, \widetilde{\Theta}(t, r_t))dt - \partial_r\widetilde{\Theta}(t, r_t))dW_t^{\sigma}. \tag{56}$$

Similarly, the generator $\mathcal{X}$ of the Lévy Hull-White short rate r driven by an IG process is given by

$$\mathcal{X}\widetilde{\Theta}\,(t,r) =(\alpha(\kappa(t)-r))\partial_r\widetilde{\Theta}+\int_{\epsilon>0}\Big(\widetilde{\Theta}(t,r+\epsilon)-\widetilde{\Theta}(t,r)\Big)F_\varsigma(d\epsilon), \tag{57}$$

where F_ς stands for the Lévy measure of Z^ς.[4] Assuming (14), the corresponding TVA Markovian BSDE writes: $\widetilde{\Theta}(T,r_T)=0$, and for $t\in[0,T]$:

$$-d\widetilde{\Theta}(t,r_t) =\widehat{g}(t,r_t,\widetilde{\Theta}(t,r_t))dt-\int_{\epsilon>0}\Big(\widetilde{\Theta}(t,r_{t-}+\epsilon)-\widetilde{\Theta}(t,r_{t-})\Big)N^\varsigma(dt,d\epsilon),$$

where N^ς stands for the compensated jump measure of Z^ς.

5.2 BSDE Scheme

Even though finding deterministic solutions of the corresponding PDEs would be possible in the above univariate setups, in this paper we nevertheless favor BSDE schemes, as they are more generic – in real-life higher-dimensional applications deterministic schemes cannot be used anymore. We solve (56) and (57) by backward regression over the time-space grids generated in Subsection 4.4; see the top panels of Figure 1. We thus approximate $\widetilde{\Theta}_t(\omega)$ in (56) and (57) by $\widehat{\Theta}_i^j$ on the corresponding time-space grid, where the time-index i runs from 1 to $n=200$ and the space-index j runs from 1 to $m=10^4$. Denoting by $\widehat{\Theta}_i=(\widehat{\Theta}_i^j)_{1\le j\le m}$ the vector of TVA values on the space grid at time i, we have $\widehat{\Theta}_n=0$, and then for every $i=n-1,\cdots,0$ and $j=1,\cdots,m$

$$\widehat{\Theta}_i^j=\widehat{\mathbb{E}}_i^j\Big(\widehat{\Theta}_{i+1}+\widehat{g}_{i+1}\big(t,\widehat{r}_{i+1},\widehat{\Theta}_{i+1}\big)h\Big)$$

for the time-step $h=\frac{T}{n}=0.02$y (one week, roughly). The conditional expectations in space at every time-step are computed by a q-nearest neighbor average non-parametric regression estimate (see, e.g., Hastie, Tibshirani, and Friedman (2009)), with $q=5$ in our numerical experiments below.

5.3 Numerics

We set the following TVA parameters: $\gamma=10\%$, $b=\bar{b}=\lambda=1.5\%$, $\bar{\lambda}=4.5\%$, $p=50\%$, $\overline{p}=70\%$ and we consider five possible CSA specifications in this order:

$$\begin{array}{lll}
(\mathfrak{r},\rho,\bar{\rho})=(40,40,40)\%, & Q=P, & \Gamma=0\\
(\mathfrak{r},\rho,\bar{\rho})=(100,40,40)\%, & Q=P, & \Gamma=0\\
(\mathfrak{r},\rho,\bar{\rho})=(100,100,40)\%, & Q=P, & \Gamma=0\\
(\mathfrak{r},\rho,\bar{\rho})=(100,100,40)\%, & Q=\Pi, & \Gamma=0\\
(\mathfrak{r},\rho,\bar{\rho})=(100,40,40)\%, & Q=P, & \Gamma=Q=P.
\end{array}$$

Note that under the first CSA specification, one has $\widetilde{\lambda}=4.5\%-0.6\times0.5\times10\%=1.5\%=\lambda$, so this is a linear TVA special case of Remark 3.1, where the TVA at time 0 can be computed through a straight Monte Carlo simulation.

[4] The integral converges in (57) under technical conditions stated in Crépey et al. (2012).

Moreover, we shall study the TVA in the two co-calibrated Vasicek and Lévy models, and for the receiver and payer swaps. We thus consider twenty cases (5 CSA specifications $\times$ 2 models $\times$ receiver versus payer swap).

Table 1 shows the time-0 TVAs and the corresponding CVA/DVA/LVA/RC decompositions (four terms on the right-hand side of (17)) in each of the twenty cases. For benchmarking the numerical BSDE results we display in Figure 3 the time-0 TVA BSDE value versus the TVA Monte Carlo mid- and 95%-lower and upper bounds in the first CSA specification (linear). In all four cases the BSDE time-0 value of the TVA is close to the middle of the confidence interval.

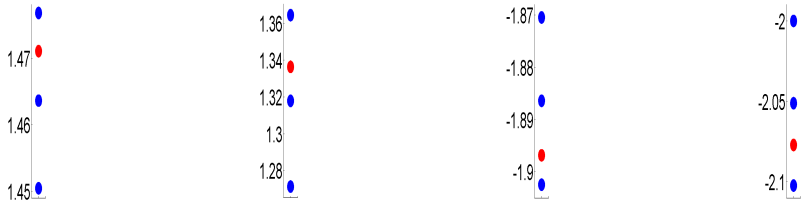


Figure 3: Time-0 linear TVA BSDE value versus Monte Carlo mid-value and 95%-bounds. *From left to right*: Receiver swap in the Vasicek model; Receiver swap in the Lévy model; Payer swap in the Vasicek model; Payer swap in the Lévy model.

The numbers of Table 1 are fully consistent with the CVA/DVA/LVA/RC interpretation of the four terms in the TVA decomposition on the right-hand side of (17), given the increasing term structure of the data discussed in Subsection 4.4. For instance, a "high" DVA of 1.75 in the Vasicek model and 2.34 in the Lévy model for the receiver swap in the first row of Table 1, is consistent with the fact that with an increasing term structure of rates, the bank is on average out-of-the-money on the receiver swap with a positive $P_t = P_t^{sw}$ (see Figure 1). The CVA, on the contrary, is moderate, as it should be for a receiver swap in an increasing term structure of interest rates, and higher ("more negative") in the Lévy than in the Vasicek model (-0.90 versus -0.06). The numbers of Table 1 are not negligible at all in view of the initial value of 100\$ of the fixed leg of the swap. In particular, the LVA terms are quite significant in case of the payer swap with $\mathfrak{r}$ and/or $\rho = 100\%$, see the corresponding terms in rows 3 and 4 in the two bottom parts of Table 1. The choice of $\mathfrak{r}$ and ρ thus has tangible operational consequences, in regard of the "deal facilitating"("deal hindering") interpretation of the positive (negative) TVA terms as explained at the end of Subsection 2.4. It is worthwhile noting that all this happens in a simplistic toy model of TVA, in which credit risk is independent from interest rates. These numbers could be even much higher (in absolute value) in a model accounting for potential wrong way risk dependence effects between interest rates and credit risk, see Remark 4.1.

Figures 4 and 5 (receiver swap in the Vasicek and Lévy model, respectively) and Figures 6 and 7 (payer swap in the Vasicek and Lévy model, respectively) show the "expected exposures" of the four right-hand side terms of the "local" TVA decompositions (16) with ϑ replaced by $\widetilde{\Theta}_t$ therein. These exposures are computed as space-averages over 10^4 paths as a function of time t. Each time-0 integrated term of the TVA in Table 1 corresponds to the surface under the corresponding curve in Figures 4 to 7 (with mappings between, respectively: Figure 4 and the upper left corner of Table 1, Figure 5 and the upper right corner of Table 1, Figure 6 and the lower left corner of Table 1, Figure 7 and the lower right

TVA	CVA	DVA	LVA	RC	TVA	CVA	DVA	LVA	RC
1.47	-0.06	1.75	0.71	-0.92	1.34	-0.90	2.34	0.72	-0.85
1.40	-0.06	1.75	0.64	-0.91	0.93	-0.90	2.34	0.15	-0.68
0.40	-0.06	0.00	0.76	-0.29	-0.45	-0.90	0.00	0.32	0.12
0.66	-0.08	0.00	0.74	0.00	-0.43	-0.76	0.00	0.32	0.00
0.43	0.00	0.00	0.72	-0.29	0.44	0.00	0.00	0.72	-0.29
TVA	CVA	DVA	LVA	RC	TVA	CVA	DVA	LVA	RC
-1.90	-2.45	0.04	-0.68	1.17	-2.08	-3.28	0.64	-0.66	1.25
-2.64	-2.45	0.04	-1.92	1.67	-3.17	-3.28	0.64	-2.41	1.92
-2.67	-2.45	0.00	-1.92	1.68	-3.59	-3.28	0.00	-2.38	2.11
-3.59	-1.77	0.00	-1.83	0.00	-4.80	-2.49	0.00	-2.26	0.00
-0.50	0.00	0.00	-0.81	0.31	-0.51	0.00	0.00	-0.81	0.31

Table 1: Time-0 TVA and its decomposition. *Top*: Receiver swap ; *Bottom*: Payer swap. *Left*: Vasicek model; *Right*: LHW Model.

corner of Table 1).

Finally, Figures 8 (receiver swap) and 9 (payer swap) show the TVA processes in the same format as the swap clean prices at the bottom of Figure 1.

Conclusion

In this paper, which is a numerical companion to Crépey (2012a, 2012b), we show on the standing example of an interest-rate swap how CVA, DVA, LVA and RC, the four "wings" (or pillars) of the TVA, can be computed for various CSA and model specifications. Positive terms such as the DVA (resp. negative terms such as the CVA) can be considered as "deal facilitating" (resp. "deal hindering") as they increase (resp. decrease) the TVA and therefore decrease (resp. increase) the price (cost of the hedge) for the bank. Beliefs regarding the tangibility of a benefit-at-own-default, which depends in reality on the ability to hedge and therefore monetize this benefit before the actual default, are controlled by the choice of the "own recovery-rate" parameters ρ and $\mathfrak{r}$. Larger ρ and $\mathfrak{r}$ mean smaller DVA and LVA and therefore smaller TVA, which in principle means less deals (or a recognition of a higher cost of the deal). This is illustrated numerically in two alternative short rate models to emphasize the model-free feature of the numerical TVA computations through nonlinear regression BSDE schemes. The results show that the TVA model risk is under reasonable control for both co-calibrated models (models calibrated to the same initial term structure, but also with the same level of volatility as imposed through calibration to cap prices). We emphasize however that the latter observation applies to the "standard" case studied in this paper without dominant wrong-way and gap risks, two important features which will be dealt with in future work.

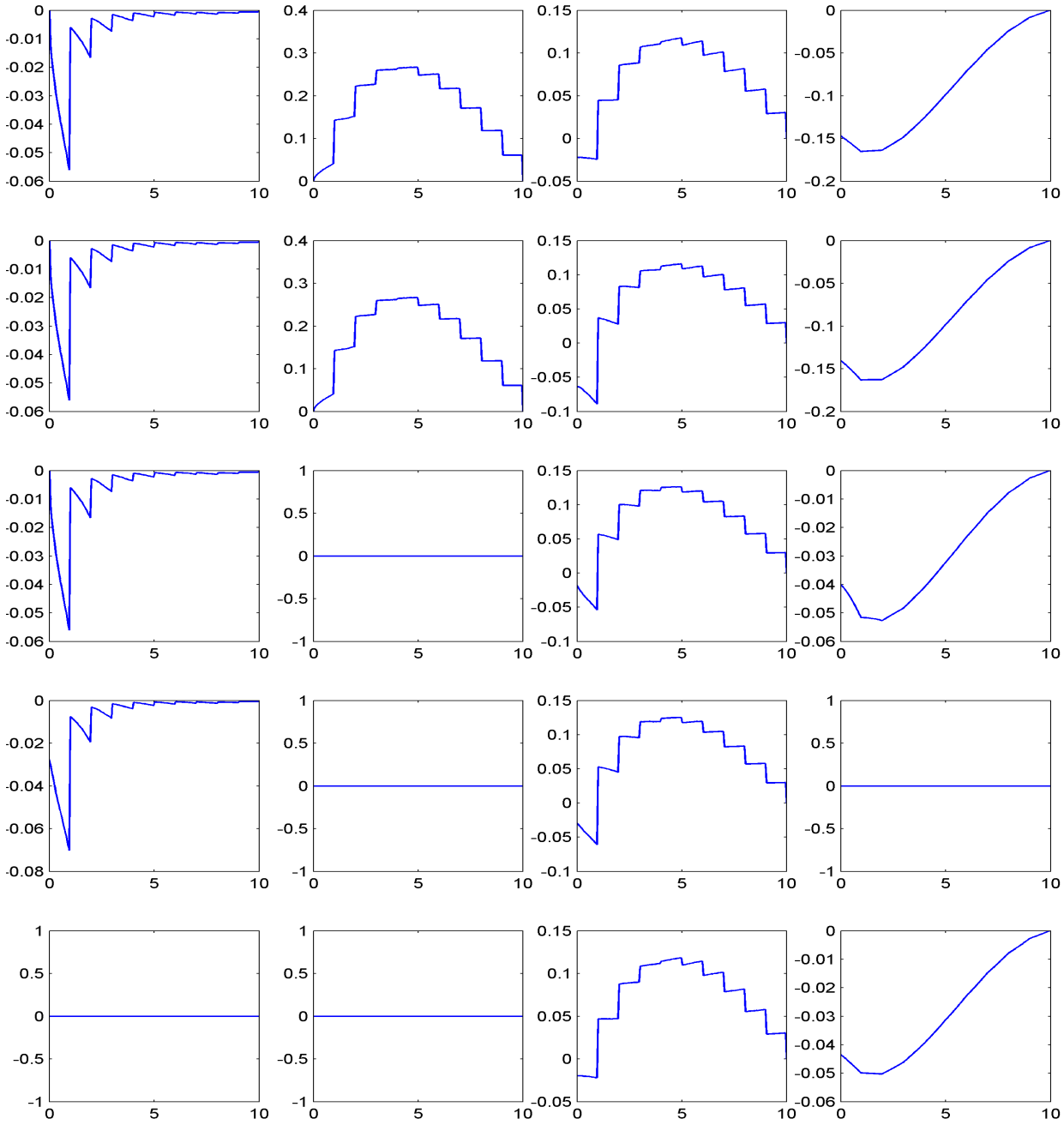


Figure 4: Receiver swap in the Vasicek model. *Columns*: CVA/DVA/LVA/RC, the 4 wings of the TVA; *Rows*: 5 CSA Specifications.

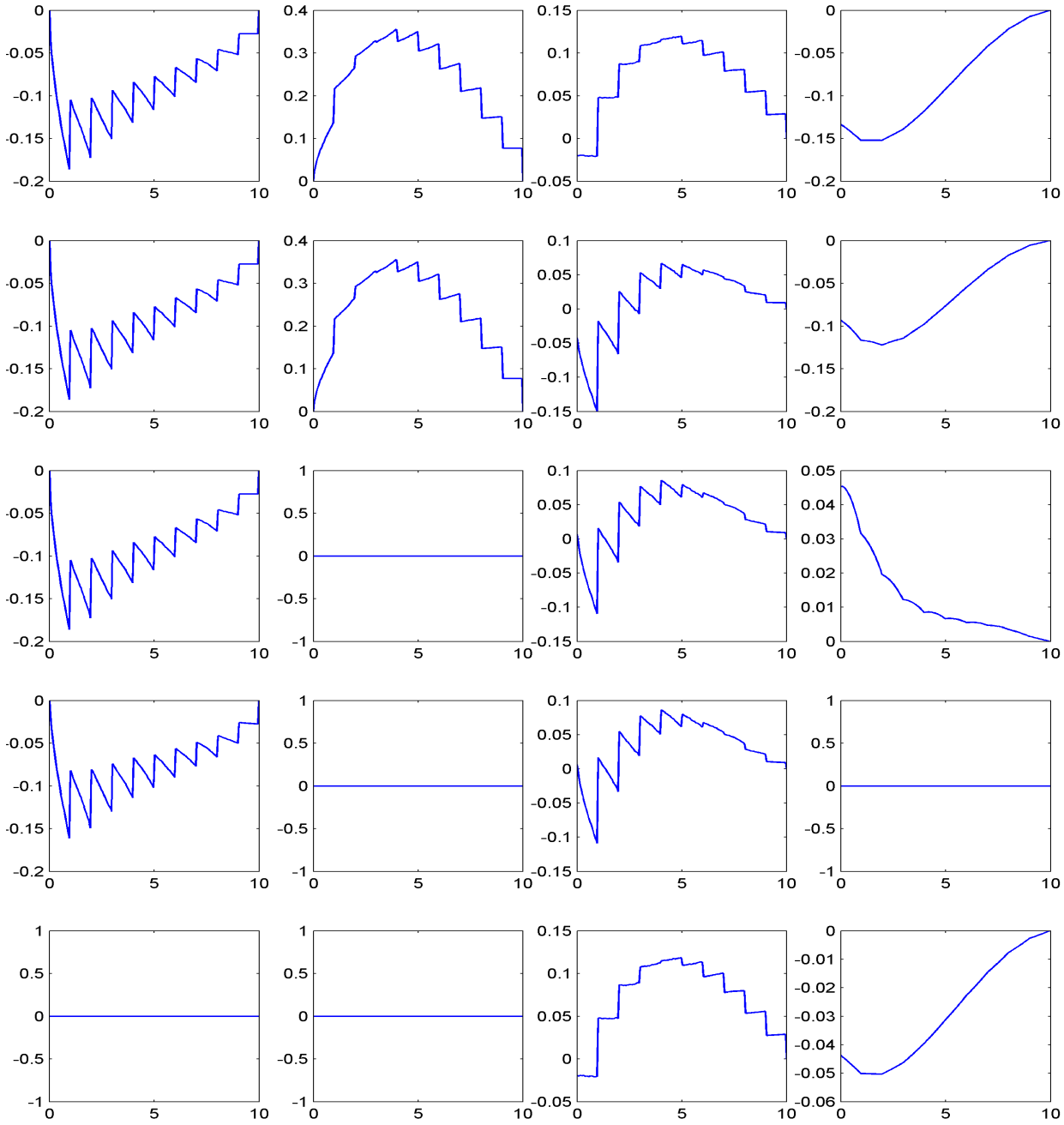


Figure 5: Receiver swap in the Lévy model. *Columns*: CVA/DVA/LVA/RC, the 4 wings of the TVA; *Rows*: 5 CSA Specifications.

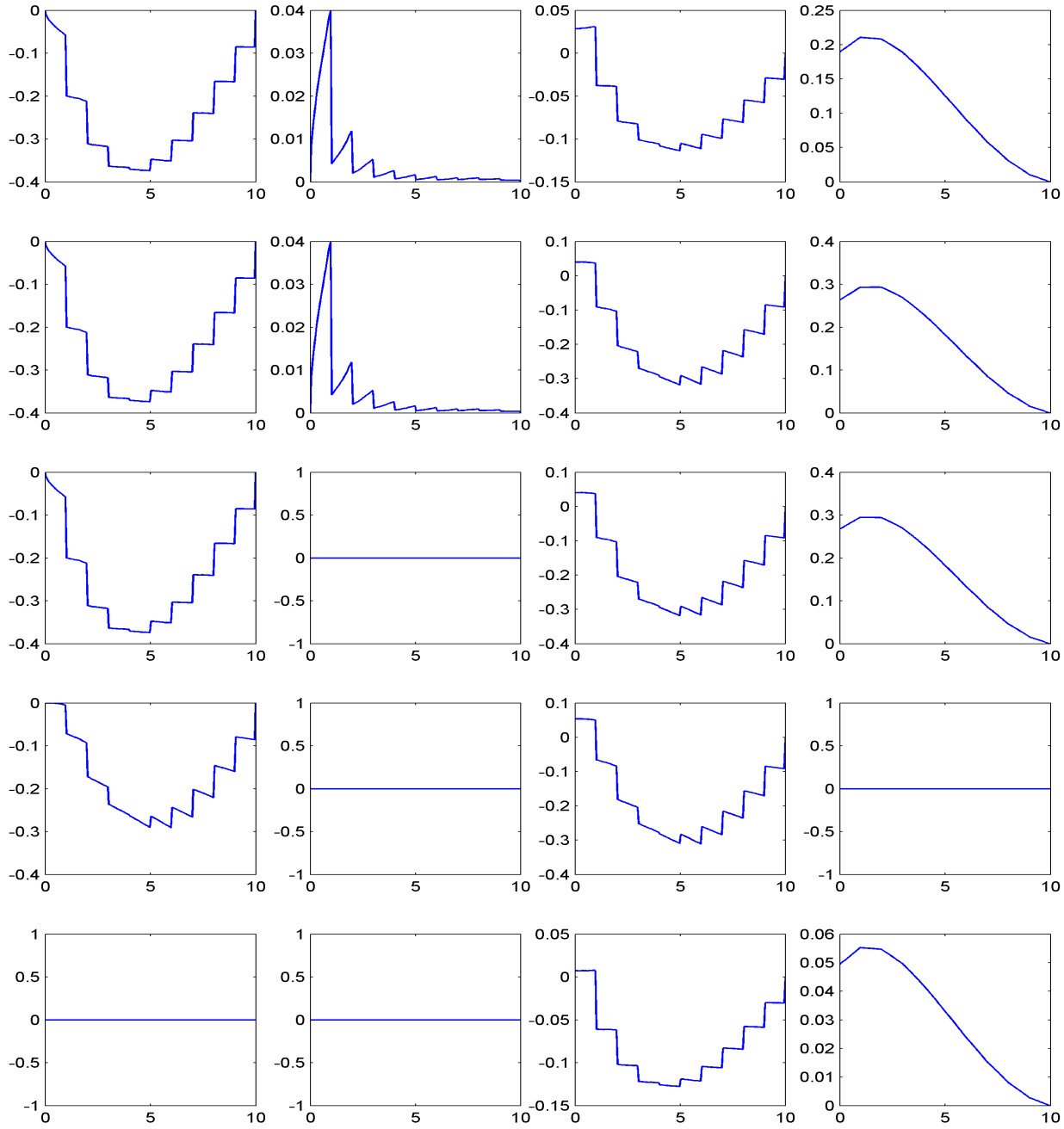


Figure 6: Payer swap in the Vasicek model. *Columns*: CVA/DVA/LVA/RC, the 4 wings of the TVA; *Rows*: 5 CSA Specifications.

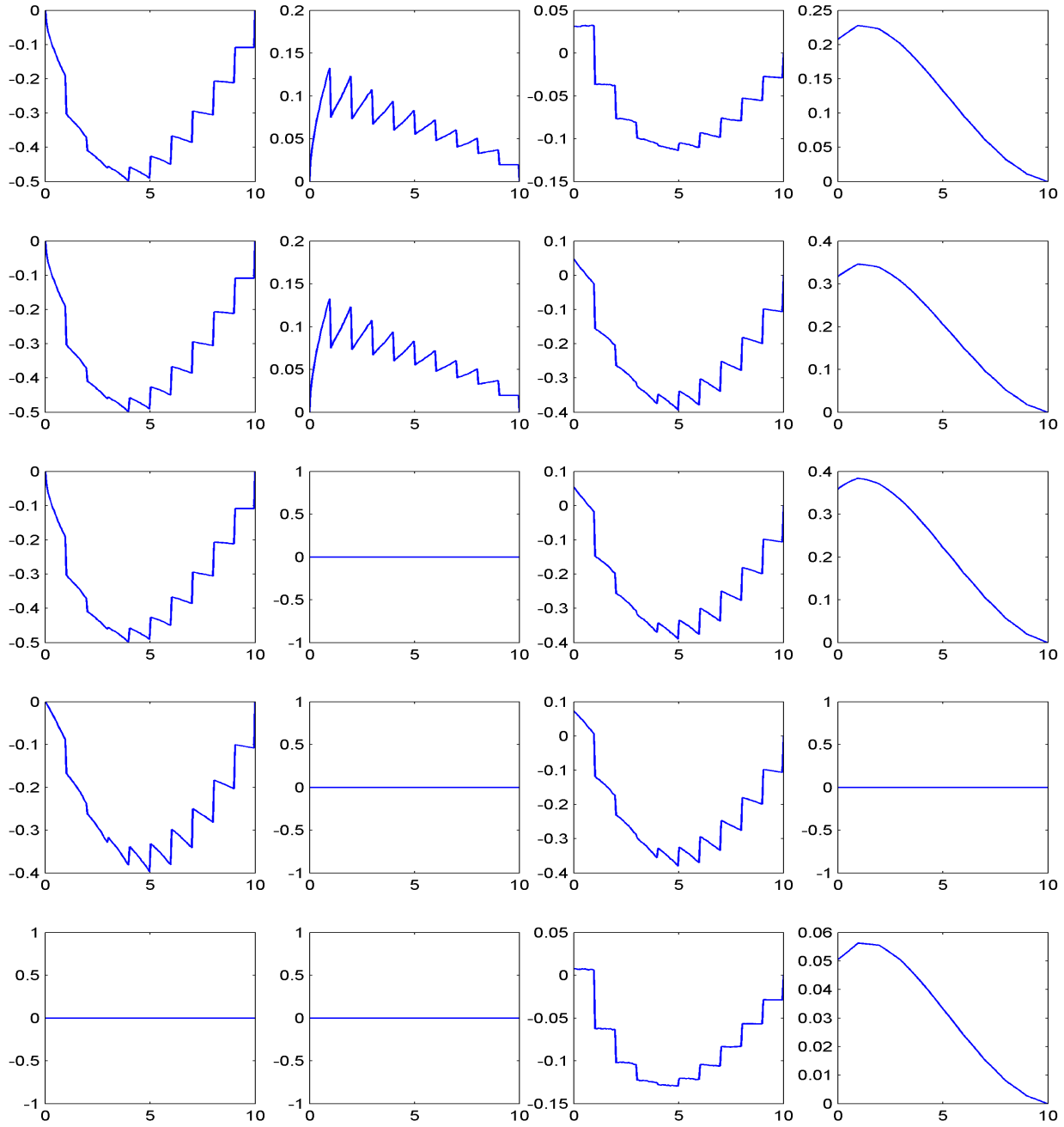


Figure 7: Payer swap in the Lévy model. *Columns*: CVA/DVA/LVA/RC, the 4 wings of the TVA; *Rows*: 5 CSA Specifications.

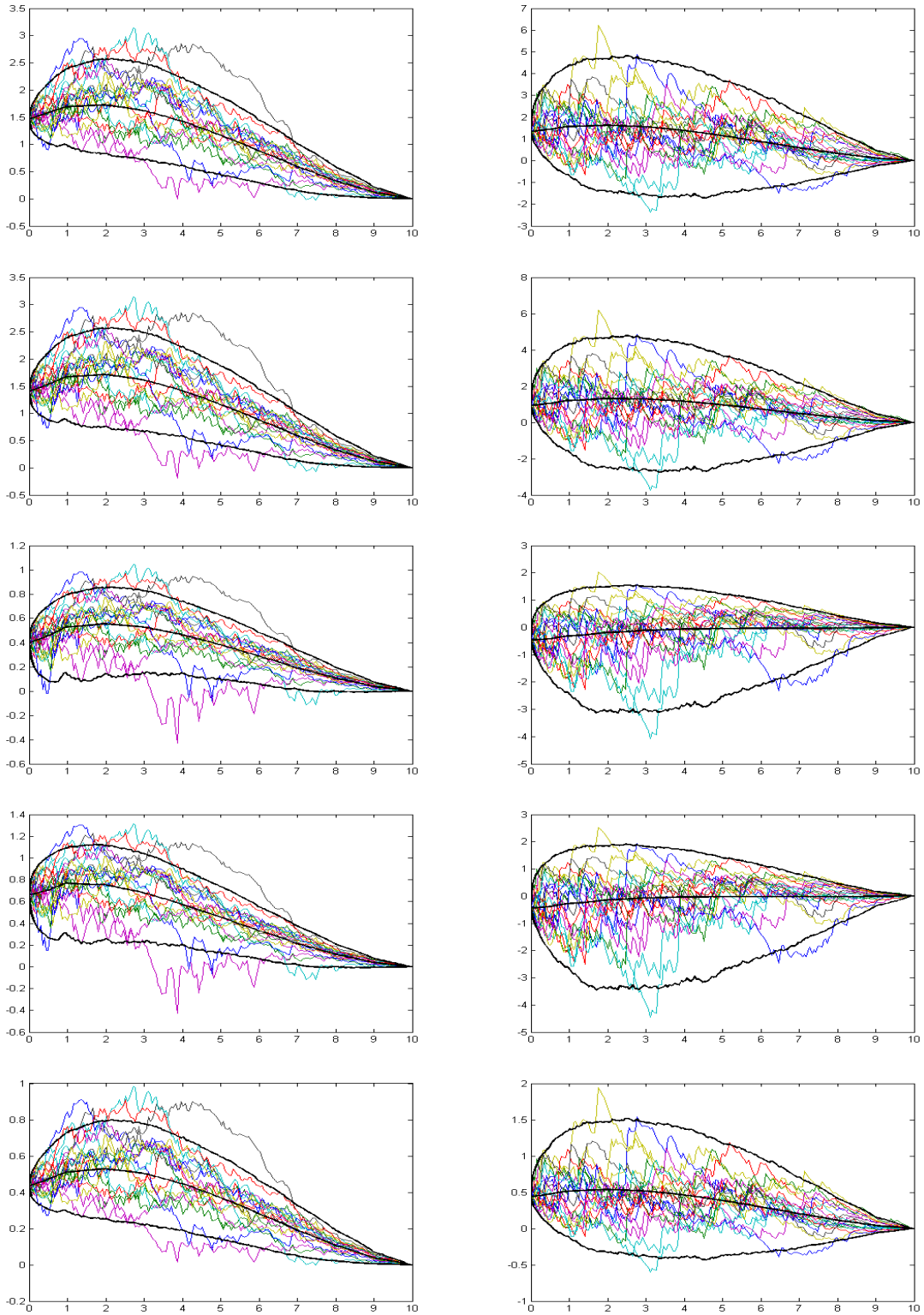


Figure 8: Receiver swap: 20 paths with 200 time points each of the TVA process $\widetilde{\Theta}_t = \widetilde{\Theta}(t, r_t)$. *Left*: Vasicek model ; *Right*: LHW Model. *Top to Bottom*: 5 CSA Specifications.

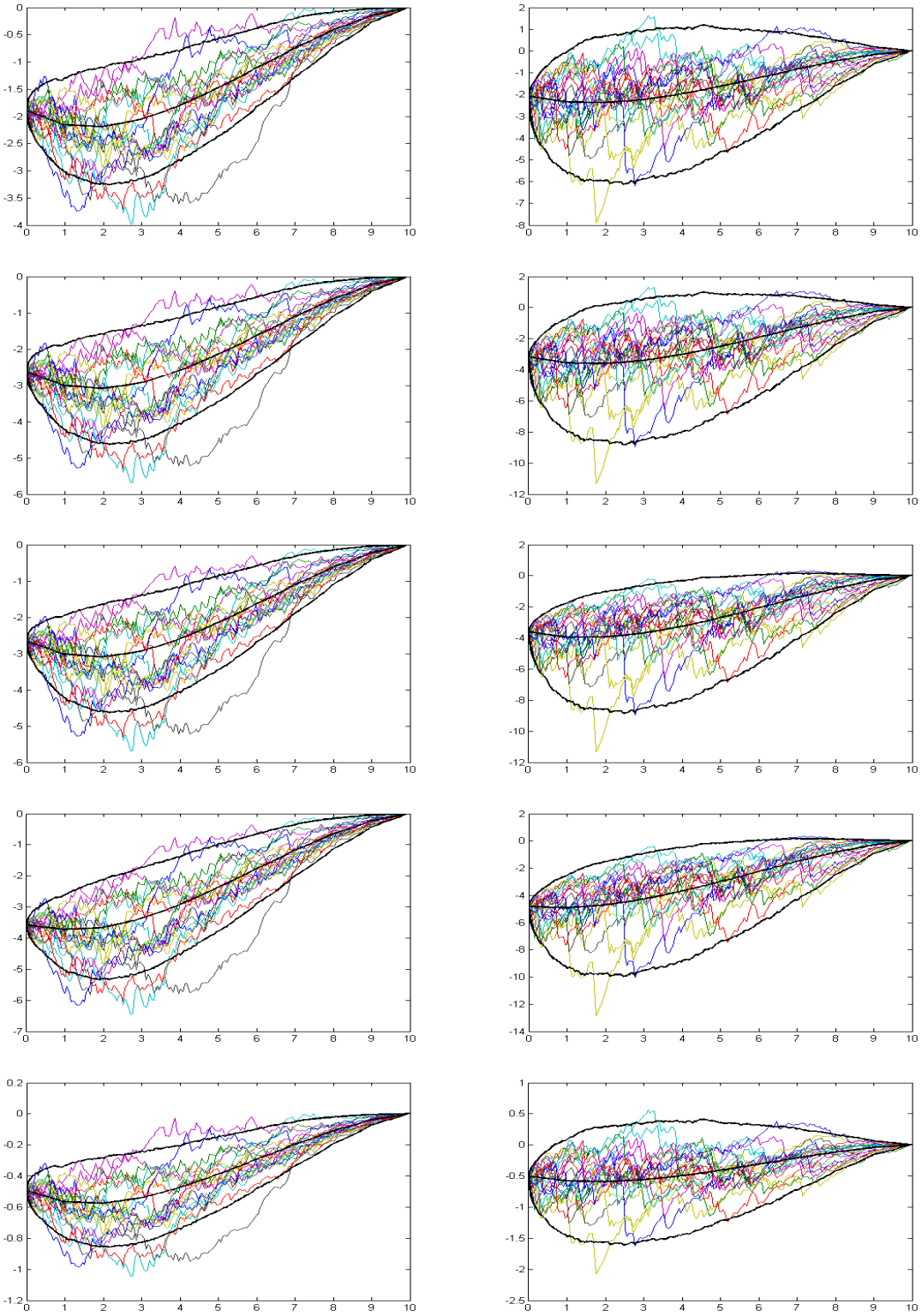


Figure 9: Payer swap: 20 paths with 200 time points each of the TVA process $\widetilde{\Theta}_t = \widetilde{\Theta}(t, r_t)$. *Left*: Vasicek model ; *Right*: LHW Model. *Top to Bottom*: 5 CSA Specifications.

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