Maximum principle for quasilinear SPDE's on a bounded domain without regularity assumptions

Laurent DENIS

Département de Mathématiques
Equipe Analyse et Probabilités
Université d'Evry-Val-d'Essonne
23 Boulevard de France
91037 EVRY Cedex -FRANCE
e-mail: ldenis@univ-evry.fr

Anis MATOUSSI

Laboratoire Manceau de Mathématiques
Université du Maine
Avenue Olivier Messiaen
72 085 LE MANS Cedex - FRANCE
e-mail: anis.matoussi@univ-lemans.fr

Abstract:
We prove a maximum principle for local solutions of quasi-linear parabolic stochastic PDEs, with non-homogeneous second order operator on a bounded domain and driven by a space-time white noise. Our method based on an approximation of the domain and the coefficients of the operator, does not require regularity assumptions. As in previous works [8, 9] the results are consequences of Itô's formula and estimates for the positive part of local solutions which are non-positive on the lateral boundary.

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1. Introduction

In the theory of deterministic Partial Differential Equations, the maximum principle plays an important role since it gives a relation between the bound of the solution on the boundary and a bound on the whole domain. In the deterministic case, the maximum principle for quasi-linear parabolic equations was proved by Aronson -Serrin (see Theorem 1 of [2]). In a previous work [9], we have adapted the method of these authors to the stochastic framework and proved maximum principle for SPDE's with homogeneous second order operator and driven by a finite dimensional Brownian motion. The aim of the present paper is to generalize these results to the case of SPDE's with non-homogeneous second order operator and driven by a noise which is white in time and colored in space. In [8] and [9], many proofs are based on the notion of semigroup associated to the second order operator and on the regularizing property of the semigroup. But now, since in this present paper the operator is non homogeneous we can not follow exactly the same proofs and so we

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work with the Green function associated to the operator and use heavily the results of Aronson [1] on the existence and the Gaussian estimates of the weak fundamental solution of a parabolic PDE.
More precisely, we study the following stochastic partial differential equation (in short SPDE) for a real-valued random field $u_t(x) = u(t,x)$,

$$du_t(x) = \Big(\sum_{i=1}^{d} \partial_i \big[\sum_{j=1}^{d} a_{i,j}(t,x)\partial_j u_t(x) + g_i(t,x,u_t(x),\nabla u_t(x)) \big] + f(t,x,u_t(x),\nabla u_t(x)) \Big) dt + \sum_{i=1}^{+\infty} h_i(t,x,u_t(x),\nabla u_t(x))\, dB_t^i, \tag{1}$$

with a given initial condition $u_0 = \xi$, where a is a time-dependant symmetric, uniformly elliptic, measurable matrix defined on some bounded open domain $\mathcal{O} \subset \mathbb{R}^d$ and $f, g_i, i = 1, \cdots, d, h_j, j = 1, 2, \cdots$ are nonlinear random functions.

This class of SPDE's has been widely studied by many authors (see [18], [4], [25],....) but in all these references, regularity assumptions are made on the boundary of the domain or on a which permit to use Sobolev embedding theorems or/and regularity of the Green function. Since in this work coefficients $a_{i,j}$ and the domain $\mathcal{O}$ are not smooth, the associated Green function is not regular enough, so one more time we follow the ideas of Aronson (see [1]). The method consists in approximating the domain by an increasing sequence of smooth domains and the matrix a by a sequence of smooth matrices. We first prove existence and uniqueness for the SPDE (1) with null Dirichlet condition on the boundary. Then we get some estimates of the positive part of a local solution which is non-negative on the boundary and this permits to get a comparison Theorem and a maximum principle, in this part only we assume that the boundary of the domain is Lipschitz . This yields for example the following result:

Theorem 1. *Let $(M_t)_{t \geqslant 0}$ be an Itô process satisfying some integrability conditions, $p \geqslant 2$ and u be a local weak solution of (1). Assume that ∂O is Lipschitz and that $u \leqslant M$ on the parabolic boundary $\{[0,T[\times\partial\mathcal{O}\} \cup \{\{0\} \times \mathcal{O}\}$, then for all $t \in [0,T]$:*

$$E \left\| (u-M)^+ \right\|_{\infty,\infty;t}^p \leq k(p,t)\, E \left(\|\xi - M_0\|_\infty^p + \left\| (f^{0,M})^+ \right\|_{\theta,t}^{*p} + \left\| |g^{0,M}|^2 \right\|_{\theta;t}^{*p/2} + \left\| |h^{0,M}|^2 \right\|_{\theta;t}^{*p/2} \right)$$

where $f^{0,M}(t,x) = f(t,x,M,0)$, $g^{0,M}(t,x) = g(t,x,M,0)$, $h^{0,M}(t,x) = h(t,x,M,0)$ and k is a function which only depends on the structure constants of the SPDE, $\|\cdot\|_{\infty,\infty;t}$ is the uniform norm on $[0,t] \times \mathcal{O}$ and $\|\cdot\|_{\theta;t}^*$ is a certain norm which is precisely defined below.

For the references concerning the study of the L^p norms w.r.t. the randomness of uniform norm on the trajectories of a stochastic PDE, see [9]. Let us also mention that some L^p-estimates have been established by Kim [15] for a class of parabolic spde's on Lipschitz domain where coefficients f, g, h are also random but depend linearly of the solution. Moreover, Krylov [17] obtained a maximum principle for the same class of SPDE's, its approach is also based on the estimate of the positive part of the solution, nevertheless it does encompass the class of SPDE's we study and our method is different.

The paper is organized as follows : in section 2 we introduce notations and hypotheses and we take care to detail the integrability conditions which are used all along the paper. In section 3 we establish an Itô formula for the solution and prove existence and uniqueness of this solution with null Dirichlet condition on the boundary. In section 4, we prove an Itô's formula and estimates for the positive part of a local solution which is non-positive on the boundary of the domain and obtain a comparison Theorem which leads to our main result: the maximum principle Theorem. The last section is an Appendix devoted to the definitions of some functional spaces that we use and to the proofs of some technical results.

2. Preliminaries

2.1. $L^{p,q}$-spaces

Let $\mathcal{O}$ be an open bounded domain in $\mathbb{R}^d$. The space $L^2(\mathcal{O})$ is the basic Hilbert space of our framework and we employ the usual notation for its scalar product and its norm,

$$(u,v)=\int_{\mathcal{O}} u(x)\,v(x)\,dx,\ \|u\|=\left(\int_{\mathcal{O}} u^2(x)\,dx\right)^{\frac{1}{2}}.$$

In general, we shall extend the notation

$$(u,v)=\int_{\mathcal{O}} u(x)v(x)\,dx,$$

where u, v are measurable functions defined on $\mathcal{O}$ such that $uv\in L^1(\mathcal{O})$.
The first order Sobolev space of functions vanishing at the boundary will be denoted as usual by $H_0^1(\mathcal{O})$. Its natural scalar product and norm are

$$(u,v)_{H_0^1(\mathcal{O})}=(u,v)+\sum_{i=1}^{d}\int_{\mathcal{O}}\partial_i u(x)\,\partial_i v(x)\,dx,\ \|u\|_{H_0^1(\mathcal{O})}=\left(\|u\|_2^2+\|\nabla u\|_2^2\right)^{\frac{1}{2}}.$$

We shall denote by $H^1_{loc}(\mathcal{O})$ the space of functions which are locally square integrable in $\mathcal{O}$ and which admit first order derivatives that are also locally square integrable.
Another Hilbert space that we use is the second order Sobolev space $H_0^2(\mathcal{O})$ of functions vanishing at the boundary and twice differentiable in the weak sense.
For each $t>0$ and for all real numbers p, $q\geqslant 1$, we denote by $L^{p,q}([0,t]\times\mathcal{O})$ the space of (classes of) measurable functions $u:[0,t]\times\mathcal{O}\longrightarrow\mathbb{R}$ such that

$$\|u\|_{p,q;\,t}:=\left(\int_0^t\left(\int_{\mathcal{O}}|u(s,x)|^p\,dx\right)^{q/p}ds\right)^{1/q}$$

is finite. The limiting cases with p or q taking the value ∞ are also considered with the use of the essential sup norm.
The space of measurable functions $u:\mathbb{R}_+\to L^2(\mathcal{O})$ such that $\|u\|_{2,2;t}<\infty$, for each $t\geq 0$, is denoted by $L^2_{loc}\left(\mathbb{R}_+;L^2(\mathcal{O})\right)$, where $\mathbb{R}_+$ denotes the set of non-negative real numbers.

Similarly, the space $L^2_{loc}\left(\mathbb{R}_+; H^1_0\left(\mathcal{O}\right)\right)$ consists of all measurable functions $u : \mathbb{R}_+ \to H^1_0\left(\mathcal{O}\right)$ such that

$$\|u\|_{2,2;t} + \|\nabla u\|_{2,2;t} < \infty,$$

for any $t \geq 0$.

We recall that the Sobolev inequality states that

$$\|u\|_{2^*} \leq c_S \|\nabla u\|_2 ,$$

for each $u \in H^1_0\left(\mathcal{O}\right)$, where $c_S > 0$ is a constant that depends on the dimension and $2^* = \frac{2d}{d-2}$ if $d > 2$, while 2^* may be any number in $]2, \infty[$ if $d = 2$ and $2^* = \infty$ if $d = 1$.
Finally, we introduce the following norm which is obtained by interpolation in $L^{p,q}$-spaces:

$$\|u\|_{\#;t} = \|u\|_{2,\infty;t} \vee \|u\|_{2^*,2;t} ,$$

and we denote by $L_{\#;t}$ the set of functions u such that $\|u\|_{\#;t}$ is finite. Its dual space is a functional space: $L^*_{\#;t}$ equipped with the norm $\| \ \|^*_{\#;t}$ and we have

$$\int_0^t \int_{\mathcal{O}} u\left(s,x\right) v\left(s,x\right) dxds \leq \|u\|_{\#;t} \|v\|^*_{\#;t} , \tag{2}$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.
See Appendix 5.1 for more details on these spaces.

2.2. Hypotheses and definitions

We consider a sequence $((B^i(t))_{t \geqslant 0})_{i \in \mathbb{N}^*}$ of independent Brownian motions defined on a standard filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t \geqslant 0}, P)$ satisfying the usual conditions. Let a be a measurable and symmetric matrix defined on $\mathbb{R}_+ \times \mathcal{O}$. We assume that there exist positive constants λ , Λ and M such that for all $\xi \in \mathbb{R}^d$ and almost all $(t,x) \in \mathbb{R}_+ \times \mathcal{O}$:

$$\lambda|\xi|^2 \leqslant \sum_{i,j} a_{i,j}(t,x)\xi^i \xi^j \leqslant \Lambda|\xi|^2 \text{ and } |a_{i,j}(t,x)| \leqslant M. \tag{3}$$

Let $\Delta = \{(t,x,s,y) \in \mathbb{R}_+ \times \mathcal{O} \times \mathbb{R}_+ \times \mathcal{O}; t > s\}$. We denote by $G : \Delta \mapsto \mathbb{R}_+$ the weak fundamental solution of the problem

$$\partial_t G(t,x;s,y) - \sum_{i,j=1}^{d} \partial_i a_{i,j}(t,x) \partial_j G(t,x;s,y) = 0 \tag{4}$$

with Dirichlet boundary condition $G(t,x,s,y) = 0, \quad$ for all $(t,x) \in \ (s, +\infty) \times \partial\mathcal{O}$ and where for $i \in \{1, \cdots, d\}$, ∂_i denotes the partial derivative of oder 1 with respect to x_i.
Sometimes, for convenience, we shall restrict ourselves to a finite time-interval, that's why we fix a time $T > 0$.
Following Aronson ([1]), Theorem 9 (iii) p. 671, we have the following estimate:

$$G(t,x,s,y) \leqslant C(t-s)^{-\frac{d}{2}} \exp\{-\varrho \frac{|x-y|^2}{8(t-s)}\}, \tag{5}$$

for all $(t,x,s,y) \in \Delta$ with $t \leqslant T$, where C and ϱ are positive constants depending only on T and the structure constants i.e. λ, Λ and the Lebesgue measure of $\mathcal{O}$. Let us point out that we do not assume that coefficients are smooth or that $\partial\mathcal{O}$ is regular.
We consider predictable random functions

$$\begin{aligned} f &: \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\,, \\ g &= (g_1, ..., g_d) : \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \\ h = (h_1, \cdots, h_i, \cdots) &: \mathbb{R}_+ \times \Omega \times \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^{\mathbb{N}^*}\,, \end{aligned}$$

where $\mathbb{N}^*$ denotes the set of positive integers.
In the sequel, $|\cdot|$ will always denote the underlying Euclidean or l^2-norm. For example

$$|h(t,w,x,y,z)|^2 = \sum_{i=1}^{+\infty} h_i(t,w,x,y,z)|^2.$$

We define

$$\begin{aligned} &f(\cdot,\cdot,\cdot,0,0) := f^0 \\ &h(\cdot,\cdot,\cdot,0,0) := h^0 = (h_1^0, ..., h_i^0, \cdots) \\ &g(\cdot,\cdot,\cdot,0,0) := g^0 = (g_1^0, ..., g_d^0). \end{aligned}$$

We still consider the quasilinear stochastic partial differential equation (1) for the real-valued random field $u_t(x)$, that we rewrite as:

$$\begin{aligned} du_t(x) = \left(\sum_{i,j=1}^{d} \partial_i a_{i,j}(t,x)\partial_j u_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \sum_{i=1}^{d} \partial_i g_i(t,x,u_t(x),\nabla u_t(x)) \right) dt \\ + \sum_{i=1}^{+\infty} h_i(t,x,u_t(x),\nabla u_t(x))\, dB_t^i, \end{aligned} \tag{6}$$

with initial condition $u(0,.) = \xi(.)$. Let us point out that in the equation (6), the divergence term $\partial_i g_i(t,x,u_t(x),\nabla u_t(x))$ has to be understod as

$$\frac{\partial}{\partial x_i}\left(g_i(t,x,u_t(x),\nabla u_t(x))\right),$$

and is defined rigorously in the weak sense (by integration by parts).
We also assume that ξ is a $\mathcal{F}_0$-measurable, $L^2(\mathcal{O})$-valued random variable. We consider the following sets of assumptions :

Assumption (H): There exist non negative constants C, α, β such that for almost all ω, the following inequalities hold for all $(x,y,z,t) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$:

(i) $|f(t,\omega,x,y,z) - f(t,\omega,x,y',z')| \leqslant C\big(|y-y'| + |z-z'|\big)$

(ii) $\Big(|h(t,\omega,x,y,z) - h(t,\omega,x,y',z')|^2\Big)^{\frac{1}{2}} \leqslant C\,|y-y'| + \beta\,|z-z'|,$

(iii) $\Big(\sum_{i=1}^d |g_i(t,\omega,x,y,z) - g_i(t,\omega,x,y',z')|^2\Big)^{\frac{1}{2}} \leqslant C\,|y-y'| + \alpha\,|z-z'|.$

(iv) the contraction property : $\alpha + \dfrac{\beta^2}{2} < \lambda$.

Moreover we introduce some integrability conditions on f^0, g^0, h^0 and the initial data ξ :

Assumption (HD#)

$$E\left(\left(\left\|f^0\right\|^*_{\#;t}\right)^2 + \left\|g^0\right\|^2_{2,2;t} + \left\|h^0\right\|^2_{2,2;t}\right) < \infty,$$

for each $t \geq 0$.

Sometimes we shall consider the following stronger conditions:

Assumption (HD2)

$$E\left(\left\|f^0\right\|^2_{2,2;t} + \left\|g^0\right\|^2_{2,2;t} + \left\|h^0\right\|^2_{2,2;t}\right) < \infty,$$

for each $t \geq 0$.

Assumption (HI2) *integrability condition on the initial condition* :

$$E\|\xi\|^2 < \infty.$$

Remark 1. *Note that* $(2,1)$ *is the pair of conjugates of the pair* $(2,\infty)$ *and so* $(2,1)$ *belongs to the set* I' *which defines the space* $L^*_{\#;t}$ *(see the Appendix for more details). Since* $\|v\|_{2,1;t} \leq \sqrt{t}\,\|v\|_{2,2;t}$ *for each* $v \in L^{2,2}\left([0,t]\times\mathcal{O}\right)$ *, it follows that*

$$L^{2,2}\left([0,t]\times\mathcal{O}\right) \subset L^{2,1;t} \subset L^*_{\#;t},$$

and $\|v\|^*_{\#;t} \leq \sqrt{t}\,\|v\|_{2,2;t}$ *, for each* $v \in L^{2,2}\left([0,t]\times\mathcal{O}\right)$ *. This shows that the condition* ***(HD#)*** *is weaker than* ***(HD2)****.*

2.3. Main example of stochastic noise

Let W be a noise white in time and colored in space, defined on a standard filtered probability space $\left(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geqslant 0},P\right)$ whose covariance function is given by:

$$\forall s,t\in\mathbb{R}_+,\ \forall x,y\in\mathcal{O},\quad E[\dot{W}(x,s)\dot{W}(y,t)] = \delta(t-s)k(x,y),$$

where $k:\mathcal{O}\times\mathcal{O}\mapsto\mathbb{R}_+$ is a symmetric and measurable function.
Consider the following SPDE driven by W:

$$\begin{aligned} du_t(x) = \big(\sum_{i,j=1}^{d}\partial_i a_{i,j}(t,x)\partial_j u_t(x) + f(t,x,u_t(x),\nabla u_t(x)) + \sum_{i=1}^{d}\partial_i g_i(t,x,u_t(x),\nabla u_t(x))\big)dt \\ + \tilde{h}(t,x,u_t(x),\nabla u_t(x))\,W(dt,x), \end{aligned} \tag{7}$$

where f and g are as above and $\tilde{h}$ is a random real valued function.
We assume that the covariance function k defines a trace class operator denoted by K in

$L^2(\mathcal{O})$. It is well known (see [23]) that there exists an orthogonal basis $(e_i)_{i\in\mathbb{N}^*}$ of $L^2(\mathcal{O})$ consisting of eigenfunctions of K with corresponding eigenvalues $(\lambda_i)_{i\in\mathbb{N}^*}$ such that

$$\sum_{i=1}^{+\infty}\lambda_i < +\infty,$$

and

$$k(x,y)=\sum_{i=1}^{+\infty}\lambda_i e_i(x)e_i(y).$$

It is also well known that there exists a sequence $((B^i(t))_{t\geqslant 0})_{i\in\mathbb{N}^*}$ of independent standard Brownian motions such that

$$W(dt,\cdot)=\sum_{i=1}^{+\infty}\lambda_i^{1/2}e_iB^i(dt).$$

So that equation (7) is equivalent to (6) with $h=(h_i)_{i\in\mathbb{N}^*}$ where

$$\forall i\in\mathbb{N}^*,\ h_i(s,x,y,z)=\sqrt{\lambda_i}\tilde{h}(s,x,y,z)e_i(x).$$

Assume as in [25] that for all $i\in\mathbb{N}^*$, $\|e_i\|_\infty<+\infty$ and

$$\sum_{i=1}^{+\infty}\lambda_i\|e_i\|_\infty^2<+\infty.$$

Since

$$\left(|h(t,\omega,x,y,z)-h(t,\omega,x,y',z')|^2\right)^{\frac{1}{2}}\leqslant\left(\sum_{i=1}^{+\infty}\lambda_i\|e_i\|_\infty^2\right)\left|\tilde{h}(t,x,y,z)-\tilde{h}(t,x,y',z')\right|^2,$$

h satisfies the Lipschitz hypothesis **(H)-(ii)** if $\tilde{h}$ satisfies a similar Lipschitz hypothesis.

2.4. Spaces of processes and notion of weak solutions

We shall denote by $\mathcal{P}$ the set of predictable processes which admit a version in $L^2_{loc}\left(\mathbb{R}_+;L^2(\mathcal{O})\right)$. We now introduce $\mathcal{H}=\mathcal{H}(\mathcal{O})$, the space of $H_0^1(\mathcal{O})$-valued predictable processes $(u_t)_{t\geqslant 0}$ such that

$$\|u\|_t=\left(E\sup_{0\leqslant s\leqslant t}\|u_s\|^2+E\int_0^t\|\nabla u_s\|^2dt\right)^{1/2}<\infty\,,\quad\text{for each } t>0\,.$$

We define $\mathcal{H}_{loc}=\mathcal{H}_{loc}(\mathcal{O})$ to be the set of $H^1_{loc}(\mathcal{O})$-valued predictable processes such that for any compact subset K in $\mathcal{O}$ and all $t>0$:

$$\left(E\sup_{0\leqslant s\leqslant t}\int_K|u_s(x)|^2\,dx+E\int_0^t\int_K|\nabla u_s(x)|^2\,dxdt\right)^{1/2}<\infty.$$

We also denote by $\hat{F}$ the subspace of elements in $\mathcal{H}$ which are $L^2(\mathcal{O})$-continuous. Moreover, we denote by $\mathcal{H}_T$ (resp. $\hat{F}_T$) the set of processes which are the restrictions to $[0,T]$

of elements in $\mathcal{H}$ (resp. $\hat{F}$). Let us remark that $(\hat{F}_T, \|\cdot\|_T)$ is a Banach space.
The space of test functions is $\mathcal{D} = \mathcal{C}_c^\infty(\mathbb{R}_+) \otimes \mathcal{C}_c^2(\mathcal{O})$, where $\mathcal{C}_c^\infty(\mathbb{R}_+)$ denotes the space of all real valued infinitely differentiable functions with compact support in $\mathbb{R}^+$ and $\mathcal{C}_c^2(\mathcal{O})$ the set of C^2-functions with compact support in $\mathcal{O}$.

Definition 2. *We say that $u \in \mathcal{H}_{loc}$ is a weak solution of equation* (6) *with initial condition ξ if the following relation holds almost surely, for each $\varphi \in \mathcal{D}$,*

$$\int_0^\infty [\,(u_s, \partial_s\varphi) - \sum_{i,j=1}^d \int_{\mathcal{O}} a_{i,j}(s,x)\partial_i u_s(x)\partial_j\varphi_s(x)dx + (f\,(s,u_s,\nabla u_s)\,,\varphi_s) \\ - \sum_{i=1}^d (g_i\,(s,u_s,\nabla u_s)\,,\partial_i\varphi_s)]ds + \sum_{i=1}^{+\infty}\int_0^\infty (h_i\,(s,u_s,\nabla u_s)\,,\varphi_s)\,dB_s^i + (\xi,\varphi_0) = 0. \tag{8}$$

We denote by $\mathcal{U}_{loc}(\xi,f,g,h)$ the set of all such solutions u.

In general we do not know much about the set $\mathcal{U}_{loc}\,(\xi,f,g,h)$. It may be empty or may contain several elements. As the Sobolev space $H_0^1\,(\mathcal{O})$ consists of functions which vanish at the boundary $\partial\mathcal{O}$, we say that a solution which belongs to $\mathcal{H}$ satisfies the zero Dirichlet conditions at the boundary of $\mathcal{O}$.
We denote by $\mathcal{U}\,(\xi,f,g,h)$ the solution of (6) with zero Dirichlet boundary conditions whenever it exists and is unique, we shall prove that this is the case for example under **(H)**, **(HI2)** and **(HD2)**.
We should also note that if the conditions **(H)**, **(HD2)** and **(HI2)** are satisfied and if u is a process in $\mathcal{H}$, the relation from this definition holds with any test function $\varphi \in \mathcal{D}$ if and only if it holds for any test function in $\mathcal{C}_c^\infty\,(\mathbf{R}_+) \otimes H_0^1\,(\mathcal{O})$. In fact, in this case, one may use as space of test functions any space of the form $\mathcal{C}_c^\infty\,(\mathbf{R}_+) \otimes V$, where V is a dense subspace of $H_0^1\,(\mathcal{O})$, obtaining equivalent definitions of the notion of solution with null Dirichlet conditions at the boundary of $\mathcal{O}$.
Let us now precise the sense in which a solution is dominated on the lateral boundary.

Definition 3. *If v belongs to $H_{loc}^1(\mathcal{O})$, we say that v is non-positive on the boundary of $\mathcal{O}$ if v^+ belongs to $H_0^1(\mathcal{O})$ and denotes it simply: $v \leqslant 0$ on $\partial\mathcal{O}$.*

3. Existence, uniqueness and estimates of the solution with null-Dirichlet condition

3.1. Notion of mild solution

We now turn out to the notion of *mild* solution:

Definition 4. *We say that $u \in \mathcal{H}$ is a mild solution of equation* (6) *with initial condition*

$\xi \in L^2(\Omega \times \mathcal{O})$, *if for all* $t \in \mathbb{R}_+$,

$$\begin{aligned} u_t(x) = &\int_{\mathcal{O}} G(t,x,0,y)\xi(y)\,dy + \int_0^t \int_{\mathcal{O}} G(t,x,s,y) f(s,y,u_s(y),\nabla u_s(y))dyds \\ &+ \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} G(t,x,s,y)\partial_i g_i(s,.,u_s,\nabla u_s)(y)dyds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t,x,s,y) h_i(s,y,u_s(y),\nabla u_s(y))dB_s^i. \end{aligned} \tag{9}$$

Let us remark that thanks to Gaussian estimate (5), all the quantities in (9) are well defined excepted the term

$$\int_0^t \int_{\mathcal{O}} G(t,x,s,y)\partial_{i,y} g_i(s,.,u_s,\nabla u_s)(y)dyds.$$

This last term has to be understood in the weak sense thanks to the following Proposition:

Proposition 5. *Let* $U : \left(C_c^\infty(\mathbb{R}_+) \otimes H_0^1(\mathcal{O})\right)^d \longrightarrow \hat{F}$ *be defined by*

$$\forall \tilde{w} \in \left(C_c^\infty(\mathbb{R}_+) \otimes H_0^1(\mathcal{O})\right)^d,\ \forall t \geqslant 0,\ (U\tilde{w})_t = \sum_{i=1}^{d} \int_0^t G(t,\cdot,s,y)\partial_{i,y}\tilde{w}_{i,s}(y)dyds.$$

The operator U *admits a uniquely determined continuous extension*

$$U : L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{O})^d) \longrightarrow \hat{F},$$

that we still denote

$$\forall t \geqslant 0,\ (U\tilde{w})_t = \sum_{i=1}^{d} \int_0^t G(t,\cdot,s,y)\partial_{i,y}\tilde{w}_{i,s}(y)dyds.$$

Moreover if $\tilde{w} \in L^2_{loc}(\mathbb{R}_+; L^2(\mathcal{O})^d)$, $u = U\tilde{w}$ *is the weak solution of*

$$du_t(x) = \sum_{i,j=1}^{d} \partial_i a_{i,j}(t,x)\partial_j u_t(x) + \sum_{i=1}^{d} \partial_i \tilde{w}_i\ ,\ u_0 = 0,$$

and it satisfies the following relation:

$$\frac{1}{2}\|u_t\|^2 + \int_0^t \sum_{i,j=1}^{d} \int_{\mathcal{O}} a_{i,j}(s,x)\partial_i u_s(x)\partial_j u_s(x)\,dx\,ds = -\sum_{i=1}^{d} \int_0^t (\tilde{w}_{i,s}, \partial_i u_s)\ ds,\ \ t \geqslant 0. \tag{10}$$

As a consequence, we have the following estimate:

$$\|u\|_T^2 \leqslant C_\lambda \int_0^T \|\tilde{w}_s\|^2\,ds, \tag{11}$$

where C_λ *is a constant depending only on* λ.

Proof. See Subsection 5.2 in the Appendix. □

3.2. The linear case

Let $\xi \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathcal{O}))$, $w = (w_i)_{i\in\mathbb{N}^*} \in \mathcal{P}^{\mathbb{N}^*}, w' \in \mathcal{P}, w'' \in \mathcal{P}^d$. We assume that

$$|w| = \left(\sum_{i=1}^{+\infty} |w_i^2|\right)^{1/2} \in \mathcal{P}.$$

We set

$$\begin{aligned} u_t(\cdot) = &\int_{\mathcal{O}} G(t,\cdot,0,y)\xi(y)\,dy + \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)w'_s(y)dyds \\ &+ \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\partial_i w''_i(y)dyds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)w_{i,s}(y)dB^i_s. \end{aligned} \tag{12}$$

The goal of this section is to prove that u is the unique solution of the *linear* equation

$$du_t(x) = \Big(\sum_{i=1}^{d} \partial_i\big[\sum_{j=1}^{d} a_{i,j}(t,x)\partial_j u_t(x) + w''_{i,s}(x)\big] + w'_s(x)\Big)dt + \sum_{i=1}^{+\infty} w_{i,s}(x)\,dB^i_t, \tag{13}$$

with initial condition $u_0 = \xi$ and zero Dirichlet condition on the boundary:

$$u(t,x) = 0 \quad \forall (t,x) \in (0,+\infty)\times\partial\mathcal{O}.$$

To this end we proceed as follows: first we prove the result in the case where all the coefficients are regular and then, using an approximation procedure, we prove it in the general case. This second part is quite long and we shall split the proof in several steps.

The regular case

We assume first that all the coefficients are regular and that $\partial\mathcal{O}$ is smooth. In this case, existence and uniqueness are well known (see for example [18]), nevertheless we give the proof in order to explicit the estimates we need to pass to the limit in the general case.

Proposition 6. *Assume that $\partial\mathcal{O}$ is smooth, all the coefficients $a_{i,j}$ belong to $C^\infty\left(\mathbb{R}_+\times\overline{\mathcal{O}}\right)$, $\xi \in C_c^\infty(\mathcal{O})$, $w, w' \in \left(L^2(\Omega)\otimes C_c([0,+\infty))\otimes C_c^\infty(\mathcal{O})\right)\bigcap\mathcal{P}$ and $w'' \in \left(L^2(\Omega)\otimes C_c([0,+\infty))\otimes C_c^\infty(\mathcal{O})\right)^d\bigcap\mathcal{P}^d$. We set*

$$\begin{aligned} u_t(\cdot) = &\int_{\mathcal{O}} G(t,\cdot,0,y)\xi(y)\,dy + \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)w'_s(y)dyds \\ &+ \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\partial_{i,y} w''_i(y)dyds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)w_{i,s}(y)dB^i_s\,. \end{aligned} \tag{14}$$

Then u has a version in $\hat{F}$ and is the unique solution in the weak sense of (13) *in $\mathcal{H}$ i.e. the unique element in $\mathcal{H}$ such that for each $\varphi \in \mathcal{D}$, the following relation holds almost surely:*

$$\begin{aligned}(\xi,\varphi_0)+\int_0^\infty[(u_s,\partial_s\varphi)-\sum_{i,j=1}^d\int_{\mathcal{O}}a_{i,j}(s,x)\partial_i u_s(x)\partial_j\varphi_s(x)dx-\left(w'_s,\varphi_s\right)]ds\\ +\int_0^{+\infty}\sum_{i=1}^d\left(w''_{i,s},\partial_i\varphi_s\right)ds+\sum_{i=1}^{+\infty}\int_0^\infty\left(w_{i,s},\varphi_s\right)dB_s^i=0.\end{aligned} \tag{15}$$

Moreover, we have the following estimates for all $t \geqslant 0$:

$$\begin{aligned}&\|u_t\|^2+2\int_0^t\sum_{i,j}\int_{\mathcal{O}}a_{i,j}(s,x)\partial_i u_s(x)\partial_j u_s(x)dxds=\|\xi\|^2+2\sum_i^{+\infty}\int_0^t(w_{i,s},u_s)\,dB_s^i\\ &+2\int_0^t\left(u_s,w'_s\right)ds-2\sum_i\int_0^t\left(\partial_i u_s,w''_{i,s}\right)ds+\int_0^t|||w_s|||^2\,ds,\end{aligned} \tag{16}$$

and

$$E[\|u\|_T^2]\leqslant cE\left[\|\xi\|^2+\int_0^T\left(|||w_t|||^2+\left\|w'_t\right\|^2+\sum_i\left\|w''_{i,t}\right\|^2\right)dt\right] \tag{17}$$

where c is a constant which only depends on T.

Proof. Following Aronson [1], we know that the weak fundamental solution G is a classical one. Moreover, it is well known that G is one time differentiable with respect to time and infinitely differentiable with respect to space variables in Δ and that we have the following estimate for all $(t,x,s,y)\in\Delta$ and $1\leqslant i,j\leqslant d$ (see [11] for example):

$$|\partial_{i,x}^l\partial_{j,y}^k G(t,x,s,y)| \leqslant C(t-s)^{-\frac{d+l+k}{2}}\exp(-\varrho\frac{|x-y|^2}{(t-s)}), \tag{18}$$

$$|\partial_t G(t,x,s,y)| \leqslant C(t-s)^{-\frac{d}{2}-1}\exp(-\varrho\frac{|x-y|^2}{(t-s)}) \tag{19}$$

with $k,l=0$, 1 or 2 and where $\partial_{i,x}^l$ denotes the partial derivative of order l with respect to the variable x_i.
Due to the regularity of G and of all the coefficients in the expression of u, one can use the fact that G is a strong solution. As a consequence, u is a $H_0^2(\mathcal{O})$-valued semi-martingale with $L^2(\mathcal{O})$-continuous trajectories (see [18], Chapter 1 or [5]) and we have the following integral representation:

$$\begin{aligned}u_t(x)=&\xi(x)+\sum_{i,j=1}^d\int_0^t\partial_i\left(a_{i,j}(s,x)\partial_j u_s(x)\right)ds+\int_0^t w'_s(x)ds+\sum_{i=1}^d\int_0^t\partial_{i,y}w''_{i,s}(x)ds\\ &+\sum_{i=1}^{+\infty}\int_0^t w_{i,s}(x)dB_s^i.\end{aligned} \tag{20}$$

Applying the Itô's formula for Hilbert-valued semimartingaled (see [18] Chapter 1, Section 3) and then integrating with respect to x, we get

$$\begin{aligned}\| u_t\|^2 + 2\int_0^t \int_{\mathcal{O}} a_{i,j}(s,x)\partial_i u_s(x)\partial_j u_s(x)dxds &= \|\xi\|^2 + 2\int_0^t \left(w'_s, u_s\right) ds \\ &+ 2\sum_i \int_0^t \left(w''_{i,s}, \partial_i v_s^{n,m}\right) ds + 2\sum_i \int_0^t \left(w_{i,s}, u_s\right) dB_s^i \\ &+ \int_0^t \| |w_s| \|^2 \, ds.\end{aligned} \tag{21}$$

Fix $\varepsilon > 0$ small. We have for all $t \in [0,T)$:

$$2\left|\int_0^t \left(w'_s, u_s\right) ds\right| \quad \leqslant \quad \varepsilon \int_0^T \|u_s\|^2 ds + \frac{1}{\varepsilon}\int_0^T \|w'_s\|^2 ds$$

and

$$2\left|\sum_i \int_0^t \left(w''_{i,s}, \partial_i u_s\right) ds\right| \quad \leqslant \quad \varepsilon \int_0^T \|\nabla u_s\|^2 ds + \frac{1}{\varepsilon}\int_0^T \| |w''_s| \|^2 ds.$$

Moreover, thanks to the Burkholder-Davies-Gundy inequality, we get

$$\begin{aligned} E[\sup_{t\in[0,T]} \left|\sum_i \int_0^t (w_{i,s}, u_s)\, dB_s^i\right|] &\leqslant c_1 E\left[\left(\int_0^T \sum_{i=1}^{+\infty} (w_{i,s}, u_s)^2 \, ds\right)^{1/2}\right] \\ &\leqslant c_1 E\left[\left(\int_0^T \sum_{i=1}^{+\infty} \sup_{t\in[0,T]} \|u_t\|^2 \, \|w_{i,s}\|^2 dt\right)^{1/2}\right] \\ &\leqslant c_1 E\left[\sup_{t\in[0,T]} \|u_t\| \left(\int_0^T \| |w_s| \|^2 dt\right)^{1/2}\right] \\ &\leqslant \varepsilon E\left[\sup_{t\in[0,T]} \|u_t\|^2\right] + \frac{c_1}{4\varepsilon} E\left[\int_0^T \| |w_s| \|^2 dt\right]. \end{aligned}$$

Then using the ellipticity assumption on a and the inequalities above, by taking the supremun in $t \in [0,T]$ in relation (21) and then the expectation, we get:

$$\begin{aligned}(1 - 2\varepsilon(T+1))E[\sup_{t\in[0,T]} \| u_t\|^2] + (2\lambda - \varepsilon)E\int_0^t \|\nabla u_s\|^2 \, ds \leqslant 2\|\xi\|^2 \\ + \frac{2}{\varepsilon}E\int_0^T \|w'_s\|^2 ds + \frac{2}{\varepsilon}E\int_0^T \| |w''_s| \|^2 ds + \frac{c_1}{2\varepsilon}E\left[\int_0^T \| |w_s| \|^2 dt\right].\end{aligned} \tag{22}$$

Taking ε small enough, we deduce that we have the following a priori estimate:

$$E\,\|u\|_T^2 \leqslant cE\left(\|\xi\|^2 + \int_0^T \left(\| |w_t| \|^2 + \left\|w'_t\right\|^2 + \sum_i \left\|w''_{i,t}\right\|^2\right) dt\right) \tag{23}$$

where c is a constant which only depends on T and λ but not on $\mathcal{O}$. This proves inequality (17).
Relation (15) and the fact that u is a weak solution are direct consequences of Itô's formula. Finally, uniqueness is clear, indeed if v is another element in $\mathcal{H} \bigcap L^2_{loc}(\mathbb{R}_+; L^2(\Omega; H^1_0(\mathcal{O})$ which satisfies (15) for all $\varphi \in C^\infty_c([0,+\infty[) \otimes C^\infty_c(\mathcal{O})$, then $\zeta = u - v$ satisfies

$$\int_0^\infty [(\zeta_s, \partial_s \varphi) - \sum_{i,j} \int_{\mathcal{O}} a_{i,j}(s,x) \partial_i \zeta_s(x) \partial_j \varphi_s(x) dx] ds = 0, \tag{24}$$

standard results on deterministic PDE's ensure that $\zeta = 0$.

□

The general case

Here, we only assume that a is measurable and satisfies assumption (3), that $\mathcal{O}$ is a bounded open domain without any condition on its boundary and we are given coefficients: $\xi \in L^2(\Omega, \mathcal{F}_0, P; L^2(\mathcal{O}))$, $w' \in \mathcal{P}$, $w'' = (w''_1, \cdots, w''_d) \in \mathcal{P}^d$ and $w = (w_i)_{i \in \mathbb{N}^*} \in \mathcal{P}^{\mathbb{N}^*}$, such that $E[\int_0^T |||w_s|||^2 \, ds] < +\infty$.
We first prove that Proposition 6 remains true in this case and then we establish Itô's formula for the solution. To do that, we approximate the coefficients, the domain and the second order operator in the following way:

1. We mollify coefficients $a_{i,j}$ and so consider sequences $(a^n_{i,j})_n$ of C^∞ functions such that for all $n \in \mathbb{N}^*$, the matrix a^n satisfies the same ellipticity and boundedness assumptions as a and

$$\forall 1 \leqslant i,j \leqslant d, \ \lim_{n \to +\infty} a^n_{i,j} = a_{i,j} \ a.e.$$

2. We approximate $\mathcal{O}$ by an increasing sequence of smooth domains $(\mathcal{O}^n)_{n \geqslant 1}$.
3. We consider a sequence (ξ^n) in $C^\infty_c(\mathcal{O})$ which converges to ξ in $L^2(\mathcal{O})$ and such that for all n, $\operatorname{supp} \xi^n \subset \mathcal{O}^n$.
4. For each $i \in \mathbb{N}^*$, we construct a sequence (w^n_i) in $\big(L^2(\Omega) \otimes C_c([0,+\infty)) \otimes C^\infty_c(\mathcal{O})\big) \bigcap \mathcal{P}$ which converges in $L^2_{loc}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O}))$ to w_i such that for all n, $\operatorname{supp} w^n_i \subset \mathcal{O}^n$ and

$$\forall t \geqslant 0, \ E[\int_0^t \|w^n_{i,s}\|^2 \, ds] \leqslant E[\int_0^t \|w_{i,s}\|^2 \, ds],$$

   so that

$$E[\int_0^t |||w^n_s|||^2 ds] \leqslant E[\int_0^t |||w_s|||^2 ds] < +\infty.$$

5. We consider a sequence $(w'^{,n})$ in $\big(L^2(\Omega) \otimes C_c([0,+\infty)) \otimes C^\infty_c(\mathcal{O})\big) \bigcap \mathcal{P}$ which converges in $L^2_{loc}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O}))$ to w' and such that for all n, $\operatorname{supp} w'^{,n} \subset \mathcal{O}^n$.
6. Finally, let $(w''^{,n})$ be a sequence in $\big(L^2(\Omega) \otimes C_c([0,+\infty)) \otimes C^\infty_c(\mathcal{O})\big)^d \bigcap \mathcal{P}^d$ which converges in $L^2_{loc}(\mathbb{R}_+; L^2(\Omega \times \mathcal{O})^d)$ to $w''$and such that for all n, $\operatorname{supp} w''^{,n} \subset \mathcal{O}^n$.

For all $n \in \mathbb{N}^*$, we put $\Delta^n = \{(t,x,s,y) \in \mathbb{R}_+ \times \mathcal{O}^n \times \mathbb{R}_+ \times \mathcal{O}^n; t > s\}$. We denote by $G^n : \Delta^n \mapsto \mathbb{R}_+$ the weak fundamental solution of the problem (4) associated to a^n and

$\mathcal{O}^n$:

$$\partial_t G^n(t,x;s,y) - \sum_{i,j=1}^{d} \partial_i a^n_{i,j}(t,x)\partial_j G^n(t,x;s,y) = 0 \tag{25}$$

with Dirichlet boundary condition $G^n(t,x,s,y) = 0, \quad \text{for all } (t,x) \in (s,+\infty)\times\partial\mathcal{O}^n$.
In a natural way we extend G^n on Δ by setting: $G^n \equiv 0$ on $\Delta \setminus \Delta^n$.
We define the process u^n by setting for all $(t,x) \in \mathbb{R}_+ \times \mathcal{O}$:

$$\begin{aligned} u^n_t(x) = &\int_{\mathcal{O}} G^n(t,x,0,y)\xi^n(y)dy + \int_0^t\int_{\mathcal{O}} G^n(t,x,s,y)w'^{,n}_s(y)dyds \\ &- \sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}} \partial_{i,y}G^n(t,x,s,y)w''^{,n}_{i,s}(y)dyds \\ &+ \sum_{i=1}^{+\infty}\int_0^t\int_{\mathcal{O}} G^n(t,x,s,y)w^n_{i,s}(y)\,dydB^i_s\,. \end{aligned} \tag{26}$$

The key Lemma is the following:

Lemma 7. *There exists a subsequence of $(G^n)_{n\geqslant 1}$ which converges everywhere to G on Δ, where G still denotes the fundamental solution of (4).*

Proof. Let K be a compact subset of Δ. There exists $\varepsilon > 0$, $\eta > 0$ such that for $(t,x) \in K$, $|t-s| \geqslant \eta$, $d(x,\partial\mathcal{O}_n) \geqslant \varepsilon$ and $d(y,\partial\mathcal{O}_n) \geqslant \varepsilon$, for n large enough. Then using Theorem C in Aronson ([1] p.616) we know that the sequence of functions $(G^n)_n$ is equicontinuous on K. Moreover thanks to the Gaussian estimates (5) and Ascoli theorem, we have that $(G^n)_n$ converges uniformly to G on K, for some subsequence. We conclude by taking an exhaustive sequence of compact subsets in Δ and a diagonalisation procedure. □

For simplicity, from now on we assume that the sequence (G^n) is chosen such that it converges to G on Δ.

Theorem 8. *Assume that the general hypotheses of Subsection 2.2 hold.*
Let $\xi \in L^2(\Omega,\mathcal{F}_0,P;L^2(\mathcal{O}))$, $w' \in \mathcal{P}$, $w'' = (w''_1,\cdots,w''_d) \in \mathcal{P}^d$ and $w = (w_i)_{i\in\mathbb{N}^*} \in \mathcal{P}^{\mathbb{N}^*}$, such that $E[\int_0^T |||w_s|||^2\,ds] < +\infty$. We set

$$\begin{aligned} u_t(\cdot) = &\int_{\mathcal{O}} G(t,\cdot,0,y)\xi(y)\,dy + \int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)w'_s(y)\,dyds \\ &+ \sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)\partial_i w''_i(y)\,dyds \\ &+ \sum_{i=1}^{+\infty}\int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)w_{i,s}(y)\,dydB^i_s, \end{aligned} \tag{27}$$

then all the results of Proposition 6 remain valid.

Proof. Let us first note that (27) is well defined thanks to Proposition (5). As $\mathcal{O}^n$ and a^n are smooth, hypotheses of the previous subsection are fulfilled so that for all n, u^n satisfies Propositions 6 and Proposition 9 with domain $\mathcal{O}^n$, operator given by a^n and coefficients w^n,

$w^{\prime,n}$ and $w^{\prime\prime,n}$. Moreover, we know that for all $n \in \mathbb{N}^*$, the restriction of u^n to $\mathcal{O}^n$ belongs to $L^2_{loc}(\mathbb{R}_+; H^1_0(\mathcal{O}^n))$ and is a $H^2_0(\mathcal{O}^n)$-valued semimartingale hence as we put $u^n_t \equiv 0$ on $\Delta \setminus \Delta^n$, u^n belongs to $\mathcal{H}$ and is a $H^2_0(\mathcal{O})$-valued semimartingale which admits the following decomposition:

$$\begin{aligned} u^n_t(x) =& \xi^n(x) + \sum_{i,j} \int_0^t \partial_i \left(a^n_{i,j}(s,x) \partial_j u^n_s(x) \right) ds + \int_0^t w^{\prime,n}_s(x) ds - \sum_{i=1}^d \int_0^t \partial_i w^{\prime\prime,n}_{i,s}(x) ds \\ &+ \sum_{i=1}^{+\infty} \int_0^t w^n_{i,s}(x) dB^i_s. \end{aligned} \tag{28}$$

Let us now pass to the limit in $\mathcal{H}$. For simplicity, we work on the finite time-interval $[0,T]$ and consider the Hilbert space

$$F_T = L^2(\Omega \times [0,T]; H^1_0(\mathcal{O})),$$

equipped with the norm

$$\|u\|_{F_T} = \left(E \int_0^T \|u_t\|^2 dt + \int_0^T E \, \|\nabla u_t\|^2 dt \right)^{1/2}.$$

Estimate (17) ensures that the sequence (u^n) is bounded in F_T, that for all $i \in \{1, \cdots, d\}$, as the a^n are uniformly bounded by M, the sequence $(\sum_{j=1}^d a^n_{i,j} \partial_j u^n)$ is bounded in $L^2(\Omega \times [0,T] \times \mathcal{O})$. As a consequence, we can extract (successively) a subsequence $(u^{n_k})_{k \geqslant 1}$ which converges weakly in F_T to an element $\tilde{u}$ and such that for each i, the sequence $(\sum_{j=1}^d a^{n_k}_{i,j} \partial_j u^{n_k})_k$ converges weakly in $L^2(\Omega \times [0,T] \times \mathcal{O})$ to an element v_i. Since clearly $(\sum_{j=1}^d (a^{n_k}_{i,j} - a_{i,j}) \partial_j u^{n_k})_k$ converges to 0 in $L^2(\Omega \times [0,T] \times \mathcal{O})$ we conclude that $v_i = \sum_{j=1}^d a_{i,j} \partial_j \tilde{u}$. Therefore, we can construct a sequence $(\tilde{u}^n)$ of convex combinations of elements in (u^{n_k}) of the form

$$\tilde{u}^n = \sum_{k=1}^{N_n} \alpha^n_k u^{n_k}$$

with $\lim_{n \to +\infty} N_n = +\infty$, $\alpha^n_k \geqslant 0$, $\sum_{k=1}^{N_n} \alpha^n_k = 1$ for all n and such that:

1. $(\tilde{u}^n)$ converges strongly in F_T to an element $\tilde{u} \in F_T$,
2. $\forall i \in \{1, \cdots, d\}$, $\tilde{A}^n_i = \sum_{j=1}^d \sum_{k=1}^{N_n} \alpha^n_k a^{n_k}_{i,j} \partial_j u^{n_k}$ converges to $\sum_{j=1}^d a_{i,j} \partial_j \tilde{u}$ in $L^2(\Omega \times [0,T] \times \mathcal{O})$ as n goes to infinity.

In a natural way, we set:

$$\tilde{\xi}^n = \sum_{k=1}^{N_n} \alpha^n_k \xi^{n_k}, \; \tilde{w}^{\prime,n} = \sum_{k=1}^{N_n} \alpha^n_k w^{\prime,n_k}, \; \tilde{w}^{\prime\prime,n} = \sum_{k=1}^{N_n} \alpha^n_k w^{\prime\prime,n_k}, \tilde{w}^n_i = \sum_{k=1}^{N_n} \alpha^n_k w^{n_k}_i \; \forall i \geqslant 1.$$

So $\tilde{u}^n$ admits the following representation for all $n \geqslant 1$:

$$\begin{aligned} \tilde{u}^n_t(x) =& \tilde{\xi}^n(x) + \sum_{i=1}^d \int_0^t \partial_i \tilde{A}^n_i(s,x) ds + \int_0^t \tilde{w}^{\prime,n}_s(x) ds - \sum_{i=1}^d \int_0^t \partial_{i,y} \tilde{w}^{\prime\prime,n}_{i,s}(x) ds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \tilde{w}^n_{i,s}(x) dB^i_s \,. \end{aligned} \tag{29}$$

Let $n, m \in \mathbb{N}^*$, we set $v^{n,m} = \tilde{u}^n - \tilde{u}^m$. Applying Itô's formula and then integrating with respect to x, we get

$$\begin{aligned}\| v_t^{n,m}\|^2 =& \|\tilde{\xi}^n - \tilde{\xi}^m\|^2 - 2\sum_i \int_0^t \int_{\mathcal{O}} \left(\tilde{A}_i^n(s,x) - \tilde{A}_i^m(s,x)\right) \partial_i v_s^{n,m}(x) dx ds \\ &+ 2\int_0^t \left(\tilde{w}_s'^{,n} - \tilde{w}_s'^{,m}, v_s^{n,m}\right) ds + 2\sum_i \int_0^t \left(\tilde{w}_{i,s}''^{,n} - \tilde{w}_{i,s}''^{,m}, \partial_i v_s^{n,m}\right) ds \qquad (30) \\ &+ 2\sum_i \int_0^t \left(\tilde{w}_{i,s}^n - \tilde{w}_{i,s}^m, v_s^{n,m}\right) dB_s^i + \sum_i \int_0^t \| \tilde{w}_{i,s}^n - \tilde{w}_{i,s}^m \|^2 \, ds.\end{aligned}$$

Let $\varepsilon > 0$, we have for almost all $t \in [0,T)$ and $x \in \mathcal{O}$:

$$\begin{aligned}\sum_i \left(\tilde{A}_i^n(t,x) - \tilde{A}_i^m(t,x)\right) \partial_i v_t^{n,m}(x) &= \sum_{i,j} a_{i,j}(t,x) \partial_j v_t^{n,m}(x) \partial_i v_t^{n,m}(x) \\ &+ \sum_i \left(\tilde{A}_i^n(t,x) - \sum_j a_{i,j}(t,x)\partial_j \tilde{u}_t^n(x)\right) \partial_i v_t^{n,m}(x) \\ &- \sum_i \left(\tilde{A}_i^m(t,x) - \sum_j a_{i,j}(t,x)\partial_j \tilde{u}_t^m(x)\right) \partial_i v_t^{n,m}(x)\end{aligned}$$

this yields, thanks to the ellipticity asumption on the matrix a:

$$\begin{aligned}-2\sum_i \int_0^t \int_{\mathcal{O}} \left(\tilde{A}_i^n(s,x) - \tilde{A}_i^m(s,x)\right) \partial_i v_s^{n,m}(x) dx ds \leqslant & -2\lambda \int_0^t \|\nabla v_s^{n,m}\|^2 ds \\ &+ 2\sum_i \int_0^t \int_{\mathcal{O}} |\tilde{A}_i^n(s,x) - \sum_j a_{i,j}(s,x)\partial_j \tilde{u}_s^n(x)| \, |\partial_i v_s^{n,m}(x)| \, ds dx \\ &+ 2\sum_i \int_0^t \int_{\mathcal{O}} |\tilde{A}_i^m(s,x) - \sum_j a_{i,j}(s,x)\partial_j \tilde{u}_s^m(x)| \, |\partial_i v_s^{n,m}(x)| \, ds dx \,.\end{aligned}$$

Using the trivial inequality $2ab \leqslant \varepsilon a^2 + \frac{1}{\varepsilon} b^2$ we get

$$2\left|\int_0^t \left(\tilde{w}_s'^{,n} - \tilde{w}_s'^{,m}, v_s^{n,m}\right) ds\right| \leqslant \varepsilon \int_0^T \|v_s^{n,m}\|^2 ds + \frac{1}{\varepsilon}\int_0^T \|\tilde{w}_s'^{,n} - \tilde{w}_s'^{,m}\|^2 ds$$

and

$$2\left|\sum_i \int_0^t \left(\tilde{w}_{i,s}''^{,n} - \tilde{w}_{i,s}''^{,m}, \partial_i v_s^{n,m}\right) ds\right| \leqslant \varepsilon \int_0^T \|\nabla v_s^{n,m}\|^2 ds + \frac{1}{\varepsilon}\int_0^T \| |\tilde{w}_s''^{,n} - \tilde{w}_s''^{,m}| \|^2 ds \,.$$

Moreover, thanks to the Burkholder-Davies-Gundy, we obtain

$$
\begin{aligned}
&E[\sup_{t\in[0,T]}\left|\sum_i\int_0^t\left(\tilde{w}^n_{i,s}-\tilde{w}^m_{i,s},v^{n,m}_s\right)dB^i_s\right|]\\
&\leqslant c_1E\left[\left(\int_0^T\sum_{i=1}^{+\infty}\left(\tilde{w}^n_{i,s}-\tilde{w}^m_{i,s},v^{n,m}_s\right)^2ds\right)^{1/2}\right]\\
&\leqslant c_1E\left[\left(\int_0^T\sum_{i=1}^{+\infty}\sup_{t\in[0,T]}\|v^{n,m}_t\|^2\,\|\tilde{w}^n_{i,s}-\tilde{w}^m_{i,s}\|^2dt\right)^{1/2}\right]\\
&\leqslant c_1E\left[\sup_{t\in[0,T]}\|v^{n,m}_t\|\left(\int_0^T|||\tilde{w}^n_s-\tilde{w}^m_s|||^2dt\right)^{1/2}\right]\\
&\leqslant \varepsilon E\left[\sup_{t\in[0,T]}\|v^{n,m}_t\|^2\right]+\frac{c_1}{4\varepsilon}E\left[\int_0^T|||\tilde{w}^n_s-\tilde{w}^m_s|||^2dt\right].
\end{aligned}
$$

Then using the inequalities above, by taking the supremun in $t\in[0,T]$ in relation (30) and then the expectation, we get:

$$
\begin{aligned}
&(1-\varepsilon(T+2))E[\sup_{t\in[0,T]}\|\,v^{n,m}_t\|^2]+(2\lambda-\varepsilon)E\int_0^T\|\nabla v^{n,m}_s\,\|^2\,ds\leqslant E[\|\xi^n-\xi^m\|^2]\\
&+2\sum_iE\left[\int_0^T\int_{\mathcal{O}}|\tilde{A}^n_i(s,x)-\sum_j a_{i,j}(s,x)\partial_j\tilde{u}^n_s(x)|\,|\partial_iv^{n,m}_s(x)|\,dsdx\right]\\
&+2\sum_iE\left[\int_0^T\int_{\mathcal{O}}|\tilde{A}^m_i(s,x)-\sum_j a_{i,j}(s,x)\partial_j\tilde{u}^m_s(x)|\,|\partial_iv^{n,m}_s(x)|\,dsdx\right]\\
&+\frac{1}{\varepsilon}E\left[\int_0^T\|\tilde{w}^{\prime,n}_s-\tilde{w}^{\prime,m}_s\|^2ds\right]+\frac{1}{\varepsilon}E\left[\int_0^T|||\tilde{w}^{\prime\prime,n}_s-\tilde{w}^{\prime\prime,m}_s|||^2ds\right]\\
&+\frac{c_1}{4\varepsilon}E\left[\int_0^T|||\tilde{w}^n_s-\tilde{w}^m_s|||^2ds\right].
\end{aligned}
\tag{31}
$$

Let us prove now that each term in the right member tends to 0 as n,m go to $+\infty$.
First of all, by construction of the approximating sequences $(\xi^n)_n$, $(w^n)_n$, $(w^{'n})_n$ and $(w^{''n})_n$ given at the beginning of this step, we have

$$
\lim_{n,m\to+\infty}E[\|\tilde{\xi}^n-\tilde{\xi}^m\|^2]=\lim_{n,m\to+\infty}E\left[\int_0^T\|\tilde{w}^{\prime,n}_s-\tilde{w}^{\prime,m}_s\|^2ds\right]=\lim_{n,m\to+\infty}E\left[\int_0^T|||\tilde{w}^{\prime\prime,n}_s-\tilde{w}^{\prime\prime,m}_s|||^2ds\right]=0,
$$

and

$$
\lim_{n\to+\infty}E\left[\int_0^T|||\tilde{w}^n_s-w_s|||^2\,ds\right]=0.
$$

Let $i\in\{1,\cdots,d\}$. As a is bounded, $\sum_j a_{i,j}\partial_j\tilde{u}^n$ tends to $\sum_j a_{i,j}\partial_j\tilde{u}$ and so $|\tilde{A}^n_i-\sum_j a_{i,j}\partial_j\tilde{u}^n_s(x)|$ tends to 0 in $L^2(\Omega\times[0,T]\times\mathcal{O})$ as n goes to $+\infty$. As $(v^{n,m})_{n,m}$ is bounded

in $L^2(\Omega \times [0,T] \times \mathcal{O})$, we deduce from this that

$$\lim_{n,m\to+\infty} E\left[\int_0^T \int_{\mathcal{O}} |\tilde{A}_i^n(s,x) - \sum_j a_{i,j}(s,x)\partial_j \tilde{u}_s^n(x)|\, |\partial_i v_s^{n,m}(x)|\; dsdx\right] = 0,$$

and in the same way

$$\lim_{n,m\to+\infty} E\left[\int_0^T \int_{\mathcal{O}} |\tilde{A}_i^m(s,x) - \sum_j a_{i,j}(s,x)\partial_j \tilde{u}_s^m(x)|\, |\partial_i v_s^{n,m}(x)|\; dsdx\right] = 0.$$

Taking ε small enough in (31), we conclude that $(\tilde{u}^n)$ is a Cauchy sequence in $\hat{F}_T$ it is clear that its limit is $\tilde{u}$ so we have

$$\lim_{n\to+\infty} \|\tilde{u}^n - \tilde{u}\|_T = 0.$$

It remains to prove that $\tilde{u} = u$.
We have for all n:

$$\begin{aligned} u_t^n(x) = &\int_{\mathcal{O}} G^n(t,x,0,y)\xi^n(y)dy + \int_0^t \int_{\mathcal{O}} G^n(t,x,s,y)w_s'^{,n}(y)dyds \\ &- \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \partial_{i,y} G^n(t,x,s,y) w_{i,s}''^{,n}(y)dyds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G^n(t,x,s,y) w_{i,s}^n(y)dydB_s^i. \end{aligned} \tag{32}$$

Thanks to Lemma 7 and the Gaussian estimates (5), we deduce by the dominated convergence Theorem that the first, second and fourth terms in the right member of (32) converge to the corresponding term in the expression (27) of u. In order to study the third one, we put for all n:

$$z^n = -\sum_{i=1}^d \int_0^t \int_{\mathcal{O}} \partial_{i,y} G^n(t,x,s,y) w_{i,s}''^{,n}(y)dyds = \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} G^n(t,x,s,y)\partial_{i,y} w_{i,s}''^{,n}(y)dyds,$$

and

$$z = \sum_{i=1}^d \int_0^t \int_{\mathcal{O}} G(t,x,s,y)\partial_{i,y} w_{i,s}''(y)dyds.$$

By the same proof as above, we can prove that, at least for a subsequence, (z^n) converges weakly in $L^2(\Omega \times [0,T]; H_0^1(\mathcal{O}))$ to an element $\tilde{z}$. But, it is easy by passing to the limit, to verify that $\tilde{z}$ is a weak solution of the equation:

$$d\tilde{z}_t = \sum_{i,j} \partial_i a_{i,j} \partial_j \tilde{z}_t + \sum_i \partial_i w_i'',\ \tilde{z}_0 = 0.$$

Since the weak solution is unique, $\tilde{z} = z$. This permits to conclude that $\tilde{u} = u$.
Finally, as

$$\lim_{n\to+\infty} \|\tilde{u}^n - u\|_T = 0,$$

to see that Proposition 6 remains valid, one just has to apply it to $\tilde{u}^n$ and then pass to the limit by making n tend to $+\infty$. □

We now prove the following version of Ito's formula which is crucial to get uniform estimates of the solution.

Proposition 9. *Let u be the solution defined in Theorem 8 with same hypotheses and $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be one time differentiable with continuous derivative with respect to the first variable and two times differentiable with continuous derivatives w.r.t. the second variable. We denote by φ' and φ'' the derivatives of φ with respect to the second variable and by $\frac{\partial \varphi}{\partial t}$ the partial derivative with respect to time. We assume that these derivatives are bounded and $\varphi'(t,0) = 0$ for all $t \geqslant 0$. Then the following relation holds a.s. for all $t \geqslant 0$:*

$$\begin{aligned}\int_{\mathcal{O}} \varphi\left(t, u_t\left(x\right)\right) dx + \int_0^t \int_{\mathcal{O}} \sum_{i,j=1}^{d} a_{i,j}(s,x)\varphi''\left(s, u_s(x)\right) \partial_i u_s(x) \partial_j u_s(x) dx ds = \int_{\mathcal{O}} \varphi\left(0, \xi\left(x\right)\right) dx \\ + \int_0^t \int_{\mathcal{O}} \frac{\partial \varphi}{\partial s}(s, u_s(x)) dx ds + \int_0^t \int_{\mathcal{O}} \varphi'\left(s, u_s(x)\right)\, w_s(x)\, dx\, ds \\ - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \partial_i \varphi'\left(s, u_s(x)\right)\, w''_{i,s}(x) dx ds + \sum_i \int_0^t \int_{\mathcal{O}} \varphi'(s, u_s(x)) w_{i,s}(x)\, dx dB_s^i \\ + \frac{1}{2} \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''(s, u_s(x)) \left(w_{i,s}(x)\right)^2\, dx\, ds.\end{aligned} \tag{33}$$

Proof. First of all, let us mention that due the boundedness of the derivatives of φ and the integrability conditions on w, w' and w'', each of the terms in (33) are well defined. We consider the same approximation $(\tilde{u}_t^n)_{t \geqslant 0}$ as in the proof of Theorem 8 and we keep the same notations. We know that $\tilde{u}^n$ is a H_0^1-valued semimartingale and that it admits the decomposition given by (29). We apply the classical Itô's formula and then integrate w.r.t. x, this yields:

$$\begin{aligned}\int_{\mathcal{O}} \varphi\left(t, \tilde{u}_t^n\left(x\right)\right) dx + \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \tilde{A}_i^n(s,x) \partial_i \varphi'\left(s, \tilde{u}_s^n(x)\right)\, dx ds = \int_{\mathcal{O}} \varphi\left(0, \xi^n\left(x\right)\right) dx \\ + \int_0^t \int_{\mathcal{O}} \frac{\partial \varphi}{\partial s}(s, \tilde{u}_s^n(x)) dx ds + \int_0^t \int_{\mathcal{O}} \varphi'\left(s, \tilde{u}_s^n(x)\right)\, \tilde{w}'^{,n}_s(x)\, dx\, ds \\ - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \partial_i \varphi'\left(s, \tilde{u}_s^n(x)\right)\, \tilde{w}''^{,n}_{i,s}(x) dx ds + \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi'(s, \tilde{u}_s^n(x)) \tilde{w}^n_{i,s}(x)\, dx dB_s^i \\ + \frac{1}{2} \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''(s, \tilde{u}_s^n(x)) \left(\tilde{w}^n_{i,s}(x)\right)^2\, dx\, ds.\end{aligned}$$

By extracting subsequences, we can assume that $(\tilde{u}^n)_n$ and $(\tilde{A}_i^n)_n$ converge in $L^2(\Omega \times [0,T] \times \mathcal{O})$ and $dt \times dx \times P$-almost everywhere respectively to $\tilde{u}$ and $\sum_{j=1}^d a_{i,j} \partial_j u$, so that we can apply the dominated convergence Theorem in each term of the previous equality and obtain the result in the general case. □

3.3. Existence and uniqueness of the solution in $\mathcal{H}$ under **(HD2)** *and* **(HI2)**

The aim of this section is to prove existence and uniqueness of the solution of (6) with zero Dirichlet condition on the boundary under usual L^2-integrability conditions and assumption **(H)**.
So, all along this section, we assume that hypotheses **(H)**, **(HD2)** and **(HI2)** hold.

Proposition 10. *Notions of* mild solution *and* weak solution *coïncide.*

Proof. The fact that any mild solution is a weak solution follows from Theorem 8.
Conversely, assume that u is a weak solution and define the process

$$\begin{aligned} v_t(\cdot) = &\int_{\mathcal{O}} G(t,\cdot,0,y)\xi(y)\,dy + \int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)f(s,y,u_s(y),\nabla u_s(y))dyds \\ &+ \sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)\partial_{i,y}g_i(s,.,u_s,\nabla u_s)(y)dyds \\ &+ \sum_{i=1}^{+\infty}\int_0^t\int_{\mathcal{O}} G(t,\cdot,s,y)h_i(s,y,u_s(y),\nabla u_s(y))dB^i_s. \end{aligned} \tag{34}$$

We should prove that $u = v$. Comparing the value of the integral

$$\int_0^\infty\int_{\mathcal{O}}[u_s(x)\partial_s\varphi(x) - \sum_{i,j=1}^{d} a_{i,j}(s,x)\partial_i u_s(x)\partial_j\varphi_s(x)]dxds,$$

obtained from the relation defining a weak solution, and the value of the same integral with v in the place of u, given by the relation (15), we observe that the two are almost surely equal. So, we deduce that

$$\int_0^\infty\int_{\mathcal{O}}(u_s(x)-v_s(x))\partial_s\varphi(x) - \sum_{i,j=1}^{d} a_{i,j}(s,x)\partial_i(u_s(x)-v_s(x))\partial_j\varphi_s(x)\,dxds = 0,$$

almost surely, for each $\varphi \in \mathcal{D}$. Since $\mathcal{D}$ contains a countable set which is dense in it, we deduce that the relation holds with arbitrary $\varphi \in \mathcal{D}$, outside of a negligeable set in Ω. From this, it is standard to conclude that $u = v$ almost surely. □

The proof of the following Theorem is given in the Appendix 5.3.

Theorem 11. *Under hypotheses* **(H)**, **(HD2)** *and* **(HI2)**, *equation* (6) *with zero Dirichlet condition on the boundary admits a unique solution, u, which belongs to $\mathcal{H}$. Moreover u admits $L^2(\mathcal{O})$-continuous trajectories and satisfies the following estimate:*

$$E[\|u\|_T^2] \leqslant cE\left[\|\xi\|^2 + \left\||f^0|\right\|^2_{2,2;T} + \left\||g^0|\right\|^2_{2,2;T} + \left\||h^0|\right\|^2_{2,2;T}\right], \tag{35}$$

where c is a constant which only depends on the structure constants.

3.4. L^p-estimate of the uniform norm of the solution

As in [8, 9], for $\theta \in [0,1)$ and $p \geqslant 2$ fixed, we consider the following assumptions:

Assumption (HI∞p)

$$E\left\|\xi\right\|_{\infty}^{p} < \infty.$$

Assumption (HDθp)

$$E\left(\left(\left\|f^0\right\|_{\theta;t}^{*}\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}\right) < \infty,$$

for each $t \geq 0$.
Here, $\|\ \|_{\theta;t}^{*}$ is the functional norm similar to $\|\ \|_{\#;T}^{*}$ (see Appendix 5.1).

In [8], in the case of a SPDE driven by a finite dimensional Brownian motion and an homogeneous second order symmetric differential operator, we have established an L^p-estimate of the uniform norm of the solution. The proof of this L^p-estimate is based on Itô's formula applied to the power function and the domination of the quadratic variation of the martingale part in this formula. However, the method and the technics involved to get this estimate do not depend on the dimension of the Brownian motion neither on the fact that the matrix a is homogeneous in time. Therefore, to generalize these results to our context, we can follow the same arguments as in [8] starting from Lemma 12 of this reference and this yields:

Theorem 12. *Assume* **(H)**, ***(HDθp)***, ***(HI∞p)*** *for some $\theta \in [0,1[$, $p \geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda$. Let $u = \mathcal{U}(\xi, f, g, h)$, then*

$$\forall t \geqslant 0,\ E\left\|u\right\|_{\infty,\infty;t}^{p} \leq k(t)\, E\left(\left\|\xi\right\|_{\infty}^{p} + \left(\left\|f^0\right\|_{\theta;t}^{*}\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;t}^{*}\right)^{\frac{p}{2}}\right),$$

where $k(t)$ is a constant which depends on the structure constants and t.

4. Maximum principle for local solutions

In [9], we have proven a maximum principle for SPDE's driven by a finite dimensional Brownian motion and homogeneous second order symmetric differential operator. To extend these results to our context, we follow the same plan as in [9]. We mention the different estimates who lead to the result and give the details of the proofs only when needed.

4.1. Estimates of the solution with null Dirichlet condition under **(HD#)**

The first step consists in establishing an estimate for the positive part of the solution with null Dirichlet condition. To get this estimate, we can adapt to our case the arguments of proofs of Theorem 3, Corollary 1 and Theorem 4 in [9] which are based only on estimate (35) and Itô's formula for the solution which do not depend on the dimension of the noise neither on the fact that the matrix a is homogeneous. This yields:

Theorem 13. *Under the conditions* **(H)**, **(HD#)** *and* **(HI2)** *there exists a unique solution u of (6) in $\mathcal{H}$. This solution has a version with $L^2(\mathcal{O})$-continuous trajectories and it satisfies the following estimates for each $t \geq 0$:*

1. $E\left(\|u\|^2_{2,\infty;t} + \|\nabla u\|^2_{2,2;t}\right) \leq k(t)\, E\left(\|\xi\|_2^2 + \left(\left\|f^0\right\|^*_{\#;t}\right)^2 + \left\|g^0\right\|^2_{2,2;t} + \left\||h^0|\right\|^2_{2,2;t}\right).$

2. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ and assume that φ'' is bounded and $\varphi'(0) = 0$. Then the following relation holds a.s. for all $t \geqslant 0$:*

$$\int_{\mathcal{O}} \varphi(u_t(x))\, dx + \int_0^t \mathcal{E}\left(\varphi'(u_s), u_s\right) ds = \int_{\mathcal{O}} \varphi(\xi(x))\, dx + \int_0^t \left(\varphi'(u_s), f_s(u_s, \nabla u_s\right) ds$$
$$- \int_0^t \sum_{i=1}^d \left(\partial_i\left(\varphi'(u_s)\right), g_{i,s}(u_s, \nabla u_s)\right) ds + \frac{1}{2}\int_0^t \left(\varphi''(u_s), |h_s(u_s, \nabla u_s)|^2\right) ds$$
$$+ \sum_{j=1}^{+\infty} \int_0^t \left(\varphi'(u_s), h_{j,s}(u_s, \nabla u_s)\right) dB^j_s.$$

3. *The positive part of the solution satisfies the following estimate*

$$E\left(\left\|u^+\right\|^2_{2,\infty;t} + \left\|\nabla u^+\right\|^2_{2,2;t}\right) \leq k(t)\, E\left(\left\|\xi^+\right\|_2^2 + \left(\left\|f^{u,0+}\right\|^*_{\#;t}\right)^2 + \left\|g^{u,0}\right\|^2_{2,2;t} + \left\||h^{u,0}|\right\|^2_{2,2;t}\right),$$

where $k(t)$ is a constant that only depends on t and the structure constants and

$$\begin{aligned} &f^{u,0} = 1_{\{u>0\}} f^0,\ g^{u,0} = 1_{\{u>0\}} g^0,\ h^{u,0} = 1_{\{u>0\}} h^0, \\ &f^u = f - f^0 + f^{u,0},\ g^u = g - g^0 + g^{u,0},\ h^u = h - h^0 + h^{u,0} \\ &f^{u,0+} = 1_{\{u>0\}}\left(f^0 \vee 0\right),\ \xi^+ = \xi \vee 0. \end{aligned} \tag{36}$$

Let us mention that a similar relations to the one of point 3. have been obtained by Krylov, under stronger conditions (see [17], Lemma 2.4 and Lemma 2.5).

4.2. Estimate of the positive part of a local solution

We first make the following remark concerning the regularity of the trajectories of any local solution.

Remark 2. *We have proved in Theorem 13 that under* **(H)**, **(HD#)** *and* **(HI2)** *the solution with null Dirichlet conditions at the boundary of $\mathcal{O}$ has a version with $L^2(\mathcal{O})$-continuous trajectories and, in particular, that $\lim_{t\to 0}\|u_t - \xi\|_2 = 0$, a.s. This property extends to the local solutions in the sense that any element of $\mathcal{U}_{loc}(\xi, f, g, h)$ has a version with the property that a.s. the trajectories are $L^2(K)$-continuous, for each compact set $K \subset \mathcal{O}$ and*

$$\lim_{t\to 0} \int_K \left(u_t(x) - \xi(x)\right)^2 dx = 0.$$

In order to see this it suffices to take a test function $\phi \in \mathcal{C}_c^\infty(\mathcal{O})$ and to verify that $v = \phi u$ satifies the equation

$$dv_t = \left(Lv_t + \overline{f}_t + div\overline{g}_t\right) + \overline{h}_t dB_t,$$

with the initial condition $v_0 = \phi\xi$, *where*

$$\overline{f}_t(x) = \phi(x) f\left(t, x, u_t(x), \nabla u_t(x)\right) - \langle\, \nabla\phi(x),\, a(x)\nabla u_t(x)\rangle - \langle \nabla\phi(x),\, g\left(t, x, u_t(x), \nabla u_t(x)\right)\rangle,$$
$$\overline{g}_t(x) = \phi(x) g\left(t, x, u_t(x), \nabla u_t(x)\right) - u_t(x) a(x) \nabla\phi(x) \quad \textit{and}$$
$$\overline{h}_t(x) = \phi(x) h\left(t, x, u_t(x), \nabla u_t(x)\right).$$

Thus $v = \mathcal{U}\left(\phi\xi, \overline{f}, \overline{g}, \overline{h}\right)$ *and the results of Theorem 13 hold for* v.

Now, we consider $u \in \mathcal{U}_{loc}(\xi, f, g, h)$ and in order to simplify the notation we put

$$f_s = f(s, x, u_s(x), \nabla u_s(x)),\ g_s = g(s, x, u_s(x), \nabla u_s(x)),\ h_s = h(s, x, u_s(x), \nabla u_s(x)).$$

So that, u is the solution of

$$du_t \;=\; \left(\sum_{i,j=1}^{d} \partial_i(a_{i,j}(t,\cdot)\partial_j u_t) + f_t + \sum_{j=1}^{d} \partial_j g_{j,t}\right) dt + \sum_{i=1}^{+\infty} h_{i,t} dB_t^i, \tag{37}$$

with initial conditiuon $u_0 = \xi$.
Let us remark that this technic has already been used by Krylov ([17], Lemma 2.5) in order to get Itô's formula for the non-negative part of the solution. We also obtain such Itô's formula in our setting:

Lemma 14. *Assume that* ∂O *is Lipschitz, conditions* **(H)**, **(HD#)** *and* **(HI2)** *hold. Let* $u \in \mathcal{U}_{loc}(\xi, f, g, h)$ *such that* $u^+ \in \mathcal{H}$, *i.e. following Definition 3,* u *is non-positive on the boundary of* $\mathcal{O}$.
Let $\varphi : \mathbb{R} \to \mathbb{R}$ *be a function of class* C^2 *with bounded second order derivative and assume that* $\varphi(0) = \varphi'(0) = 0$. *Then with the notations introduced above:*

$$\begin{aligned}
\int_{\mathcal{O}} \varphi\left(u_t^+(x)\right) dx + \sum_{i,j=1}^{d} \int_0^t \int_{\mathcal{O}} \varphi''\left(u_s^+(x)\right) a_{i,j}(s,x) \partial_i u_s^+(x) \partial_j u_s^+(x) dx ds = \int_{\mathcal{O}} \varphi\left(\xi^+(x)\right) dx \\
+ \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x))\, f_s(x)\, dx\, ds - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \varphi''\left(u_s^+(x)\right) \partial_i u_s^+(x) g_{i,s}(x) dx ds \\
+ \sum_i \int_0^t \int_{\mathcal{O}} \varphi'(u_s^+(x)) h_{i,s}(x)\, dx dB_s^i + \frac{1}{2} \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \varphi''\left(u_s^+(x)\right) \mathbb{1}_{\{u_s > 0\}} |h_{i,s}(x)|^2\, dx\, ds.
\end{aligned} \tag{38}$$

Proof. For the moment, we consider $\phi \in C_c^\infty(\mathcal{O})$, $0 \leqslant \phi \leqslant 1$ and put

$$\forall t \in [0, T],\ v_t = \phi u_t.$$

By a direct calculation, we see that the process v satisfies the following equation with $\phi\xi$ as initial data and zero Dirichlet boundary conditions,

$$dv_t = \left(\sum_{i,j=1}^{d} \partial_i(a_{i,j}(t,\cdot)\partial_j v_t) + \widetilde{f}_t + \sum_{i=1}^{d} \partial_i \widetilde{g_{i,t}}\right) dt + \sum_{j=1}^{+\infty} \widetilde{h_{j,t}} dB_t^j,$$

where

$$\widetilde{f}_t = \phi f_t - \sum_{i,j=1}^{d} a^{i,j}(t,\cdot)\,(\partial_i\phi)\,(\partial_j u_t) - \sum_{i=1}^{d}(\partial_i\phi)\,g_{i,t},$$

$$\widetilde{g_{i,t}} = \phi g_{i,t} - u_t \sum_{j=1}^{d} a^{i,j}(t,\cdot)\partial_j\phi, i = 1,...d, \quad \widetilde{h_{j,t}} = \phi h_{j,t}, j \in \mathbb{N}^*.$$

Let us note that

$$E[(\left\|\tilde{f}\right\|^*_{\#;t})^2 + \|\tilde{g}\|^2_{2,2;t} + \left\|\tilde{h}\right\|^2_{2,2;t}] < \infty,$$

so we'll be able to apply Itô's formula (point 2. of the previous Theorem).
Now, we approximate the function $\psi : y \in \mathbb{R} \to \varphi(y^+)$ by a sequence $(\psi_n)_{n\in\mathbb{N}^*}$ of smooth functions constructed as follows:
So, let ζ be a C^∞ increasing function such that

$$\forall y \in]-\infty, 1],\ \zeta(y) = 0 \text{ and } \forall y \in [2, +\infty[,\ \zeta(y) = 1.$$

We set for all $n \in \mathbb{N}^*$:

$$\forall y \in \mathbb{R},\ \psi_n(y) = \varphi(y)\zeta(ny).$$

It is easy to verify that $(\psi_n)_{n\in\mathbb{N}^*}$ converges uniformly to the function ψ, $(\psi_n')_n$ converges everywhere to the function $(y \mapsto \varphi'(y^+))$ and $(\psi_n'')_n$ converges everywhere to the function $(y \mapsto \mathbb{I}_{\{y>0\}}\varphi''(y^+))$. Moreover we have the estimates:

$$\forall y \in \mathbb{R}_+, \forall n \in \mathbb{N}^*,\ \ 0 \leqslant \psi_n(y) \leqslant \psi(y),\ 0 \leqslant \psi_n'(y) \leqslant Cy,\ \ \|\psi_n''(y)\| \leqslant C, \tag{39}$$

where C is a constant.
Thanks to the previous Theorem, we have for all n and all $t \geqslant 0$:

$$\begin{aligned}\int_{\mathcal{O}} \psi_n\left(v_t\left(x\right)\right) dx + \sum_{i,j=1}^{d} \int_0^t \int_{\mathcal{O}} a_{i,j}(s,x)\psi_n''\left(v_s(x)\right)\partial_i v_s(x)\partial_j v_s(x) dx ds = \int_{\mathcal{O}} \psi_n\left(\phi(x)\xi\left(x\right)\right) dx \\ + \int_0^t \int_{\mathcal{O}} \psi_n'\left(v_s(x)\right)\ \widetilde{f}_s(x)\, dx\, ds - \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} \psi_n''\left(v_s(x)\right)\partial_i v_s(x)\widetilde{g}_{i,s}(x) dx ds \\ + \sum_i \int_0^t \int_{\mathcal{O}} \psi_n'(v_s(x))\widetilde{h}_{i,s}(x)\, dx dB_s^i + \frac{1}{2}\sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} \psi_n''(v_s(x))\left|\widetilde{h}_{i,s}(x)\right|^2 dx\, ds.\end{aligned} \tag{40}$$

Let us remark that $v \in \mathcal{H}$ so that thanks to estimates (39), each term in the previous equality is well defined and even dominated in L^1. Let us focus on the particular term $\int_0^t \int_{\mathcal{O}} \psi_n'\left(v_s(x)\right)\ \widetilde{f}_s(x)\, dx\, ds$. We have for all $n \in \mathbb{N}^*$:

$$|\psi_n'\left(v_s\right)\ \widetilde{f}_s| \leqslant C|v_s||\widetilde{f}_s|,$$

and as $v \in \mathcal{H} \bigcap L_{\#;t}$ and $\widetilde{f} \in L^*_{\#;t}$ the Hölder inequality (2) ensures that $|v\widetilde{f}|$ belongs to $L^1(\Omega \times [0,T] \times \mathcal{O})$. The other terms being easier to dominate, by the dominated convergence

Theorem and using the fact that $\mathbb{1}_{\{v_s>0\}}\partial_i v_s = \partial_i v_s^+$, we get as n tends to $+\infty$:

$$\int_{\mathcal{O}} \varphi\left(v_t^+(x)\right)dx + \sum_{i,j=1}^{d}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_s^+(x)\right)a_{i,j}(s,x)\partial_i v_s^+(x)\partial_j v_s^+(x)dxds = \int_{\mathcal{O}} \varphi\left(\phi(x)\xi^+(x)\right)dx$$
$$+\int_0^t\int_{\mathcal{O}} \varphi'(v_s^+(x))\,\widetilde{f}_s(x)\,dx\,ds - \sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_s^+(x)\right)\partial_i v_s^+(x)\widetilde{g}_{i,s}(x)dxds$$
$$+\sum_i\int_0^t\int_{\mathcal{O}} \varphi'(v_s^+(x))\widetilde{h}_{i,s}(x)\,dxdB_s^i + \frac{1}{2}\sum_{i=1}^{+\infty}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_s^+(x)\right)\mathbb{1}_{\{v_s>0\}}\left|\widetilde{h}_{i,s}(x)\right|^2 dx\,ds. \tag{41}$$

Consider now a sequence $(\phi_n)_n$ of non-negative functions in $C_0^\infty(\mathcal{O})$, $0\leqslant\phi_n\leqslant 1\ \forall n\in\mathbb{N}^*$ converging to 1 everywhere on $\mathcal{O}$ and such that for any $w\in H_0^1(\mathcal{O})$ the sequence $(\phi_n w)_n$ tends to w in $H_0^1(\mathcal{O})$ and

$$\sup_n \|\phi_n w\|_{H_0^1(\mathcal{O})}\leqslant C\|w\|_{H_0^1(\mathcal{O})},$$

where C is a constant which does not depend on w.
The existence of such a sequence is proved in the Appendix, Lemma 19.
Let us remark that if $i\in\{1,\cdots,d\}$ and $w\in H_0^1(\mathcal{O})$, then $(w\partial_i\phi_n)_n$ tends to 0 in $L^2(\mathcal{O})$.
We set $v_n=\phi_n u$ and

$$\widetilde{f_t^n} = \phi_n f_t - \sum_{i,j=1}^{d} a^{i,j}(t,\cdot)\left(\partial_i\phi_n\right)\left(\partial_j u_t\right) - \sum_{i=1}^{d}\left(\partial_i\phi_n\right)g_{i,t},$$

$$\widetilde{g_{i,t}^n} = \phi_n g_{i,t} - u_t\sum_{j=1}^{d} a^{i,j}(t,\cdot)\partial_j\phi_n, i=1,...d,\ \ \widetilde{h_{j,t}^n} = \phi_n h_{j,t}, j\in\mathbb{N}^*.$$

We now apply relation (41) to v_n and get

$$\int_{\mathcal{O}} \varphi\left(v_{n,t}^+(x)\right)dx + \sum_{i,j=1}^{d}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_{n,s}^+(x)\right)a_{i,j}(s,x)\partial_i v_{n,s}^+(x)\partial_j v_{n,s}^+(x)dxds = \int_{\mathcal{O}} \varphi\left(\phi_n(x)\xi^+(x)\right)dx$$
$$+\int_0^t\int_{\mathcal{O}} \varphi'(v_{n,s}^+(x))\,\widetilde{f}_s^n(x)\,dx\,ds - \sum_{i=1}^{d}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_{n,s}^+(x)\right)\partial_i v_s^+(x)\widetilde{g}_{i,s}^n(x)dxds$$
$$+\sum_i\int_0^t\int_{\mathcal{O}} \varphi'(v_{n,s}^+(x))\widetilde{h}_{i,s}^n(x)\,dxdB_s^i + \frac{1}{2}\sum_{i=1}^{+\infty}\int_0^t\int_{\mathcal{O}} \varphi''\left(v_{n,s}^+(x)\right)\mathbb{1}_{\{v_{n,s}>0\}}\left|\widetilde{h}_{i,s}^n(x)\right|^2 dx\,ds. \tag{42}$$

We have

$$\begin{aligned}\varphi'(v_{n,s}^+)\widetilde{f}_s^n - \sum_{i=1}^{d}\varphi''(v_{n,s}^+)\partial_i v_{n,s}^+\widetilde{g^n}_{i,s} &= \varphi'(v_{n,s}^+)\phi_n f_s - \sum_{i,j}a_{i,j}(s)\varphi'(v_{n,s}^+)\partial_j\phi_n\partial_i u_s^+\\ &+\sum_{i,j}a_{i,j}(s)\varphi''(v_{n,s}^+)u_s^+\partial_i v_{n,s}^+\partial_j\phi_n - \sum_i(\varphi'(v_{n,s}^+)g_{i,s}\partial_i\phi + \varphi''(v_{n,s}^+)\phi_n g_{i,s}\partial_i v_{n,s}^+).\end{aligned}$$

By remarking for example that for all $s \in (0,T]$ $(\phi_n \varphi'(v^+_{n,s})_n$ (resp. $(\partial_i \phi_n \varphi'(v^+_{n,s}))_n)$ tends to $(\varphi'(u^+_s))$ (resp. 0) in $H^1_0(\mathcal{O})$ (resp. in $L^2(\mathcal{O})$) we conclude, thanks to the dominated convergence Theorem, by making n tend to $+\infty$ in (42). □

4.3. A comparison Theorem

By applying the previous Itô's formula for $\varphi(x) = x^2$, as in Theorem 13, the above Proposition leads to the following generalization of the estimate of the positive part:

Corollary 15. *Under the hypotheses of Lemma 14 with same notations, one has the following estimate:*

$$E\left(\left\|u^+\right\|^2_{2,\infty;t} + \left\|\nabla u^+\right\|^2_{2,2;t}\right) \leq k(t)\, E\left(\left\|\xi^+\right\|^2_2 + \left(\left\|f^{u,0+}\right\|^*_{\#;t}\right)^2 + \left\|g^{u,0}\right\|^2_{2,2;t} + \left\|h^{u,0}\right\|^2_{2,2;t}\right).$$

The key point of the proof of the maximum principle is the following comparison Theorem which is an immediate consequence of the previous estimate:

Theorem 16. *Assume that ∂O is Lipschitz. Let f^1, f^2 be two functions similar to f which satisfy the Lipschitz condition **(H)-(i)**, g (resp. h) satisfies **(H)-(ii)** (resp. **(H)-(iii)**) and assume that both triples $\left(f^1, g, h\right)$ and $\left(f^2, g, h\right)$ satisfy **(HD#)**. Let ξ^1, ξ^2 two random variables similar to ξ satisfying **(HI)**. Let $u^i \in \mathcal{U}_{loc}\left(\xi^i, f^i, g, h\right), i = 1, 2$ and suppose that the process $\left(u^1 - u^2\right)^+$ belongs to $\mathcal{H}$ and that one has*

$$E\left(\left\|f^1\left(.,.,u^2,\nabla u^2\right) - f^2\left(.,.,u^2,\nabla u^2\right)\right\|^*_{\#;t}\right)^2 < \infty, \quad \textit{for all} \quad t \geq 0.$$

If $\xi^1 \leq \xi^2$ a.s. and $f^1\left(t,\omega,u^2,\nabla u^2\right) \leq f^2\left(t,\omega,u^2,\nabla u^2\right)$, $dt \otimes dx \otimes dP$-a.e., then one has

$$u^1(t,x) \leq u^2(t,x) \qquad dt \otimes dx \otimes P\text{-a.e.}$$

4.4. The maximum principle

As in Subsection 3.4, we work under assumptions (**HD**θp) and (**HI**∞p). The following property has been proved in [9], Lemma 2:

$$\|u\|_{1,1;T} \leq c\, \|u\|^*_{\theta;T},$$

for some constant $c > 0$. As a consequence, (**HD**θp)is stronger than (**HD**#).
We first consider the case of a solution u such that $u \leqslant 0$ on $\partial\mathcal{O}$.

Theorem 17. *Assume that ∂O is Lipschitz, that* (**H**)*, **(HD**θp**)**, **(HI**∞p**)** hold for some $\theta \in [0,1[$, $p \geq 2$, and that the constants of the Lipschitz conditions satisfy $\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda$. Let $u \in \mathcal{U}_{loc}(\xi, f, g, h)$ be such that $u^+ \in \mathcal{H}$. Then one has*

$$E\left\|u^+\right\|^p_{\infty,\infty;t} \leq k(t)\, E\left(\left\|\xi^+\right\|^p_\infty + \left(\left\|f^{0,+}\right\|^*_{\theta;t}\right)^p + \left(\left\|\left|g^0\right|^2\right\|^*_{\theta;t}\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|^*_{\theta;t}\right)^{\frac{p}{2}}\right),$$

where $k(t)$ is constant that depends of the structure constants and $t \geq 0$.

Proof. Set $v = \mathcal{U}\left(\xi^+, \widehat{f}, g, h\right)$ the solution with zero Dirichlet boundary conditions, where the function $\widehat{f}$ is defined by $\widehat{f} = f + f^{0,-}$, with $f^{0,-} = 0 \vee \left(-f^0\right)$. The assumption on the Lipschitz constants ensure the applicability of Theorem 12, which gives the estimate

$$E\left\|v\right\|_{\infty,\infty;t}^p \leq k\left(t\right) E\left(\left\|\xi^+\right\|_\infty^p + \left(\left\|f^{0,+}\right\|_{\theta;t}^*\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;t}^*\right)^{\frac{p}{2}}\right),$$

because $\widehat{f^0} = f^{0,+}$. Then $\left(u-v\right)^+ \in \mathcal{H}$ and we observe that all the conditions of the preceding theorem are satisfied so that we may apply it and deduce that $u \leq v$. This implies $u^+ \leq v^+$ and the above estimate of v leads to the asserted estimate. □

Let us generalize the previous result by considering a real Itô process of the form

$$M_t = m + \int_0^t b_s ds + \sum_{j=1}^{+\infty} \int_0^t \sigma_{j,s} dB_s^j,$$

where m is a real random variable and $b = \left(b_t\right)_{t\geq 0}$, $\sigma = \left(\sigma_{1,t}, \cdots, \sigma_{n,t} \cdots\right)_{t\geq 0}$ are adapted processes.

Theorem 18. *Assume* **(H)**, ***(HDθp)***, ***(HI∞p)*** *for some* $\theta \in [0,1[$, $p \geq 2$, *and that the constants of the Lipschitz conditions satisfy* $\alpha + \frac{\beta^2}{2} + 72\beta^2 < \lambda$. *Assume also that* m *and the processes* b *and* σ *satisfy the following integrability conditions*

$$E\left|m\right|^p < \infty,\ E\left(\int_0^t \left|b_s\right|^{\frac{1}{1-\theta}} ds\right)^{p(1-\theta)} < \infty,\ E\left(\int_0^t \left|\sigma_s\right|^{\frac{2}{1-\theta}} ds\right)^{\frac{p(1-\theta)}{2}} < \infty,$$

for each $t \geq 0$. *Let* $u \in \mathcal{U}_{loc}\left(\xi, f, g, h\right)$ *be such that* $\left(u-M\right)^+$ *belongs to* $\mathcal{H}$. *Then one has*

$$E\left\|\left(u-M\right)^+\right\|_{\infty,\infty;t}^p \leq k\left(t\right) E\Big[\left\|\left(\xi-m\right)^+\right\|_\infty^p + \left(\left\|\left(f(\cdot,\cdot,M,0) - b\right)^+\right\|_{\theta;t}^*\right)^p + \left(\left\|\left|g(\cdot,\cdot,M,0)\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} + \left(\left\|\left|h(\cdot,\cdot,M,0) - \sigma\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}}\Big]$$

where $k\left(t\right)$ *is the constant from the preceding corollary.*

Remark 3. *The right hand side of this estimate is dominated by the following quantity which is expressed directly in terms of the characteristics of the process* M,

$$k\left(t\right) E\Big[\left\|\left(\xi-m\right)^+\right\|_\infty^p + \left|m\right|^p + \left(\left\|f^{0,+}\right\|_{\theta;t}^*\right)^p + \left(\left\|\left|g^0\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} + \left(\left\|\left|h^0\right|^2\right\|_{\theta;T}^*\right)^{\frac{p}{2}} + \left(\int_0^t \left|b_s\right|^{\frac{1}{1-\theta}} ds\right)^{p(1-\theta)} + \left(\int_0^t \left|\sigma_s\right|^{\frac{2}{1-\theta}} ds\right)^{\frac{p(1-\theta)}{2}}\Big].$$

5. Appendix

5.1. Functional spaces

We just recall the main definitions, all the details may be found in [8] and [9].
Let $(p_1, q_1), (p_2, q_2) \in [1, \infty]^2$ be fixed and set

$$I = I(p_1, q_1, p_2, q_2) := \Big\{ (p, q) \in [1, \infty]^2 \,/\, \exists\, \rho \in [0, 1]\, s.t.$$

$$\frac{1}{p} = \rho \frac{1}{p_1} + (1 - \rho) \frac{1}{p_2}, \frac{1}{q} = \rho \frac{1}{q_1} + (1 - \rho) \frac{1}{q_2} \Big\}.$$

This means that the set of inverse pairs $\left(\frac{1}{p}, \frac{1}{q}\right)$, (p, q) belonging to I, is a segment contained in the square $[0, 1]^2$, with the extremities $\left(\frac{1}{p_1}, \frac{1}{q_1}\right)$ and $\left(\frac{1}{p_2}, \frac{1}{q_2}\right)$.
We introduce:

$$L_{I;t} = \bigcap_{(p,q)\in I} L^{p,q}([0, t] \times \mathcal{O}).$$

We know that this space coincides with the intersection of the extreme spaces,

$$L_{I;t} = L^{p_1,q_1}([0, t] \times \mathcal{O}) \cap L^{p_2,q_2}([0, t] \times \mathcal{O})$$

and that it is a Banach space with the following norm

$$\|u\|_{I;t} := \|u\|_{p_1,q_1;t} \vee \|u\|_{p_2,q_2;t}.$$

we also need the algebraic sum

$$L^{I;t} := \sum_{(p,q)\in I} L^{p,q}([0, t] \times \mathcal{O}).$$

It is a normed vector space with the norm

$$\|u\|^{I;t} := \inf \left\{ \sum_{i=1}^{n} \|u_i\|_{r_i,s_i;t} \,/\, u = \sum_{i=1}^{n} u_i, u_i \in L^{r_i,s_i}([0, t] \times \mathcal{O}), (r_i, s_i) \in I, i = 1, ...n; n \in \mathbb{N}^* \right\}.$$

Clearly one has $L^{I;t} \subset L^{1,1}([0, t] \times \mathcal{O})$ and $\|u\|_{1,1;t} \leq c \|u\|^{I;t}$, for each $u \in L^{I;t}$, with a certain constant $c > 0$.
We also remark that if $(p, q) \in I$, then the conjugate pair (p', q'), with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$, belongs to another set, I', of the same type. This set may be described by

$$I' = I'(p_1, q_1, p_2, q_2) := \left\{ (p', q') \,/\, \exists\, (p, q) \in I \; s.t. \; \frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1 \right\}$$

and it is not difficult to check that $I'(p_1, q_1, p_2, q_2) = I(p_1', q_1', p_2', q_2')$, where p_1', q_1', p_2' and q_2' are defined by $\frac{1}{p_1} + \frac{1}{p_1'} = \frac{1}{q_1} + \frac{1}{q_1'} = \frac{1}{p_2} + \frac{1}{p_2'} = \frac{1}{q_2} + \frac{1}{q_2'} = 1$.
Moreover, by Hölder's inequality, it follows that one has

$$\int_0^t \int_{\mathcal{O}} u(s, x)\, v(s, x)\, dxds \leq \|u\|_{I;t} \|v\|^{I';t}, \tag{43}$$

for any $u \in L_{I;t}$ and $v \in L^{I';t}$. This inequality shows that the scalar product of $L^2([0,t] \times \mathcal{O})$ extends to a duality relation for the spaces $L_{I;t}$ and $L^{I';t}$.
Now let us recall that the Sobolev inequality states that

$$\|u\|_{2^*} \leq c_S \|\nabla u\|_2 ,$$

for each $u \in H_0^1(\mathcal{O})$, where $c_S > 0$ is a constant that depends on the dimension and $2^* = \frac{2d}{d-2}$ if $d > 2$, while 2^* may be any number in $]2, \infty[$ if $d = 2$ and $2^* = \infty$ if $d = 1$ (see for example [12], Chapter 5). Therefore one has

$$\|u\|_{2^*,2;t} \leq c_S \|\nabla u\|_{2,2;t} ,$$

for each $t \geq 0$ and each $u \in L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$. And if $u \in L^\infty_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$, one has

$$\|u\|_{2,\infty;t} \vee \|u\|_{2^*,2;t} \leq c_1 \left(\|u\|^2_{2,\infty;t} + \|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}} ,$$

with $c_1 = c_S \vee 1$.
One particular case of interest for us in relation with this inequality is when $p_1 = 2, q_1 = +\infty$ and $p_2 = 2^*, q_2 = 2$. If $I = I(2, \infty, 2^*, 2)$, then the corresponding set of associated conjugate numbers is $I' = I'(2, \infty, 2^*, 2) = I\left(2, 1, \frac{2^*}{2^*-1}, 2\right)$, where for $d = 1$ we make the convention that $\frac{2^*}{2^*-1} = 1$. In this particular case we shall use the notation $L_{\#;t} := L_{I;t}$ and $L^*_{\#;t} := L^{I';t}$ and we recall that we have introduced the following norms

$$\|u\|_{\#;t} := \|u\|_{I;t} = \|u\|_{2,\infty;t} \vee \|u\|_{2^*,2;t} , \ \|u\|^*_{\#;t} := \|u\|^{I';t} .$$

Thus we may write

$$\|u\|_{\#;t} \leq c_1 \left(\|u\|^2_{2,\infty;t} + \|\nabla u\|^2_{2,2;t}\right)^{\frac{1}{2}} , \tag{44}$$

for any $u \in L^\infty_{loc}\left(\mathbb{R}_+; L^2(\mathcal{O})\right) \bigcap L^2_{loc}\left(\mathbb{R}_+; H_0^1(\mathcal{O})\right)$ and $t \geq 0$ and the duality inequality becomes

$$\int_0^t \int_{\mathcal{O}} u(s,x)\, v(s,x)\, dxds \leq \|u\|_{\#;t} \|v\|^*_{\#;t} ,$$

for any $u \in L_{\#;t}$ and $v \in L^*_{\#;t}$.
For $d \geq 3$ and some parameter $\theta \in [0, 1[$ we used the notation

$$\Gamma^*_\theta = \left\{(p,q) \in [1,\infty]^2 \,/\, \frac{d}{2p} + \frac{1}{q} = 1 - \theta\right\},$$

$$L^*_\theta = \sum_{(p,q) \in \Gamma^*_\theta} L^{p,q}([0,t] \times \mathcal{O})$$

$$\|u\|^*_{\theta;t} := \inf\left\{\sum_{i=1}^n \|u_i\|_{p_i,q_i;t} \,/\, u = \sum_{i=1}^n u_i, u_i \in L^{p_i,q_i}([0,t] \times \mathcal{O}),\right.$$

$$\left.(p_i, q_i) \in \Gamma^*_\theta,\ i = 1, ...n;\ n \in \mathbf{N}^*\right\}.$$

If $d = 1, 2$. we put

$$\Gamma_\theta^* = \left\{ (p,q) \in [1,\infty]^2 \,/\, \frac{2^*}{2^*-2}\frac{1}{p} + \frac{1}{q} = 1-\theta \right\}$$

with the convention $\frac{2^*}{2^*-2} = 1$ for $d = 1$.
We want to express these quantities in the new notation introduced in the subsection 5.1 and to compare the norms $\|u\|^*_{\theta;t}$ and $\|u\|^*_{\#;t}$. So, we first remark that $\Gamma_\theta^* = I\left(\infty, \frac{1}{1-\theta}, \frac{d}{2(1-\theta)}, \infty\right)$ and that the norm $\|u\|^*_{\theta;t}$ coincides with $\|u\|^{\Gamma^*_\theta;t} = \|u\|^{I\left(\infty,\frac{1}{1-\theta},\frac{d}{2(1-\theta)},\infty\right);t}$. On the other hand, we recall that the norm $\|u\|^*_{\#;t}$ is associated to the set $I\left(2,1,\frac{2^*}{2^*-1},2\right)$, i.e. $\|u\|^*_{\#;t}$ coincides with $\|u\|^{I\left(2,1,\frac{2^*}{2^*-1},2\right);t}$.

5.2. Proof of Proposition 5

Assume first that $w \in \left(C_c^\infty(\mathbb{R}_+) \otimes H_0^1(\mathcal{O})\right)^d$. In this case, the fact that u is the weak solution of the given equation and satisfies equality (10) i.e.

$$\frac{1}{2}\|u_t\|^2 + \int_0^t \sum_{i,j=1}^d \int_{\mathcal{O}} a_{i,j}(s,x)\partial_i u_s(x)\partial_j u_s(x)\,dx\,ds = -\sum_{i=1}^d \int_0^t (\tilde{w}_{i,s}, \partial_i u_s)\; ds, \quad t \geqslant 0$$

is a consequence of Theorem 8 and Proposition 9 with $w' = \partial\tilde{w}$ and $\xi = w'' = w = 0$.
Then, thanks to the ellipticity assumptions, we get:

$$\begin{aligned} \frac{1}{2}\|u_t\|^2 + \lambda\int_0^t \|\nabla u_s\|^2 ds &\leqslant \int_0^t |\sum_{i=1}^d (w_{i,s}, \partial_i u_s)|\, ds \\ &\leqslant \frac{\lambda}{2}\int_0^t \|\nabla u_s\|^2 ds + \frac{8}{\lambda}\int_0^t |||w_s|||^2\, ds. \end{aligned}$$

from this we clearly get estimate (11).
The general case is obtained by approximating $\tilde{w}$ by a sequence of elements in $\left(C_c^\infty(\mathbb{R}_+) \otimes H_0^1(\mathcal{O})\right)^d$.

5.3. Proof of Theorem 11

We keep the same notations as in Theorem 8.
Let γ and δ be 2 positive constants. On F_T, we introduce the norm

$$\forall u \in F_T,\ \|\ u\ \|_{\gamma,\delta} = E\left(\int_0^T e^{-\gamma t}\left(\delta\ \|\ u_t\ \|^2 + \|\nabla u_t\|^2\right)\ dt\right).$$

It is clear that $\|\cdot\|_{\gamma,\delta}$ is equivalent to $\|\cdot\|_{F_T}$. We consider the map, Λ, from F_T into F_T defined by:

$\forall u \in F_T, \forall (t,x) \in [0,T] \times \mathcal{O}$:

$$\begin{aligned}\Lambda(u)(t,x) = &\int_{\mathcal{O}} G(t,\cdot,0,y)\xi(y)\,dy + \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y) f(s,y,u_s(y),\nabla u_s(y))dyds \\ &+ \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\partial_{i,y} g_i(s,.,u_s,\nabla u_s)(y)dyds \\ &+ \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y) h_i(s,y,u_s(y),\nabla u_s(y))dB_s^i.\end{aligned} \tag{45}$$

Let u and v be in F_T. We put:

$$\forall s \in [0,T],\ \bar{f}_s = f(s,\cdot,u_s,\nabla u_s) - f(s,\cdot,v_s,\nabla v_s),$$

$$\forall s \in [0,T],\ \bar{g}_s = g(s,\cdot,u_s,\nabla u_s) - g(s,\cdot,v_s,\nabla v_s),$$

$$\forall s \in [0,T],\ \bar{h}_s = h(s,\cdot,u_s,\nabla u_s) - h(s,\cdot,v_s,\nabla v_s),$$

and $\forall t \in [0,T]$,

$$\begin{aligned}\bar{u}_t &= \Lambda(u)_t - \Lambda(v)_t \\ &= \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\bar{f}_s(y)\,dy\,ds + \sum_{i=1}^{d} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\partial_{i,y}\bar{g}_{i,s}(y)\,dy\,ds \\ &\quad + \sum_{i=1}^{+\infty} \int_0^t \int_{\mathcal{O}} G(t,\cdot,s,y)\bar{h}_{i,s}(y)\,dy\,dB_s^i.\end{aligned}$$

By Itô's formula (33), we get

$$\begin{aligned}e^{-\gamma T}\|\bar{u}_T\|^2 + 2\int_0^T e^{-\gamma s}\int_{\mathcal{O}} \sum_{i,j} a_{i,j}(s,x)\partial_i\bar{u}_s(x)\partial_j\bar{u}_s(x)\,dxds = -\gamma\int_0^T e^{-\gamma s}\|\bar{u}_s\|^2 ds \\ + 2\int_0^T e^{-\gamma s}\left(\bar{u}_s,\bar{f}_s\right)ds - 2\sum_{i=1}^{d}\int_0^T e^{-\gamma s}\left(\partial_i\bar{u}_s,\bar{g}_{i,s}\right)ds \\ + 2\sum_{i=1}^{+\infty}\int_0^T e^{-\gamma s}\left(\bar{u}_s,\bar{h}_{i,s}\right)dB_s + \int_0^T e^{-\gamma s}\|\bar{h}_s\|^2 ds.\end{aligned}$$

Using hypotheses on f, g, h we have for all $\varepsilon > 0$

$$\begin{aligned}2\int_0^T e^{-\gamma s}(\bar{u}_s,\bar{f}_s)\,ds &\leqslant 1/\varepsilon\int_0^T e^{-\gamma s}\parallel\bar{u}_s\parallel^2\,ds + \varepsilon\int_0^T e^{-\gamma s}\parallel\bar{f}_s\parallel^2\,ds \\ &\leqslant 1/\varepsilon\int_0^T e^{-\gamma s}\parallel\bar{u}_s\parallel^2\,ds + C\varepsilon\int_0^T e^{-\gamma s}\parallel u_s - v_s\parallel^2\,ds \\ &\quad + C\varepsilon\int_0^T e^{-\gamma s}\|\nabla u_s - \nabla v_s\|^2\,ds,\end{aligned}$$

$$
\begin{aligned}
-2\sum_{i=1}^{d}\int_0^T e^{-\gamma s}\left(\partial_i\bar{u}_s,\bar{g}_{i,s}\right)\,ds &\leqslant 2\int_0^T e^{-\gamma s}\|\nabla\bar{u}_s\|(C\|u-v\|+\alpha\|\nabla u_s-\nabla v_s\|)\,ds\\
&\leqslant C\varepsilon\int_0^T e^{-\gamma s}\|\nabla\bar{u}_s\|^2\,ds+\frac{C}{\varepsilon}\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2\,ds\\
&\quad+\alpha\int_0^T e^{-\gamma s}\|\nabla\bar{u}_s\|^2\,ds+\alpha\int_0^T e^{-\gamma s}\|\nabla u_s-\nabla v_s\|^2\,ds,
\end{aligned}
$$

and

$$
\int_0^T e^{-\gamma s}\|\bar{h}_s\|^2\,ds \leqslant C(1+1/\varepsilon)\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2\,ds+\beta^2(1+\varepsilon)\int_0^T e^{-\gamma s}\|\nabla u_s-\nabla v_s\|^2\,ds,
$$

where C , α and β are the constants which appear in the hypotheses of section 2.2.
Using the ellipticity assumption and taking the expectation, we obtain:

$$
\begin{aligned}
&(\gamma-1/\varepsilon)E(\int_0^T e^{-\gamma s}\parallel\bar{u}_s\parallel^2\,ds)+(2\lambda-\alpha)E(\int_0^T e^{-\gamma s}\|\nabla\bar{u}_s\|^2\,ds)\leqslant\\
&\quad C(1+\varepsilon+2/\varepsilon)E(\int_0^T e^{-\gamma s}\parallel u_s-v_s\parallel^2\,ds)+(C\varepsilon+\alpha+\beta^2(1+\varepsilon))E(\int_0^T e^{-\gamma s}\|\nabla u_s-\nabla v_s\|^2\,ds).
\end{aligned}
$$

Now, we choose ε small enough and then γ such that

$$
C\varepsilon+\alpha+\beta^2(1+\varepsilon)<2\lambda-\alpha \text{ and } \frac{\gamma-1/\varepsilon}{2\lambda-\alpha}=\frac{C(1+\varepsilon+2/\varepsilon)}{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}.
$$

If we set $\delta=\frac{\gamma-1/\varepsilon}{2\lambda-\alpha}$, we have the following inequality:

$$
\forall u,v\in F_T^2,\ \parallel\Lambda(u)-\Lambda(v)\parallel_{\gamma,\delta}\ \leqslant\frac{C\varepsilon+\alpha+\beta^2(1+\varepsilon)}{2\lambda-\alpha}\parallel u-v\parallel_{\gamma,\delta}.
$$

We conclude thanks to the fixed point Theorem and estimate (17).

5.4. The truncation sequence $(\phi_n)_n$

We first denote $\rho(x)$ the distance from a point $x\in\mathcal{O}$ to the boundary of $\mathcal{O}$, $\partial\mathcal{O}$.
Following standard construction, for any $n\in\mathbb{N}^*$, we can construct a function $\phi_n\in C_0^\infty(\mathcal{O})$ satisfying the following properties:

1. $0\leqslant\phi_n\leqslant 1$;
2. $\phi_n=1$ on $\{x\in\mathcal{O},\ \rho(x)\geqslant\frac{1}{n}\}$;
3. $\phi_n=0$ on $\{x\in\mathcal{O},\ \rho(x)\leqslant\frac{1}{2n}\}$;
4. $|\partial_x\phi_n|\leqslant 3n$.

Lemma 19. *Assume that $\partial\mathcal{O}$ is Lipschitz. Let $w\in H_0^1(\mathcal{O})$, then $(\phi_n w)_n$ tends to w in $H_0^1(\mathcal{O})$.*
Moreover there exists a constant $C>0$ such that

$$
\forall w\in H_0^1(\mathcal{O}),\ \sup_n\|\phi_n w\|_{H_0^1(\mathcal{O})}\leqslant C\|w\|_{H_0^1(\mathcal{O})}.
$$

Proof. Let us prove the first assertion. Let $w \in H_0^1(\mathcal{O})$.
It is clear that we juste have to prove that $(w\partial_x\phi_n)_n$ tends to 0 in $L^2(\mathcal{O})$.
But, we know that $\dfrac{w}{\rho}$ belongs to $L^2(\mathcal{O})$ (see Theorem 1.4.4.4 p.29 in [14]). So, we have

$$\begin{aligned}
\lim_{n\to+\infty}\int_{\mathcal{O}} |w(x)\partial\phi_n(x)|^2\,dx &= \lim_{n\to+\infty}\int_{\{\rho(x)\leqslant\frac{1}{n}\}} |w(x)\partial_x\phi_n(x)|^2\,dx \\
&\leqslant \lim_{n\to+\infty} 9n^2\int_{\{\rho(x)\leqslant\frac{1}{n}\}} |w(x)|^2\,dx \\
&\leqslant \lim_{n\to+\infty} 9\int_{\{\rho(x)\leqslant\frac{1}{n}\}} \left|\frac{w(x)}{\rho(x)}\right|^2 dx \\
&= 0,
\end{aligned}$$

which proves the first part of the Lemma.
The second assertion is a consequence of the Banach-Steinhaus Theorem.

□

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